# An introduction to the eXtended Finite Element Method (X-FEM)

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  - Then Liège...
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# Extended Finite Elements

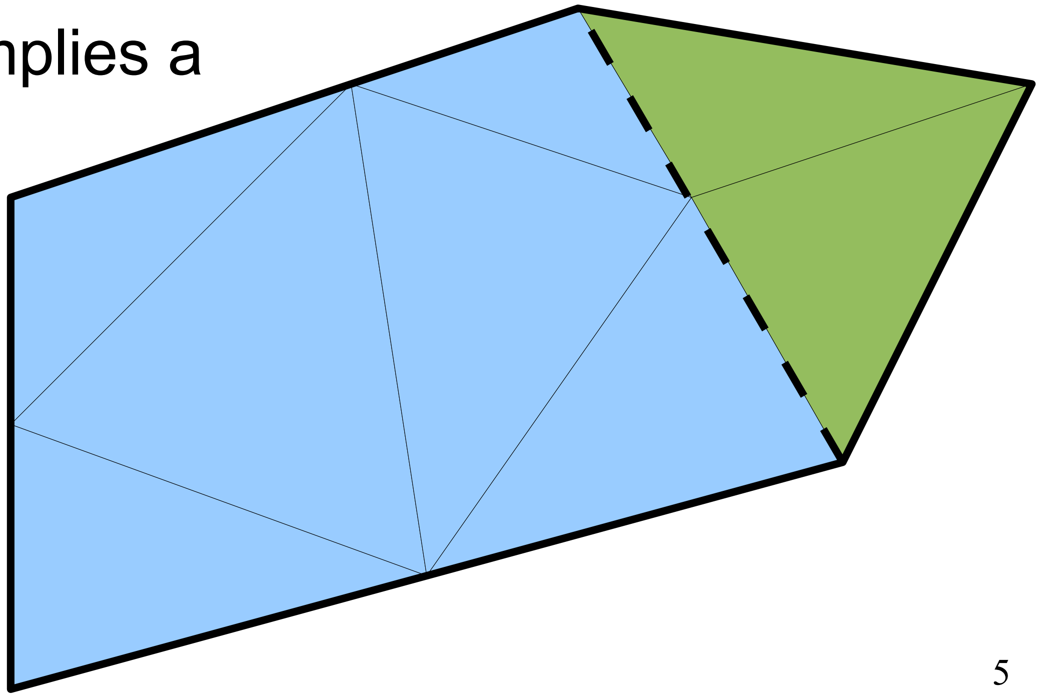
## Lecture plan

- Introduction
- Reminder
- Simple problems (jump on the primal variable)
- Extensions in 2D / 3D
- Other types of problems (jump on the derivatives)
- Other applications and current research
- Boundary conditions
- References

Course Notes available at :

https://www.cgeo.uliege.be/X-FEM

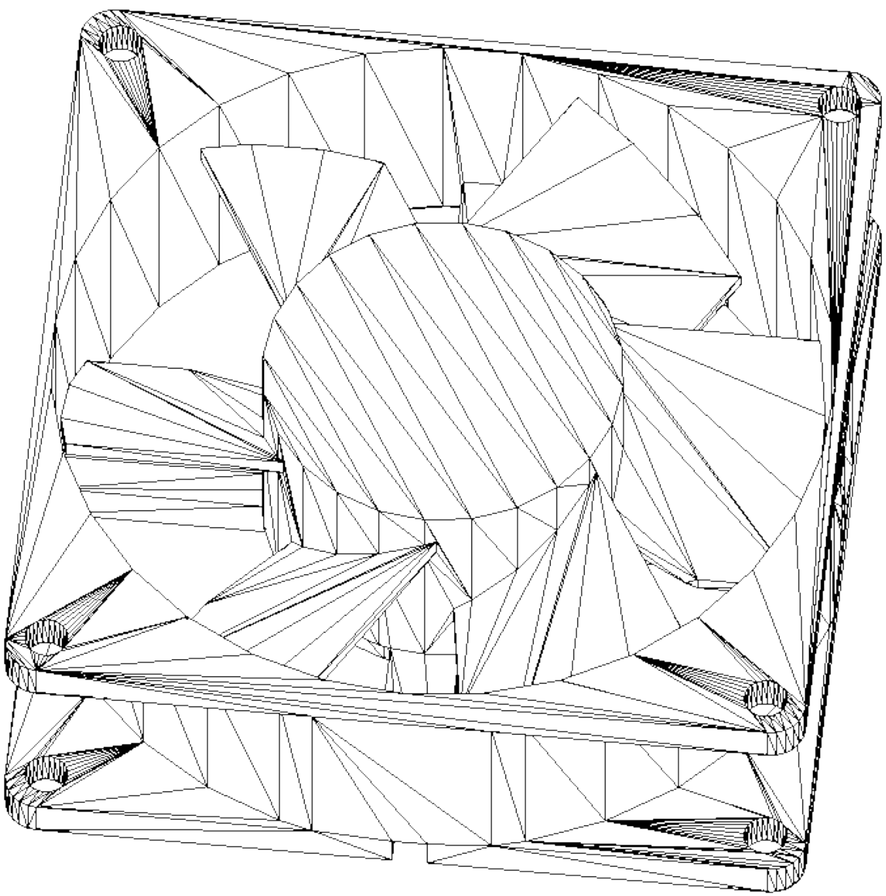
- "Classical" finite element computation
  - The geometry is bounded by element sides
    - Bounds the computation domain
    - Bounds the interface between zones of dissimilar properties
    - A change in geometry implies a change in the mesh
    - Time evolving problems may induce remeshing at each time step in the computation

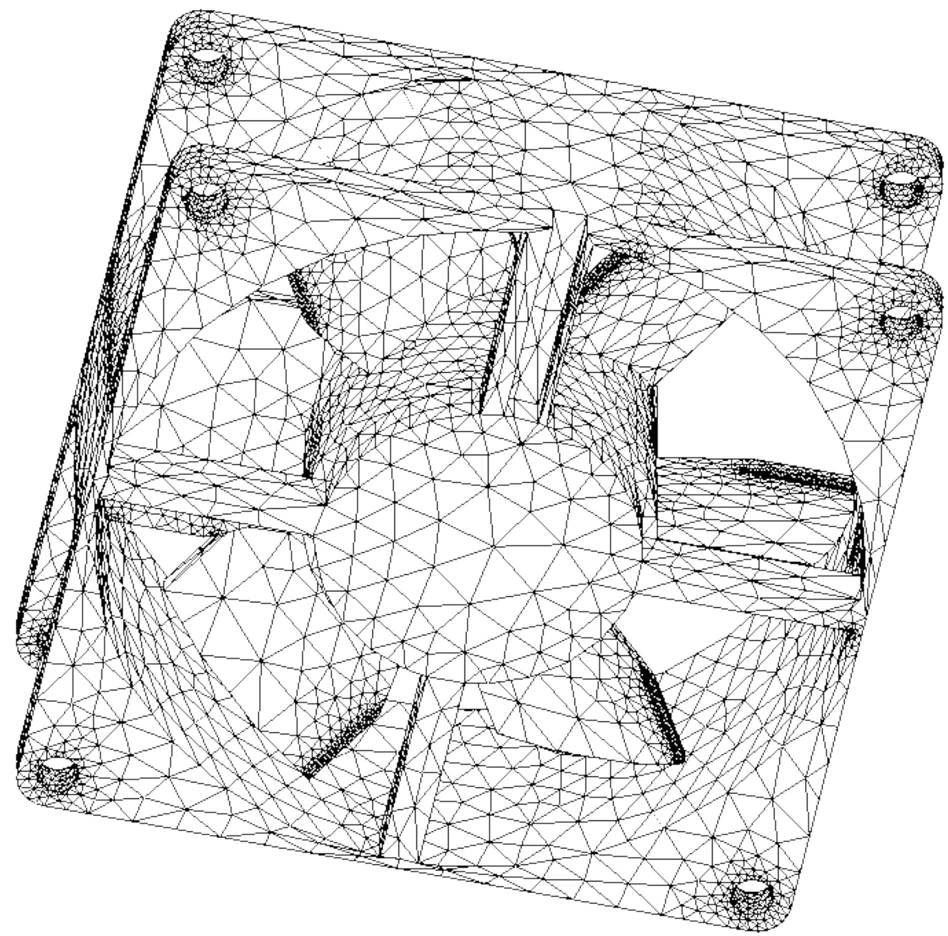


# Introduction

- Mesh generation techniques
  - May be costlier than the sole finite element computation
  - (Often) necessitates a strong human interaction
  - Are a potential source of mistakes
    - Of human origin
    - Or from the lack of robustness of remeshing algorithms

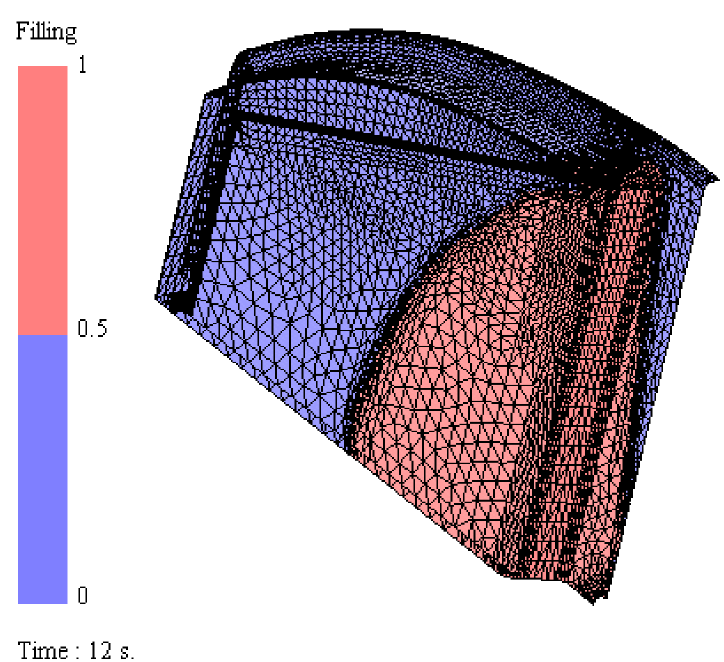
# Introduction





- The idea here:
  - Minimize the constraints on the mesh that is used in the FEM simulations
  - However, mesh generations is still necessary
    - e.g. the accuracy of the computation depents on the quality of the mesh

      → mesh adaptation



# Extended Finite Elements

## Reminder

The method relies on the classical FEM; starting with the weak form of a physical problem :

$$\text{Find } u \in H^1(\Omega) \text{ such that}$$

$$\int_\Omega a(u, v)\, d\Omega = \int_\Omega b(v)\, d\Omega \quad \forall v \in H_0^1(\Omega)$$

Discretization: One look for $u$ in a discrete function space $V_h \subset H^1(\Omega)$ (trial functions $v$ belong to the same space $V_{0h} \subset H_0^1(\Omega)$ )

$$u_h(x) = \sum_i \lambda_i N_i(x) \quad , \; x \in \Omega$$

# Extended Finite Elements

## Reminder

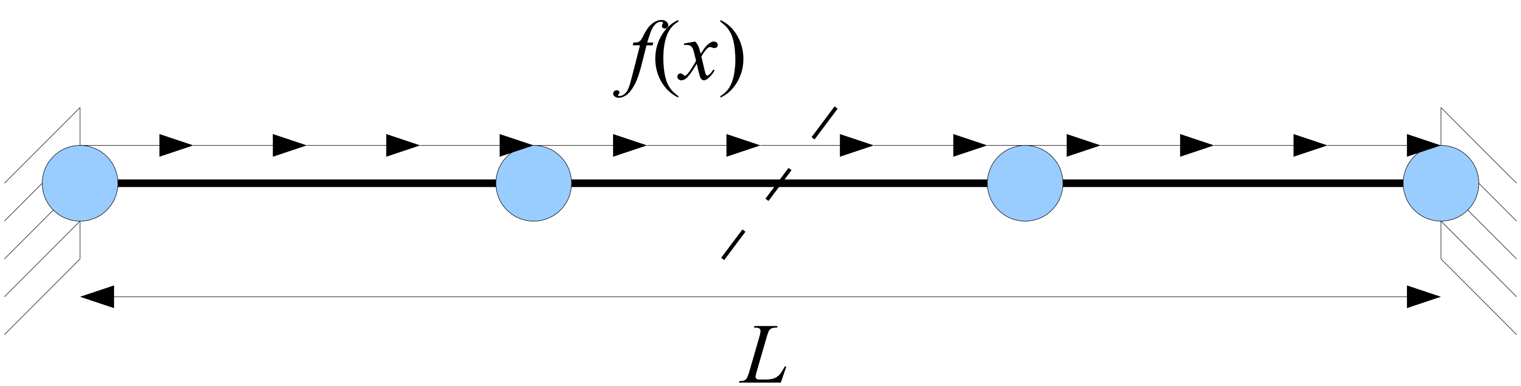
- A space-conforming mesh is used to define the shape functions SFs $u(x)=\sum_k \lambda_k N_k \text{ for } x\in T_j$
- They have a compact support
- Partition of unity $\sum_i N_i=1$
- Interpolation $u(x_i)=\lambda_i$

# Reminder

- SFs with a compact support
  - Allows to have banded or hollow matrices (low memory imprint)
- Partition of unity
  - One is able to represent a constant field !
- Interpolation
  - Easy to impose Dirichlet boundary conditions
- Use of conforming meshes
  - Pre-computations of many operators is possible at an elementary level

## Simple problem

- Clamped 1D rod (L, E, S) with a variable load $f(x)$
- One wants to get the displacement $u(x)$ and assume that the rod is cut at some place
  - With the classical FEM
  - With the eXtended Finite Element Method



# Simple problem

- Weak form, with homog. boundary conditions

$$\text{find } u \in H_0^1(\Omega) \text{ such that}$$
$$a(u,v) = b(v)\ \forall v \in H_0^1(\Omega)$$

with

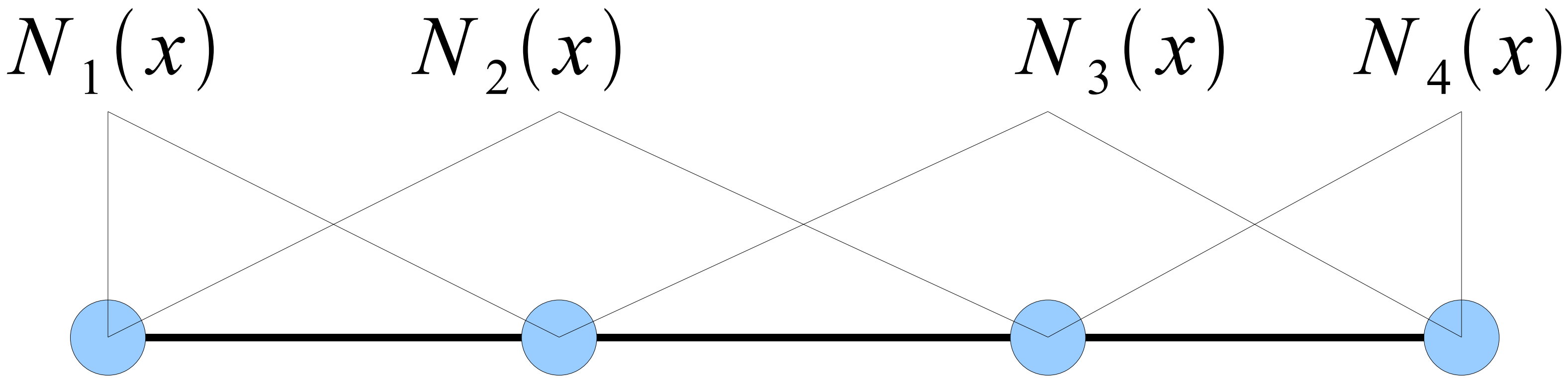
Elementary (stiffness) matrix

$$a(u,v) = \int_0^L ES\left(\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x}\right) dx \qquad b(v) = \int_0^L f(v) dx$$

Elementary vector (loads)

## Simple problem

- Discretization : Linear elements, nodal shape functions.




$$u_h(x) = \sum_i \lambda_i N_i(x)$$

## Simple problem

- By reporting the discrete form of $u$ and $v$ in the weak form, one gets the following linear system :

$$\begin{bmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{bmatrix} \cdot \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} f_2 \\ f_3 \end{pmatrix}$$

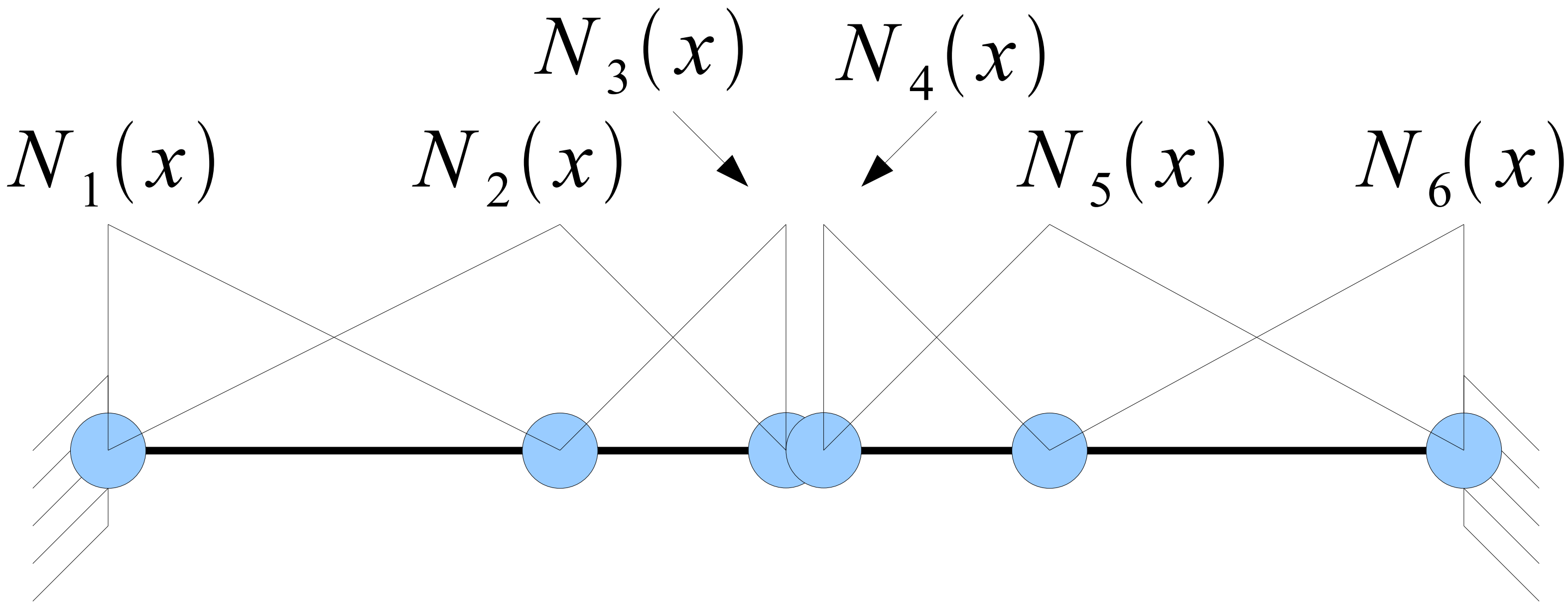
$$k_{ij} = \int_0^L ES \frac{\partial N_i}{\partial x} \cdot \frac{\partial N_j}{\partial x} dx$$

$$f_i = \int_0^L N_i \cdot f(x) dx$$

- Here, coefficients $\lambda_1$ and $\lambda_4$ vanish (clamped extremities)

## Cut the rod : FEM case

- Add two nodes and do the same
  - This is “remeshing”, it is simple, fast and robust in 1D, less 2D and much less in 3D



# Cut the rod : FEM case

- After discretizing, one gets :

$$\begin{bmatrix} k_{22} & k_{23} & 0 & 0 \\ k_{32} & k_{33} & 0 & 0 \\ 0 & 0 & k_{44} & k_{45} \\ 0 & 0 & k_{54} & k_{55} \end{bmatrix} \cdot \begin{pmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix}$$

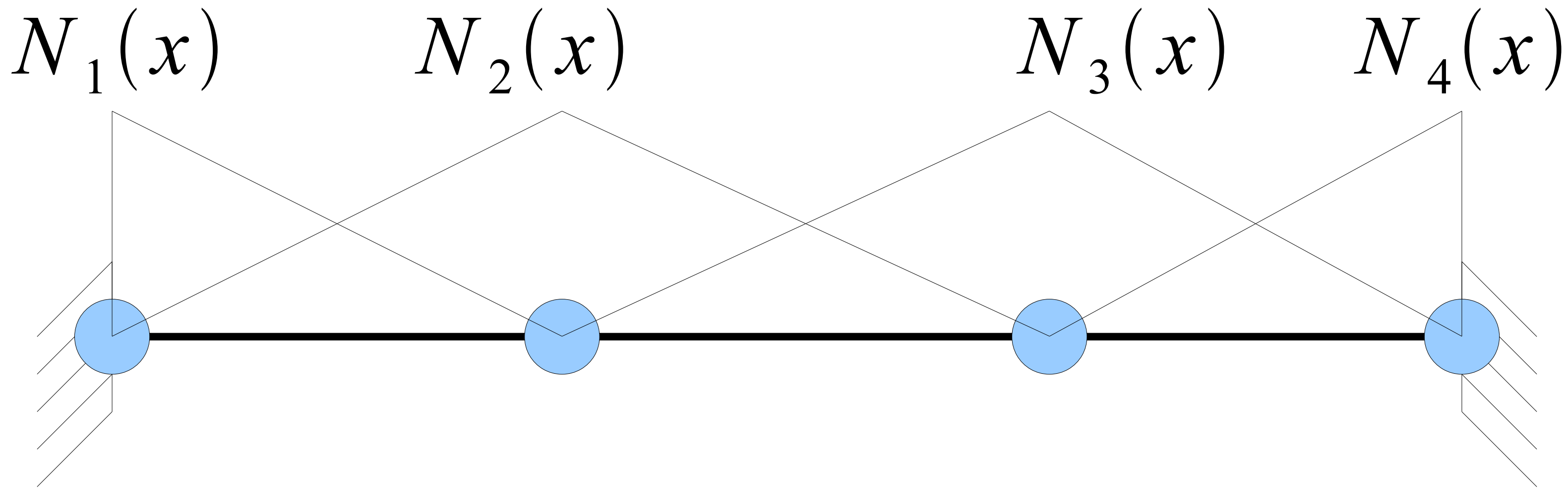
  - The two circled parts are independent
  - One could solve the linear system separately for each sub-problem

## Cut the rod : FEM case

- The meaning of the DoFs is kept
  ( $\lambda_i$ means the displacement of node $i$.)
- There is indeed a discontinuity in the displacement at nodes 3 and 4
- Nothing changes in the implementation – only the mesh and its topology are modified

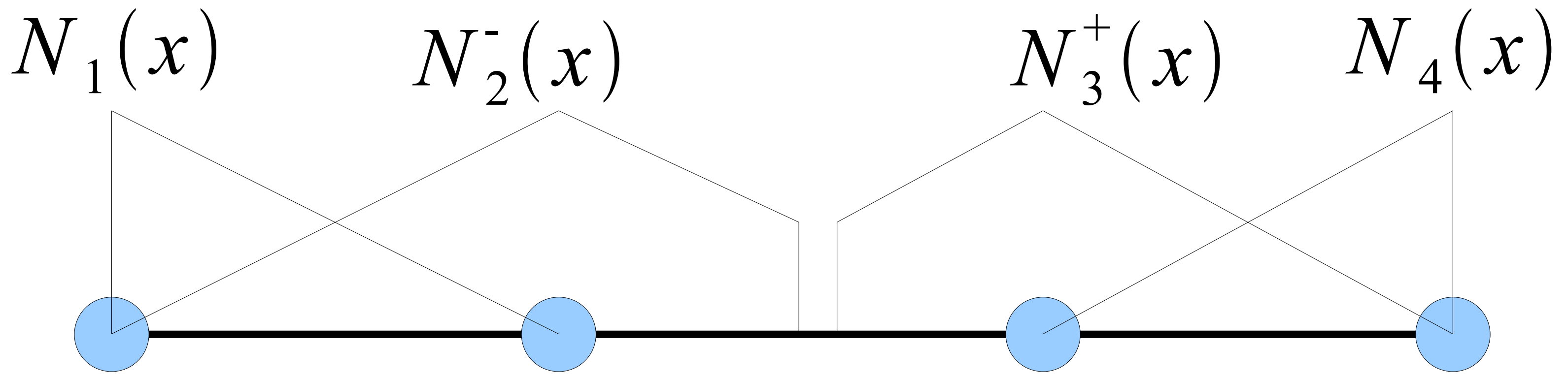
# Cut the rod : X-FEM case

- Now : we don’t change the mesh !
- But one can add/modify shape functions



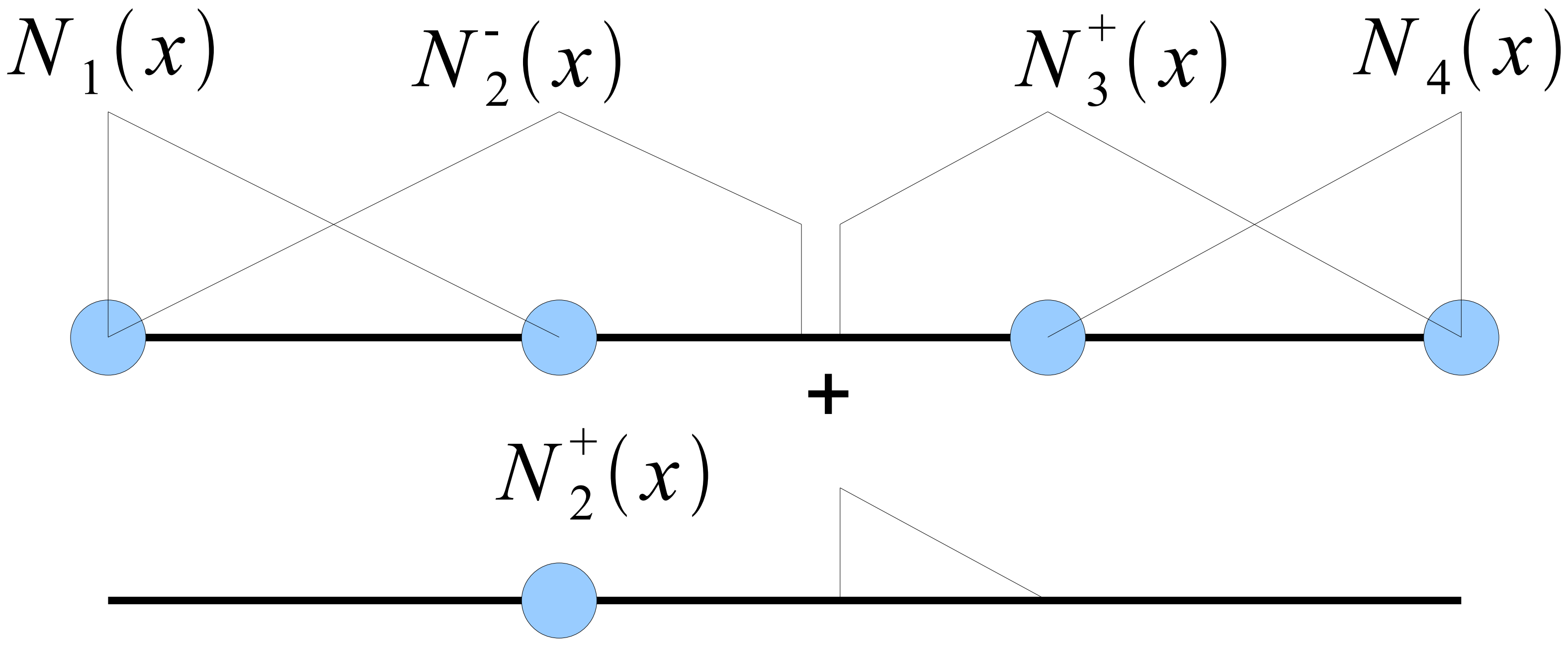
## Cut the rod : X-FEM case (I)

- Case (I) :



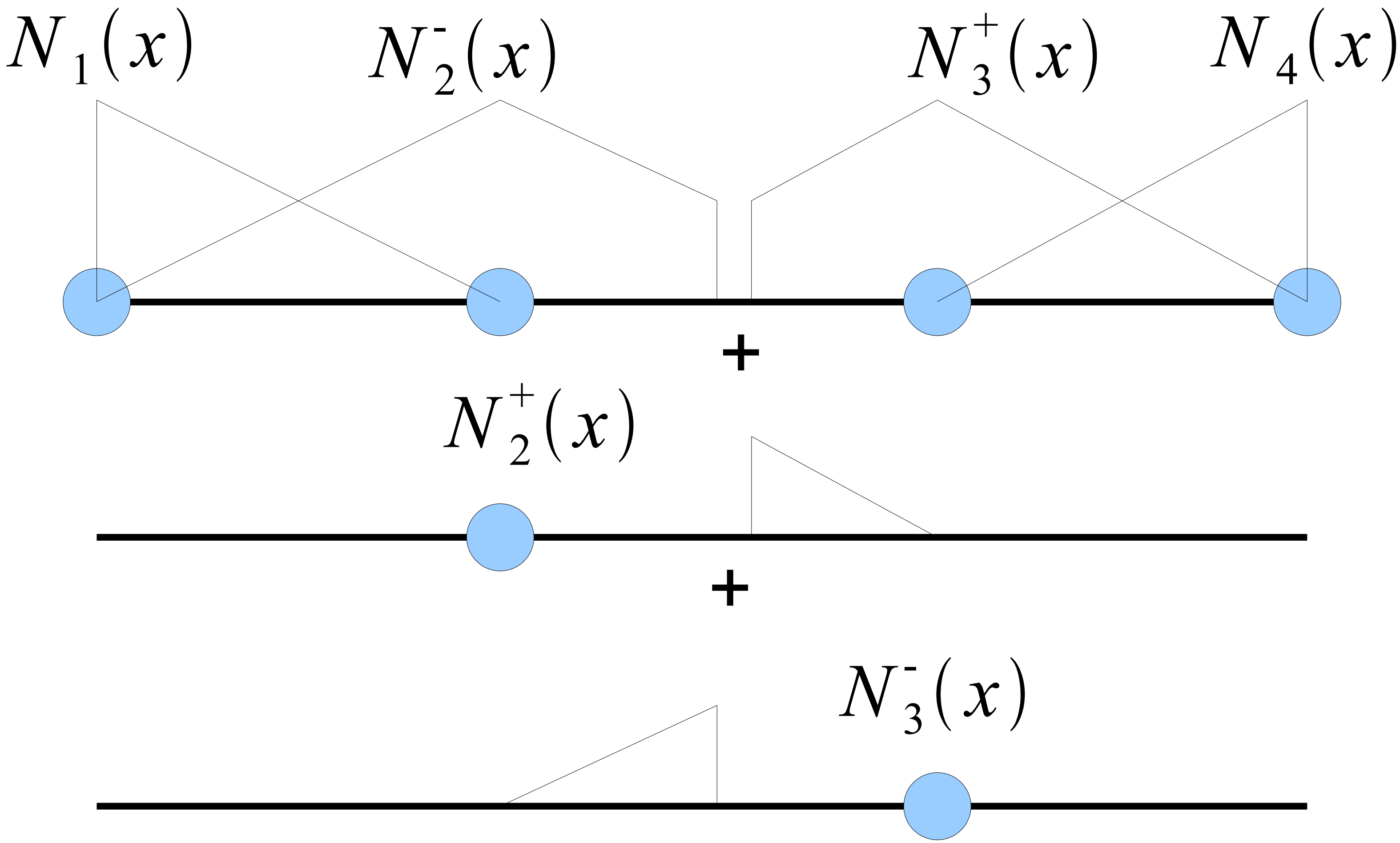
## Cut the rod : X-FEM case (I)

- Case (I) :



## Cut the rod : X-FEM case (I)

- Case (I) :



## Cut the rod : X-FEM case (I)

- How to compute the $N_j^{+,-}$ from the $N_i$ ?
  - Let's introduce the Heaviside function :

$$H(s)=\begin{cases} 0 & \text{if } s\leq 0 \\ 1 & \text{if } s>0 \end{cases}$$

  - This is its complement :

$$\bar{H}(s)=\begin{cases} 1 & \text{if } s\leq 0 \\ 0 & \text{if } s>0 \end{cases}$$

  - $s$ is the distance to the cut (here, $s=x-\dfrac{L}{2}$)

## Cut the rod : X-FEM case (I)

- With these notations, one have :

$$\begin{cases} N_i^+(x) = N_i(x) \cdot H(s) \\ N_i^-(x) = N_i(x) \cdot \bar{H}(s) \end{cases}$$

- One may notice that the partition of unity is preserved

# Extended Finite Elements

## Cut the rod : X-FEM case (I)

- One has to sort the mesh nodes
  - Those which have “regular” degrees of freedom go into set $\mathrm{N}$
  - Those which have modified degrees of freedom go into set $\mathrm{C}$
- The solution field $u$ is written as :

$$u(x)=\sum_{i\in \mathrm{N}} \lambda_i N_i(x)+\sum_{j\in \mathrm{C}} \lambda_j^+ N_j^+(x)+\sum_{k\in \mathrm{C}} \lambda_k^- N_k^-(x)$$

# Cut the rod : X-FEM case (I)

- Linear system
  - We number the DoFs as follows :

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \lambda_1 & \lambda_2^- & \lambda_3^- & \lambda_2^+ & \lambda_3^+ & \lambda_4 \end{bmatrix}$$

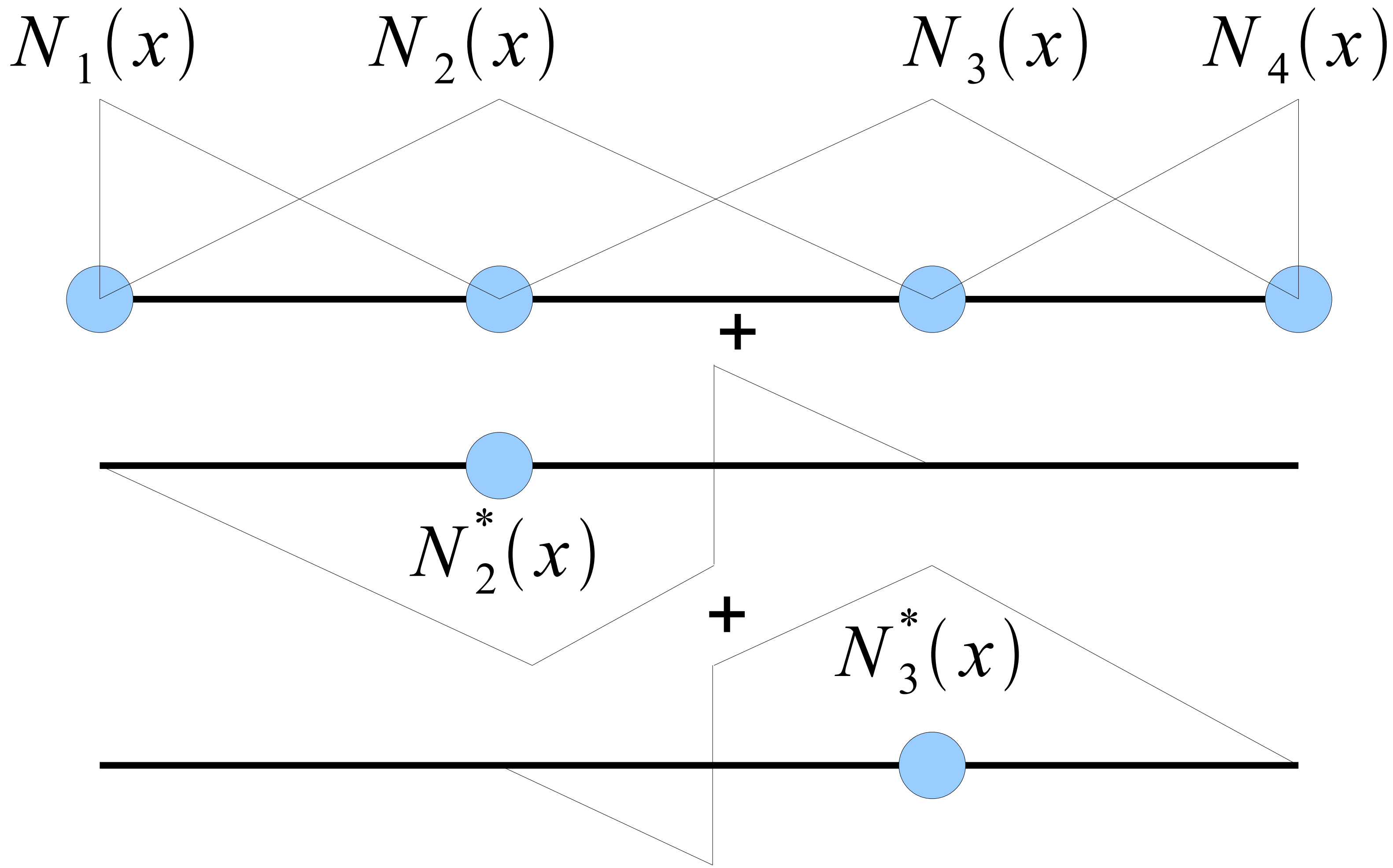
$$\begin{bmatrix} k_{22}^- & k_{23}^- & 0 & 0 \\ k_{32}^- & k_{33}^- & 0 & 0 \\ 0 & 0 & k_{22}^+ & k_{23}^+ \\ 0 & 0 & k_{32}^+ & k_{33}^+ \end{bmatrix} \cdot \begin{pmatrix} \lambda_2^- \\ \lambda_3^- \\ \lambda_2^+ \\ \lambda_3^+ \end{pmatrix} = \begin{pmatrix} f_2^- \\ f_3^- \\ f_2^+ \\ f_3^+ \end{pmatrix}$$

## Cut the rod : X-FEM case (I)

- Again, we manage to separate the domain in two parts
- The signification of the degrees of freedom is partly lost
- Some shape functions have to be modified
- Two “Heaviside” functions are needed to modify the shape functions

# Cut the rod : X-FEM case (II)

- Without changing the shape functions ! (case II)



## Cut the rod : X-FEM case (II)

- How to compute the $N_j^*$ from the $N_i$ ?
  - Lets introduce the modified Heaviside function :

$$H^*(s) = 2H(s) - 1 = \begin{cases} -1 & \text{if} \quad s \leq 0 \\ 1 & \text{if} \quad s > 0 \end{cases}$$

- With this notation, one finds that :

$$N_i^*(x) = N_i(x) \cdot H^*(s)$$

## Cut the rod : X-FEM case (II)

- One should again sort the mesh nodes
  - Those which have modified DoFs go into set $\mathrm{C}$
  - “regular” shape functions are still everywhere (no change with regular FEM in that case)
- The solution field $u$ is written as :

$$u(x)=\sum_{i\in\Omega}\lambda_i N_i(x)+\sum_{j\in \mathrm{C}}\lambda_j^* N_j^*(x)$$

## Cut the rod : X-FEM case (II)

- Linear system
  - We number the DoFs as follows :

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \lambda_1 & \lambda_2 & \lambda_2^* & \lambda_3 & \lambda_3^* & \lambda_4 \end{bmatrix}$$

$$\begin{bmatrix} k_{22} & k_{22^*} & k_{23} & k_{23^*} \\ k_{2^*2} & k_{2^*2^*} & k_{2^*3} & k_{2^*3^*} \\ k_{32} & k_{32^*} & k_{33} & k_{33^*} \\ k_{3^*2} & k_{3^*2^*} & k_{3^*3} & k_{3^*3^*} \end{bmatrix} \cdot \begin{pmatrix} \lambda_2 \\ \lambda_2^* \\ \lambda_3 \\ \lambda_3^* \end{pmatrix} = \begin{pmatrix} f_2 \\ f_2^* \\ f_3 \\ f_3^* \end{pmatrix}$$

## Cut the rod: X-FEM case (II)

- At the matrix level, the two parts are linked
- Are there two physically separated parts ?
  - Lets assemble the matrix without taking care of the boundary conditions, and then determine the number of vanishing (singular) eigenvalues of this matrix.
    - If there is only one entity, there will be only one singular eigenvalue (corresponding to the missing Dirichlet BC to get a non singular system)
    - Two singular values → the rod is indeed cut in two, and two Dirichlet boundary conditions are needed.

# Cut the rod : X-FEM case (II)

- Case without cut and without BC : typical matrix

$$K^s = k \cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\det\left(K^s - \alpha\, \mathrm{I}\right) = 0$$

$$k = \frac{3ES}{L}$$

```
octave:27> K
K =

   1  -1   0   0
  -1   2  -1   0
   0  -1   2  -1
   0   0  -1   1

octave:28> E=eig(K)
E =

  -2.67429966923143e-17
   5.85786437626905e-01
   2.00000000000000e+00
   3.41421356237310e+00

octave:29>
```

One eigenvalue vanishes.

# Cut the rod : X-FEM case (II)

- Case with a cut and without BC : typical matrix

$$K^c = k \cdot \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 & 0 \\ 1 & -1 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$$

$$\det\left(K^c - \alpha \mathrm{I}\right) = 0$$

```
octave:17> K
K =

   1  -1   1   0   0   0
  -1   2  -1  -1   0   0
   1  -1   2   0  -1   0
   0  -1   0   2   1  -1
   0   0  -1   1   2  -1
   0   0   0  -1  -1   1

octave:18> E=eig(K)
E =

  -1.30983391108489e-16
  -4.76470136000792e-17
   1.00000000000000e+00
   1.00000000000000e+00
   4.00000000000000e+00
   4.00000000000000e+00

octave:19>
```

Two eigenvalues vanished : it is OK !

# Cut the rod : X-FEM case (II)

- The meaning of the degrees of freedom is lost
- One keeps classical FE basis functions and add others by enrichment
  - A kind of hierachical FE basis is built
- Only one enrichment function (simpler !)

## Cut the rod : X-FEM case

- Cases (I) and (II) are equivalent (the results are exactly identical)

  We indeed have a linear combination between shape functions of (I) and those of (II) :

$$N_2(x) = N_2^+(x) + N_2^-(x) \qquad N_3(x) = N_3^+(x) + N_3^-(x)$$

$$N_2^*(x) = N_2^+(x) - N_2^-(x) \qquad N_3^*(x) = N_3^+(x) - N_3^-(x)$$

- The case (II) is part of the more theoretical frame – use of a given enrichment function and “constructive” synthesis.

## Definition

- eXtended Finite Element Method
  - It is based on classical FEM basis functions
  - The product between these functions and a given enrichment function $E_k(x)$ is then added
  - These enriched functions are able to represent a specific behavior of the solution field that classical shape functions are unable to represent efficiently. (e.g. a discontinuity)

$$u(x) = \sum_{i \in \Omega} \lambda_i N_i(x) + \sum_k \sum_{j \in C} \lambda^*_{jk} N_j(x) \cdot E_k(x)$$

## Lecture plan

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- Extensions in 2D / 3D
- Other types of problems (jump on the derivatives)
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- Boundary conditions
- References

- Case of linear elasticity
- Representation of cracks
- Level-sets
- Crack propagation

# 2D Example

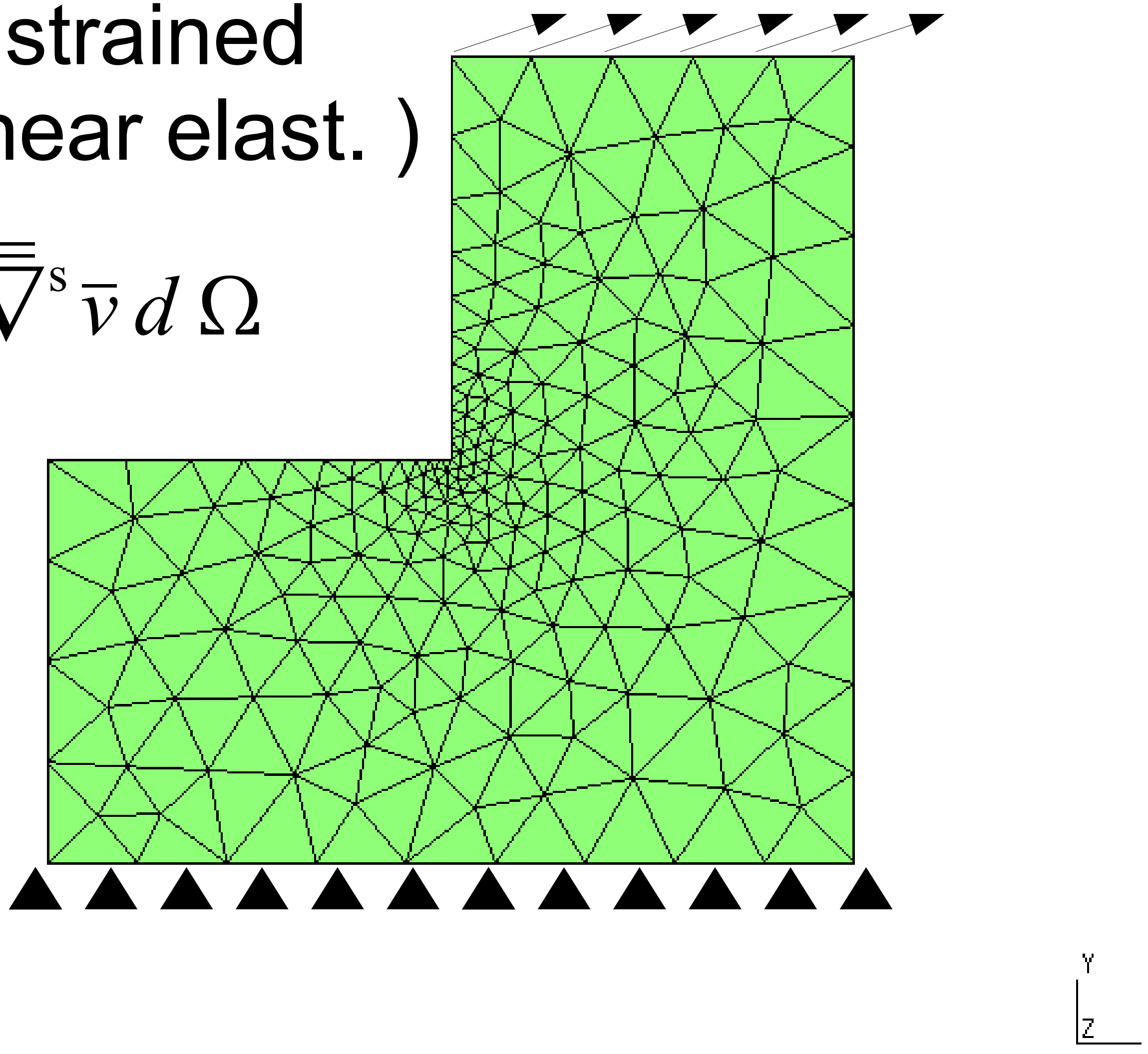
- A wedge with constrained displacements (linear elast. )

$$a(\bar{u},\bar{v})=\int_{\Omega} \bar{\bar{\nabla}}^{s}\bar{u}:\bar{\bar{\bar{\bar{D}}}}:\bar{\bar{\nabla}}^{s}\bar{v}\,d\Omega$$

$$b(\bar{v})=\int_{\Omega}\bar{f}\cdot\bar{v}\,d\Omega$$

find $\bar{u}$ such that

$$a(\bar{u},\bar{v})=b(\bar{v})\quad\forall\bar{v}$$

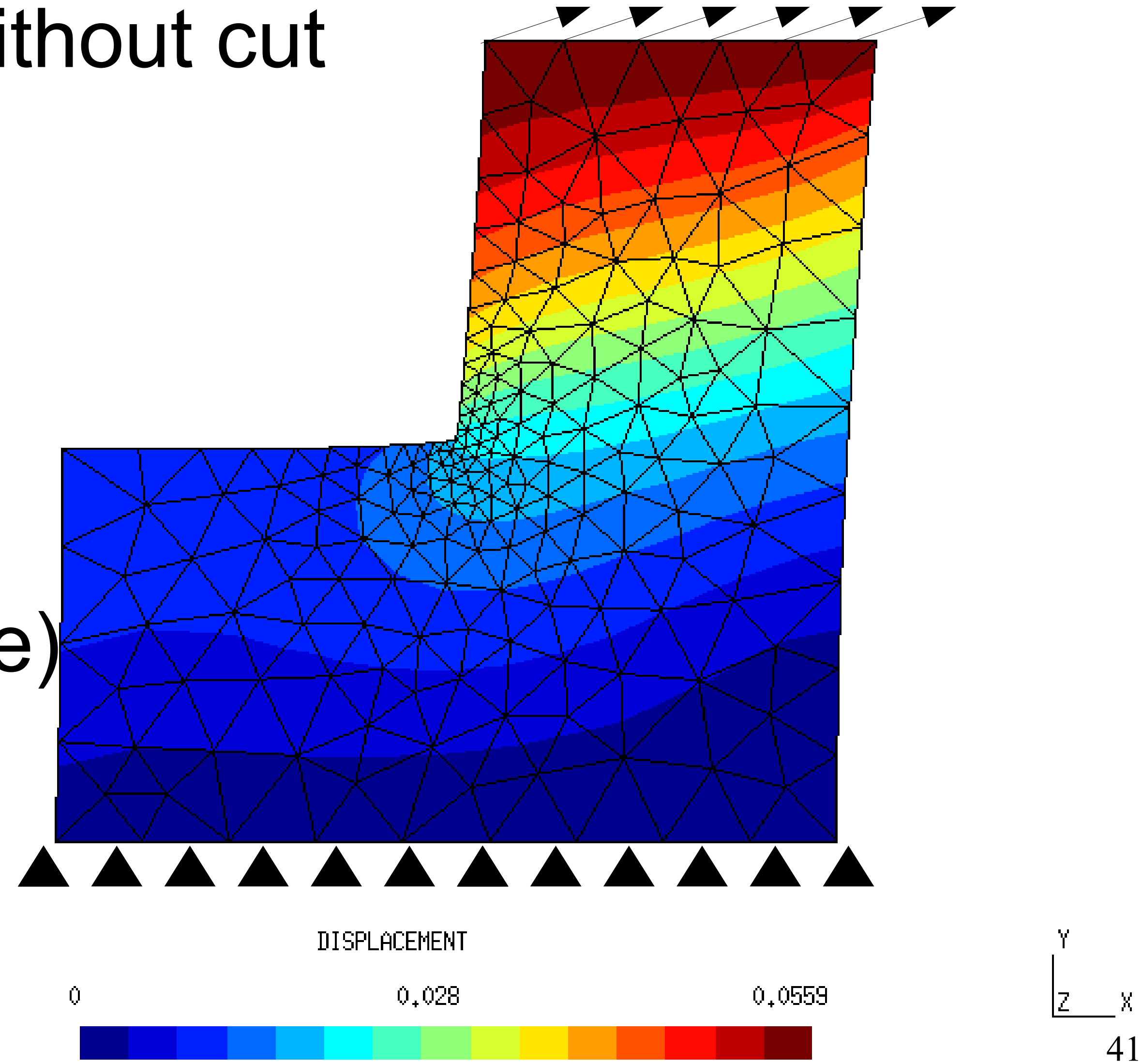


# 2D Example

- Displacements without cut (standard FEM)

$$\bar{u}(x) = \sum \lambda_i \cdot \bar{N}_i(x)$$

The $\bar{N}_i(x)$ are the classical linear shape functions (order 1 Lagrange)



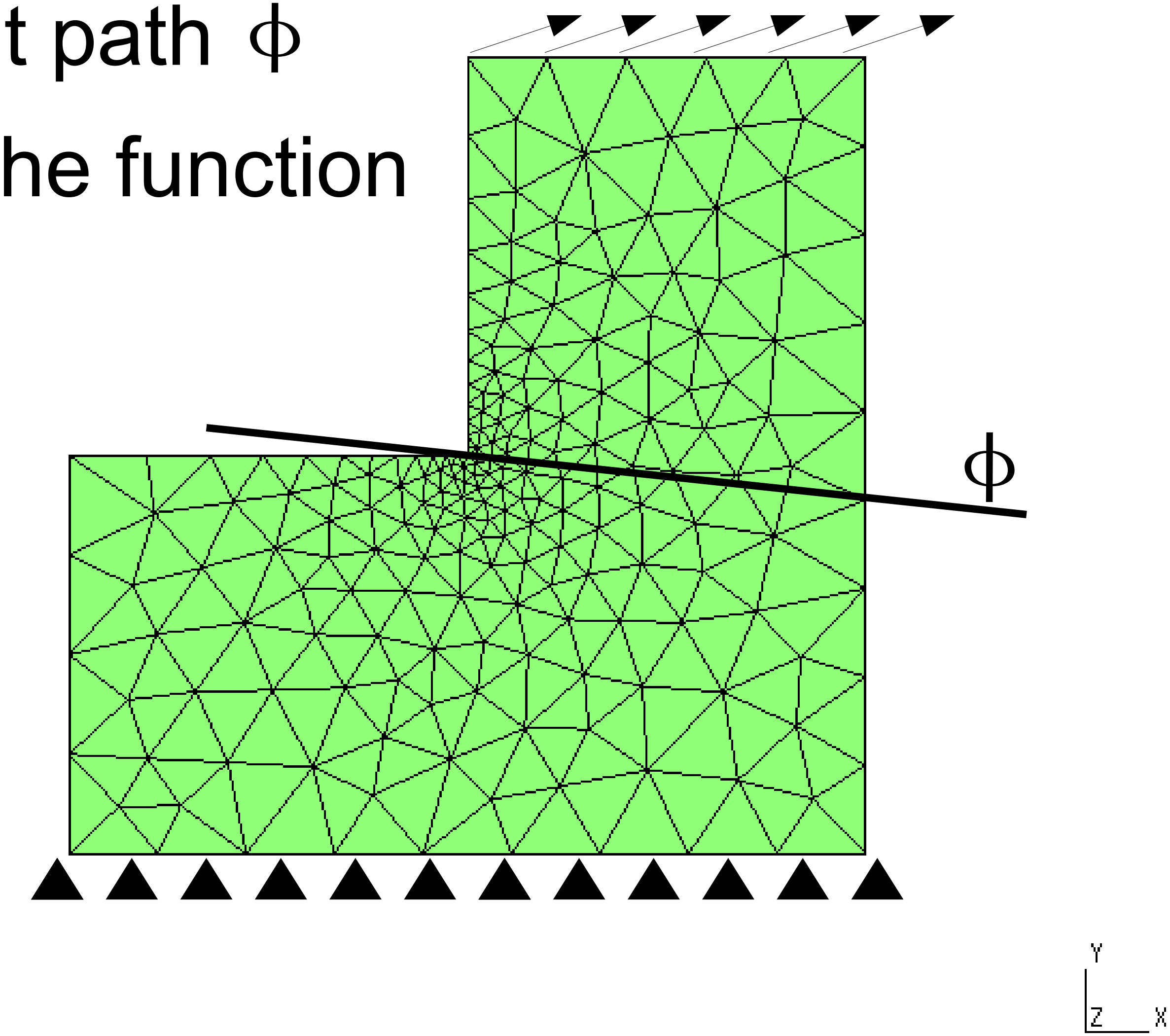
# Extended Finite Elements

## 2D Example

- Lets impose a cut path $\phi$
- Modifications of the function space :

$$\bar{u}(x)=\sum_{i\in\Omega}\lambda_i\cdot\bar{N}_i(x)+\sum_{i\in C}\lambda_i^*\cdot\bar{N}_i(x)\cdot H^*(s)$$

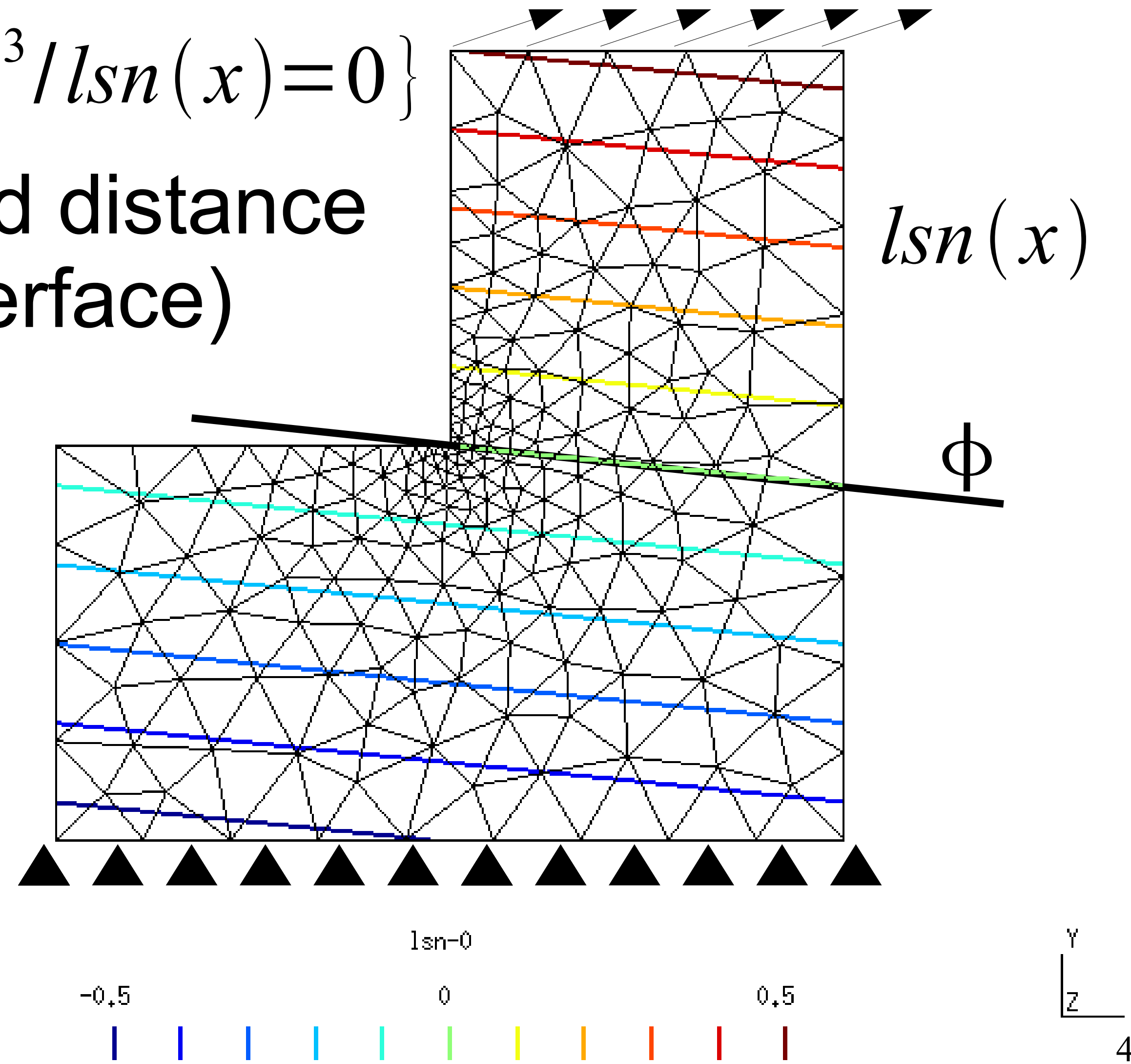
- How to define $H^*(s)$ and the set $C$ ?



# Extended Finite Elements

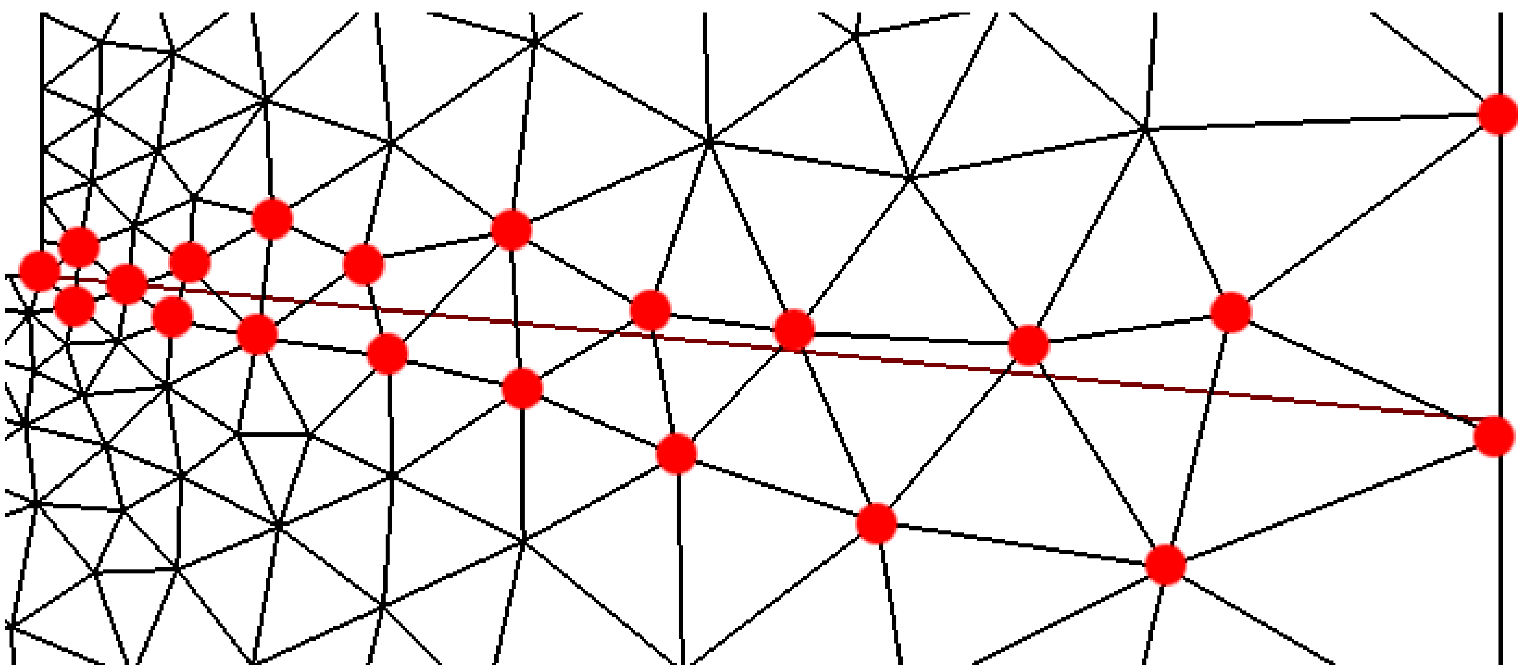
## 2D Example

- The cutting path may be defined with a “level-set” $\phi$
  We have $\phi = \{ x \in \mathbb{R}^3 / lsn(x) = 0 \}$
- $lsn(x)$ is the signed distance function (to the interface)
- One simply takes :
  $s = lsn(x)$
  $H^*(s) = H^*(lsn(x))$



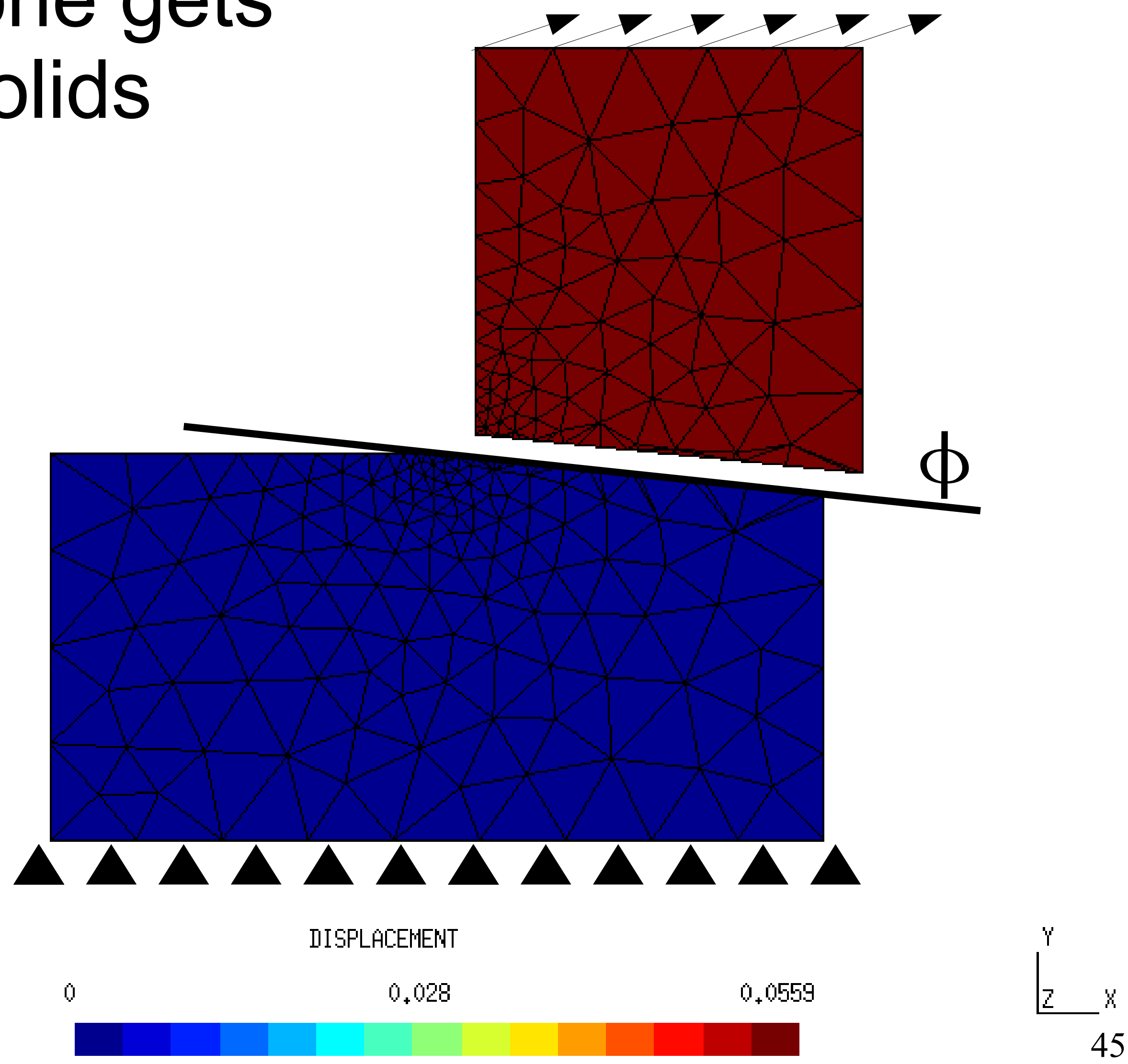
## 2D Example

- Definition of the enriched degrees of freedom (the set $C$)
  - Those are the nodes of the elements completely cut by $\phi$ (iso-0 of the level-set )



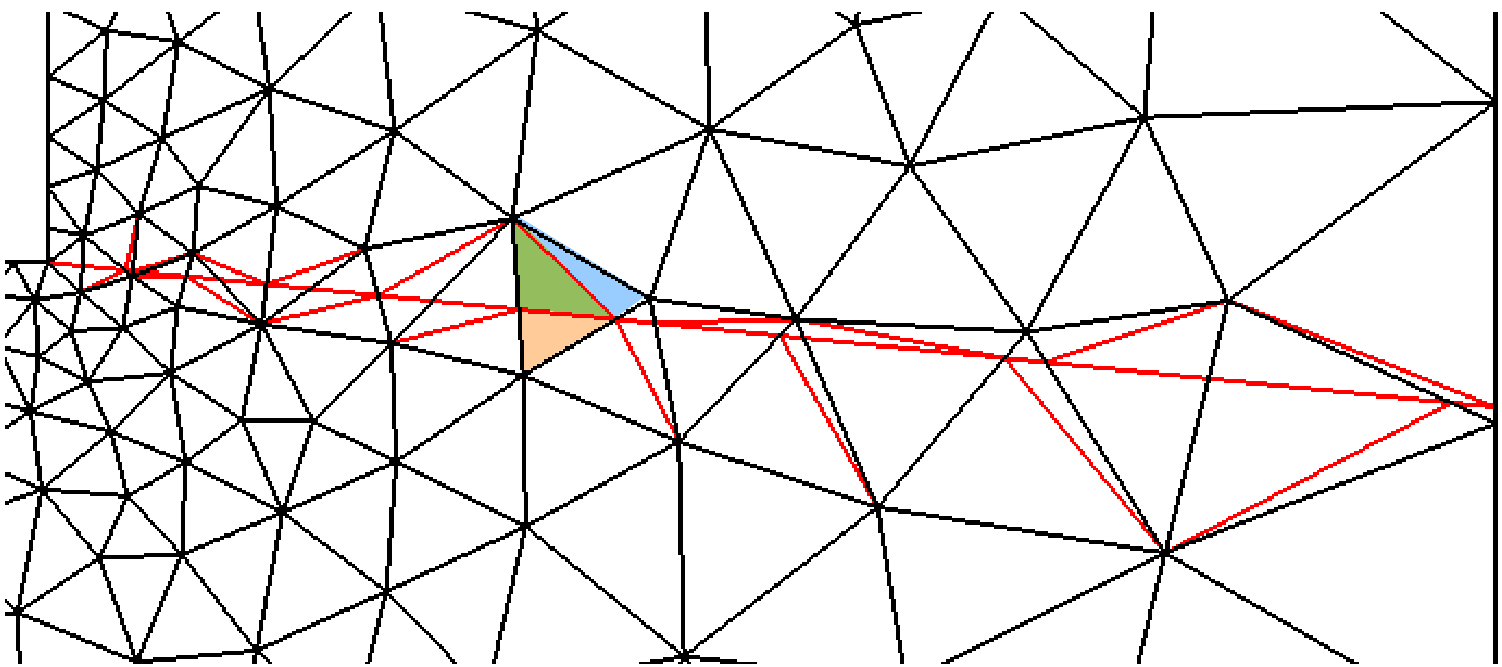
## 2D Example

- After assembly and solving the linear system one gets two independent solids (as expected)
- The geometry of ϕ may be arbitrary.
- No need of any remeshing



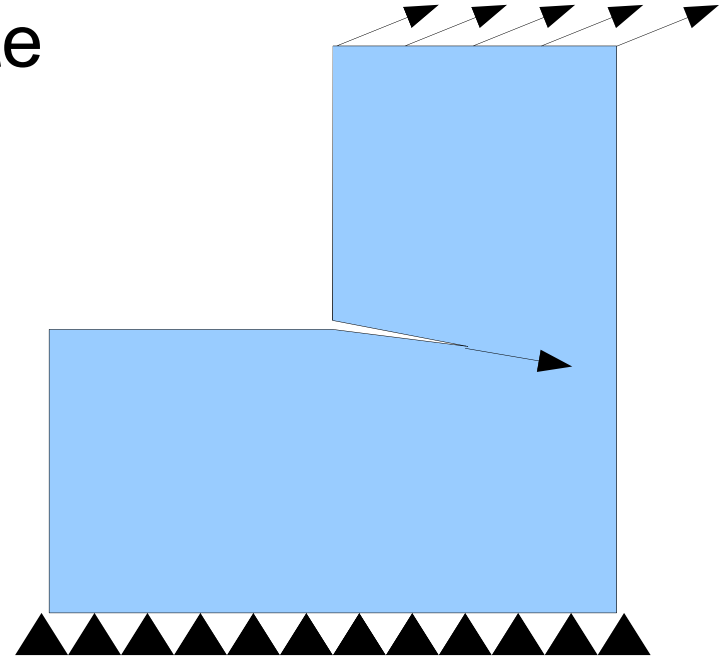
## Integration issues

- Integration
  - One need to subdivide elements that are cut by the interface (discontinuous functions to integrate)
    - On each sub triangle (in red here), a classical Gaussian quadrature is used because the integrand is a polynomial.



# Cracks

- Crack modelling
  - Historically, this is the first application of the extended finite element method
  - The crack propagates, and one does not want to generate a new mesh at each time step
  - A crack is in fact an incomplete cutting in the domain

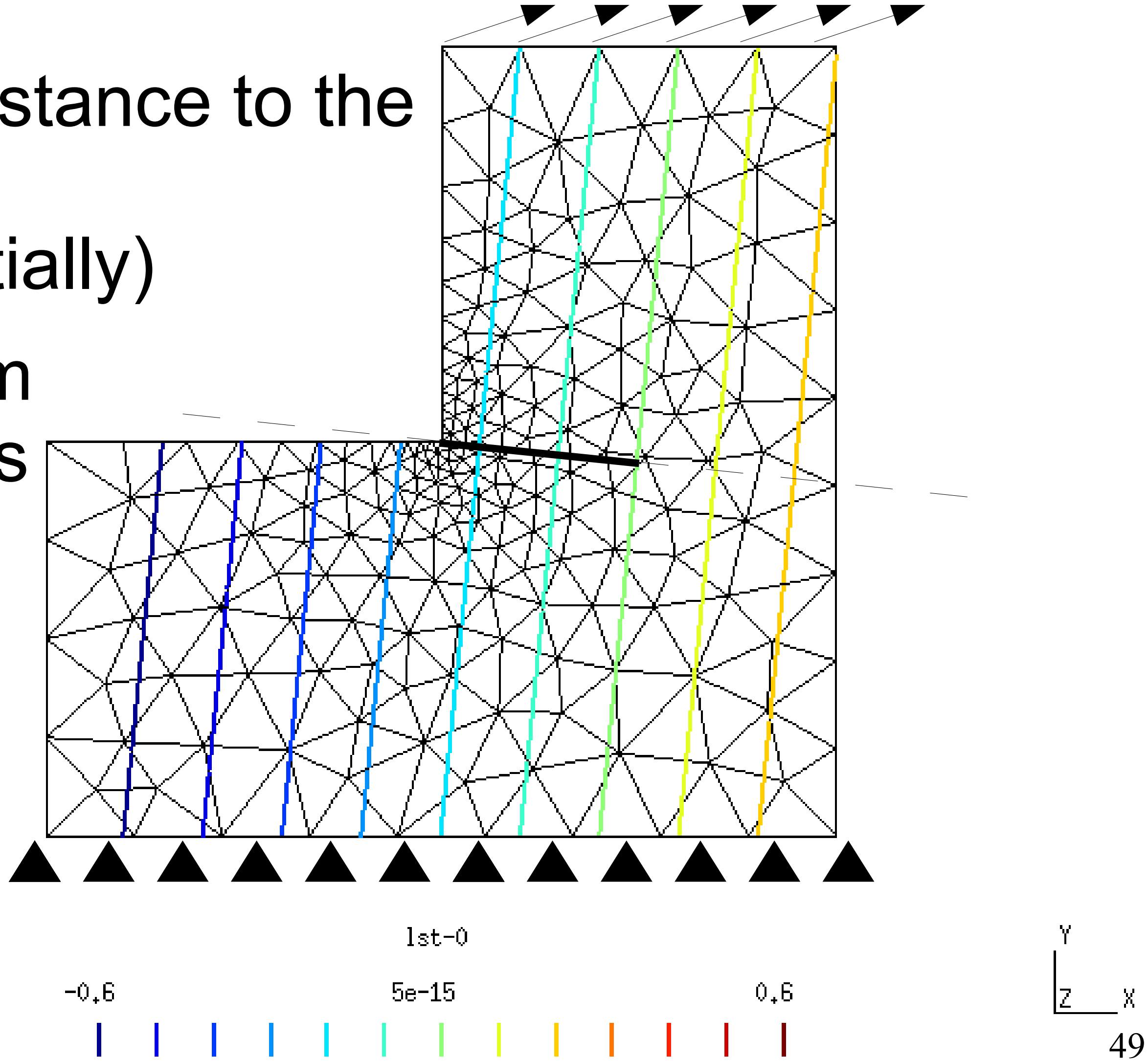


- Geometrical representation of the crack
  - It is not part of the mesh (by definition)
  - Its surface is therefore defined, as before, with a level set $lsn$ that represents the normal distance to the surface.
  - One also needs the location where it stops (on its surface)
    - Crack tip (or front in 3D )

- We make use of another level set

  $lst(x)$

  - It represents the distance to the crack front (measured tangentially)
  - Both level sets form an orthogonal basis at the crack tip



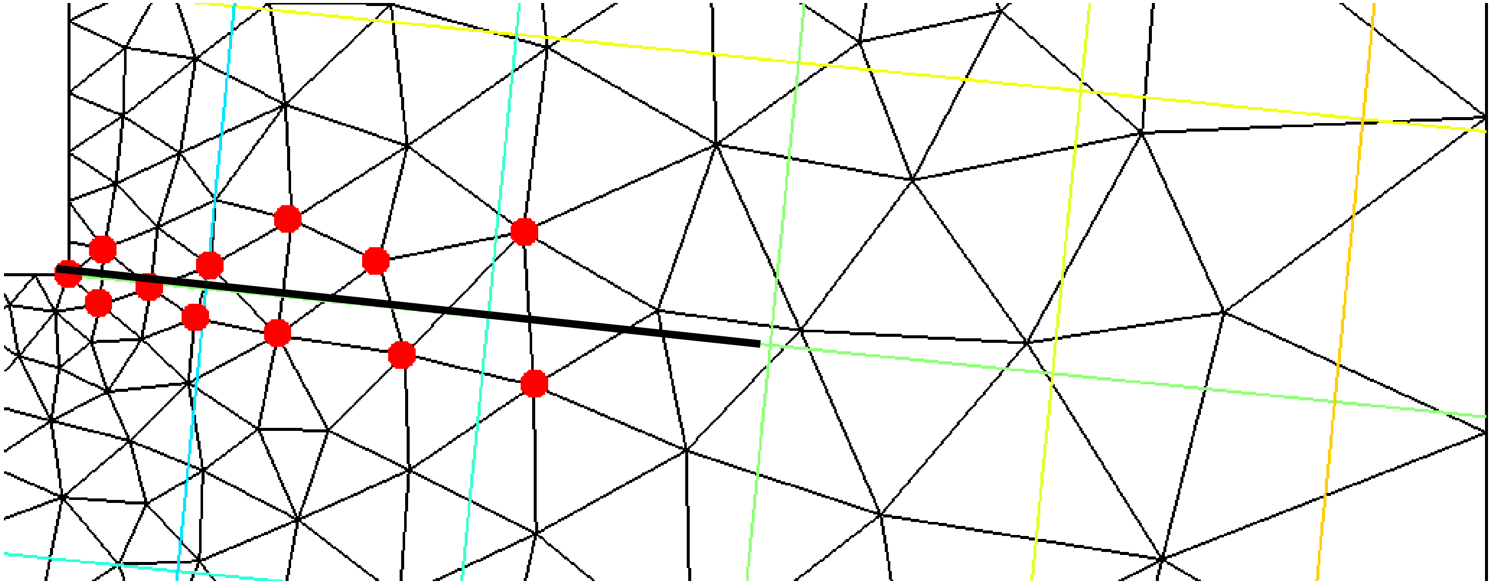
# Extended Finite Elements

## Cracks

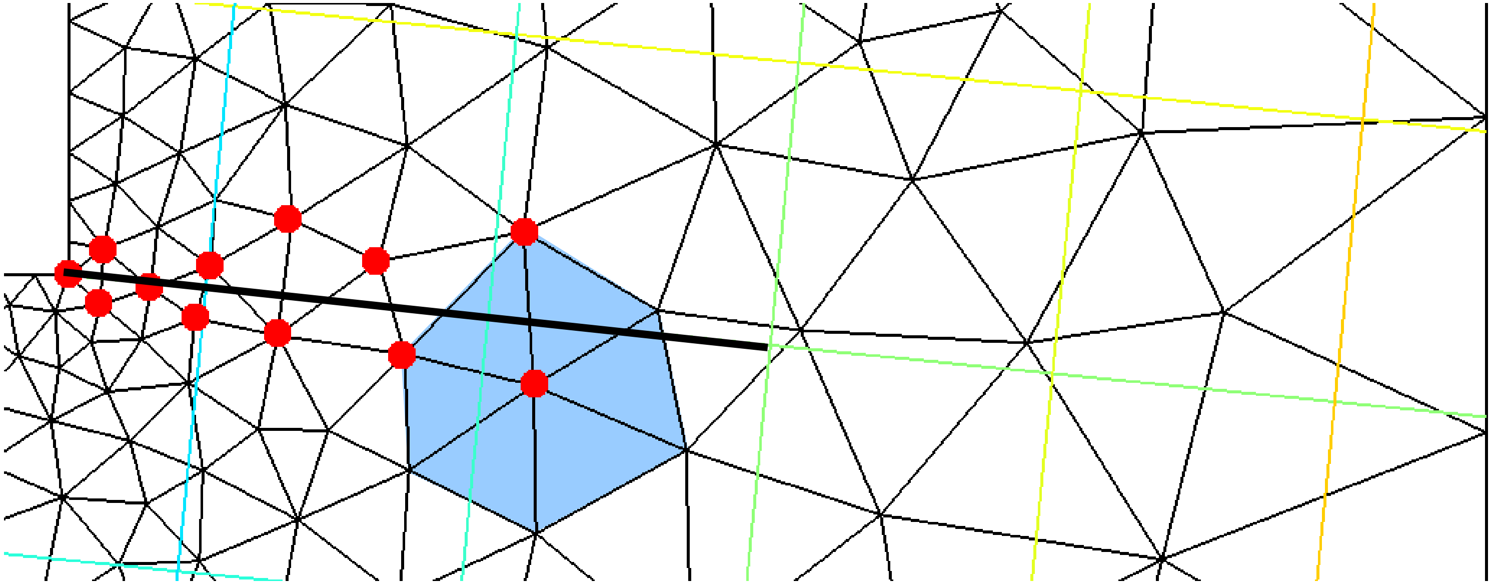
- The locus of the crack is therefore defined as :

$$\phi = \{ x \in \mathbb{R}^3 / lsn(x) = 0, lst(x) \le 0 \}$$

- The enrichment set $C$ is also modified :



- The enrichment set $\mathcal{C}$ is also modified :
  - Zone of influence of the new shape functions



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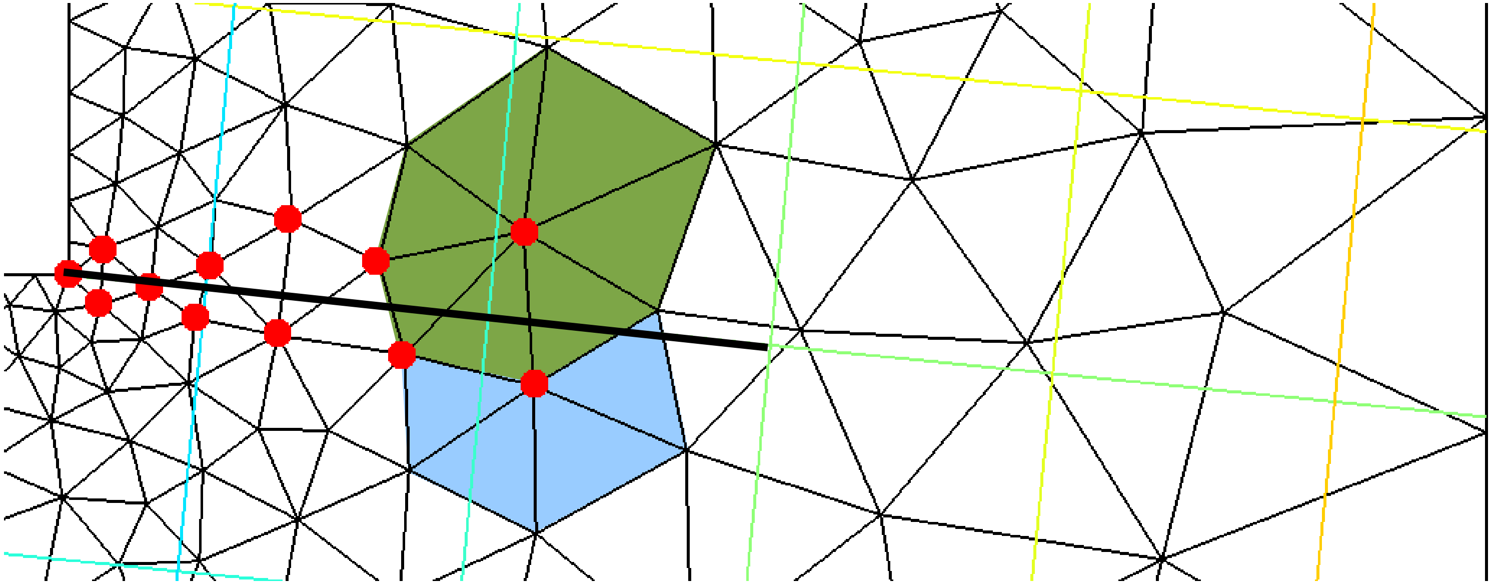
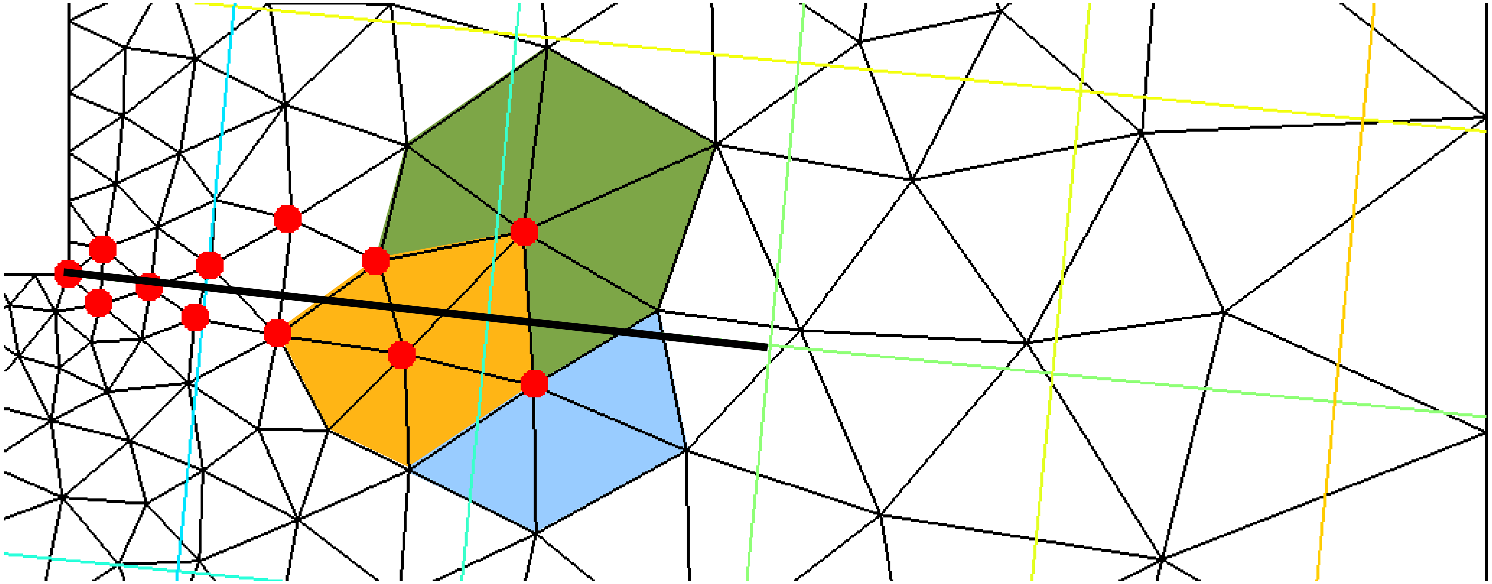
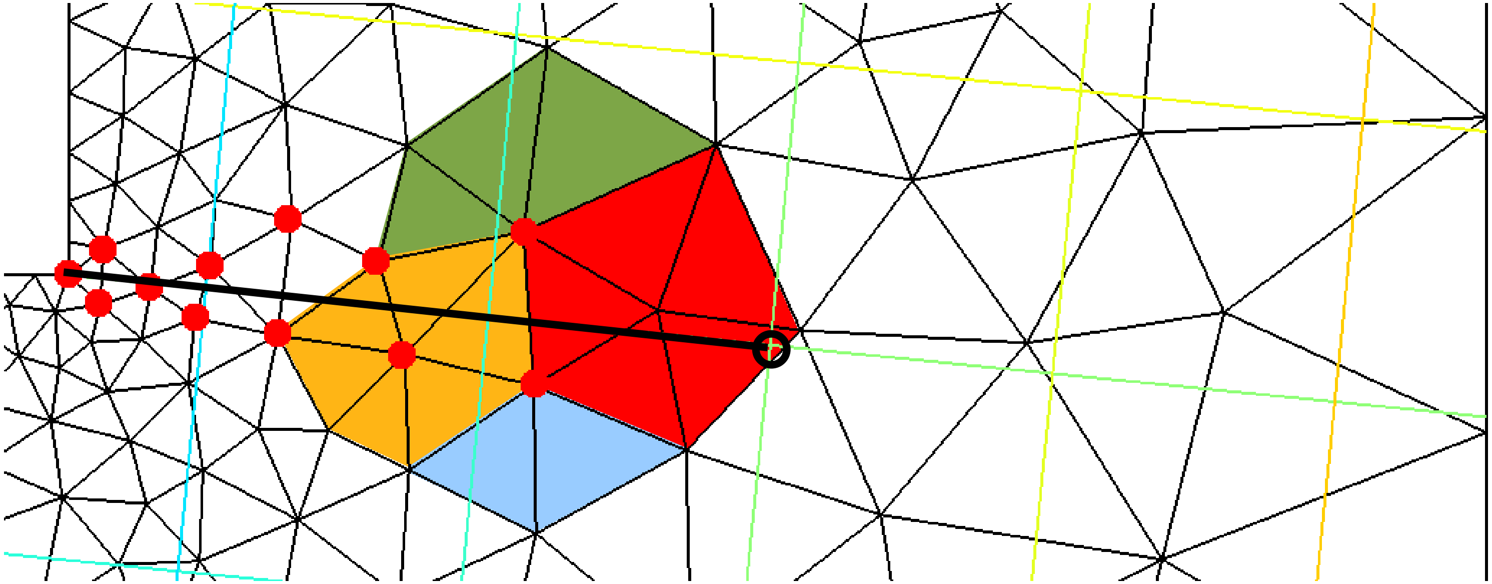


- The enrichment set $\mathrm{C}$ is also modified :
  - Zone of influence of the new shape functions
    - Either it cannot cover the crack until its tip or front...

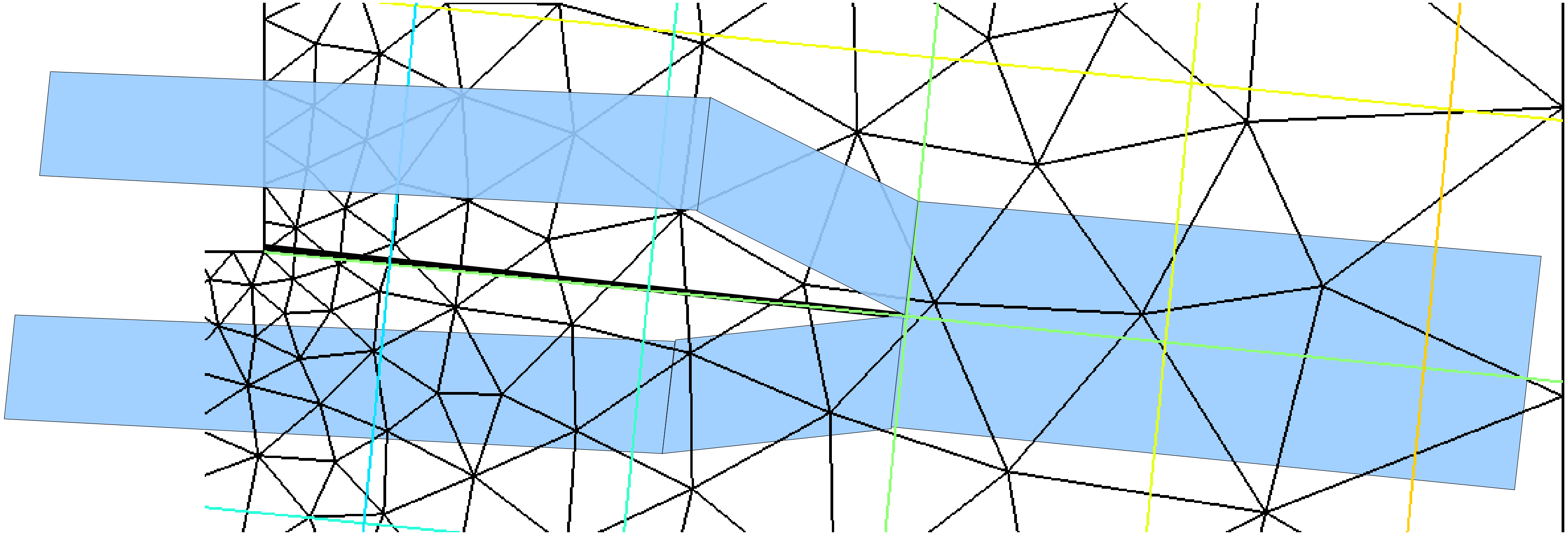


## Cracks

- The enrichment set $\mathrm{C}$ is also modified :
  - Zone of influence of the new shape functions
    - Either it cannot cover the crack until its tip or front... or it goes a bit too far

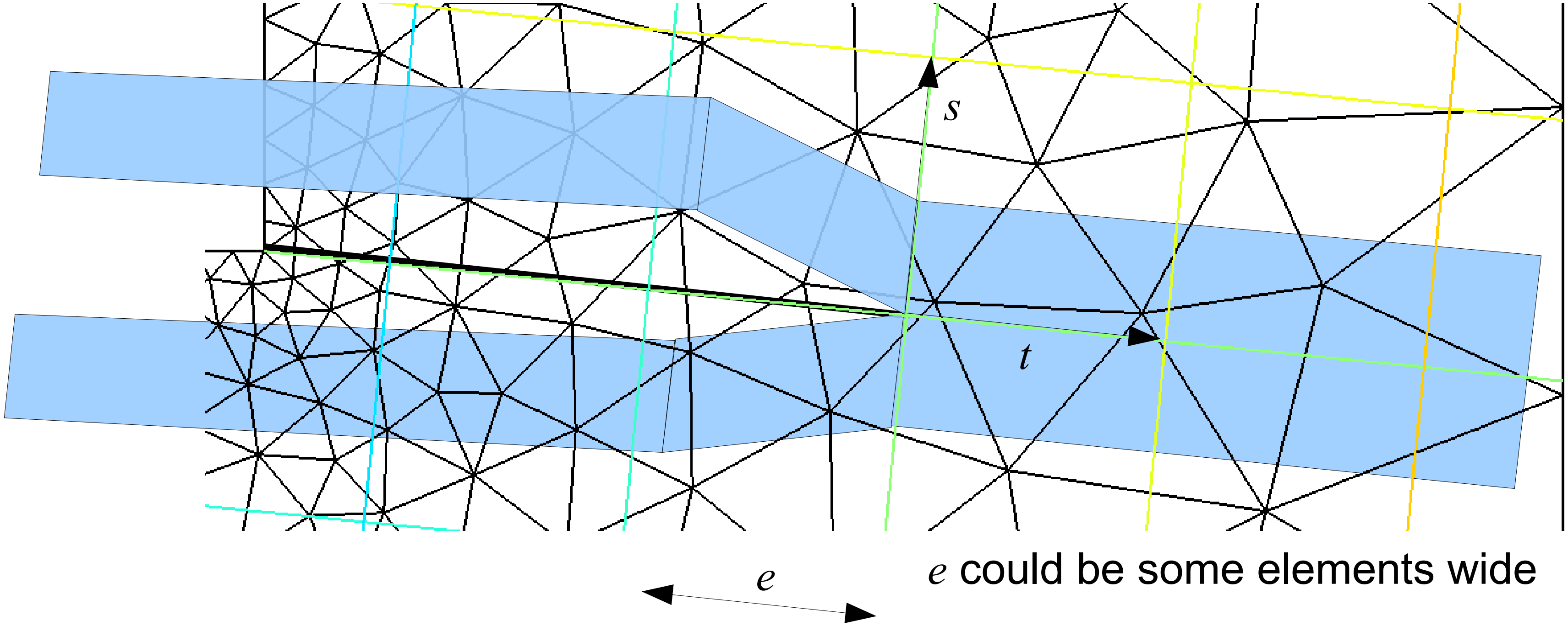


- A special procedure is needed at the crack tip
  - The enrichment function should be discontinuous until the crack tip; continuous beyond.



## Cracks

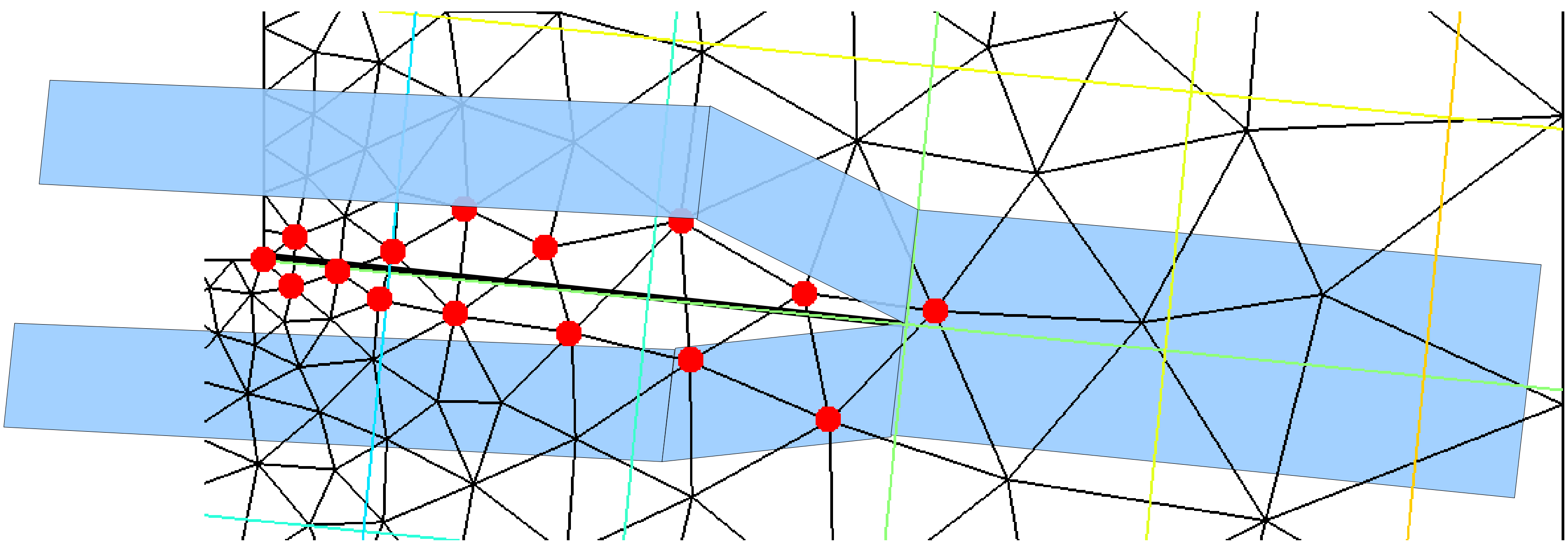
$$T(s,t)=\begin{cases} 0 & \text{if } t \geq 0 \\ H^*(s) & \text{if } t \leq -e \\ \dfrac{-t\,H^*(s)}{e} & \text{if } -e < t < 0 \end{cases} \quad \text{with} \quad \begin{cases} s = lsn(x) \\ t = lst(x) \end{cases}$$




$e$ could be some elements wide

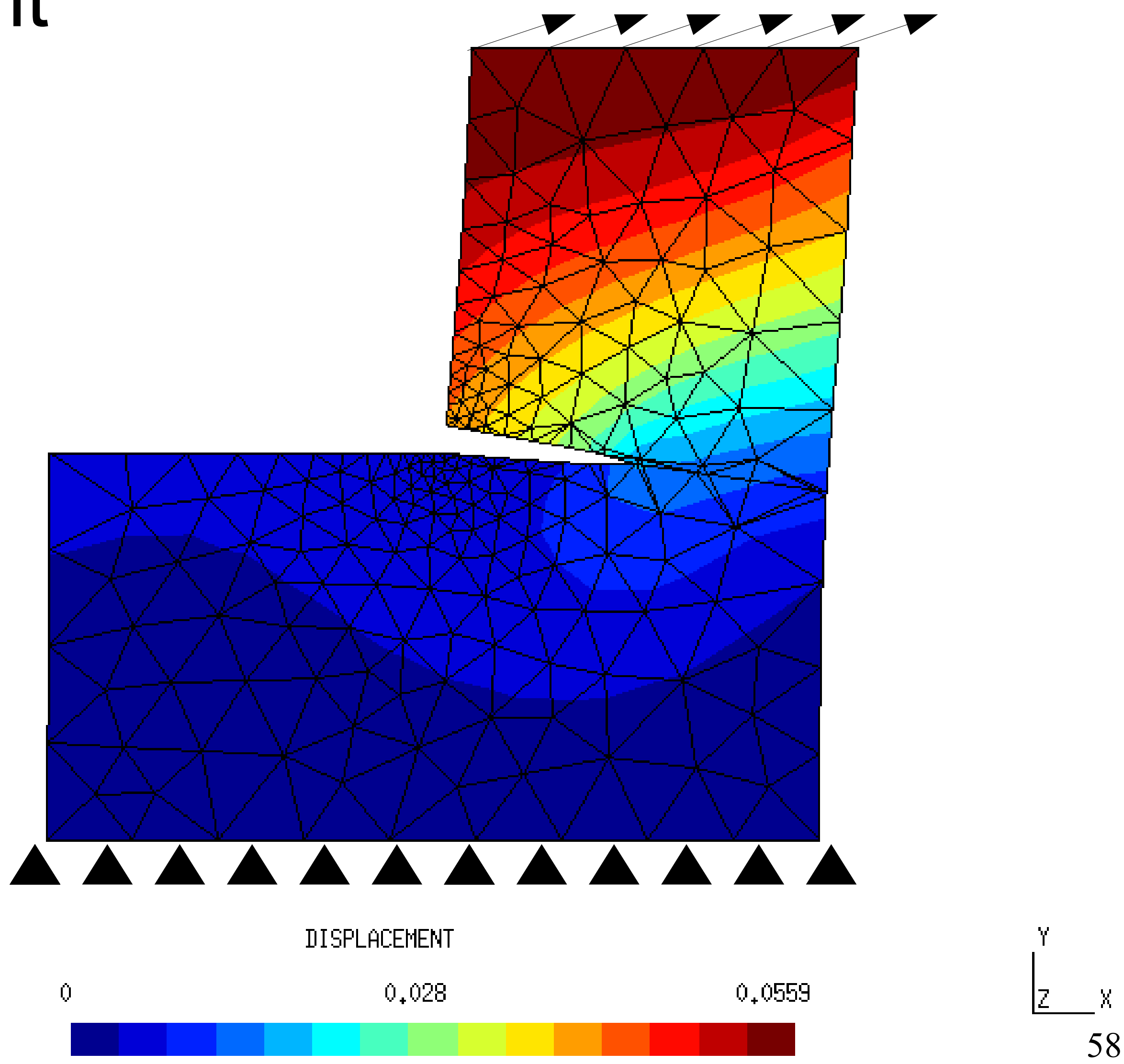
- Alternate set of enriched elements $\mathrm{C}'$
  - It includes every node for which the support is cut (at least partly) by the crack.

$$\bar{u}(x) = \sum_{i \in \Omega} \lambda_i \cdot \bar{N}_i(x) + \sum_{i \in \mathrm{C}'} \lambda_i^* \cdot \bar{N}_i(x) \cdot T(t, s)$$



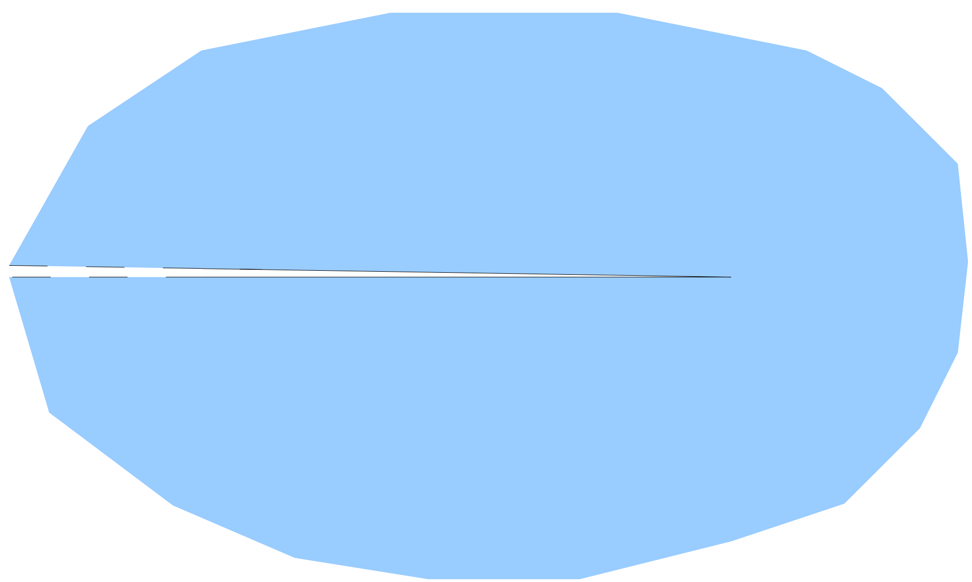
# Cracks

- Displacements with a crack tip enrichment



## Cracks

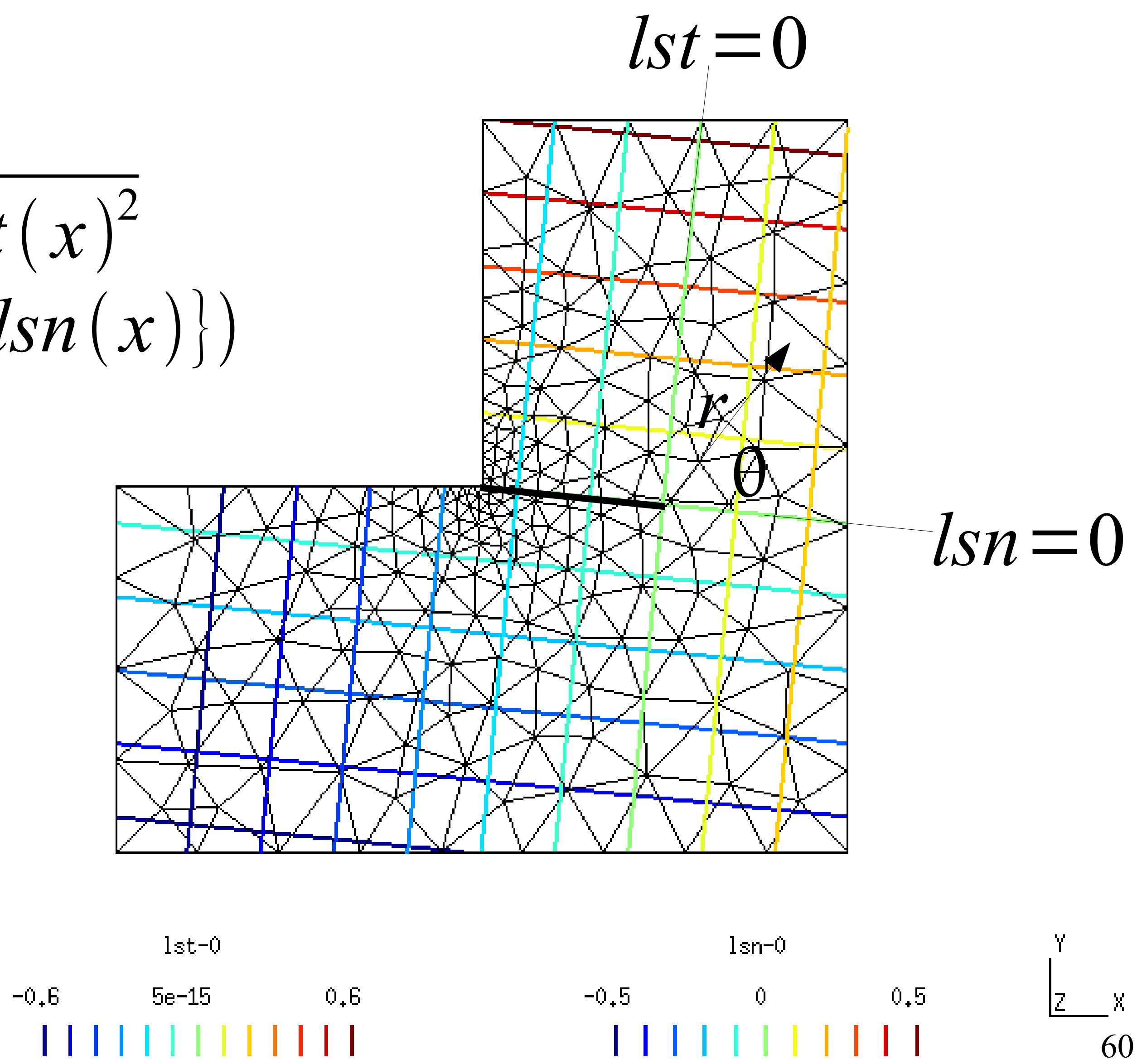
- In fact, the form of the exact solution is known at the crack tip
  - Why not use this directly as a crack enrichment function ?
  - It is readily available for a crack in an infinite medium → see any good fracture mechanics course !



- A polar basis is defined

$$r = \sqrt{lsn(x)^2 + lst(x)^2}$$
$$\theta = \arg(\{lst(x), lsn(x)\})$$

$$\theta = \arctan \frac{lsn(x)}{lst(x)}$$



- Exact asymptotic fields at the crack tip (crack in an infinite domain)

$$u_1 = \frac{1}{2\mu}\sqrt{\frac{r}{2\pi}}\left\{K_1 \cos\frac{\theta}{2}\left(\kappa - \cos\theta\right) + K_2 \sin\frac{\theta}{2}\left(\kappa + 2 + \cos\theta\right)\right\}$$

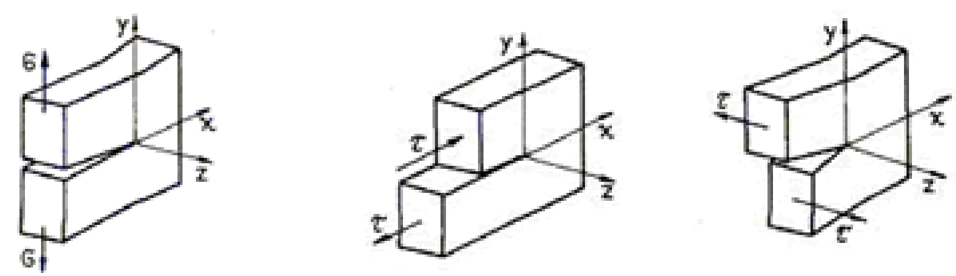
$$u_2 = \frac{1}{2\mu}\sqrt{\frac{r}{2\pi}}\left\{K_1 \sin\frac{\theta}{2}\left(\kappa - \sin\theta\right) + K_2 \cos\frac{\theta}{2}\left(\kappa - 2 + \cos\theta\right)\right\}$$

$$u_3 = \frac{2}{2\mu}\sqrt{\frac{r}{2\pi}}\left\{K_3 \sin\frac{\theta}{2}\right\}$$

$$\mu = \frac{E}{2(1+\nu)}$$

$$\kappa = 3 - 4\nu$$

$K_1$ , $K_2$ and $K_3$ are constants which depend only on boundary conditions: they are called Stress Intensity Factors (SIFs)



- Some analytical manipulations lead to :

$$u_1 = a_1 \sqrt{r} \sin\frac{\theta}{2} + a_2 \sqrt{r} \cos\frac{\theta}{2} + a_3 \sqrt{r} \sin\frac{\theta}{2} \sin\theta + a_4 \sqrt{r} \cos\frac{\theta}{2} \sin\theta + BC(x)$$

$$u_2 = b_1 \sqrt{r} \sin\frac{\theta}{2} + b_2 \sqrt{r} \cos\frac{\theta}{2} + b_3 \sqrt{r} \sin\frac{\theta}{2} \sin\theta + b_4 \sqrt{r} \cos\frac{\theta}{2} \sin\theta + BC(x)$$

$$u_3 = c_1 \sqrt{r} \sin\frac{\theta}{2} + c_2 \sqrt{r} \cos\frac{\theta}{2} + c_3 \sqrt{r} \sin\frac{\theta}{2} \sin\theta + c_4 \sqrt{r} \cos\frac{\theta}{2} \sin\theta + BC(x)$$

- One can therefore use only 4 enrichment functions (they span the whole function space)

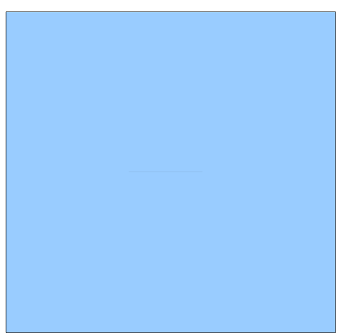
$$\begin{cases} f_1 = \sqrt{r} \sin\frac{\theta}{2} & f_3 = \sqrt{r} \sin\frac{\theta}{2} \sin\theta \\ f_2 = \sqrt{r} \cos\frac{\theta}{2} & f_4 = \sqrt{r} \cos\frac{\theta}{2} \sin\theta \end{cases}$$

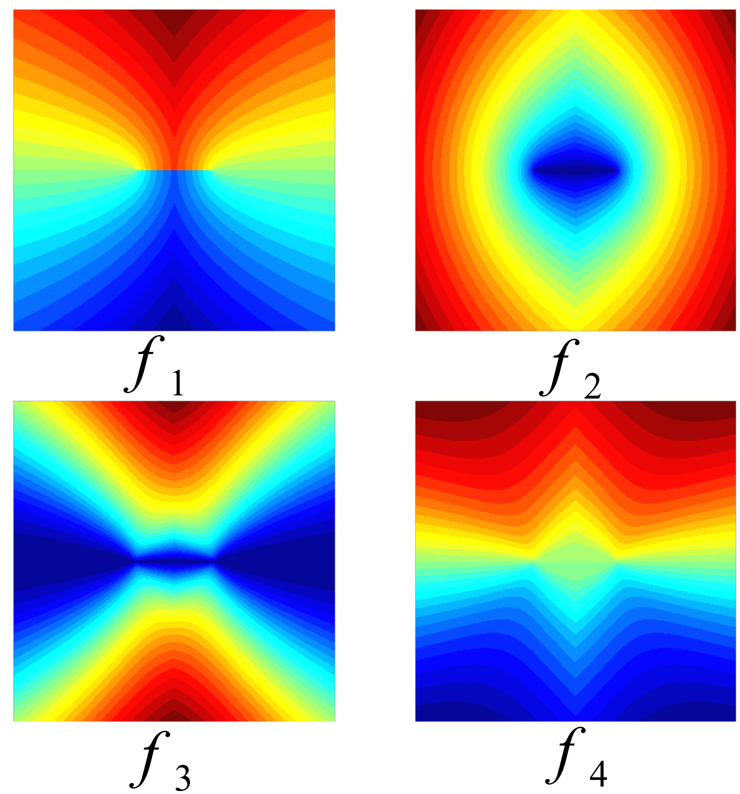
- One may notice that only $f_1$ is discontinuous; the others are simply $C^0$.

# Extended Finite Elements

## Cracks

- Shape of the enrichment functions in the case of an Irwin crack





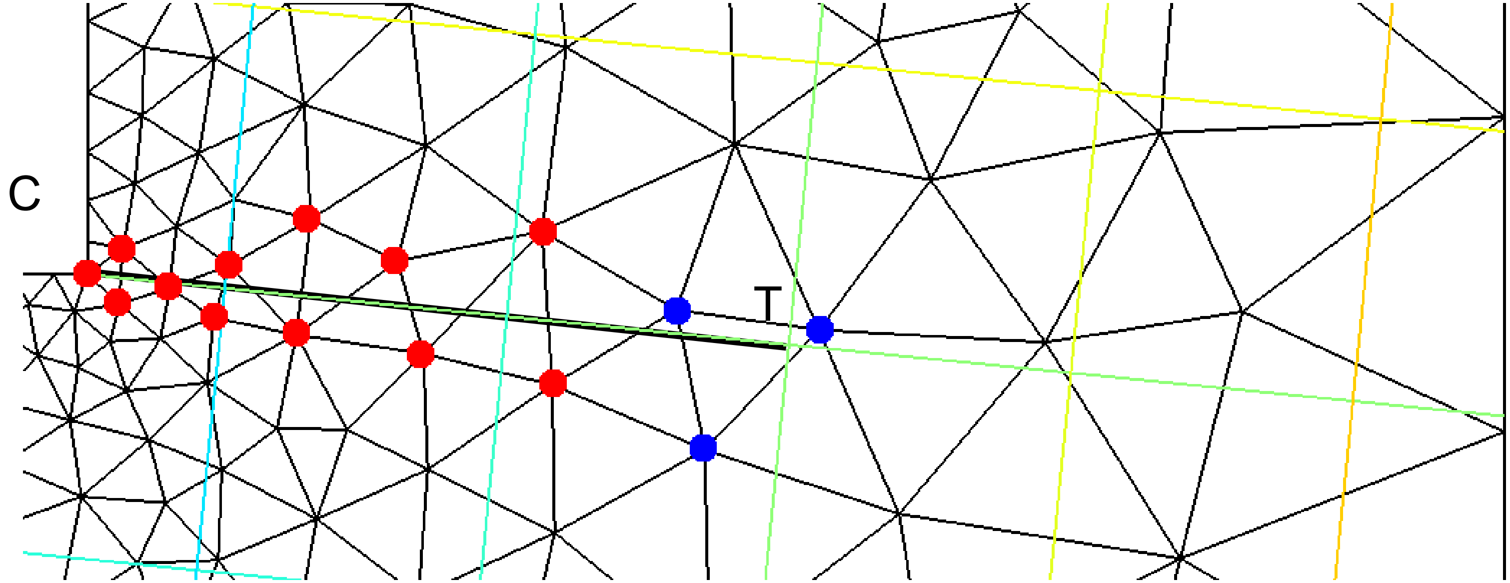
- A new function space

$$\bar{u}(x)=\sum_{i\in\Omega}\lambda_i\cdot\bar{N}_i(x)$$

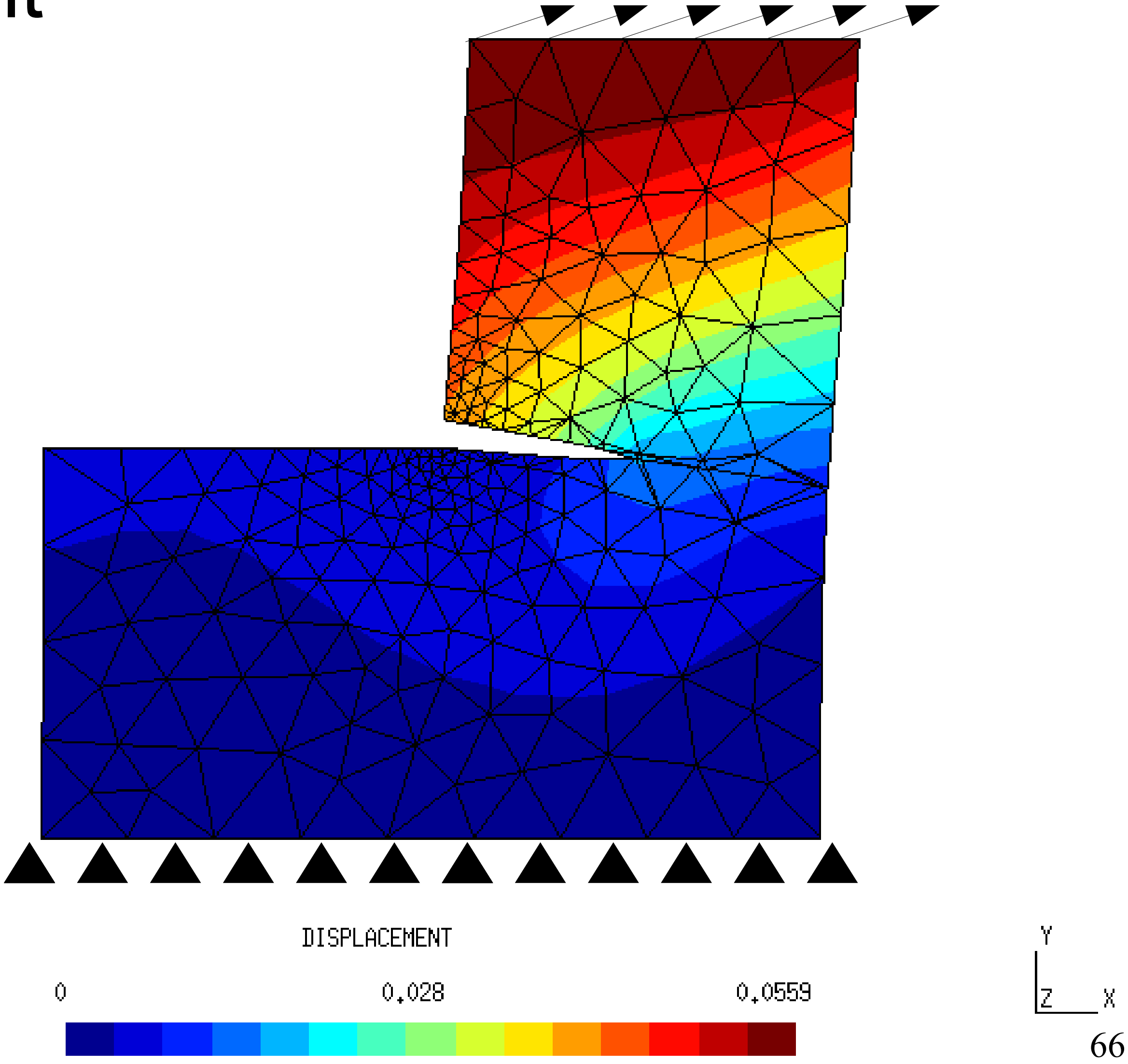
$$+\sum_{i\in C}\lambda_i^{*}\cdot\bar{N}_i(x)\cdot H^{*}(s)+\sum_{i\in T}\sum_{j\in 1..4}\lambda_i^{j}\cdot\bar{N}_i(x)\cdot f_j(r,\theta)$$

- Where to enrich ?
  - At the crack tip (T), because the rest of the cracked domain is already covered by the Heaviside enrichment
  - The analytical solution used to build the $f_j(r,\theta)$ is only valid around the crack tip.

- Choice of the nodes to enrich
  - The set C contains nodes for which the support is completely cut by the crack
  - The set T contains the nodes for which the support contains or touches the crack tip



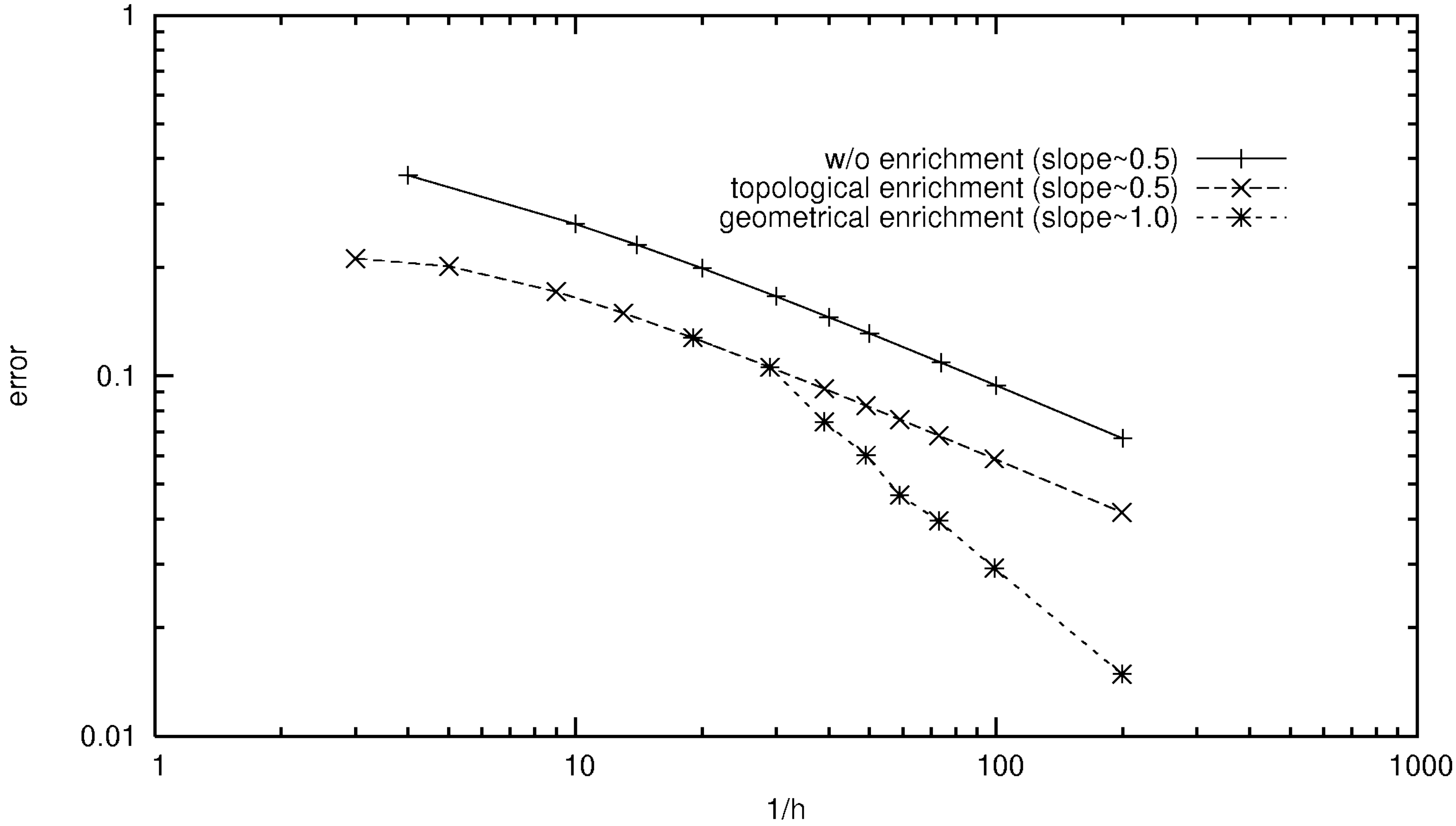
- Displacements with the new crack tip enrichment



# Extended Finite Elements

## Cracks

- If one chooses a good enrichment procedure, one may get a better convergence rate than observed with regular finite elements.



## Cracks

- To be able to propagate a crack, it is needed to :

  Perform the assembly of the linear system
  ↓ Solve the linear system
  ↓ Compute adequate propagation parameters
  ↓ Update level-sets $lsn$ and $lst$

- Crack propagation obeys to well defined physical laws
  - Fatigue
  - Fragile fracture etc...

## Cracks

- What are the adequate parameters of crack propagation
  - Charge coefficients (stress intensity factors) that are linked to the geometry of the problem and the boundary conditions.
    - Intrisic parameters having effects on the material just in front of the crack path.
  - Material behaviour with respect to these SIFs : ductile propagation (mild steel) or fragile (glass, cast iron)
  - For ductile fracture, one often uses the ratio (number of loading cycle) w.r. to (crack advance)

# Extended Finite Elements

## Cracks

- Computation of the stress intensity factors
  - Depend only on stress field around the crack
  - J integrals and interaction integrals

    (do not recall these, see further)

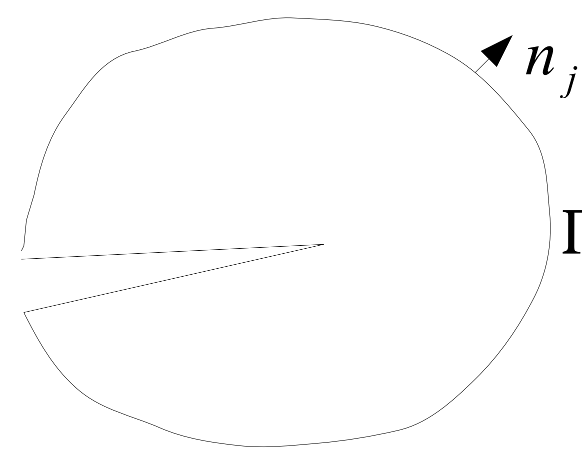
$$J=\int_{\Gamma}\left[\frac{1}{2}\sigma_{ij}\epsilon_{ij}\delta_{1j}-\sigma_{ij}\frac{\partial u_i}{\partial x_1}\right]n_j\,d\Gamma$$

$$J^{(1+2)}=\int_{\Gamma}\left[\frac{1}{2}(\sigma_{ij}^{(1)}+\sigma_{ij}^{(2)})(\epsilon_{ij}^{(1)}+\epsilon_{ij}^{(2)})\delta_{1j}-(\sigma_{ij}^{(1)}+\sigma_{ij}^{(2)})\frac{\partial(u_i^{(1)}+u_i^{(2)})}{\partial x_1}\right]n_j\,d\Gamma$$

$$=J^{(1)}+J^{(2)}+I^{(1+2)}$$

$$I^{(1+2)}=\int_{\Gamma}\left[\sigma_{ij}^{(1)}\epsilon_{ij}^{(2)}\delta_{1j}-\sigma_{ij}^{(1)}\frac{\partial u_i^{(2)}}{\partial x_1}-\sigma_{ij}^{(2)}\frac{\partial u_i^{(1)}}{\partial x_1}\right]n_j\,d\Gamma$$

$$I^{(1+2)}=2\frac{(1-\nu^2)}{E}(K_1^{(1)}K_1^{(2)}+K_2^{(1)}K_2^{(2)})+\frac{1}{\mu}K_3^{(1)}K_3^{(2)}$$



# Cracks

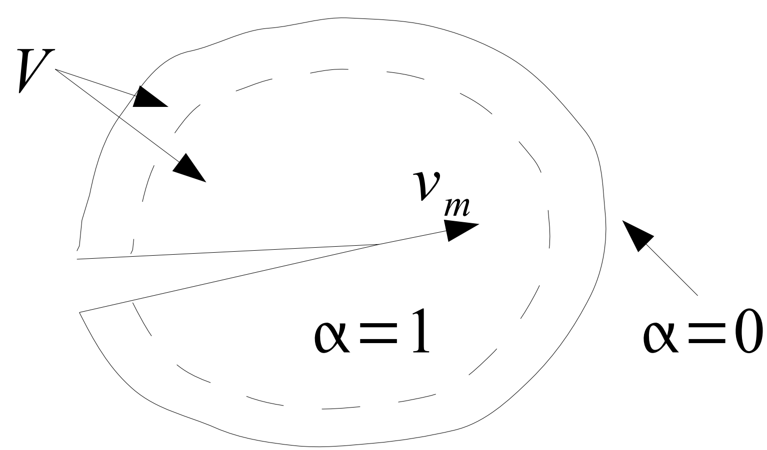
- Going from a contour integral to a volume integral (unloaded crack)

$$I^{(1+2)}=\int_V \frac{\partial q_m}{\partial x_j}\left(\sigma_{kl}^{(1)}\epsilon_{kl}^{(2)}\delta_{mj}-\sigma_{ij}^{(1)}\frac{\partial u_i^{(2)}}{\partial x_m}-\sigma_{ij}^{(2)}\frac{\partial u_i^{(1)}}{\partial x_m}\right)dV$$

One have $q_m=\alpha\cdot v_m$ and $\alpha$ is equal to 1 inside the domain and vaniqhes on the boundary $\Gamma$.

$v_m$ is the virtual crack propagation speed (norm=1)

One interpolates $\alpha$ on the mesh.



## Cracks

- The interaction integrals allows to compute the stress intensity factors
  - Robust
  - Same good properties as the J- integral
  - See fracture mechanics course(s) for more info.

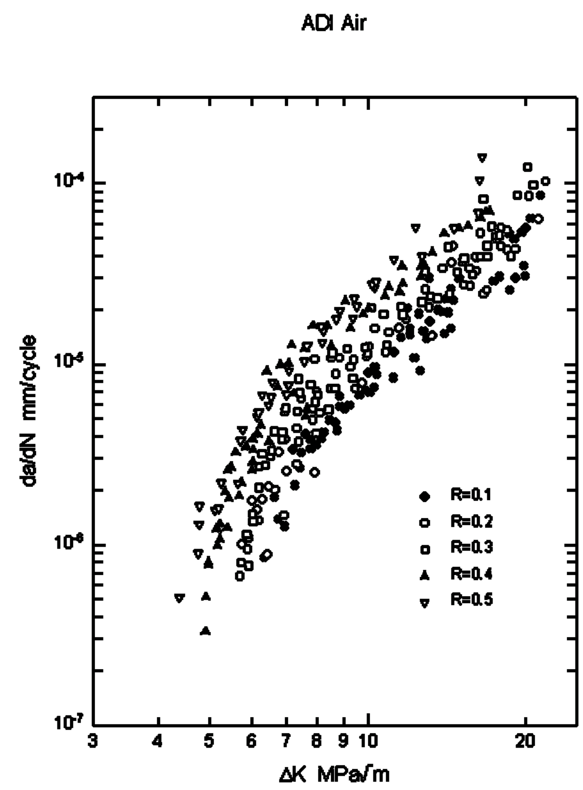
- Propagation speed

  Example : Alloys under cyclic loadings

  Paris law for the speed of propagation :

$$\frac{da}{dN} = C \cdot \Delta K^{m}$$

| Alloy | $m$ | $C$ (m/cycle) |
|---|---|---|
| Steel | 3 | $10^{-11}$ |
| Aluminium | 3 | $10^{-12}$ |
| Nickel | 3.3 | $4 \cdot 10^{-12}$ |
| Titanium | 5 | $10^{-11}$ |



# Extended Finite Elements

## Cracks

- Direction is along the maximal tangent stress $\sigma_{\theta\theta}$

$$\begin{Bmatrix} \sigma_{\theta\theta} \\ \sigma_{r\theta} \end{Bmatrix} = \frac{K_1}{4\sqrt{2\pi r}} \begin{Bmatrix} 3\cos\frac{\theta}{2} + \cos\frac{3\theta}{2} \\ \sin\frac{\theta}{2} + \sin\frac{3\theta}{2} \end{Bmatrix} + \frac{K_2}{4\sqrt{2\pi r}} \begin{Bmatrix} -3\sin\frac{\theta}{2} - 3\sin\frac{3\theta}{2} \\ \cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2} \end{Bmatrix}$$

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0 \quad \rightarrow \quad \cos\frac{\theta_c}{2}\left[\frac{1}{2} K_1 \sin\theta_c + \frac{1}{2} K_2 (3\cos\theta_c - 1)\right] = 0$$

$$\theta_c = 2\arctan\frac{1}{4}\left(\frac{K_1}{K_2} \pm \sqrt{\left(\frac{K_1}{K_2}\right)^2 + 8}\right)$$

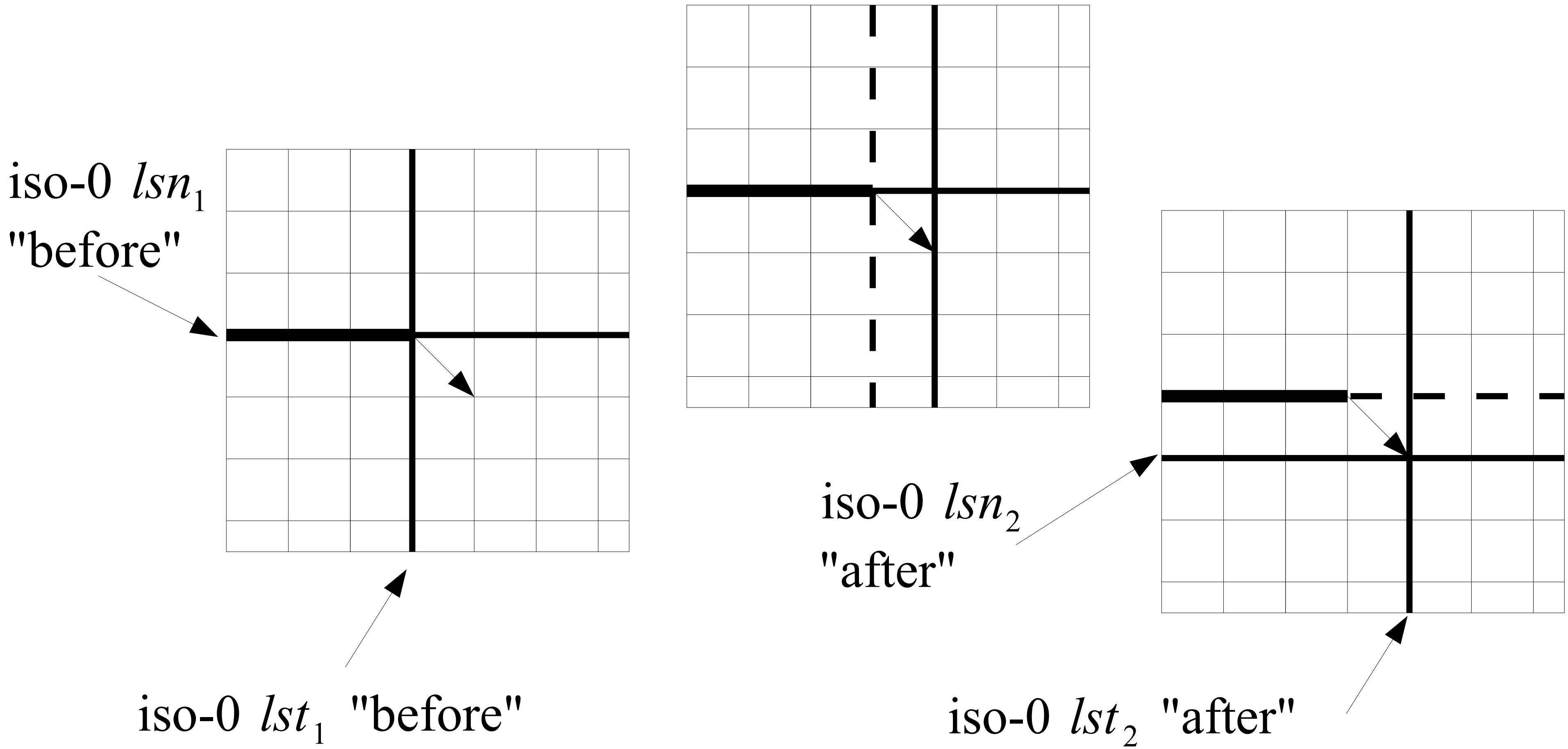
- One chooses $\theta_c$ that correspond to a maximal value of $\sigma_{\theta\theta}$ (in traction)

## Level set update

- There exists many algorithms but the essential part is to :
  - Conserve the notion of signed distance function at the interface for $lsn$
  - Have an orthonormed frame in the vincinity of the crack tip $(lst, lsn)$

## Level set update

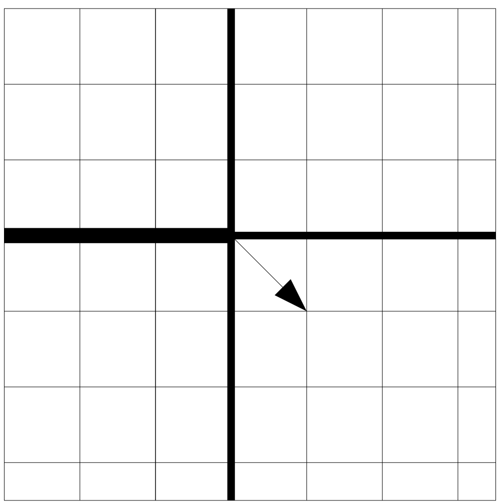
- Transport of *lsn* and *lst*



# Extended Finite Elements

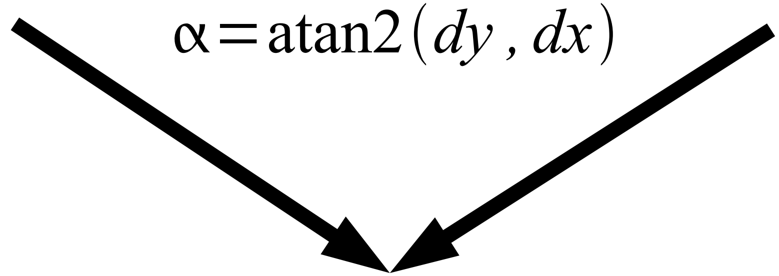
## Level set update

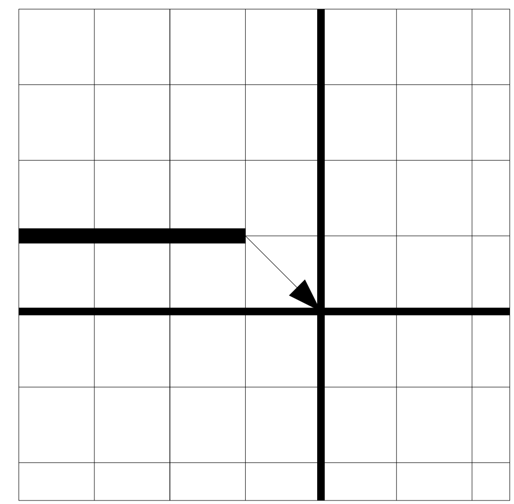
- Rebuilding of *lsn* and *lst*



$lsn_1$ & $lst_1$ "before"

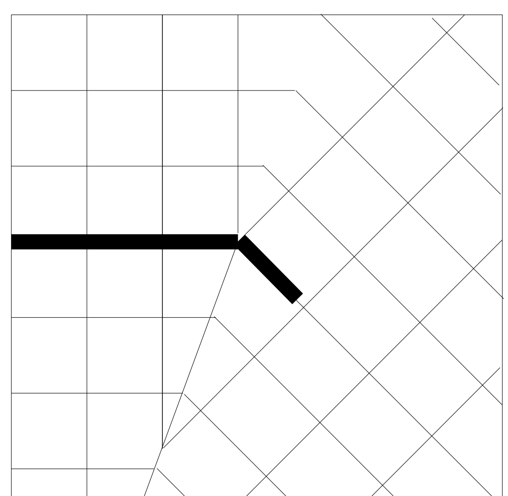
$$dx = lst_1 - lst_2$$
$$dy = lsn_1 - lsn_2$$
$$\alpha = \mathrm{atan2}(dy, dx)$$





$lsn_2$ & $lst_2$ "after"

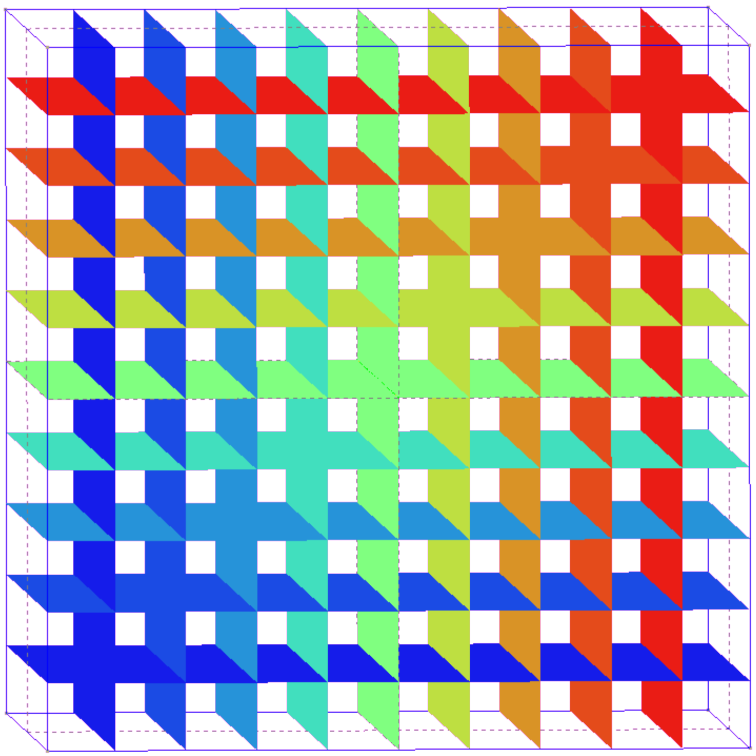
$$lst = lst_2$$
$$lsn = lsn_1$$

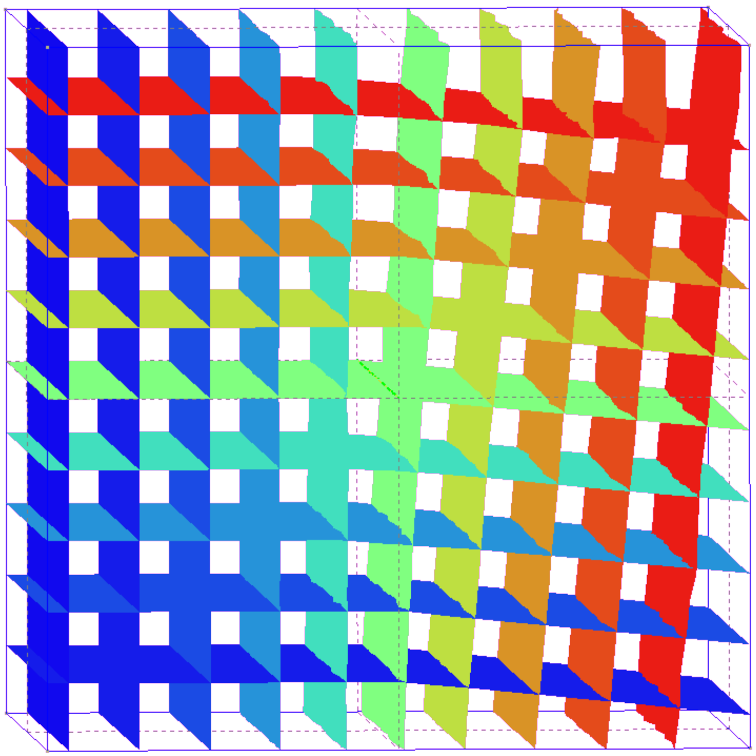


$$lst = \cos(\alpha) \cdot lst_2 + \sin(\alpha) \cdot lsn_2$$
$$lsn = -\sin(\alpha) \cdot lst_2 + \cos(\alpha) \cdot lsn_2$$

# Extended Finite Elements

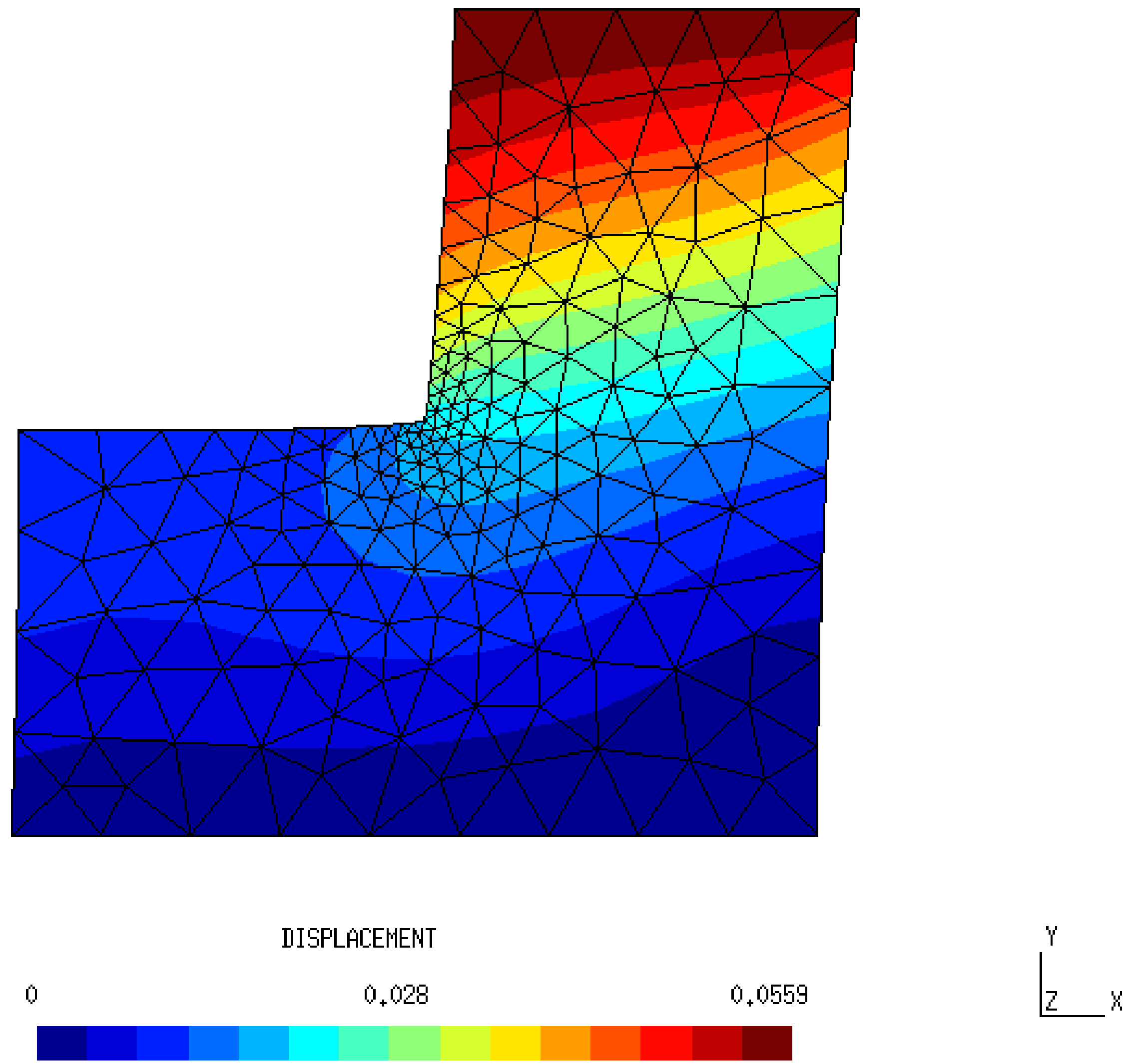
## Level set update





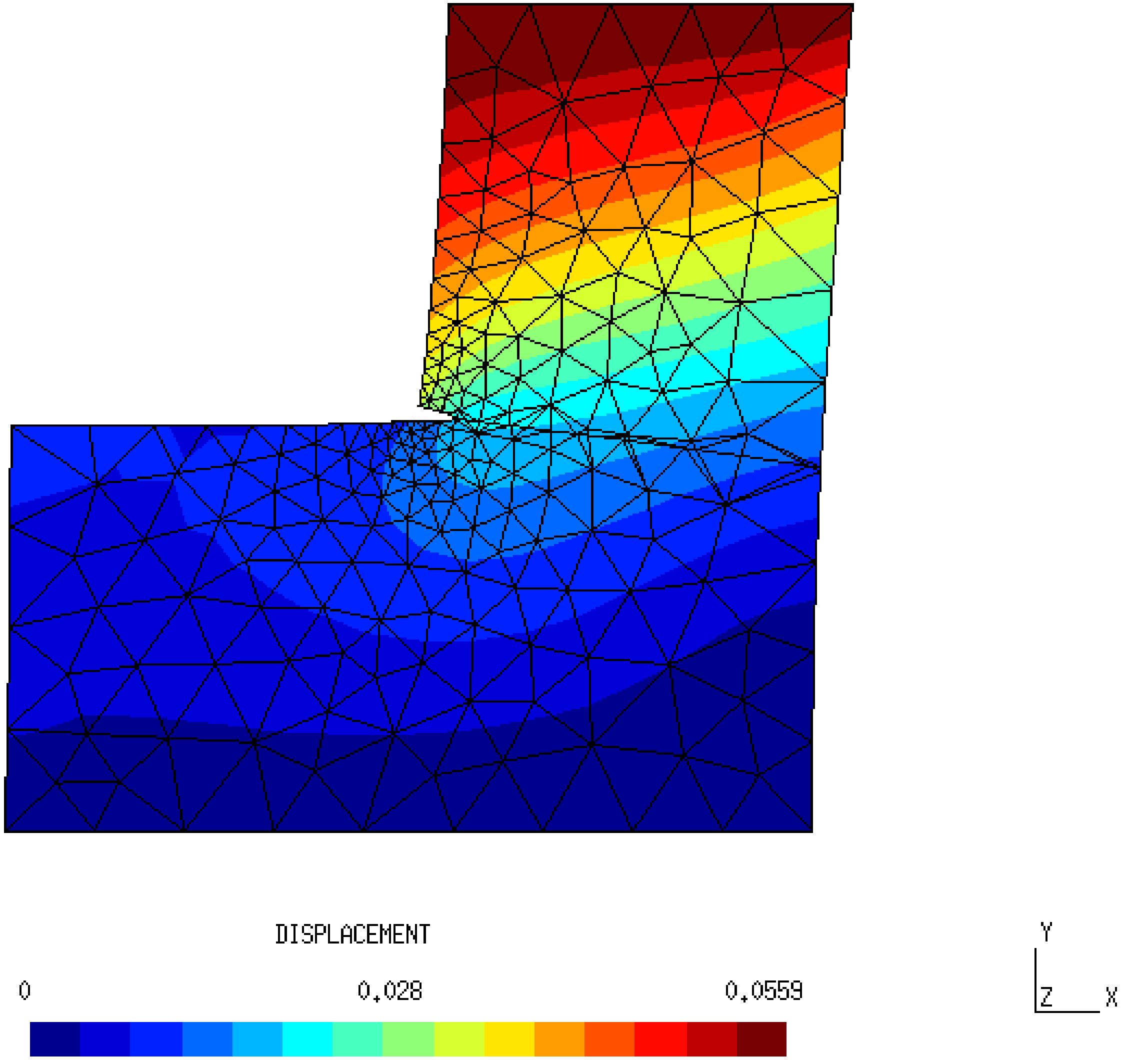
# Extended Finite Elements

## Propagation



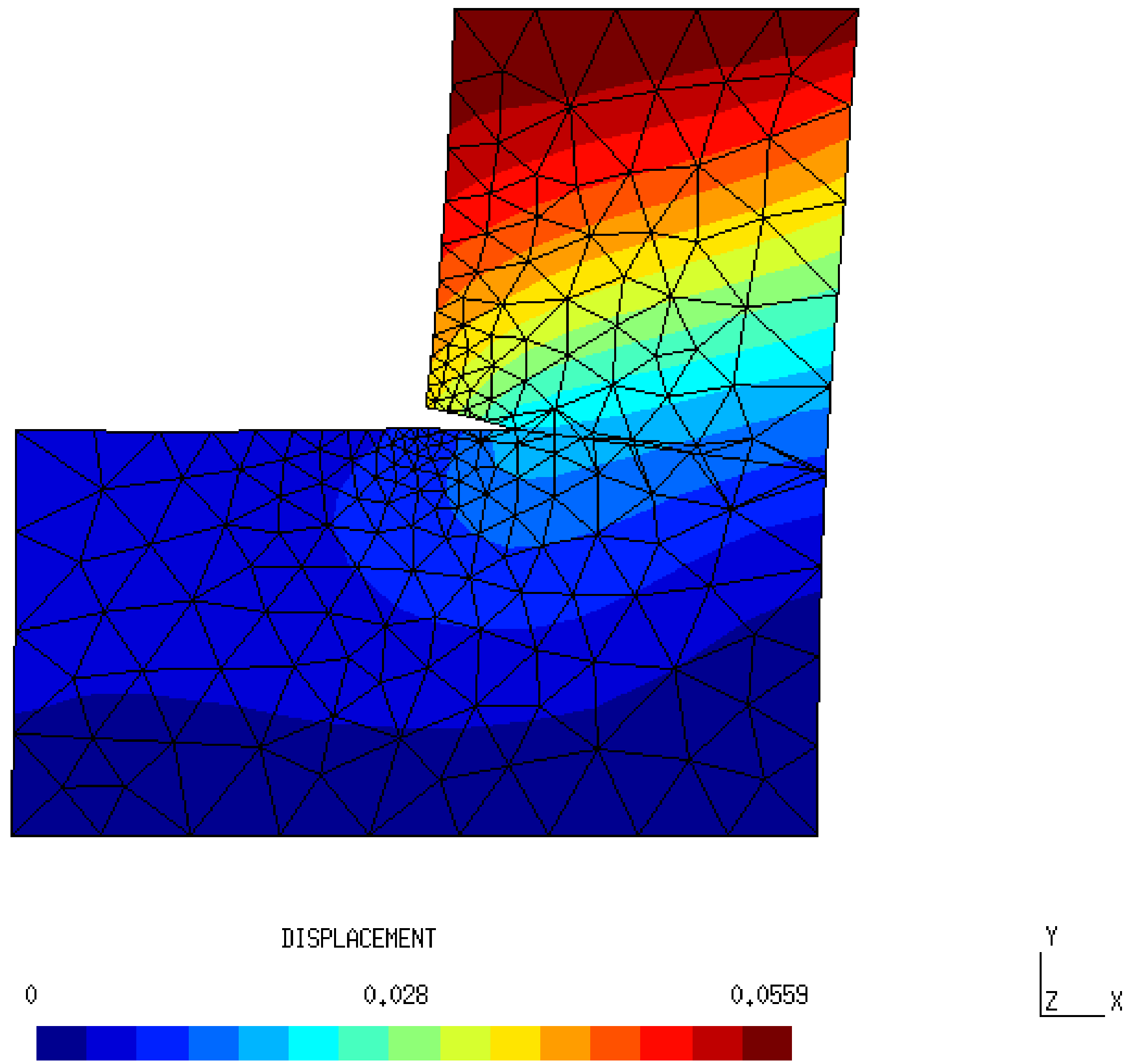
# Extended Finite Elements

## Propagation



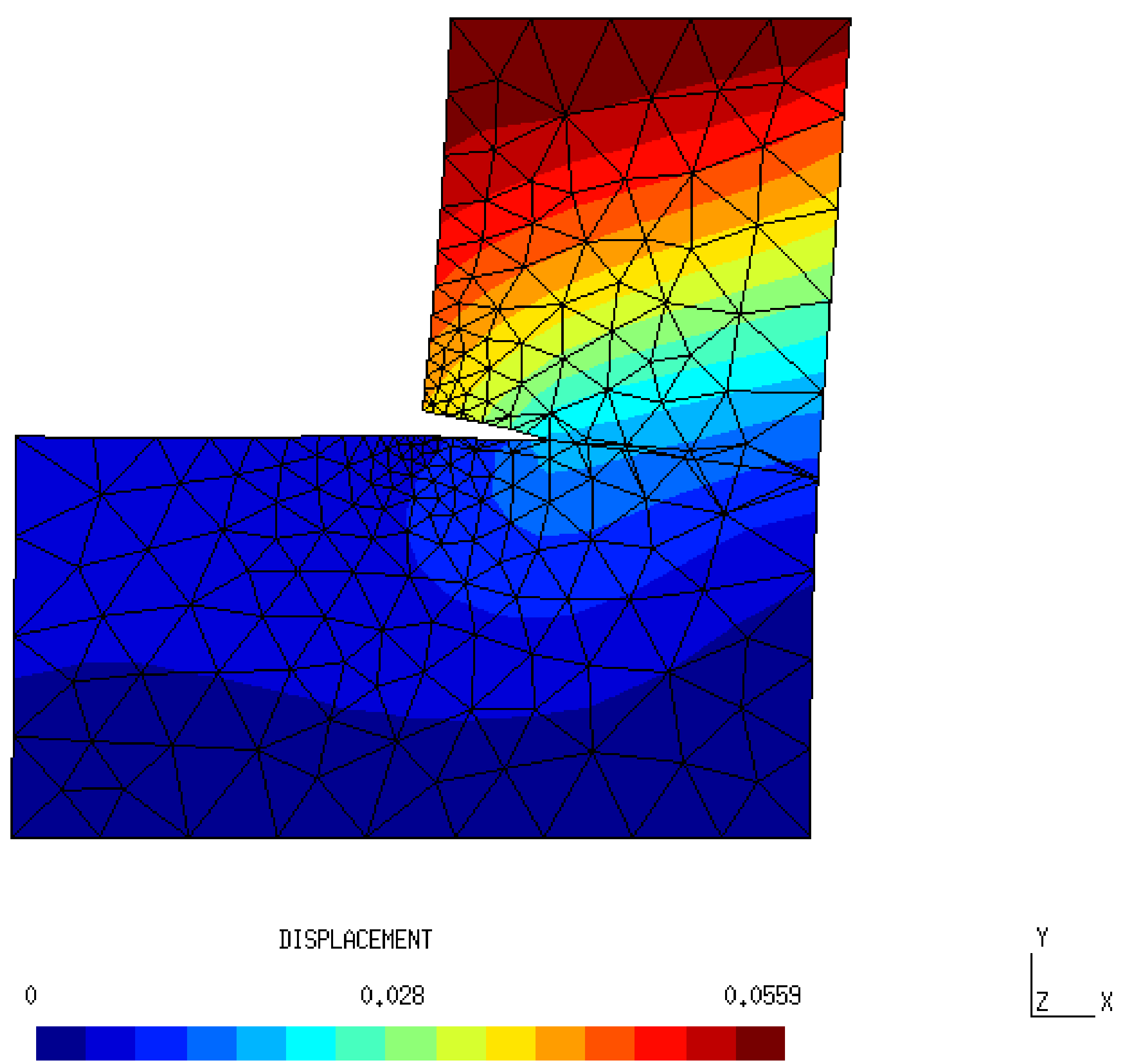
# Extended Finite Elements

## Propagation

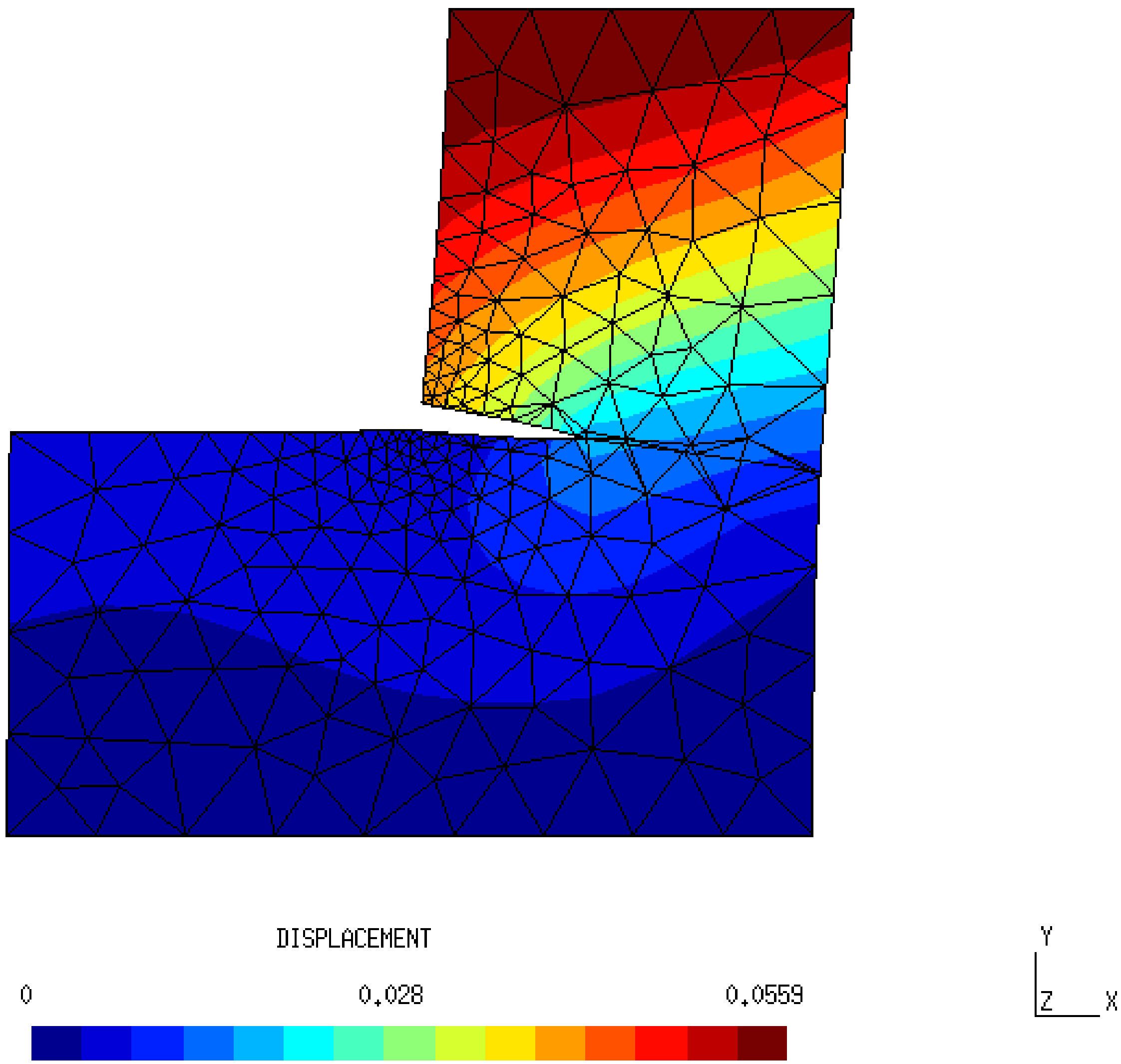


# Extended Finite Elements

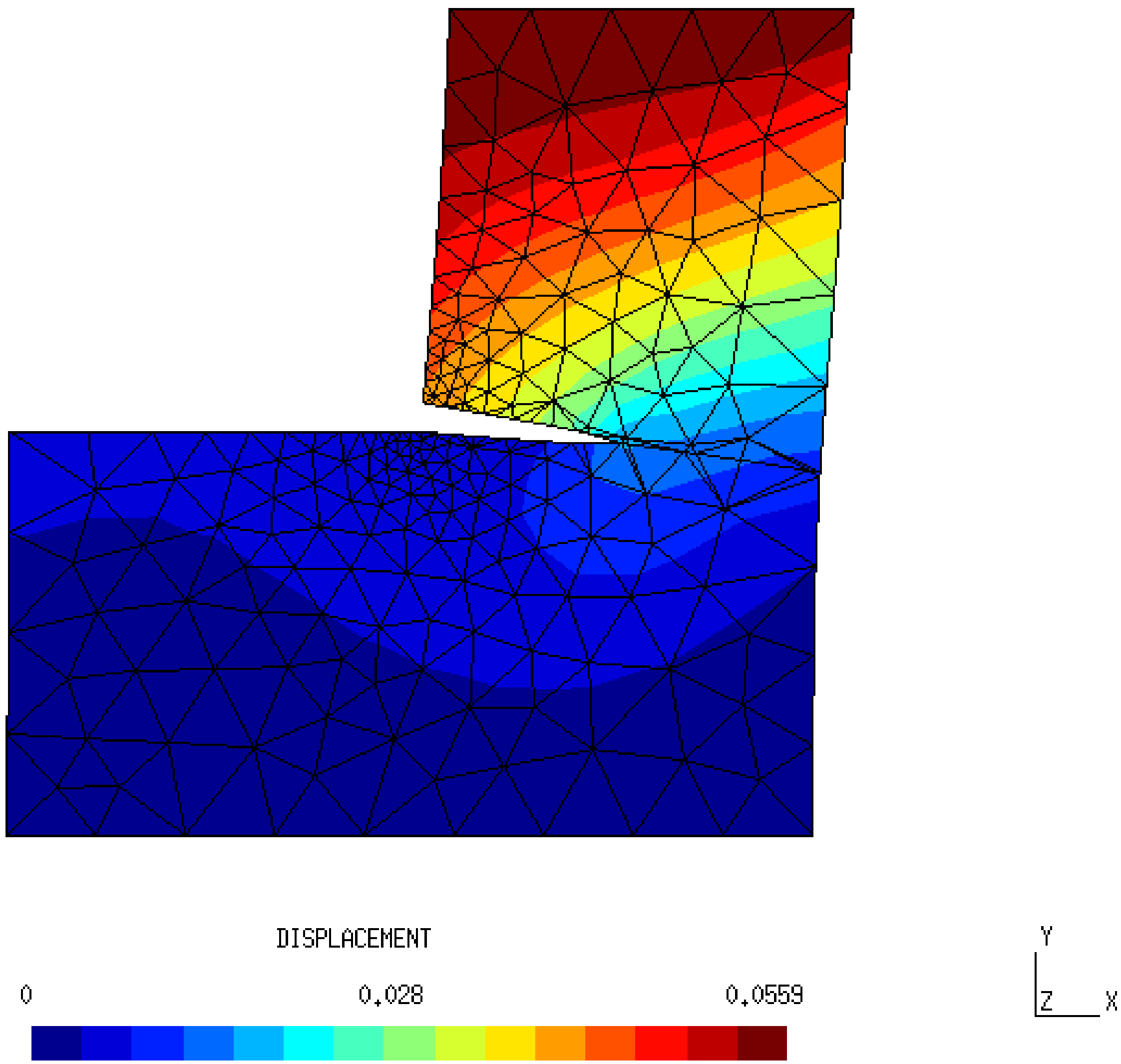
## Propagation



# Propagation

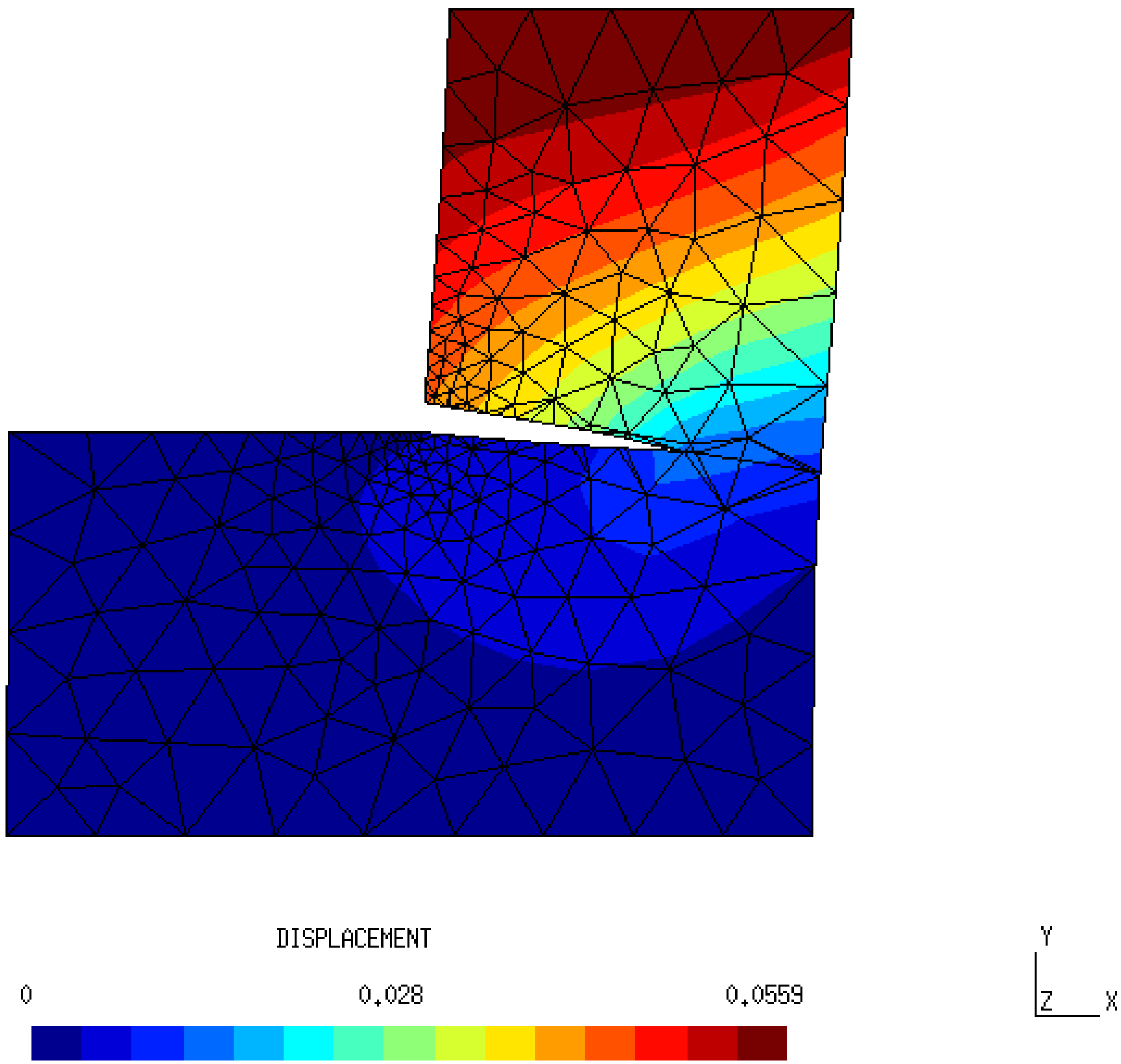


# Propagation



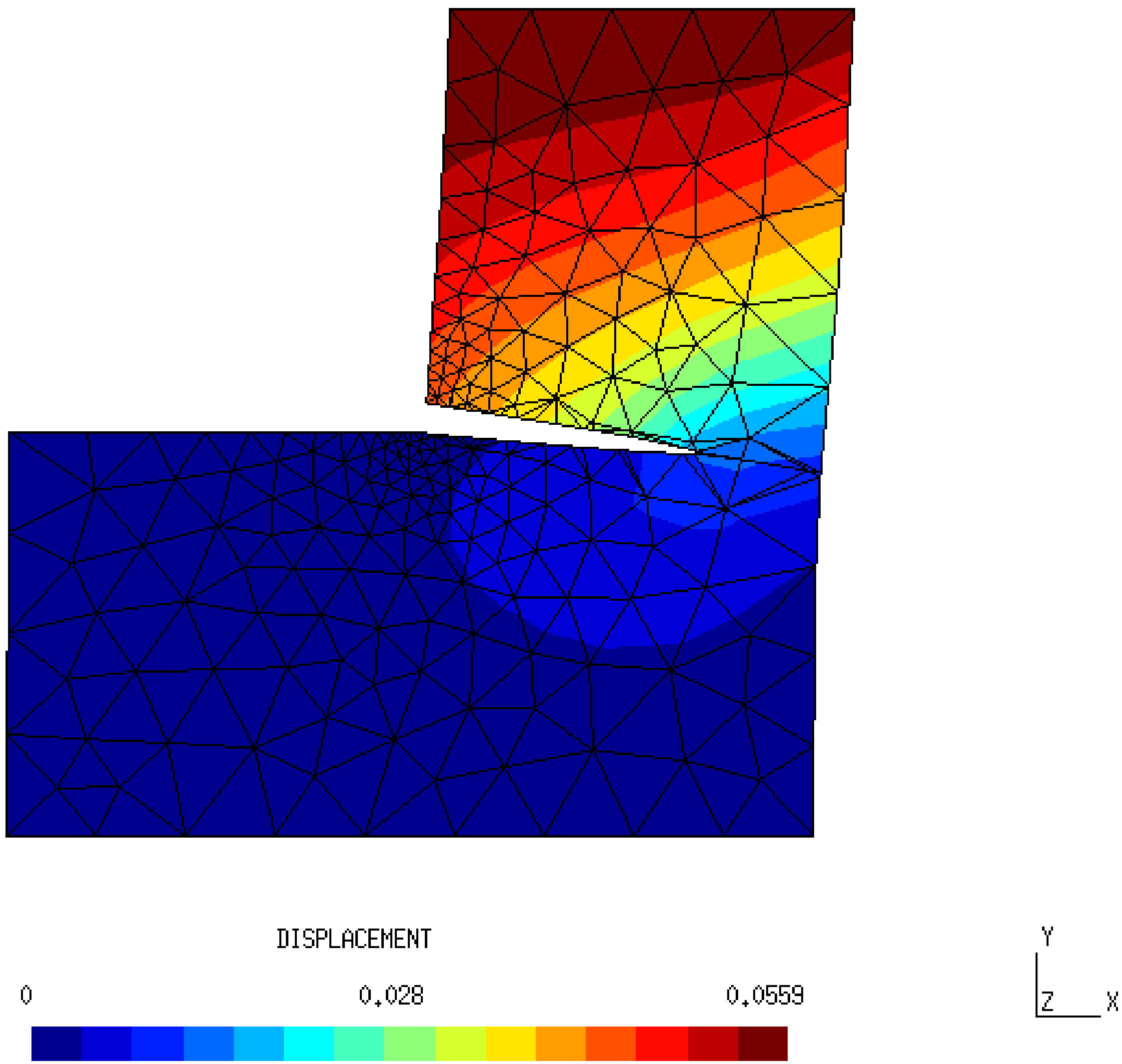
# Extended Finite Elements

## Propagation



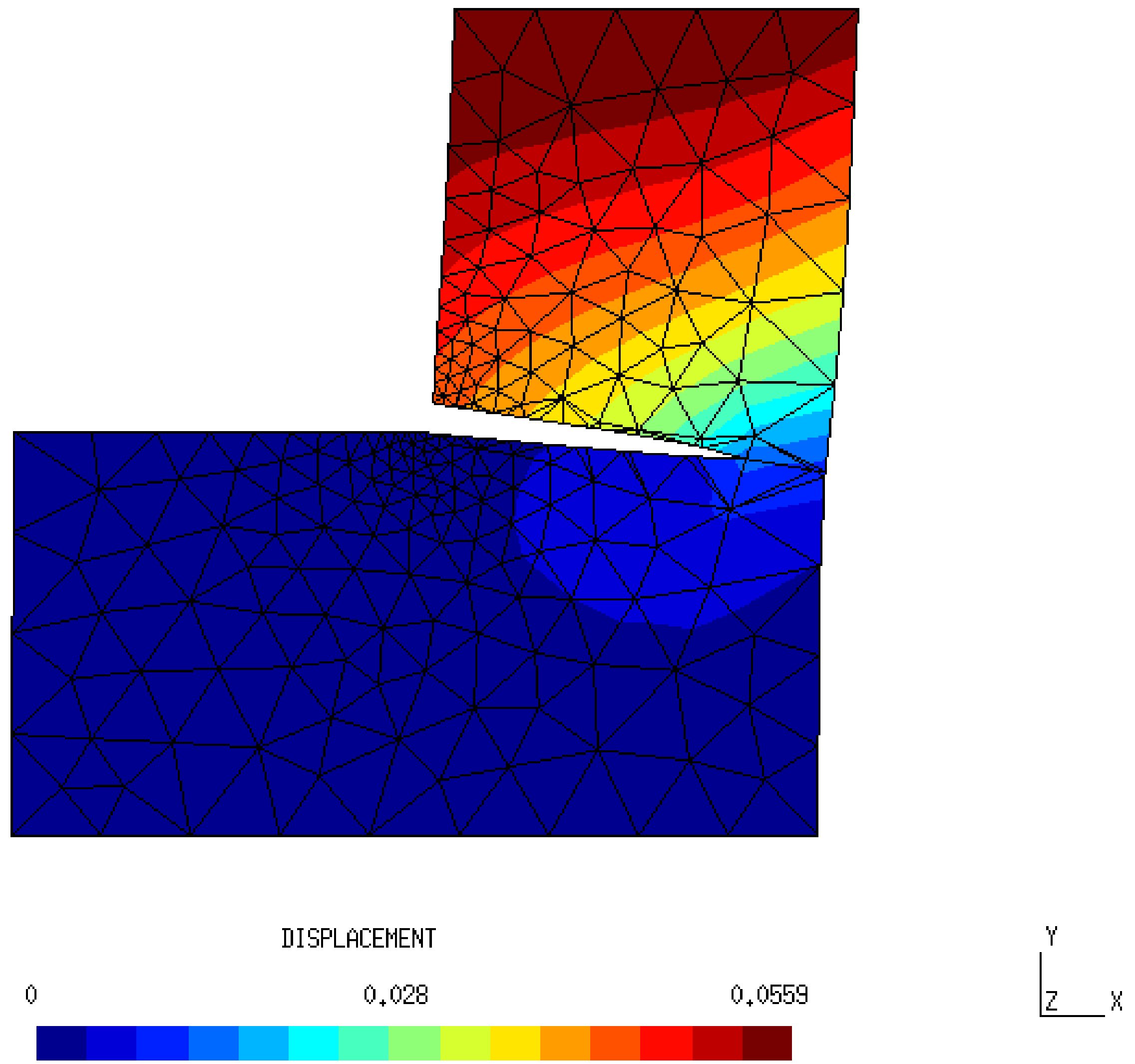
# Extended Finite Elements

## Propagation



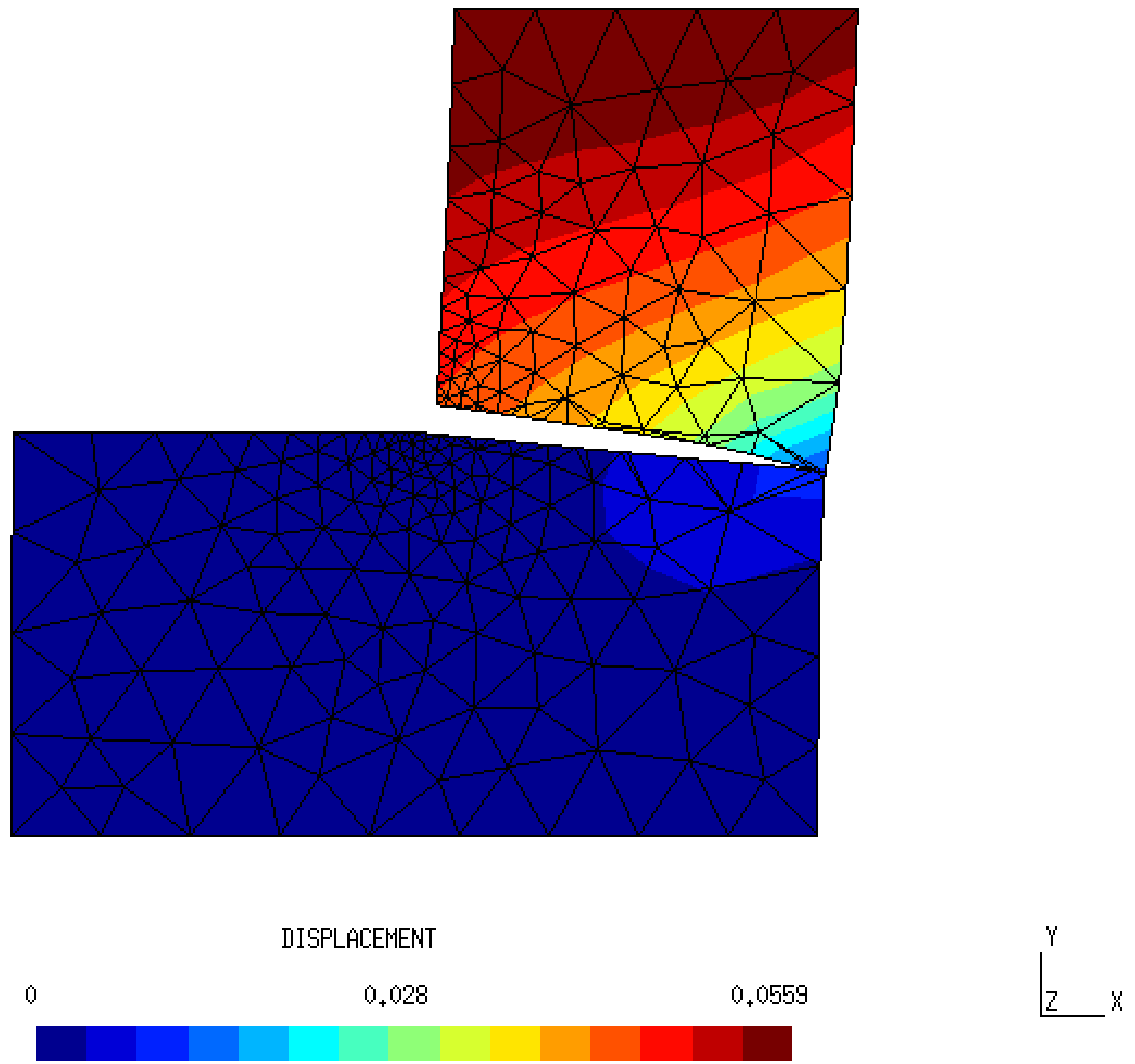
# Extended Finite Elements

## Propagation



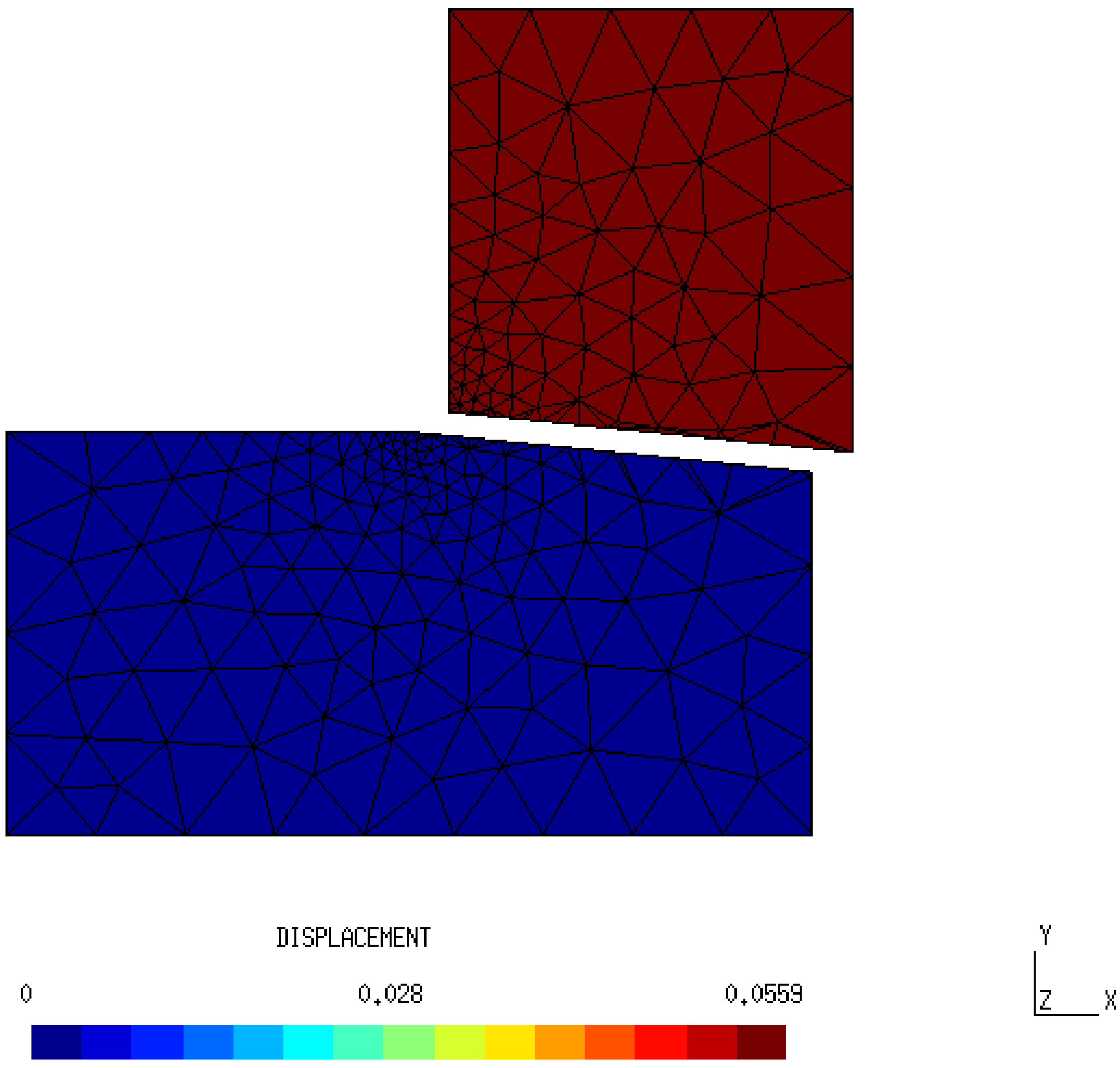
# Extended Finite Elements

## Propagation



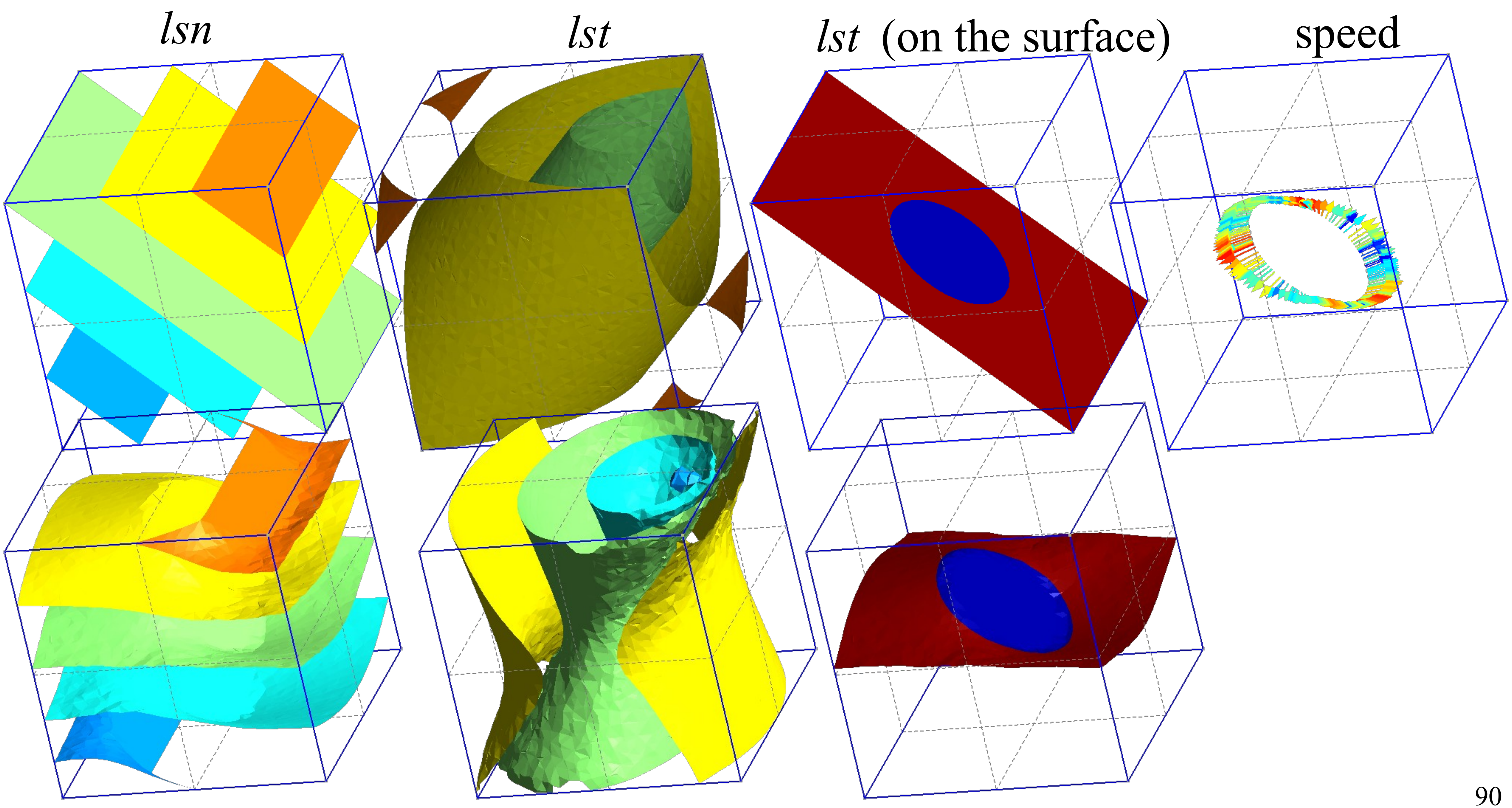
# Extended Finite Elements

## Propagation



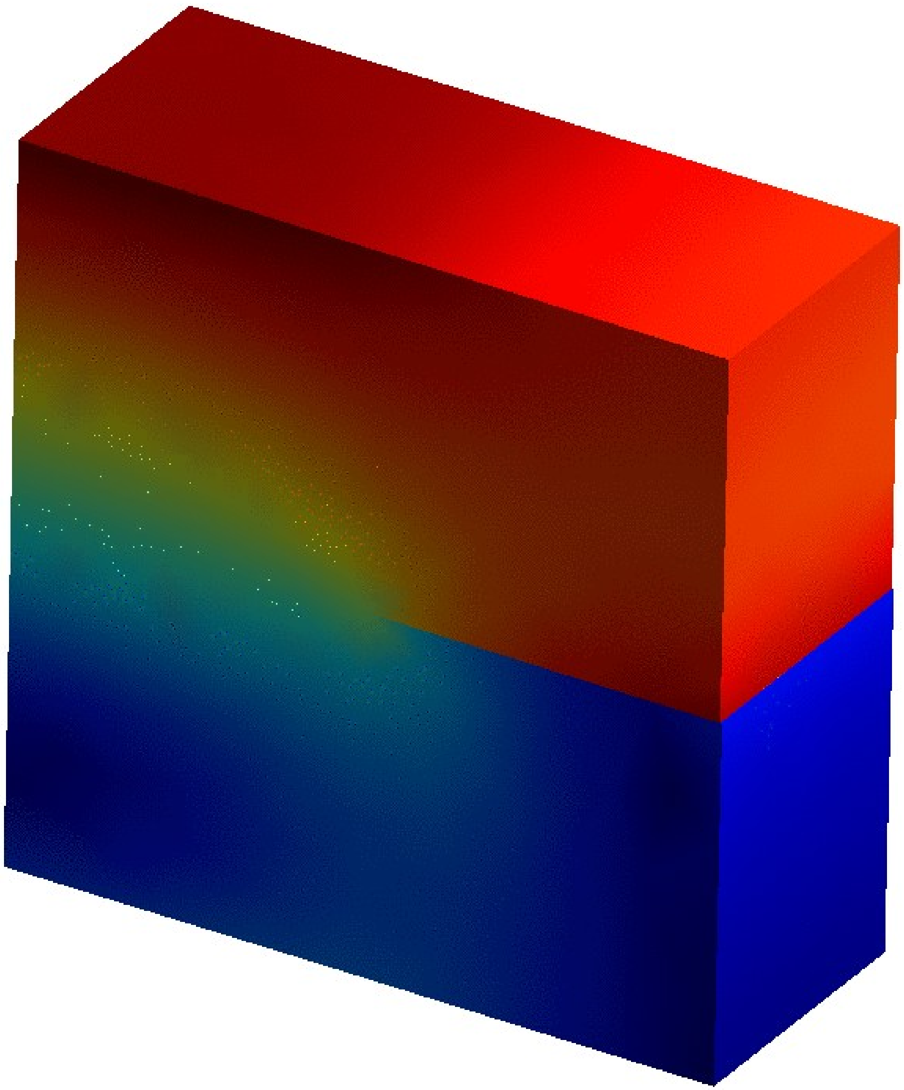
# Extended Finite Elements

## 3D Propagation



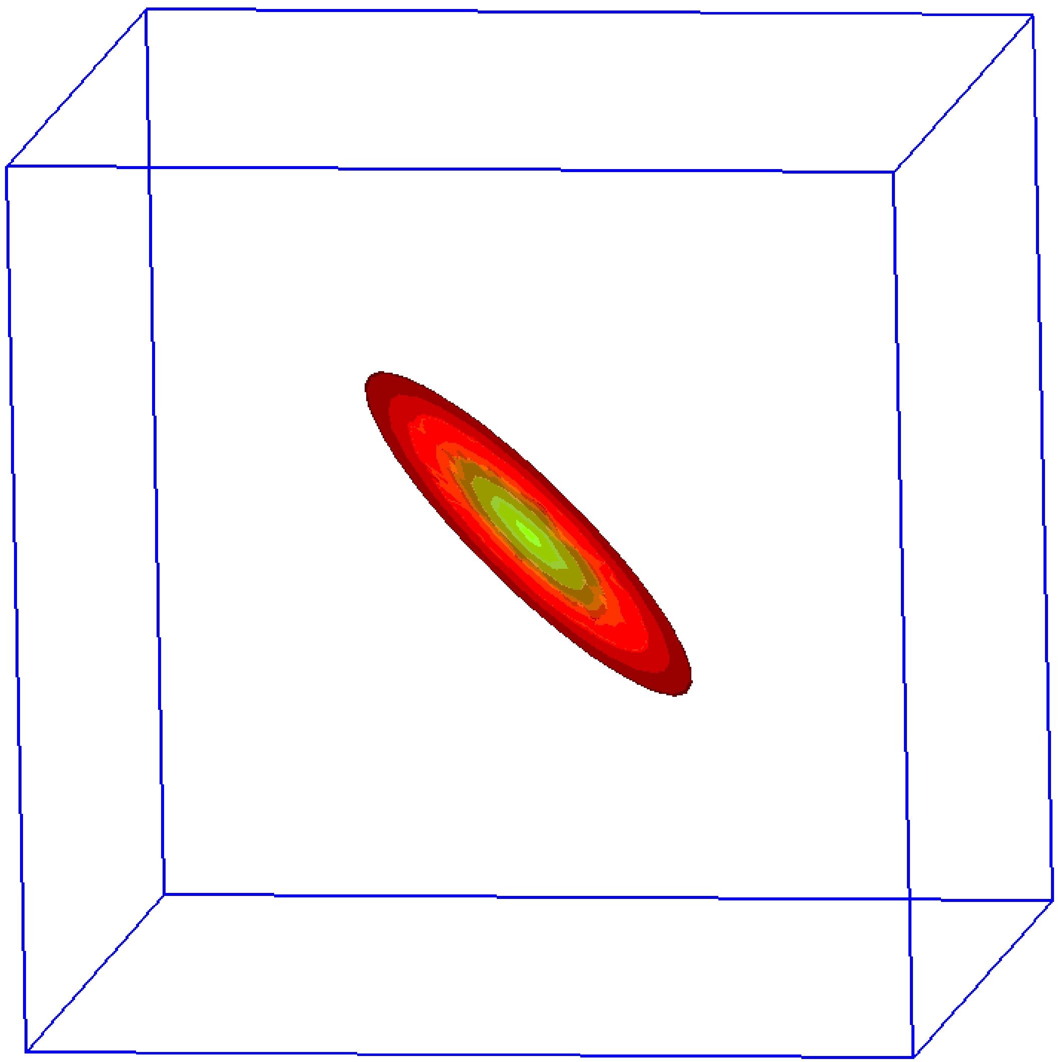
# Extended Finite Elements

## 3D Propagation

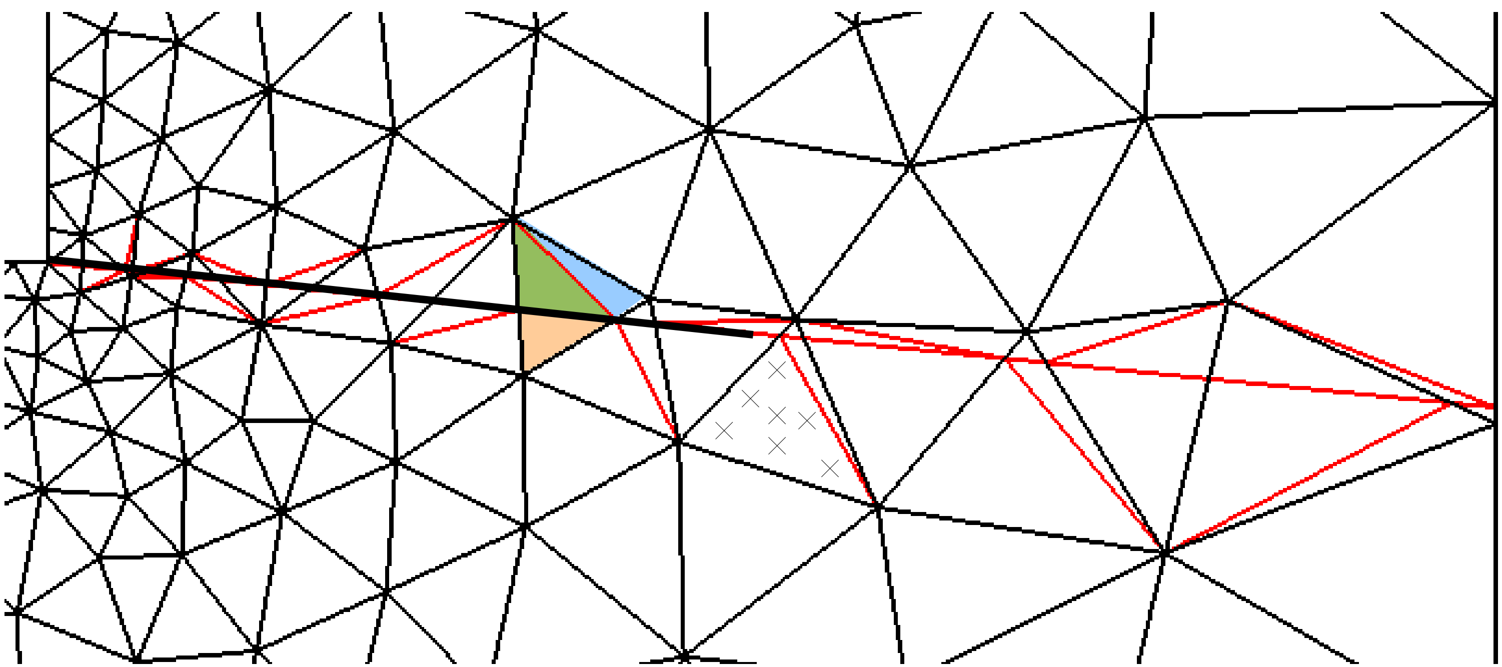


# Extended Finite Elements

## 3D Propagation



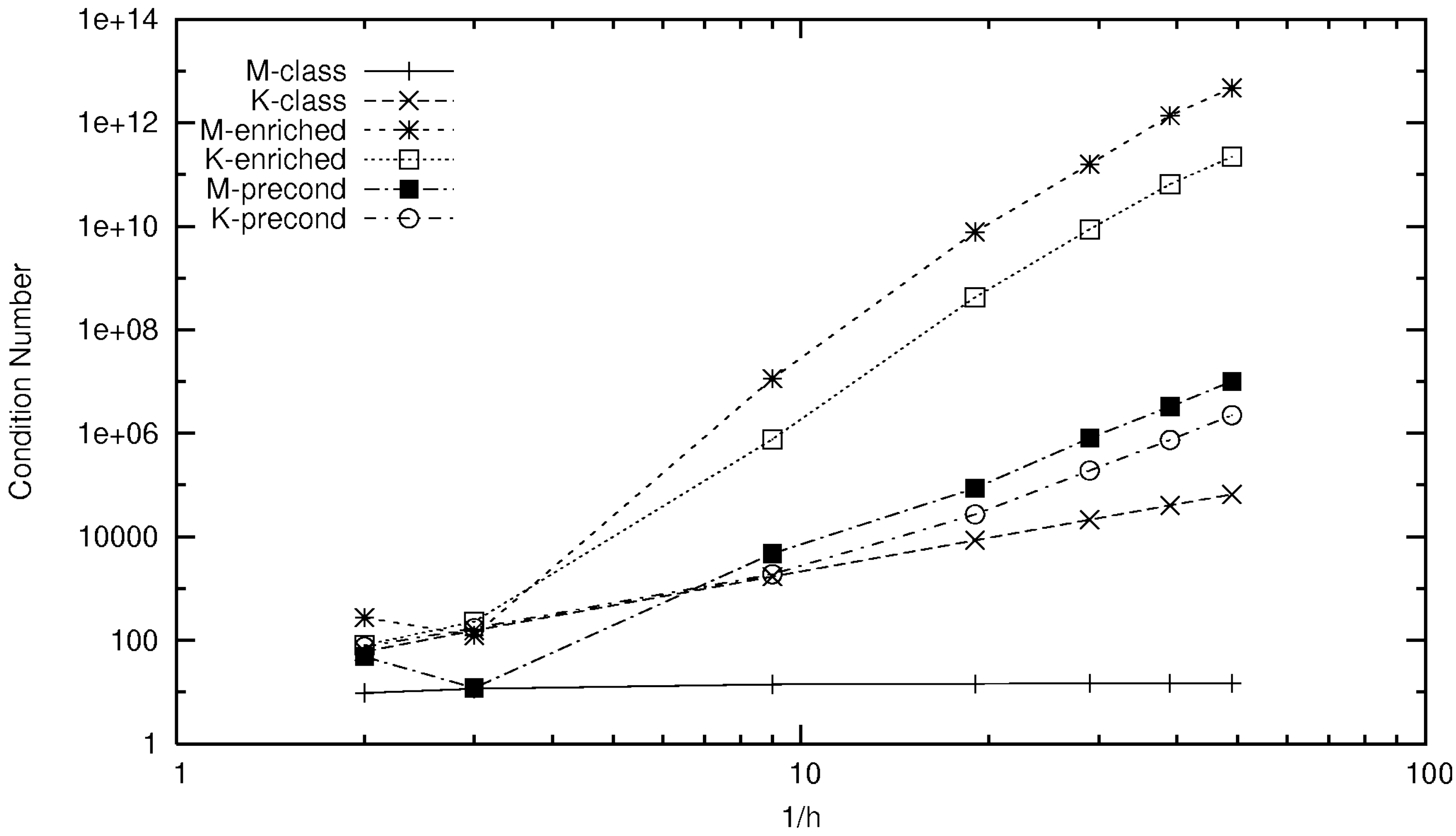
- Integration
  - One should cut elements along the interface… but one should also change the quadrature or increase the number of quadrature points because the integrand is no more polynomial



## Tricky points

- Condition number
  - If the choice of the enriched DoFs is wrongly made, then the condition number will be close to 0 (this yields a singular linear system)
    - If the crack goes close to a node $\rightarrow$ then it goes through it (at least virtually)
  - The enriched shape functions at crack tip may induce a bad condition number (they “look alike”)
    - Use of a specialized preconditionner

- Condition number



## Tricky points

- Representation valid for cracks in relatively fragile materials
  - The crack is represented using a infinitely thin line
  - The crack tip is a mathematical point, with non bounded values for the various mechanical fields
  - Propagation laws based on global energy-based laws (e.g. energy release rate G...)
- It is too restrictive to lead to good results for ductile materials

## Ductile materials

New properties

- Crack shape absolutely non trivial
- The propagation is made via a damage variable
- The level sets are used to represent at the same time

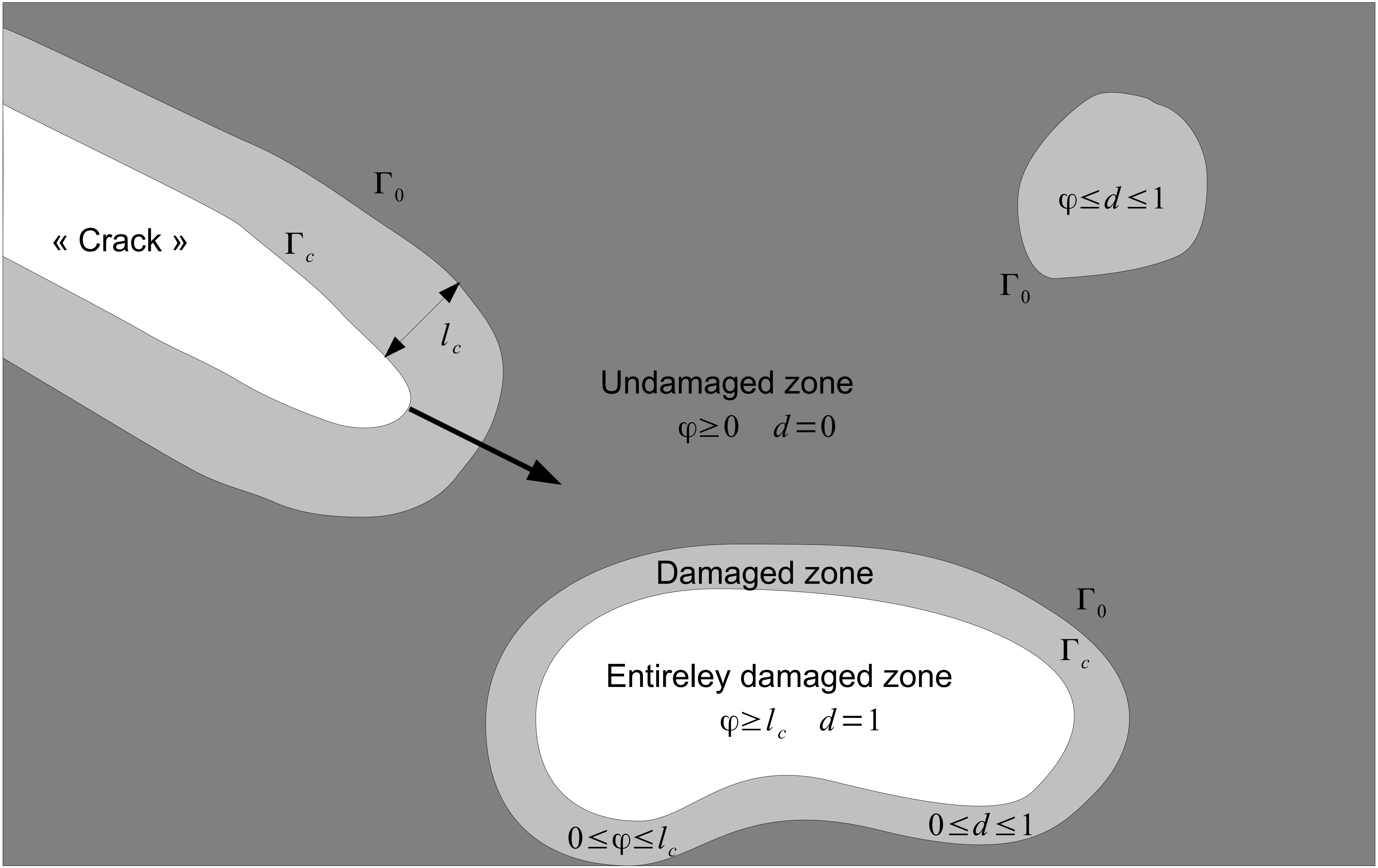
  - the damage variable d

  - the crack front (where d=1)

  - the boundary between the damaged zone (d>0) and the rest of the domain where the behavior is elastic

- Notion of "Thick" Level Set

# Thick Level Set



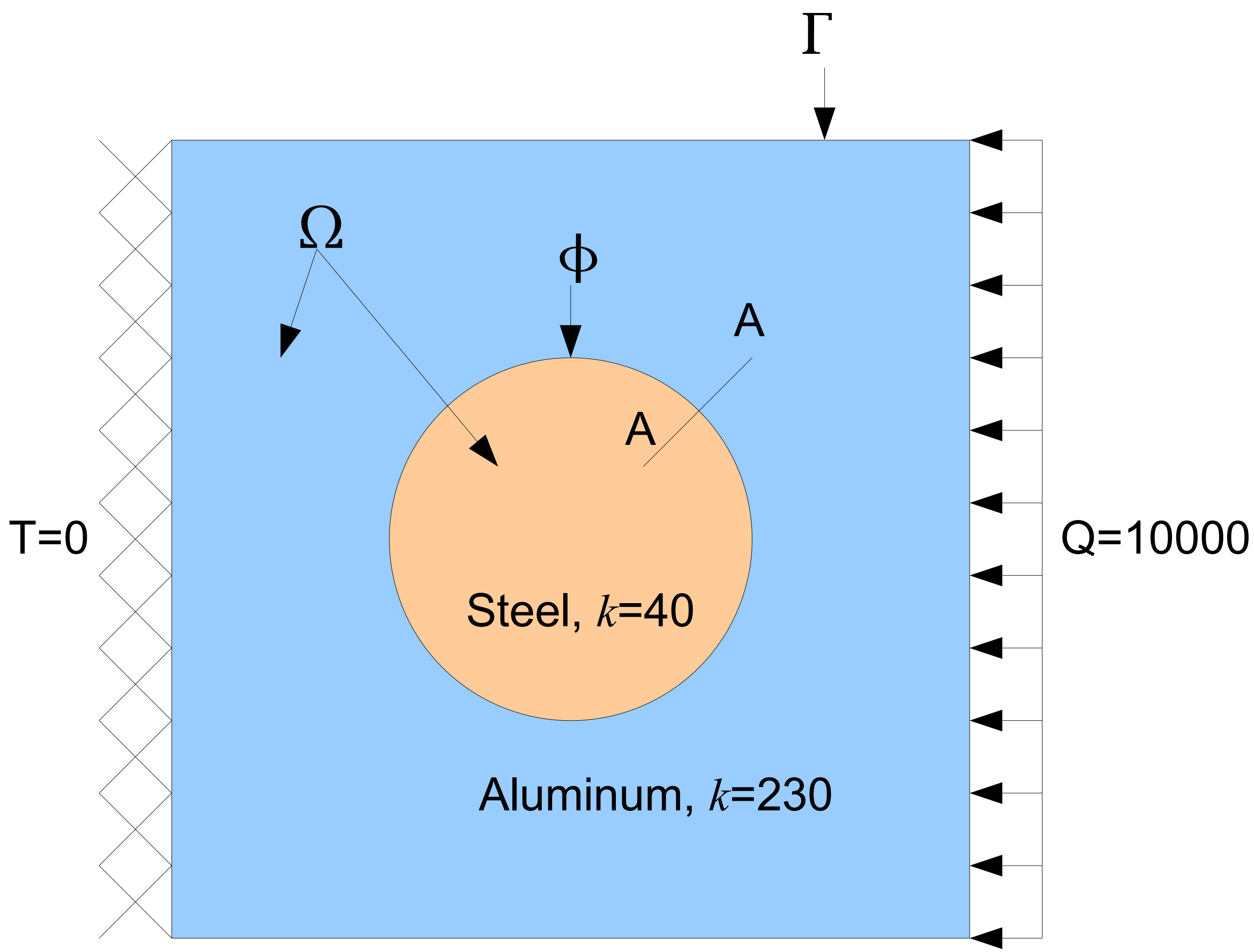
# Extended Finite Elements

## Thick Level Set

N. Moës, C. Stolz, P.-E. Bernard, and N. Chevaugeon
*A level set based model for damage growth: The thick level set Approach Int. J. Numer. Meth. Engng 2011; 86:358–380* DOI: 10.1002/nme.3069

# Problems with a jump in the gradient ("dual" variable)

- Thermal transfer model problem



## Bi-material interface

- The interface is represented by the following level-set :

  $$\phi = \{ x \in \Omega \;\; / \;\; ls(x) = 0 \}$$

- This interface can be of complex geometrical shape and / or changing in time
  - Again, no mesh conformity

## Bi-material interface

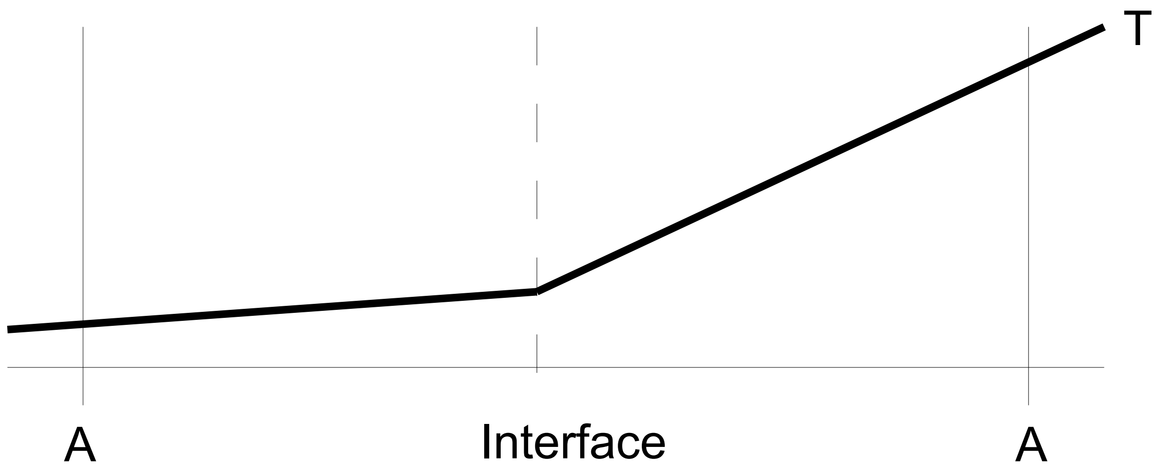
- Finite element model (again, homogeneous boundary conditions)

$$\text{Find } u \in H_0^1(\Omega) \text{ s.t.}$$
$$a(u, v) = b(v) \quad \forall v \in H_0^1(\Omega)$$

with

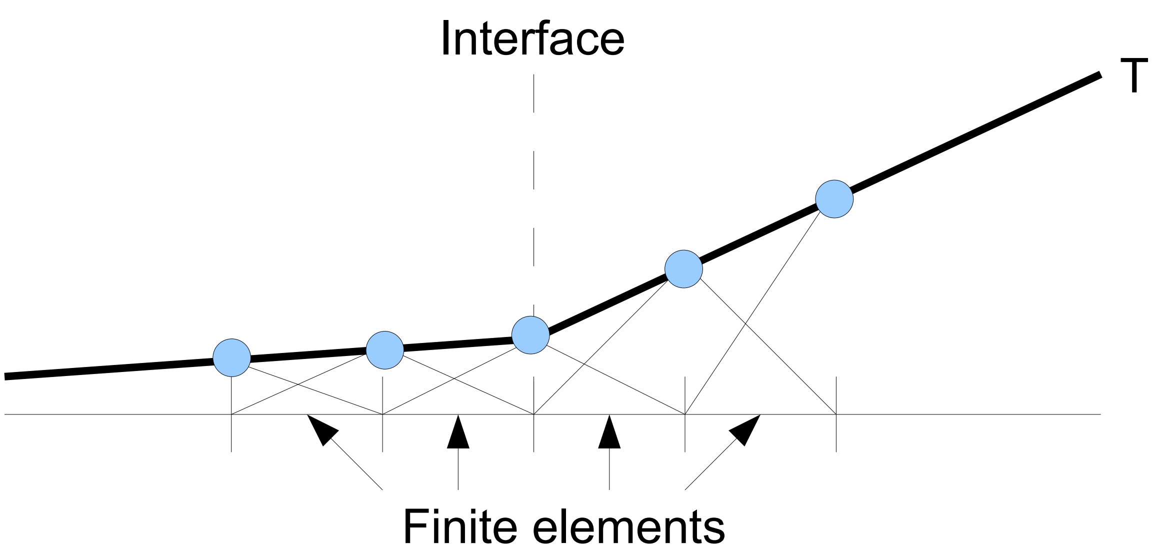
$$a(u, v) = \int_\Omega k (\nabla u \cdot \nabla v) \, d\Omega \qquad b(v) = \int_\Gamma f(x) \cdot v \, d\Gamma$$

- We want to be able to represent the right temperature profile along the interface
  - A-A cut : Theoretical temperature profile



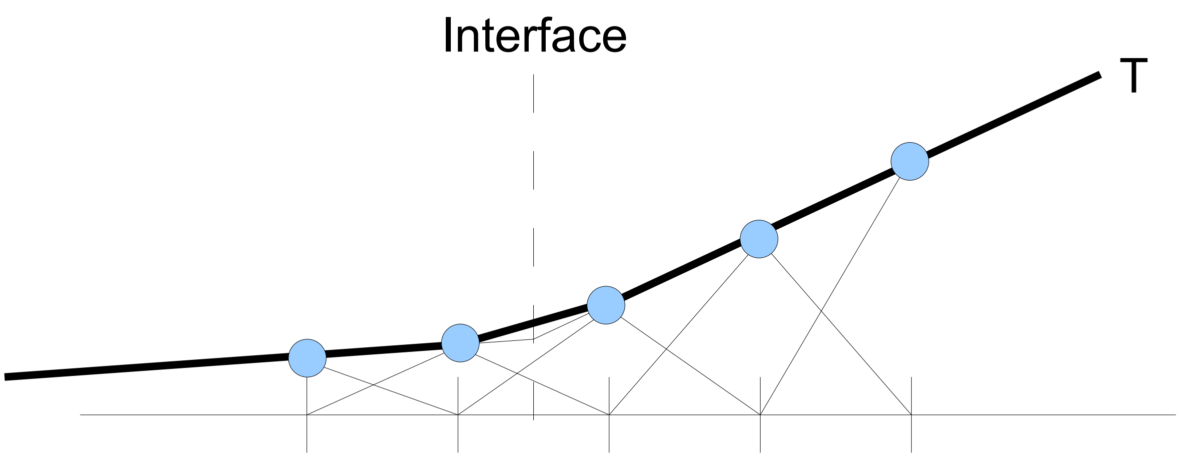
## Bi-material interface

- The discontinuity is on the derivative of T
- If the interface is exactly on element boundaries, then the discontinuity is naturally belonging to the function space



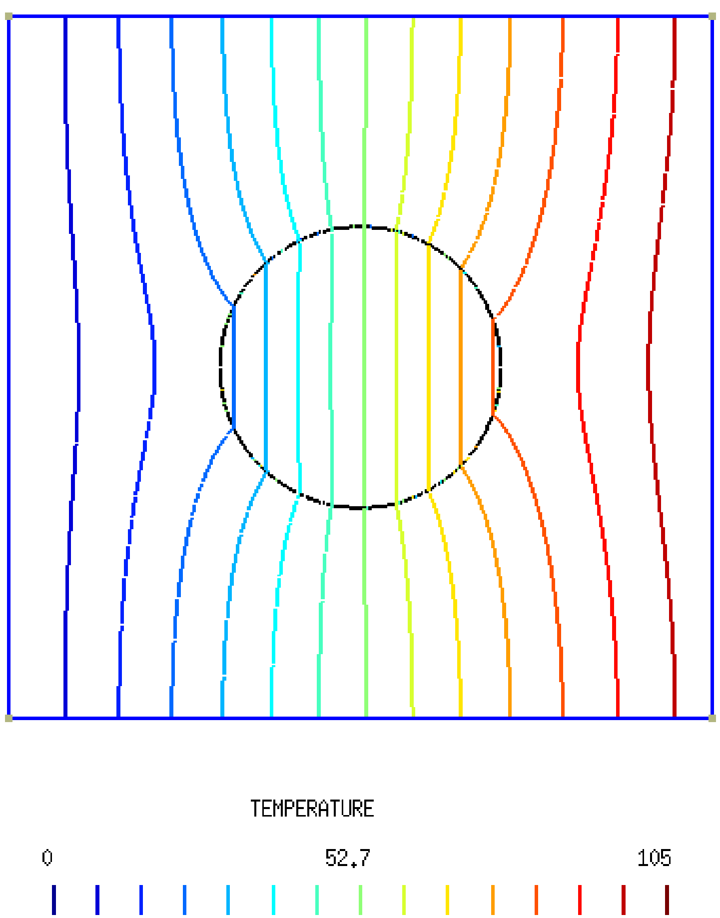
## Bi-material interface

- The discontinuity is on the derivative of T
- If the interface is not exactly on element boundaries, then ...

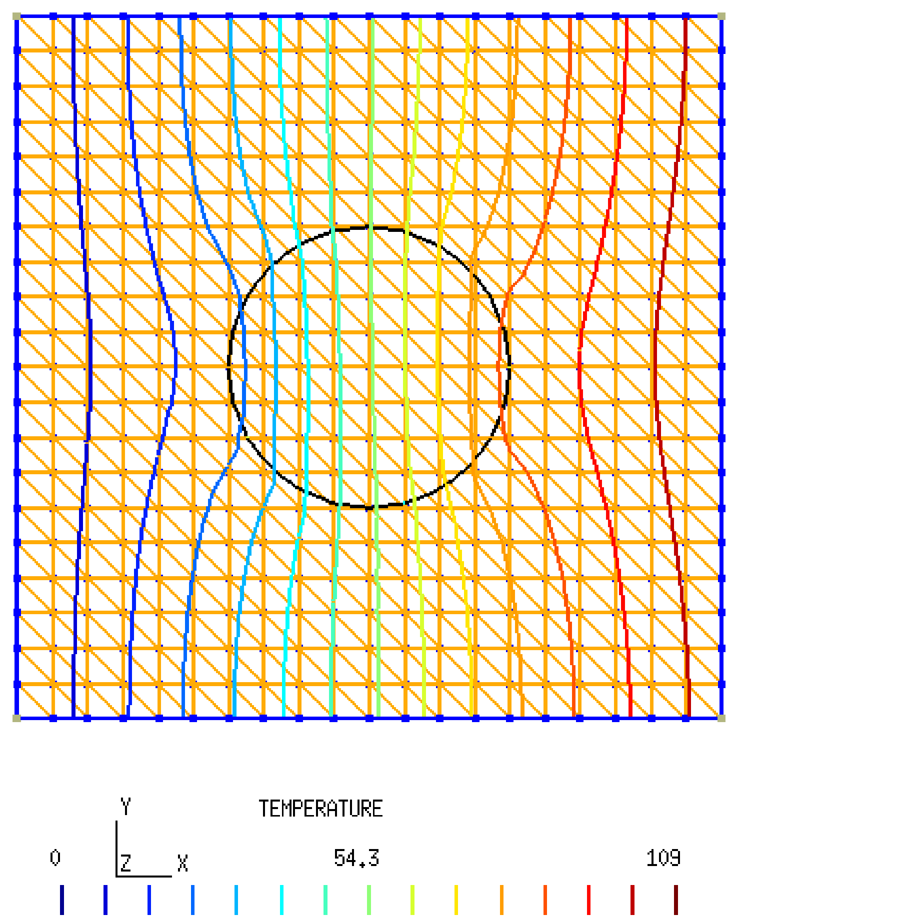


- This explains the very approximate solution...

Exact solution




Standard F.E. solution



# Extended Finite Elements

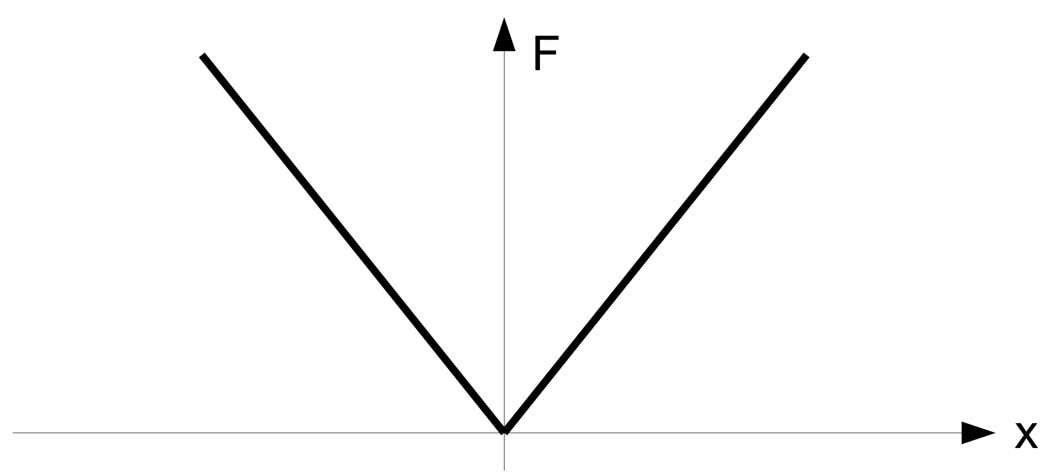
## Bi-material interface

- The idea here is to enrich the function space so that the discontinuity (in the gradient) belong to it.

$$u(x)=\sum_{i\in\Omega}\lambda_i N_i(x)+\sum_{j\in C}\lambda_j^* N_j(x)\cdot F(x)$$

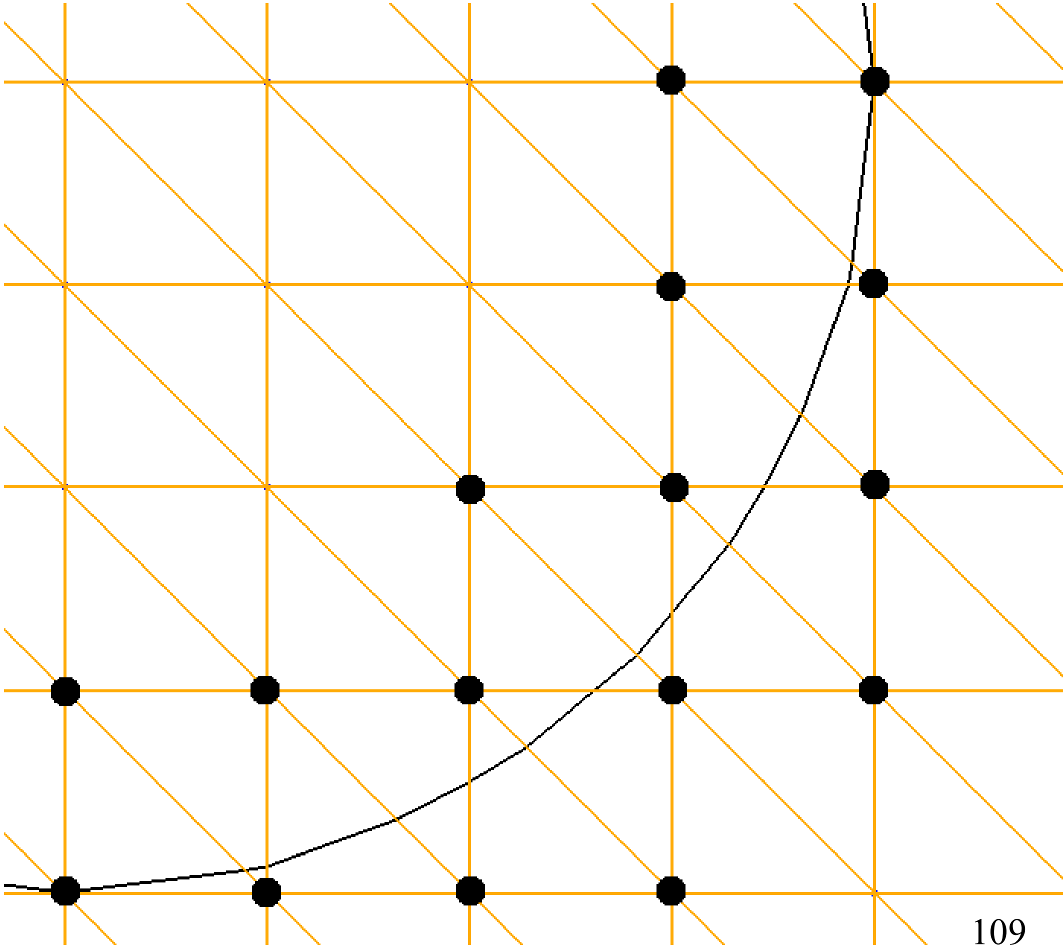
There exists many possibilities. One very simple is using directly the absolute value of the level-set.

$$F_1(x)=|ls(x)|$$



## Bi-material interface

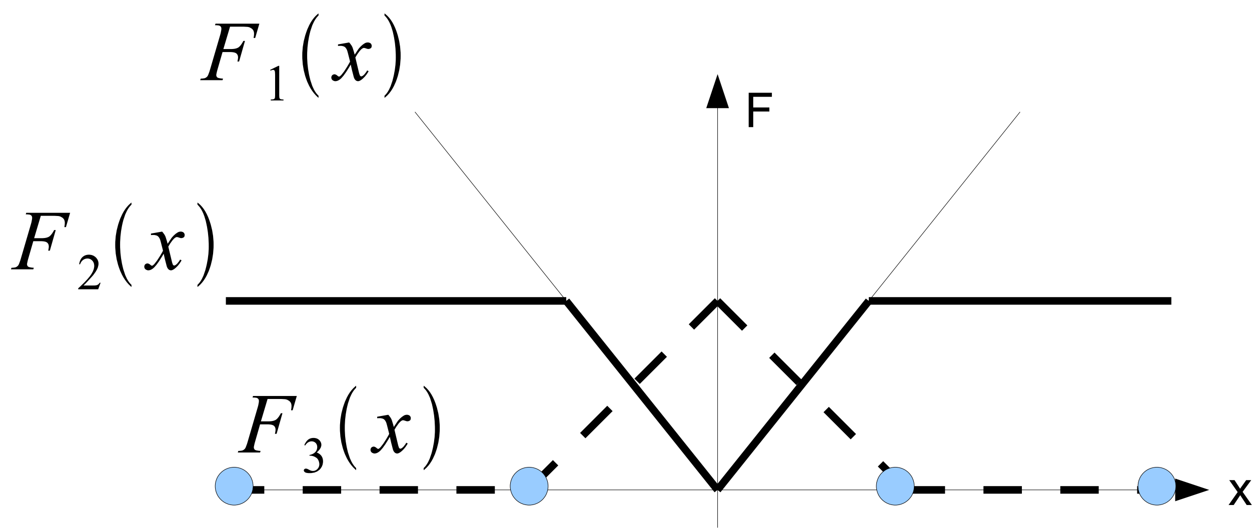
- Definition of the set C of the enriched nodes
  - This time, the nodes where at least one element of the support are cut must be enriched
  - In particular, if the interface is along edges, there is no enrichment



- Here are some other enrichment functions

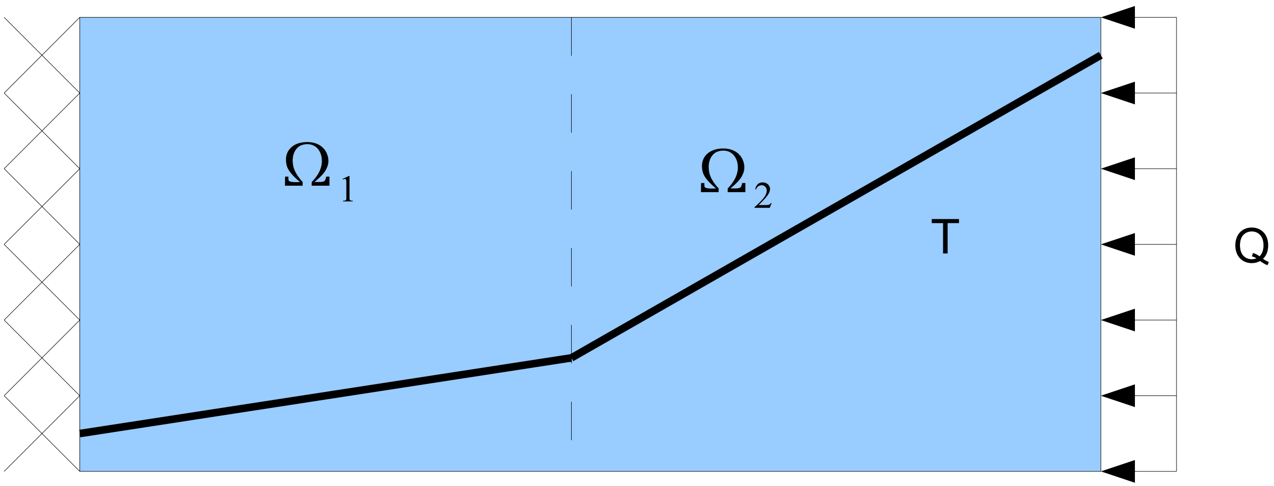
$$F_2(x) = \begin{cases} |ls(x)| \text{ in cut elements} \\ 1 \text{ else} \end{cases}$$

$$F_3(x) = \sum_i |ls_i| \cdot N_i(x) - \left| \sum_i ls_i \cdot N_i(x) \right|$$



## Bi-material interface

- Practically speaking, $F_3(x)$ gives the best results
  - On a simple model problem, the functions $F_2(x)$ and $F_3(x)$ are unable to give back the exact solution (which is linear by parts) when the interface does not belong to the mesh, but $F_1(x)$ does.

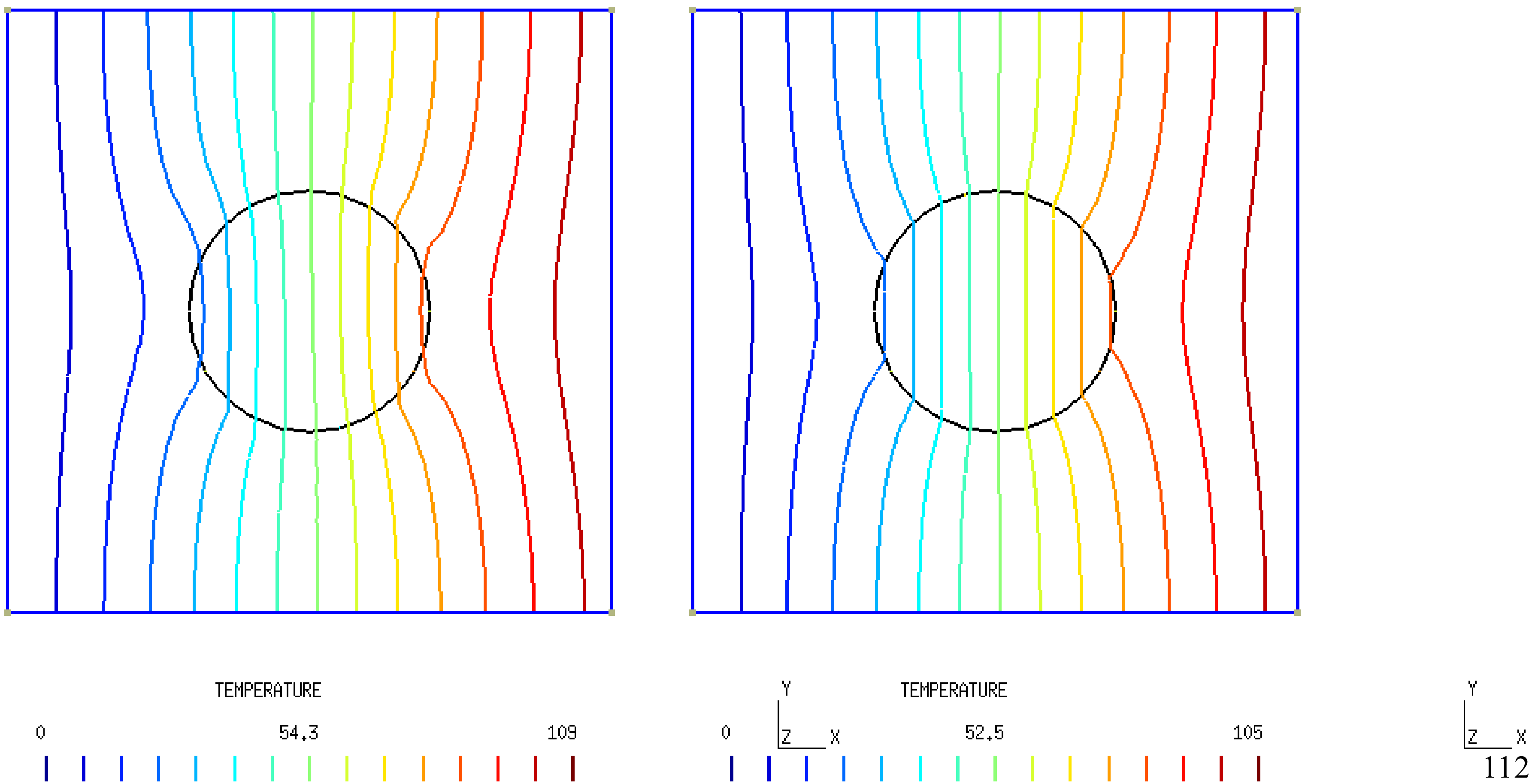


# Bi-material interface

- Comparison of the solution with the right enrichment function

Solution without enrichment

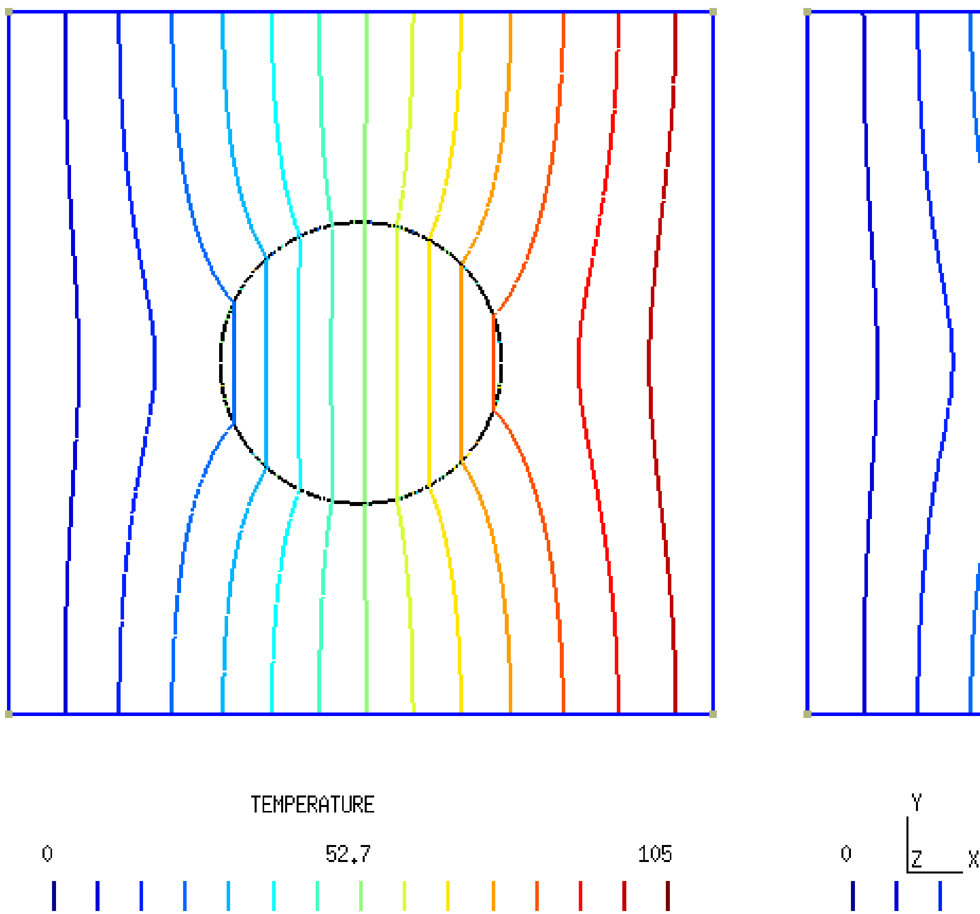
Solution with enrichment



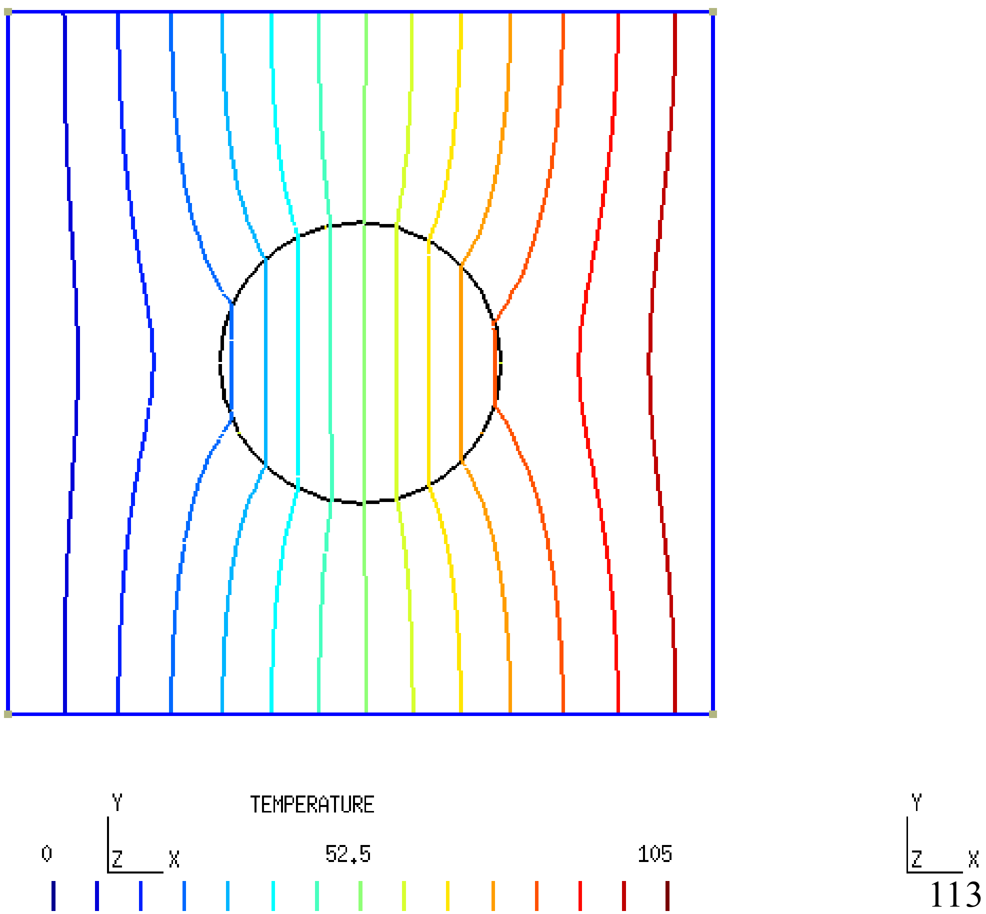
## Bi-material interface

- Comparison of the solution with the right enrichment function

Exact solution



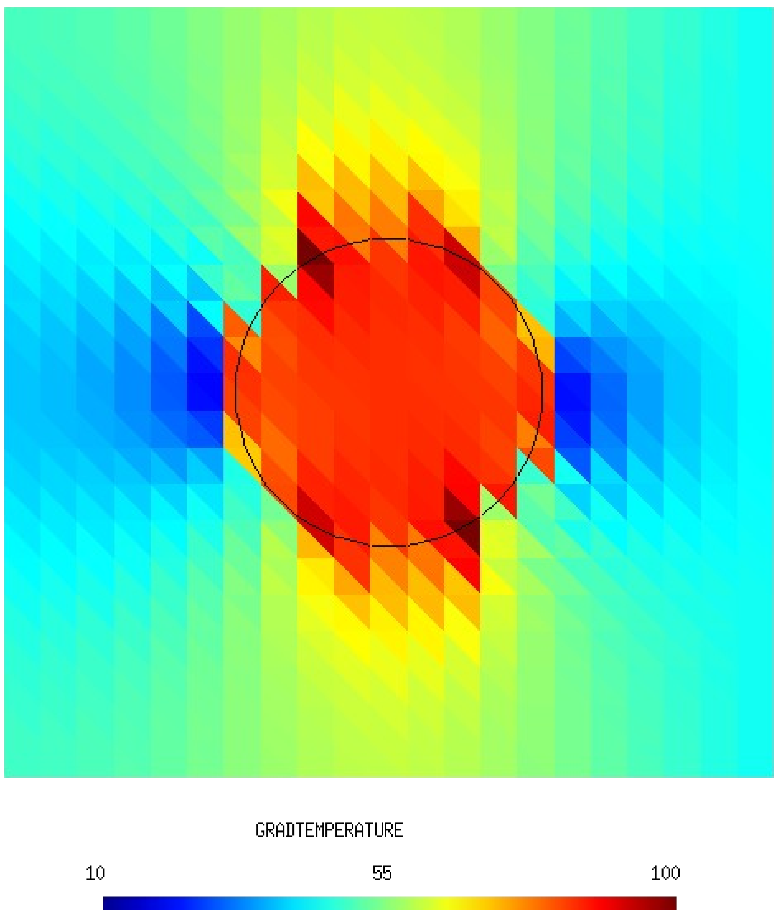

Solution with enrichment

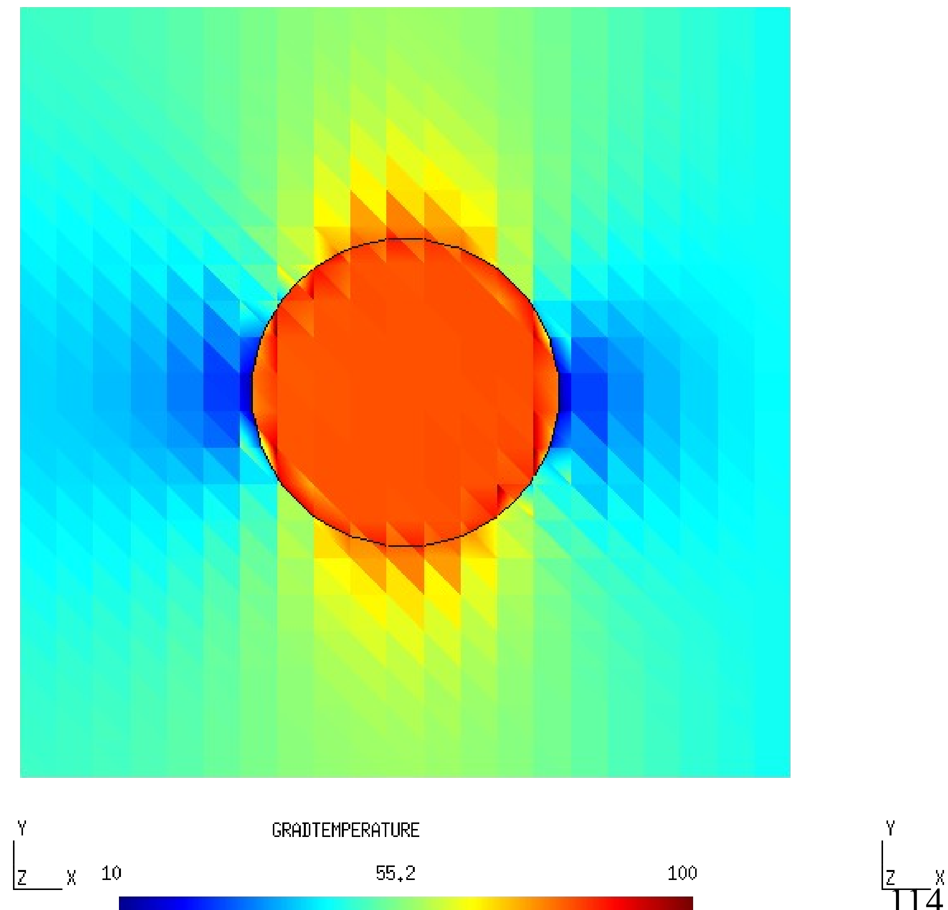


- Comparison of the gradient

Solution without enrichment

Solution with enrichment



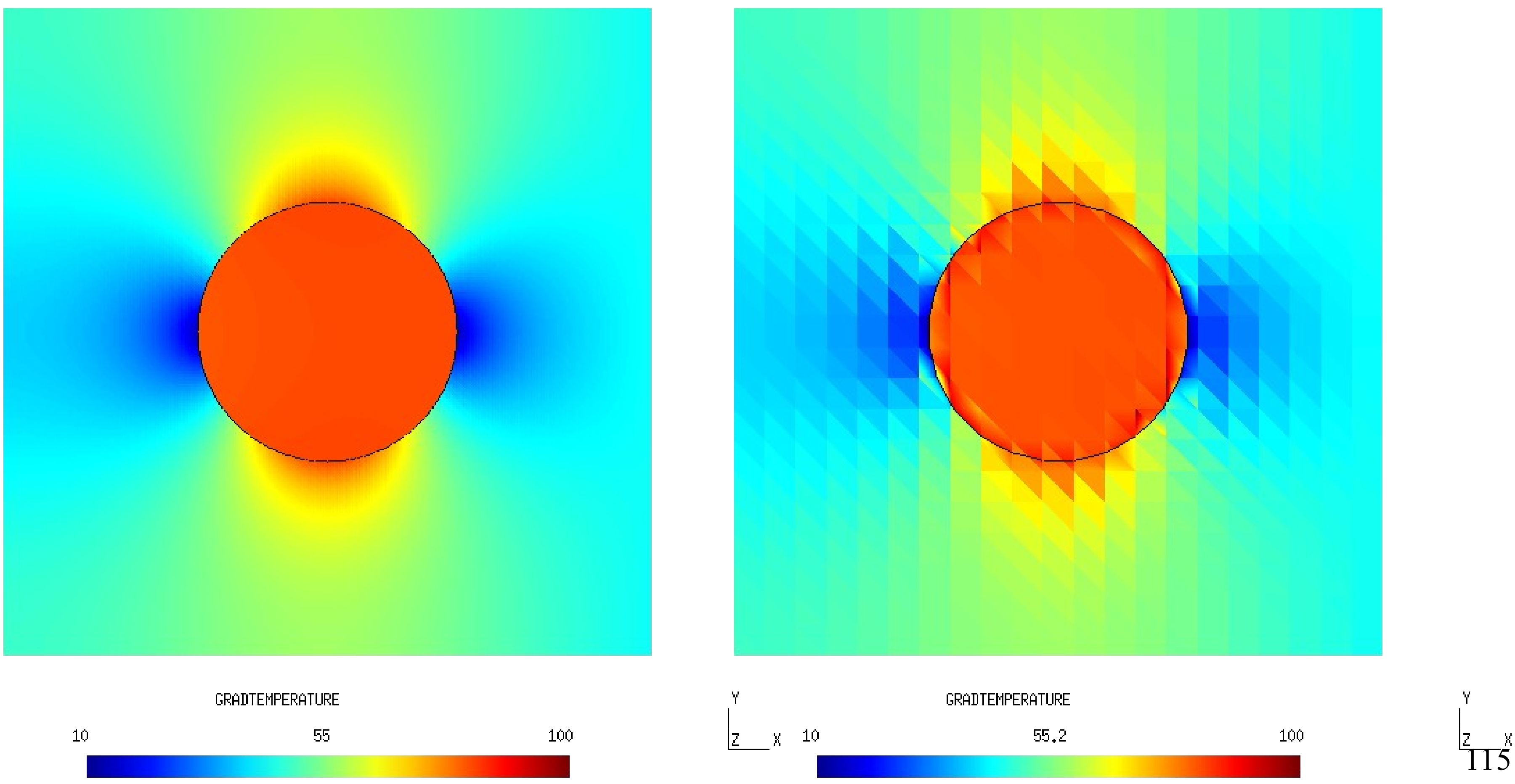
# Extended Finite Elements

## Bi-material interface

- Comparaison du gradient

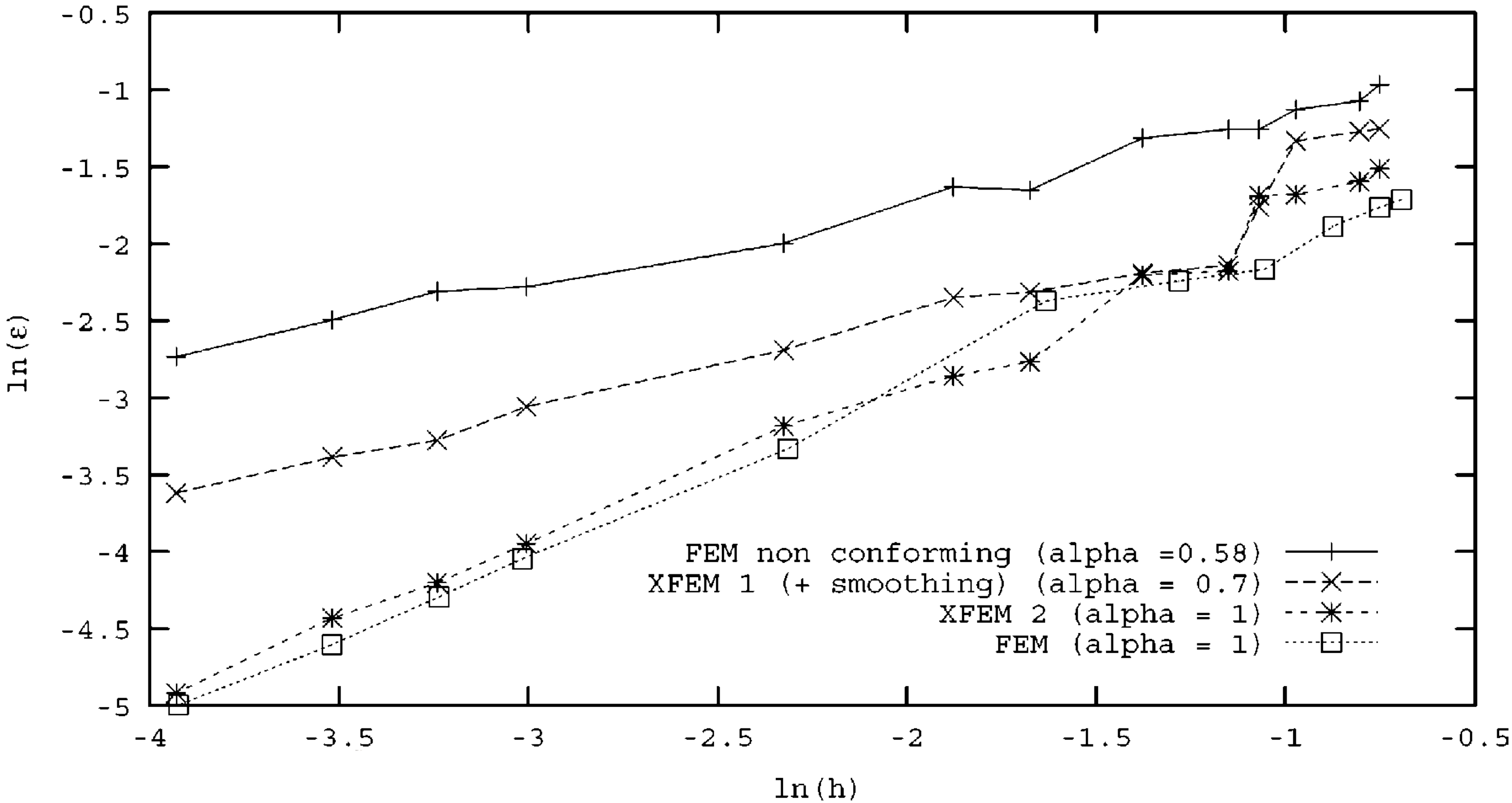
Exact solution

Solution with enrichment



# Bi-material interface

- Convergence



## More applications

- Discontinuities in the primal variable
  - Cracks
    - non linearities, plasticity
    - Dynamic effects (fast propagation)
  - Solidification front propagation
    - hydrogels
  - Discontinuities in the derivatives
    - Homogeneization

## More applications

- Applications to other materials
  - Confined plasticity
  - Composites materials
  - Piezoelectric materials
  - Etc...

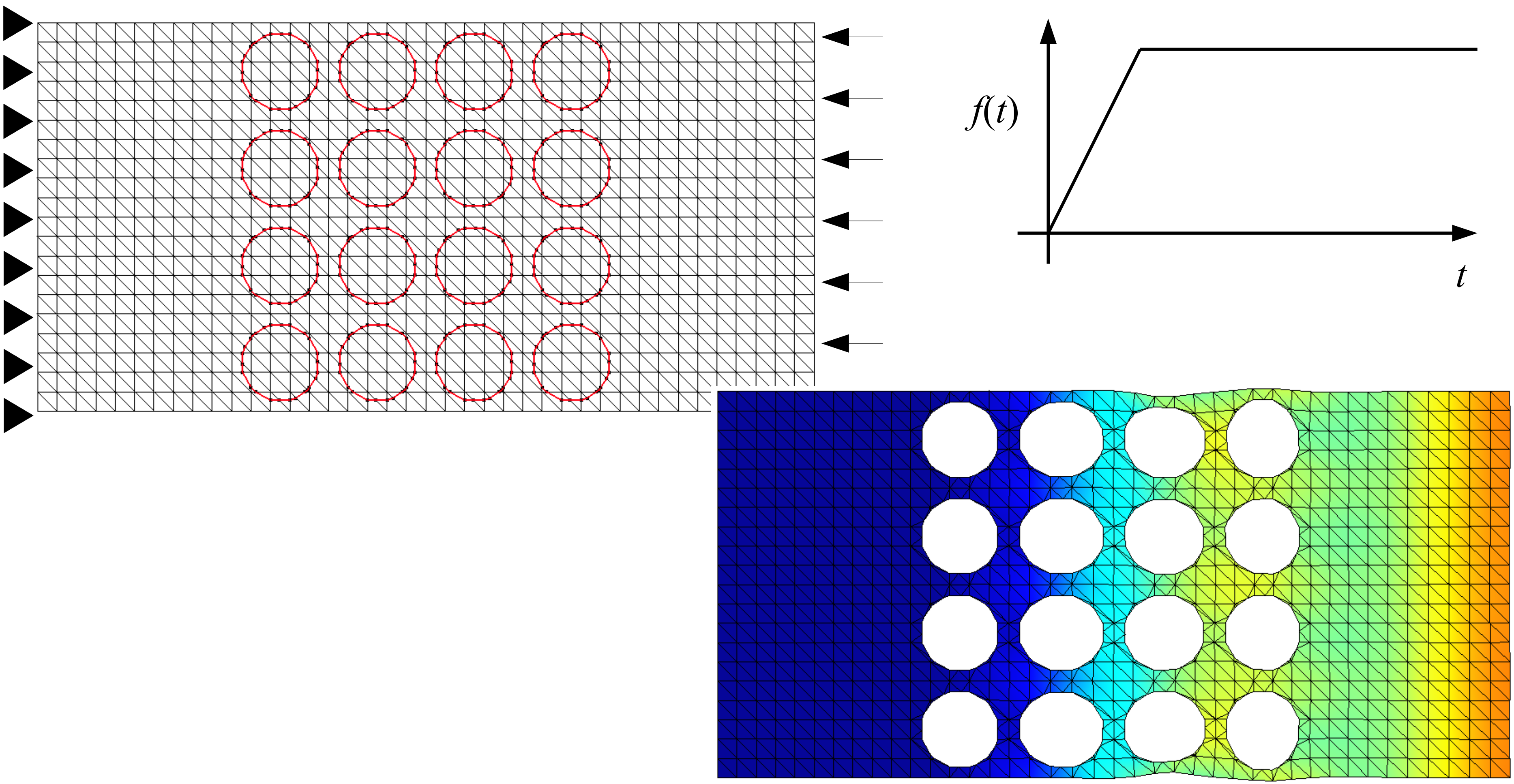
## More applications

- Direct interfaces with CAD for numerical simulations
  - From an explicit representation to an implicit representation
  - Non conforming boundaries
  - Imposition of boundary conditions
  - Non conforming material interfaces

## More applications

- Applications in explicit dynamics
  - Non conforming geometry → issue with the critical time step
  - Propagation of unstable cracks (change of function space at the crack tip → leads to problems of energy conservation)

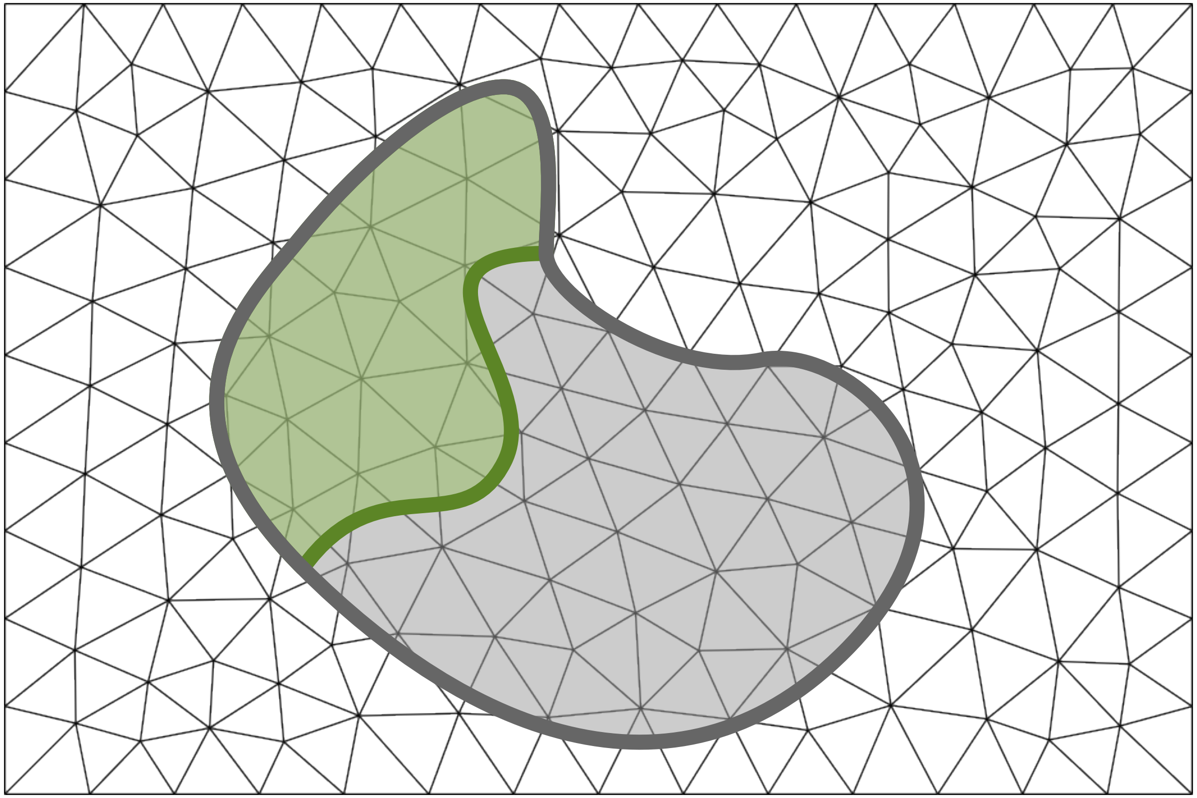
- Explicit dynamics : case without enrichment



The issue of boundary conditions on implicit volumes

## Goal

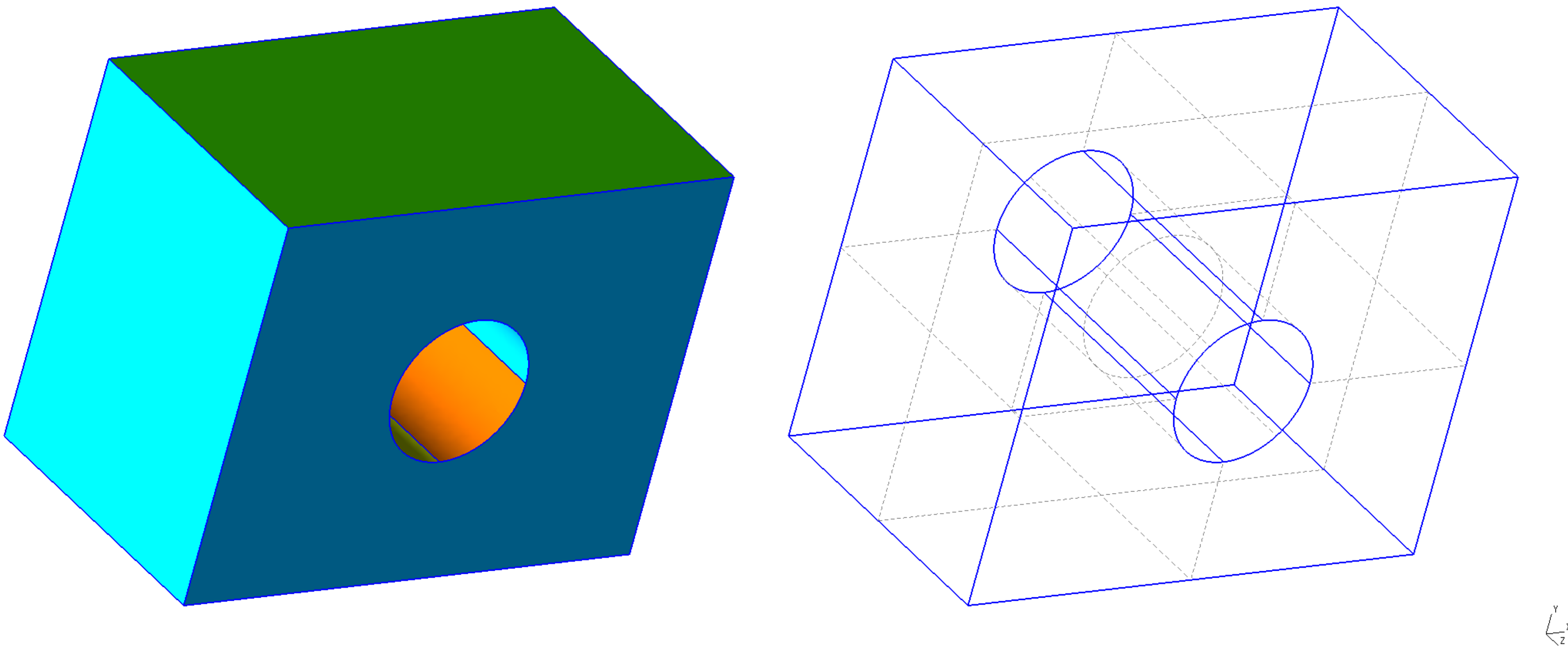
- Free the mesh from geometrical constraints
  - Boundaries of the problem
  - And/or interfaces between different materials



## Applications

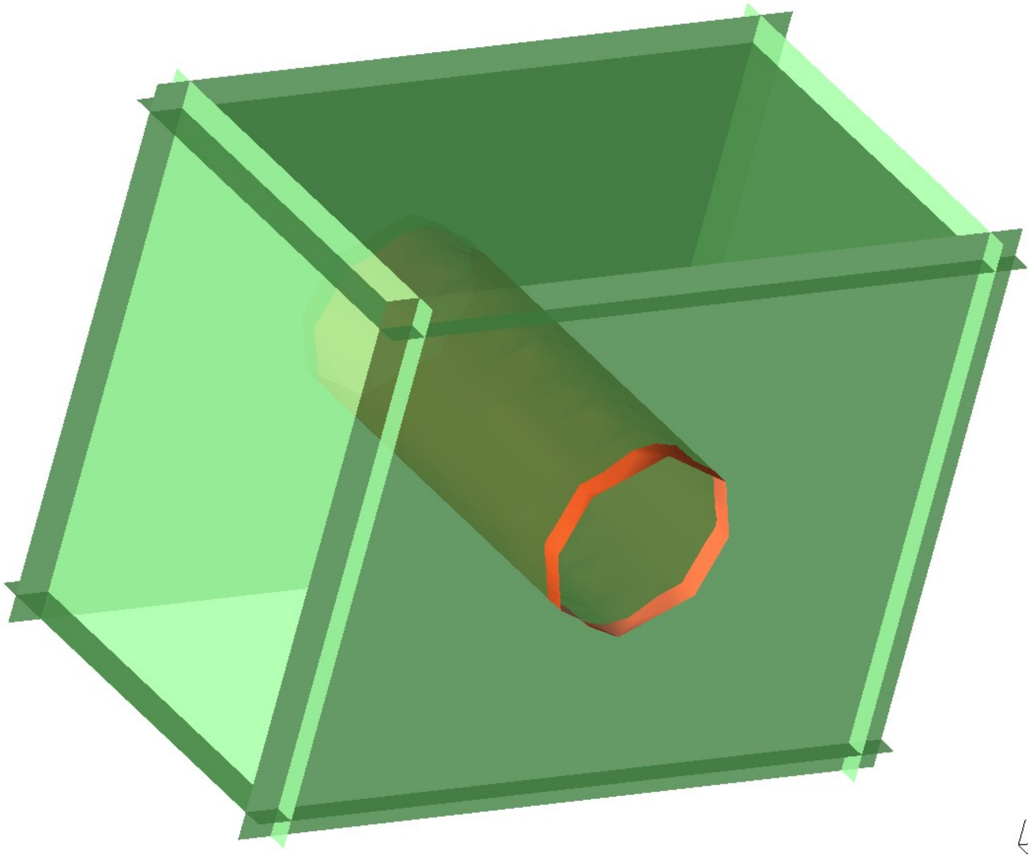
- Direct use of CAD models for the analysis
  - Mesh generation shall be minimalistic
- Use of “dirty” geometrical date not usually adapted to mesh generation
  - Tomography, biomedical applications
- Mobile interfaces
  - Thermoplastic mold filling
  - Topological shape optimization
- Contact problems in mechanical engineering

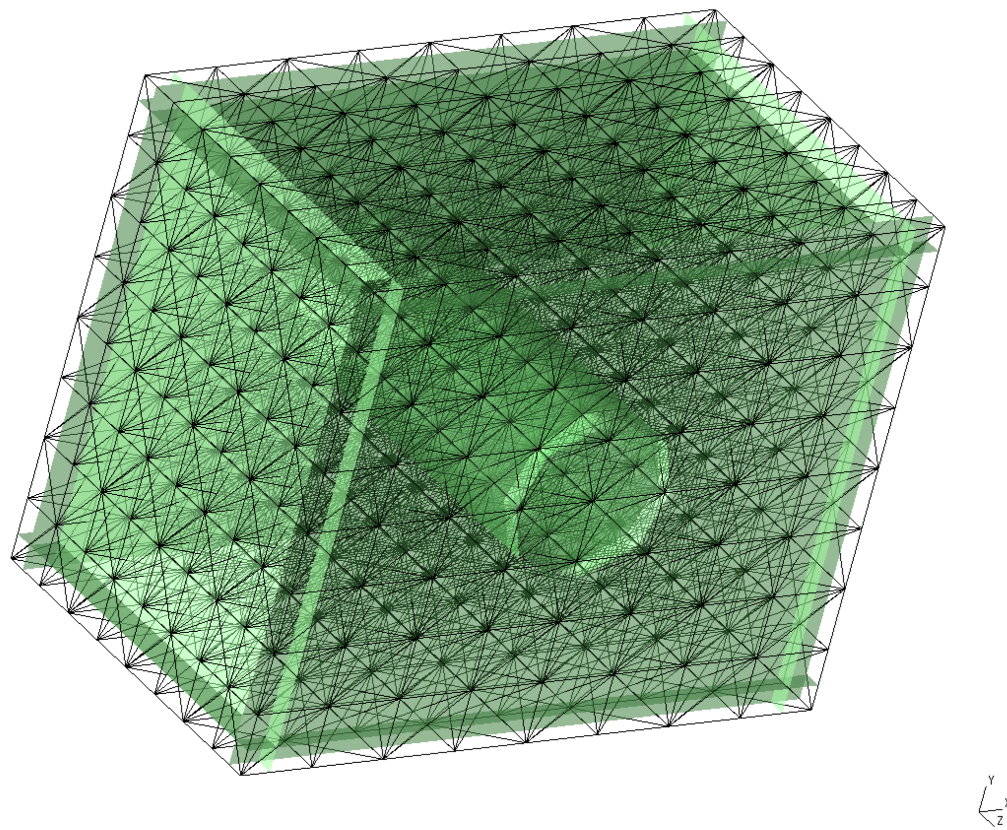
- From a traditional CAD (B-rep) representation ...



## CAD interface

- … To an implicit representation with level-sets





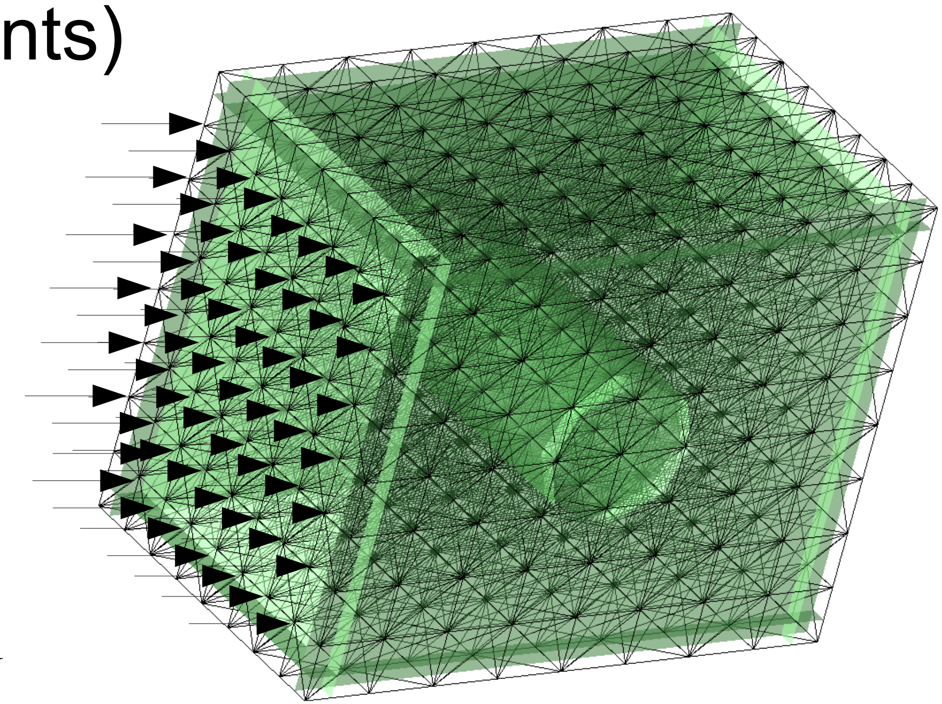
- How to apply boundary conditions
  - Neumann/natural boundary conditions (e.g. pressure, forces, gradients)
  - Using integration (it is a linear form)

Find $\bar{u}$ s.t.

$$a(\bar{u},\bar{v})=b(\bar{v})\quad \forall \bar{v}$$

$$a(\bar{u},\bar{v})=\int_{\Omega} \bar{\bar{\nabla}}^{s}\bar{u}:\bar{\bar{\bar{\bar{D}}}}:\bar{\bar{\nabla}}^{s}\bar{v}\,d\Omega$$

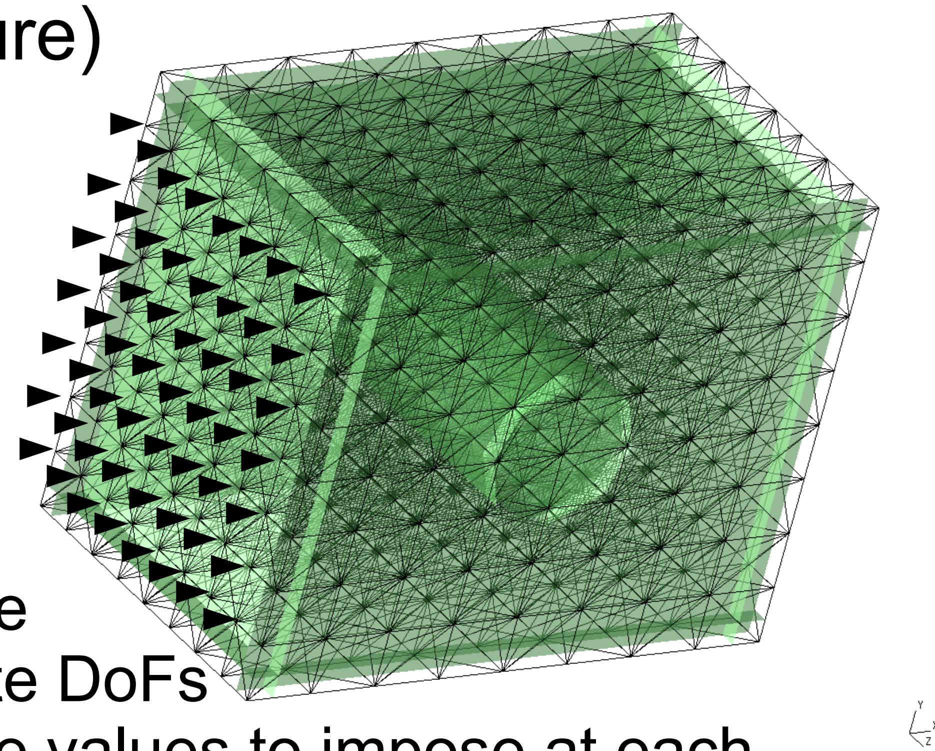
$$b(\bar{v})=\int_{\Omega} \bar{f}\cdot\bar{v}\,d\Omega+\int_{\Gamma_N} \hat{\bar{f}}\cdot\bar{v}\,d\Gamma_N$$



  - Beware ! The integration is made on a domain $\Gamma_N$ (or $\Omega$) that cut elements in the mesh

# Boundary conditions

- How to apply boundary conditions
  - Dirichlet/essential boundary conditions (e.g. : displacements, temperature)
    - “standard” FEM elimination of DoFs and adding a contribution in the right hand side
    - Here, the domain $\Gamma_D$ on which to apply this method is non conforming therefore one cannot simply eliminate DoFs - one needs to compute the values to impose at each concerned DoF; so that the “right” Dirichlet BC is obtained on the boundary



# Dirichlet boundary conditions

- Example 1: a simple Laplacian

$$\text{Find } u \in V_1 = \left\{ v \in H^1(\Omega),\, v_{|\Gamma_D} = u_D \right\} \text{s.t.}$$

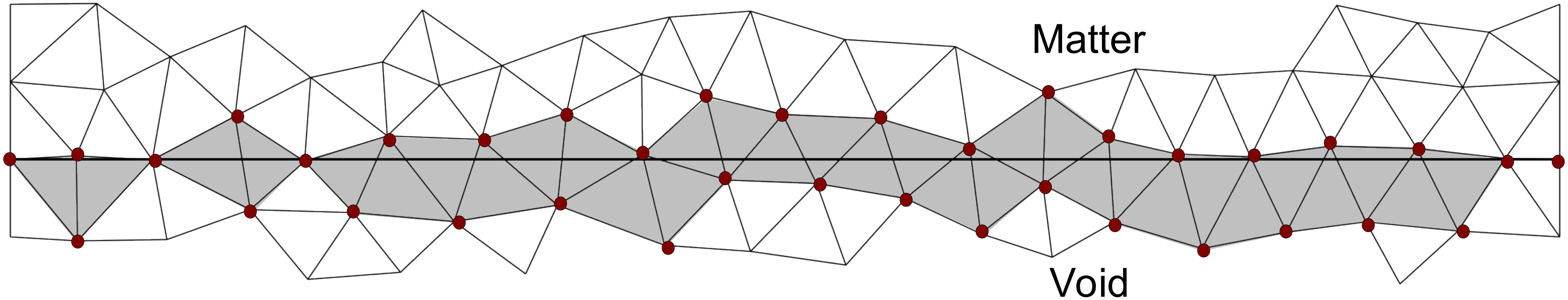
$$a(u,v) = b(v) \quad \forall v \in V_0 = \left\{ v \in H^1(\Omega),\, v_{|\Gamma_D} = 0 \right\}$$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, d\Omega$$

$$b(v) = \int_{\Gamma_N} f \cdot v \, d\Gamma$$

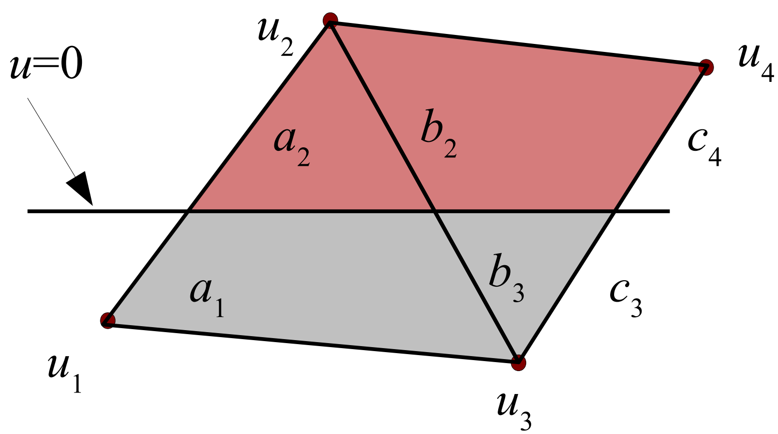
## Dirichlet boundary conditions

- Example 1
  - Dofs which are concerned : those where the support cuts the boundary...



## Dirichlet boundary conditions

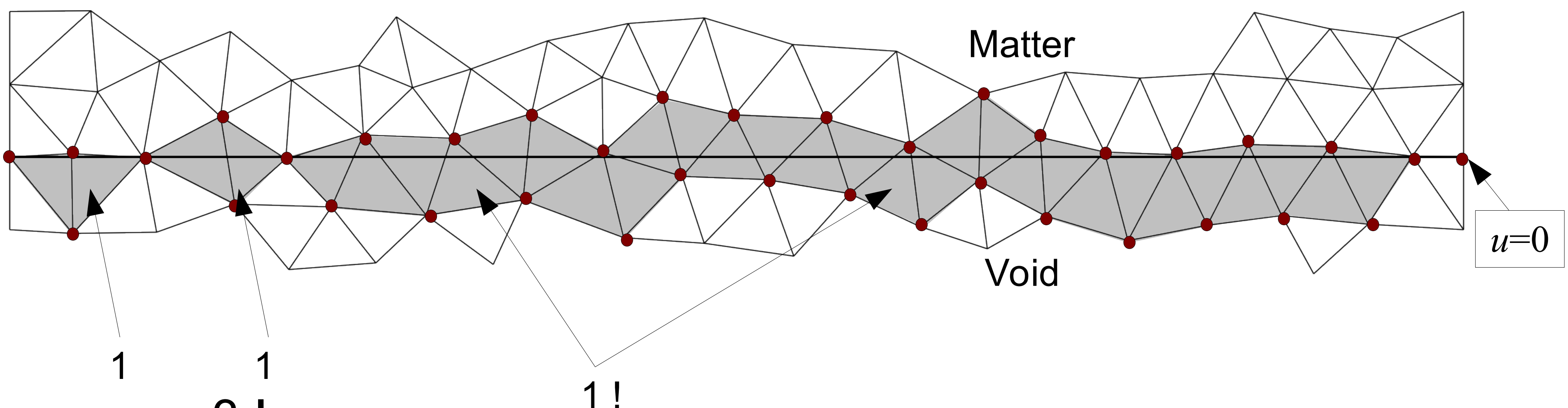
- With only two linear elements ?
  - Without Dirichlet B.S. : 4 DoFs , $u$ has some freedom in the red part
  - If one imposes *exactly* $u=0$ on the boundary …




  - $\frac{u_1}{a_1}=\frac{u_2}{a_2}; \quad \frac{u_2}{b_2}=\frac{u_3}{b_3}; \quad \frac{u_3}{c_3}=\frac{u_4}{c_4}$
  - How many DoFs left for the red part of the domain ?

# Dirichlet boundary conditions

- Concrete example
  - Number of available DoFs after imposing *exactly* the Dirichlet B.C. :




  - 3 !
  - The function space is very poor in the elements crossed by the interface, therefore the F.E. solution will be far from accurate.

# Dirichlet boundary conditions

- One cannot impose exactly a Dirichlet B.C. by elimination as long as it is crossing through finite elements !
- For this, an interpolation is preferred and the B.C. must be along element edges.
- This is the reason why Lagrange F.E. are so widely used.
  - (One) solution : the use of lagrange multipliers, see an article of Babuska (1973) - in the bibliography)

## Lagrange multipliers

On wants to minimize $\pi(u,v)=u^2+v^2$

$$\delta\pi(u,v)=2u\,\delta u+2v\,\delta v=0 \qquad \forall\,\delta u,\delta v$$
$$u=v=0 \qquad \pi(0,0)=0$$

If one sets an additional condition :

$$g(u,v)=u-v+2=0$$

Method 1 : elimination of $v$ :

$$\pi'(u)=2u^2+4u+4\equiv\pi(u,v)$$

$$\delta\pi'(u)=4(u+1)\,\delta u=0 \qquad \forall\,\delta u$$
$$u=-1 \qquad \pi'(-1)=2\rightarrow v=1$$

This is the method used just before ...

# Lagrange multipliers

Method 2 : Introduction of an additional variable

$$\tilde{\pi}(u,v,\lambda)=\pi(u,v)+\lambda g(u,v)=u^2+v^2+\lambda(u-v+2)$$

$$\delta\tilde{\pi}(u,v,\lambda)=0$$
$$=(2u+\lambda)\delta u+(2v-\lambda)\delta v+(u-v+2)\delta\lambda \qquad \forall\,\delta u,\delta v,\delta\lambda$$

$$\begin{cases}2u+\lambda=0\\2v-\lambda=0\\u-v+2=0\end{cases}\Leftrightarrow\begin{cases}u=-1\\v=1\\\lambda=2\end{cases}$$

## Lagrange multipliers

- In finite elements, this gives us :

$$a(u,\delta u)=l(\delta u)$$

$$-\Delta u=f \ \text{ in } \Omega$$
$$u=u_D \quad \text{ on } \Gamma_D$$

of weak form:
find $u$ s.t.

$$\int_\Omega \nabla u \cdot \nabla \delta u \, d\Omega=\int_\Omega f \, \delta u \, d\Omega \quad \forall \delta u$$

Equivalent to minimize $F(u)=\frac{1}{2}\int_\Omega \nabla u \cdot \nabla u \, d\Omega-\int_\Omega fu \, d\Omega$
if the conditions of
Lax-Milgram's theorem are satisfied.

$$=\frac{1}{2}a(u,u)-l(u)$$

, for all $u$ satisfying the B.C. on $\Gamma_D$. By using Lagrange multipliers for the BC's, one gets a new functional to minimize:

$$\tilde{F}(u,\lambda)=\frac{1}{2}\int_\Omega \nabla u \cdot \nabla u \, d\Omega-\int_{\Gamma_D} \lambda (u-u_D) \, d\Gamma_D-\int_\Omega fu \, d\Omega$$

# Lagrange multipliers

- Associated weak form :

$$\widetilde{F}(u,\lambda)=\frac{1}{2}\int_{\Omega}\nabla u\cdot\nabla u\,d\Omega-\int_{\Gamma_D}\lambda(u-u_D)\,d\Gamma_D-\int_{\Omega}fu\,d\Omega$$

$$=\frac{1}{2}A(U,U)-L(U)\qquad U=\begin{pmatrix}u\\ \lambda\end{pmatrix}$$

$$A(U,U)=(u,\lambda)\cdot\begin{pmatrix}a & b\\ b & 0\end{pmatrix}\cdot\begin{pmatrix}u\\ \lambda\end{pmatrix}=a(u,u)+b(u,\lambda)+b(\lambda,u)$$

$$L(U)=l(u)+c(\lambda)$$

$$a(u,v)=\int_{\Omega}\nabla u\cdot\nabla v\,d\Omega$$

$$b(u,\lambda)=b(\lambda,u)=-\int_{\Gamma_D}u\cdot\lambda\,\Gamma_D$$

$$l(u)=\int_{\Omega}fu\,d\Omega$$

$$c(\lambda)=-\int_{\Gamma_D}u_D\,d\Gamma_D$$

$$A(U,\delta U)=L(\delta U)$$

$$\downarrow$$

$$a(u,\delta u)+b(\lambda,\delta u)=l(\delta u)$$
$$b(\delta\lambda,u)=c(\delta\lambda)$$

# Dirichlet boundary conditions

To simplify notations, lets assign $v=\delta u, \mu=\delta\lambda$

Find $\begin{array}{l} u\in V=\{v\in H^1(\Omega)\} \\ \lambda\in L=\{\mu\in H^{1/2}(\Gamma_D)'\} \end{array}$ s. t.

$$\int_\Omega \nabla u\cdot\nabla v\, d\Omega - \int_{\Gamma_D} \lambda\cdot v\, d\Gamma = \int_{\Gamma_N} f\cdot v\, d\Gamma \qquad \forall v\in V$$

$$-\int_{\Gamma_D} \mu\cdot u\, d\Gamma = \boxed{-\int_{\Gamma_D} \mu\cdot u_D\, d\Gamma} \qquad \forall \mu\in L$$

The Dirichlet B.C. has been "dualized".
This is now a Neumann B.C. on the lagrange multipliers

## Dirichlet boundary conditions

- The Lagrange multipliers have a physical meaning
  - In mechanics, it is the force to impose so that the condition on the primal variable is ensured (here, displacements).
  - In our case, it is the gradient of the solution (flux) to impose so that $u=u_D$ on $\Gamma_D$.
- We have now a saddle point problem (min-max) – the matrix of the linear system is not definite positive anymore (but still has an inverse and is symmetric)
- Not all solvers are able to handle that – mostly direct solvers and very few iterative solvers.

# Extended Finite Elements

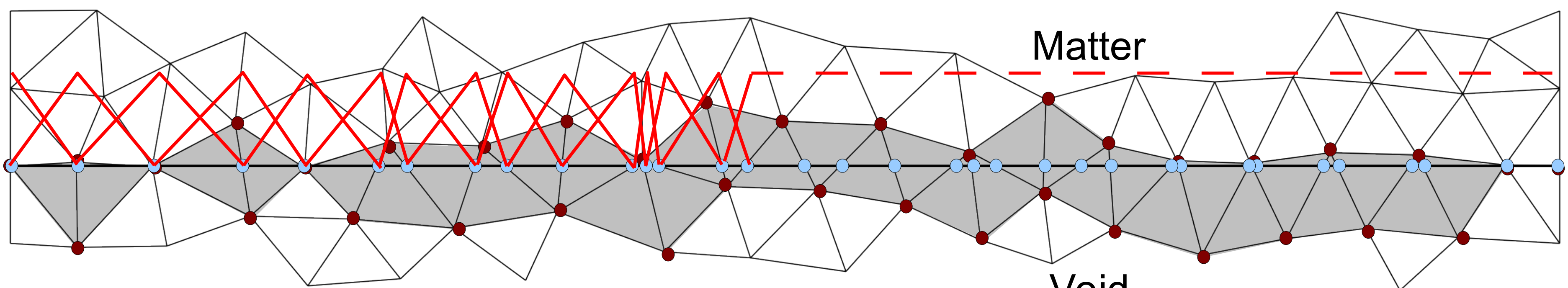
## Dirichlet boundary conditions

- How to build adequate discrete function spaces
  - Find $\begin{matrix} u_h \in V_h \subset V = \{ v \in H^1(\Omega) \} \\ \lambda_h \in L_h \subset L = \{ \mu \in H^{1/2}(\Gamma_D)' \} \end{matrix}$ s. t. ...
  - One do not change the primal functional space (for $u$). It is the usual finite element space using nodal hat functions
  - One need to build a function space for $\lambda$.
    - Lets try to use an identical function space $L_h$ for $\lambda$ (or the restriction to the boundary of tsuch a space… (the trace)

# Extended Finite Elements

## Dirichlet boundary conditions

- Lets try to use an identical function space $L_h$ for $\lambda$ (or the restriction to the boundary of tsuch a space… (the trace)




- Lets perform a computation. The linear system has the following shape :

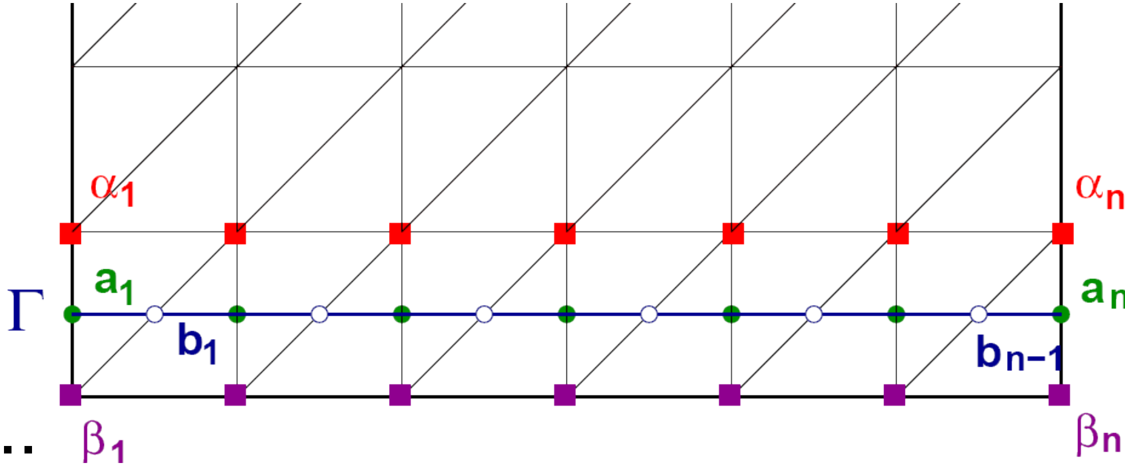
$$\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma_D} \lambda \cdot v \, d\Gamma = \int_{\Gamma_N} f \cdot v \, d\Gamma \qquad \forall v \in V_h$$

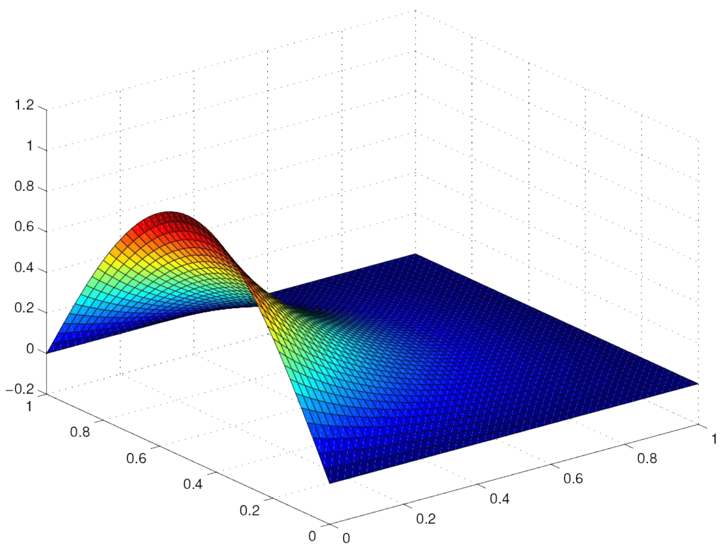
$$-\int_{\Gamma_D} \mu \cdot u \, d\Gamma = -\int_{\Gamma_D} \mu \cdot u_D \, d\Gamma \qquad \forall \mu \in L_h$$

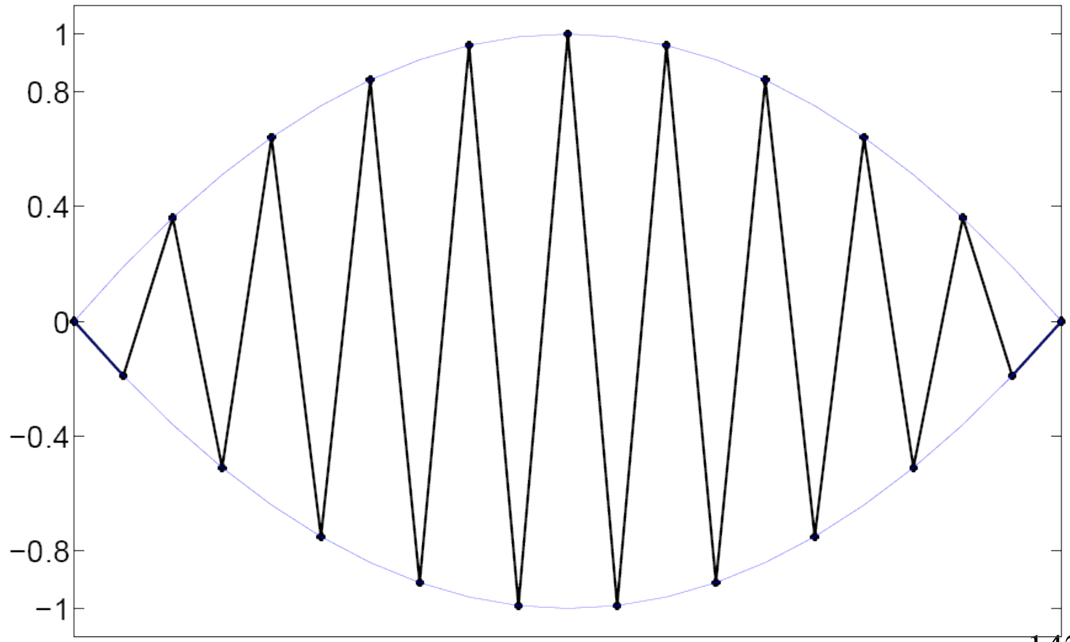
$$\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} F_h \\ D_h \end{pmatrix}$$

# Dirichlet boundary conditions

- The we solve it …
  - Lagrange multipliers are oscillating.
  - The more $h$ (element size) shrinks, the more it oscillates…





## Dirichlet boundary conditions

- What happens ?
  - The discrete spaces for $u$ et $\lambda$ are incompatible.
  - Those do not satisfy the Ladyzhenskaya-Babuška-Brezzi (LBB) condition, or inf-sup condition :

$$\inf_{\mu \in L_h} \sup_{u \in V_h} \frac{\int_\Gamma \lambda_h u_h \, d\Gamma}{h^{1/2} \|\lambda\|_{0,\Gamma_D} \|u\|_{1,\Omega}} \geq \alpha > 0$$

  - This condition is often difficult to check analytically.

O. Ladyzhensakya, Global solvability of a boundary value problem for the Navier–Stokes equations in the case of two spatial variables. *Proc. Ac. Sc. USSR* 123 (3) (1958) 427–429.
I. Babuska, Error bounds in the finite element method, *Numer. Math.*, 16 (1971), pp. 322-33.
F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, RAIRO, Anal. Num., 8, R2 (1974), pp. 129-151

## Dirichlet boundary conditions

- Numerical validation of the LBB condition.
  - There exists a “simple” numerical test; see Chapelle, Bathe, 1993 and KJ Bathe 2001 (in the bibliography)
    - One considers a more general problem with an added “stiffness” on the Dirichlet boundary condition (becomes a Robin B.C.) – if $k \to \infty$, back to a “hard” Dirichlet B.C.

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma_D} \lambda \cdot v \, d\Gamma = \int_{\Gamma_N} f \cdot v \, d\Gamma \qquad \forall v \in V_h$$

$$-\int_{\Gamma_D} \mu \cdot u \, d\Gamma - \boxed{\int_{\Gamma_D} \frac{1}{k} \lambda \mu \, d\Gamma} = -\int_{\Gamma_D} \mu \cdot u_D \, d\Gamma \qquad \forall \mu \in L_h$$

$$\begin{pmatrix} A_h & B_h^T \\ B_h & \boxed{-\frac{1}{k} M_h} \end{pmatrix} \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} F_h \\ D_h \end{pmatrix}$$

# Dirichlet boundary conditions

- Chapelle – Bathe numerical test

$$\begin{pmatrix} A_h & B_h^T \\ B_h & -\frac{1}{k} M_h \end{pmatrix} \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} F_h \\ D_h \end{pmatrix}$$

  - It amounts to check the first non vanishing eigenvalue ($\beta_0$) of the following eigenproblem :

    $$\frac{1}{h}\left(B_h A_h^{-1} B_h^{\mathrm{T}}\right) W_h = \beta M_h W_h \quad \text{ou} \quad \frac{1}{h}\left(B_h^{\mathrm{T}} M_h^{-1} B_h\right) W'_h = \beta' A_h W'_h$$

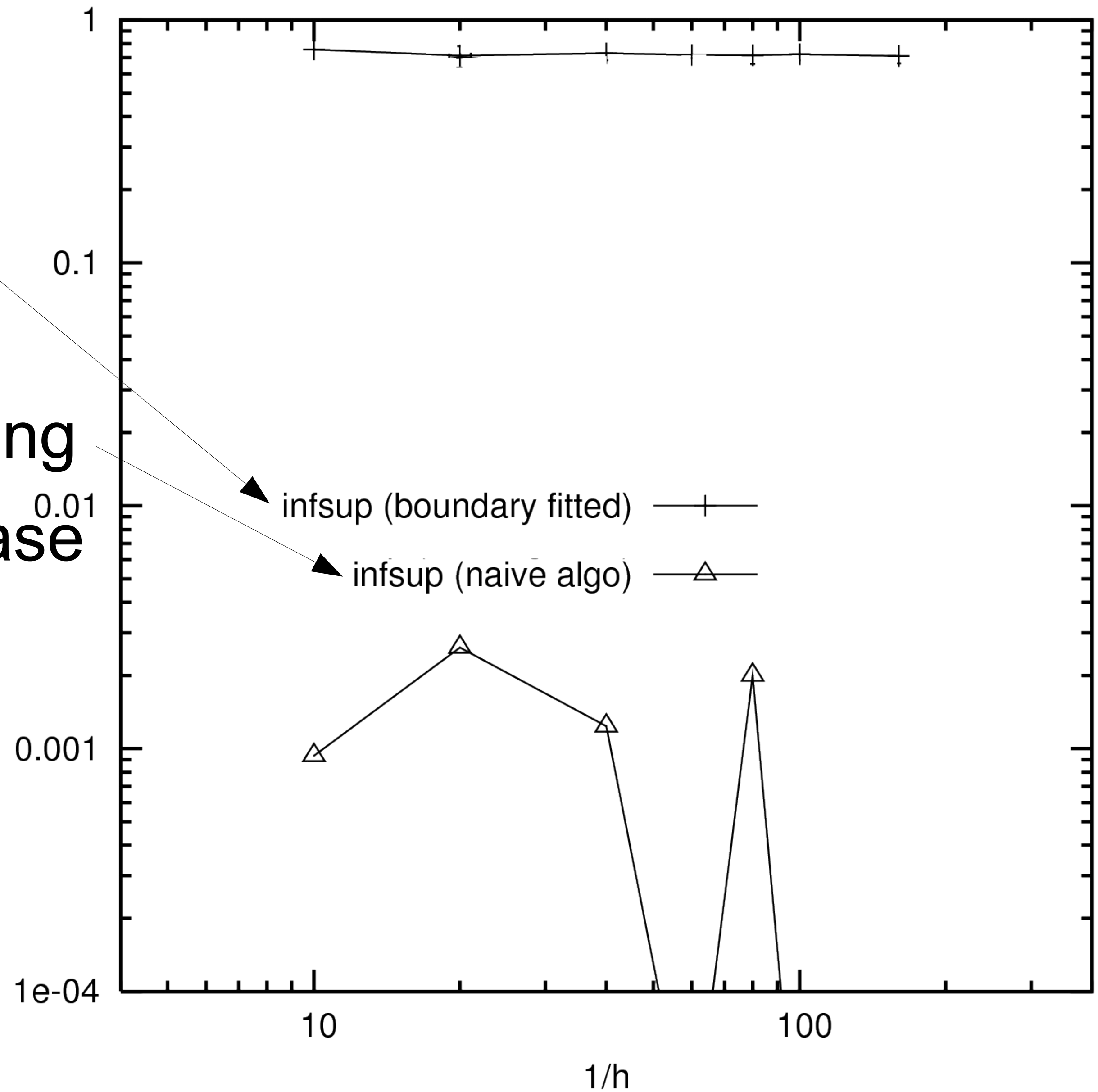
  - $A_h$ must have an inverse
  - Does not depend on $k$ !
  - One checks that $\beta_0$ does not vanish for a sequence of meshes with an increasing density
  - Here, $\alpha = \sqrt{\beta_0}$ (and for $\alpha$ : see slides before)

## Dirichlet boundary conditions

- Results
  - Two cases :

    - aligned with the mesh

    - non conforming
  - The second case does not work at all.



## Dirichlet boundary conditions

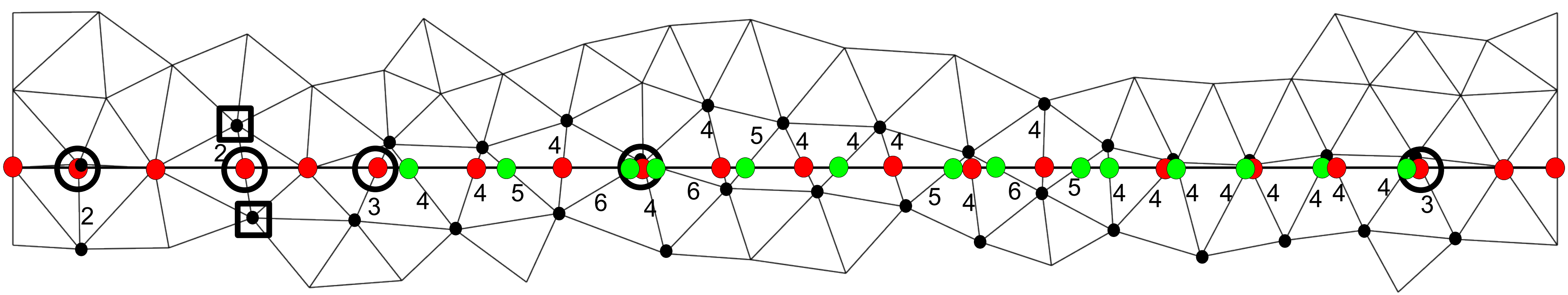
- What we have are incompatibles functional spaces...
  - The space for the Lagrange multipliers is way too “rich” with respect to the one for the primal variable.
  - It amounts to impose exactly the Dirichlet B.C., which has been already shown to be a bad idea.

  $\rightarrow$ We have to “decimate” $L_h$

# Extended Finite Elements

## Dirichlet boundary conditions

- From the mesh of the interface, take each node and put it in a set N
- If a node of N is also part of the mesh, mark it as Vital (set V) , and delete it from  N
- Take each edge incident to N and count each intersecting edge going from end nodes with the interface
- Sort N. The sorting key is the number defined above (smallest first)

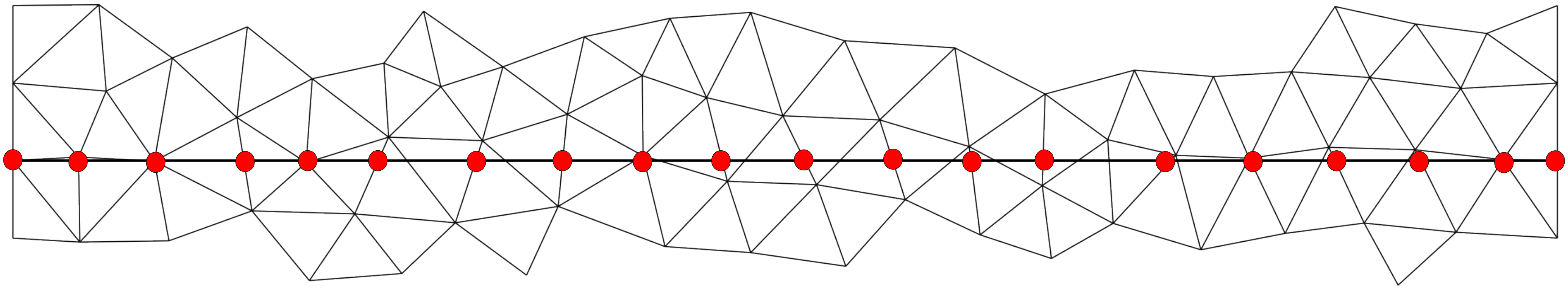


- Loop over the sorted set N, take ni
  - Take the end nodes of ni, and from those, the connected nodes in N (may be many)
  - If ni is not yet NV (non vital), mark it as Vital (V) and all the other connected nodes as (NV)
- EndLoop

## Dirichlet boundary conditions

What remains,

- An approximately uniform distribution of nodes

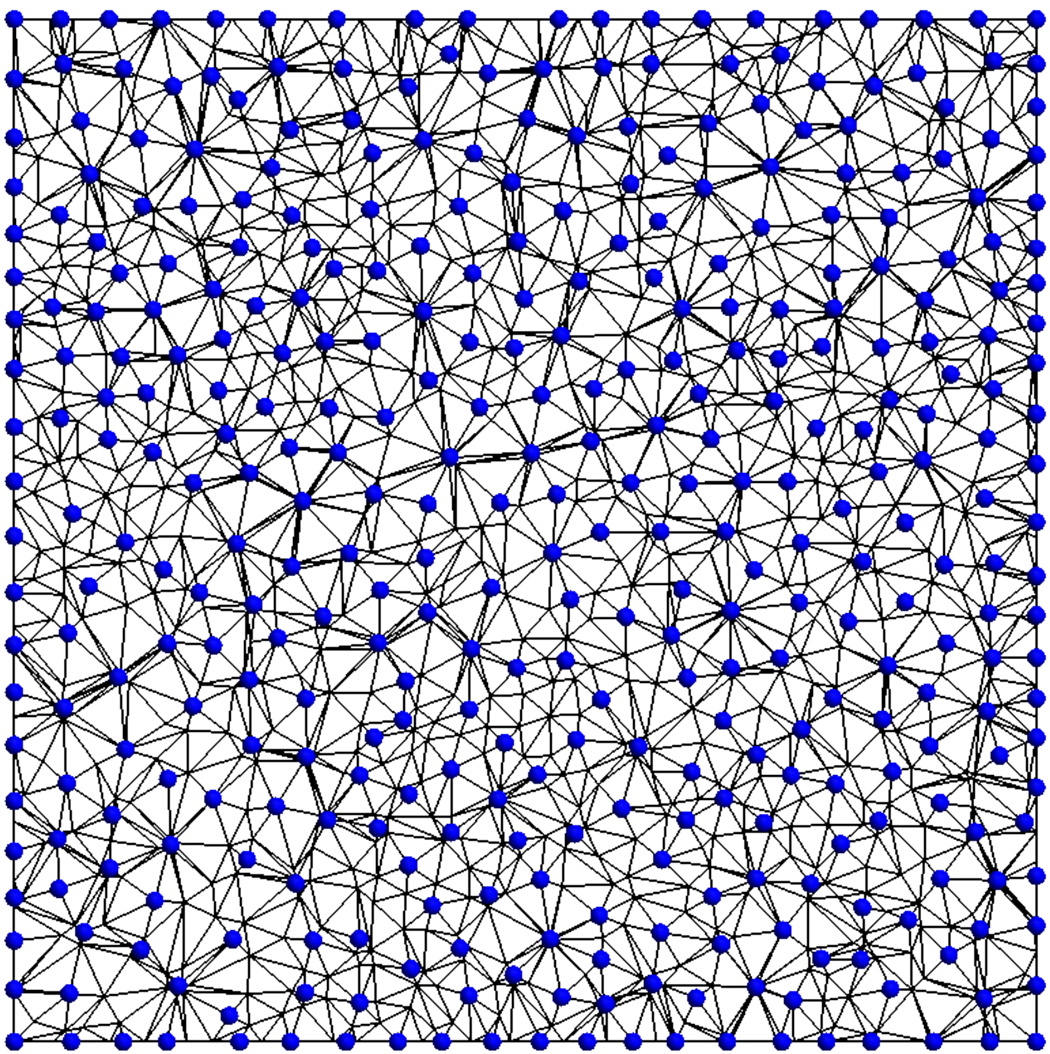


- The density is same as the initial mesh (2D here, 3D in general
- Works in 3D !

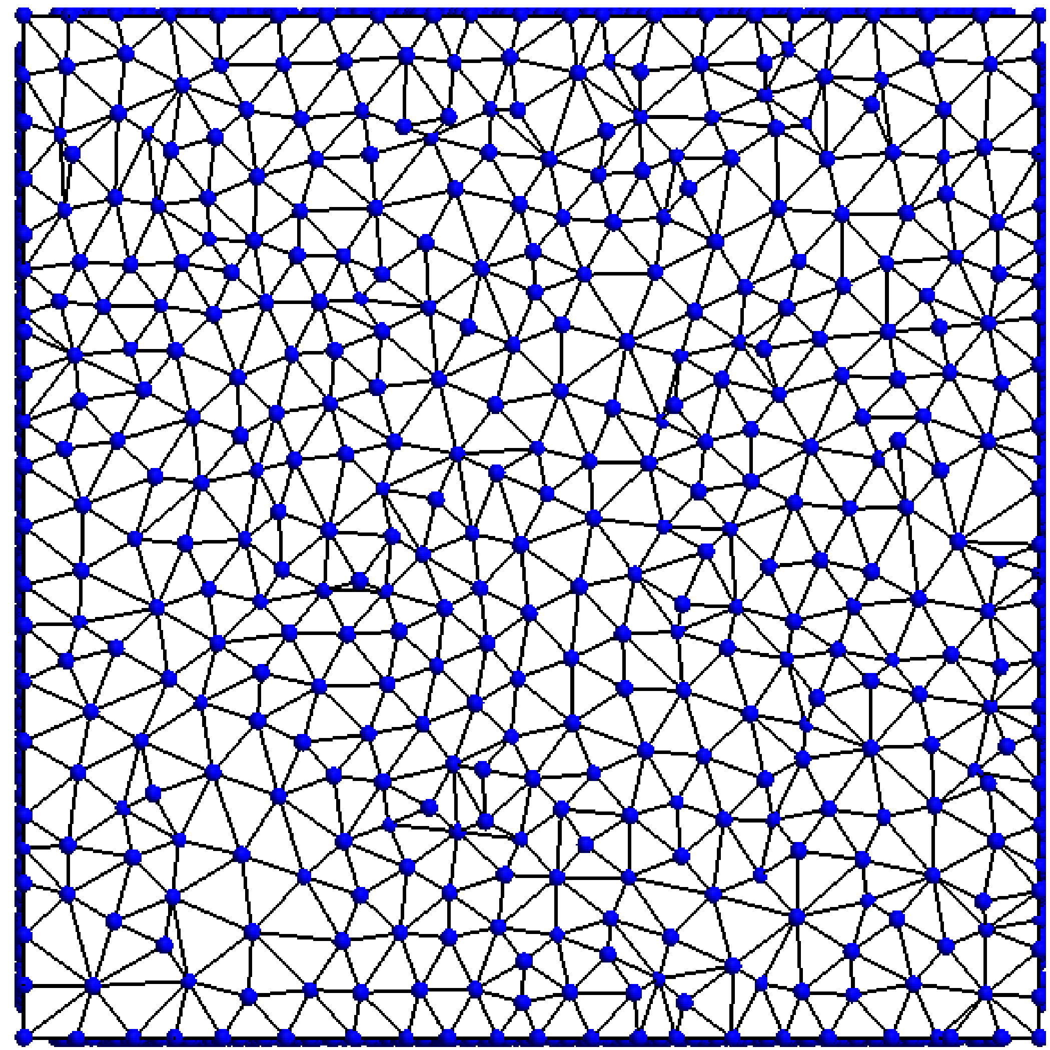
# Extended Finite Elements

## Dirichlet boundary conditions

Result of the decimation



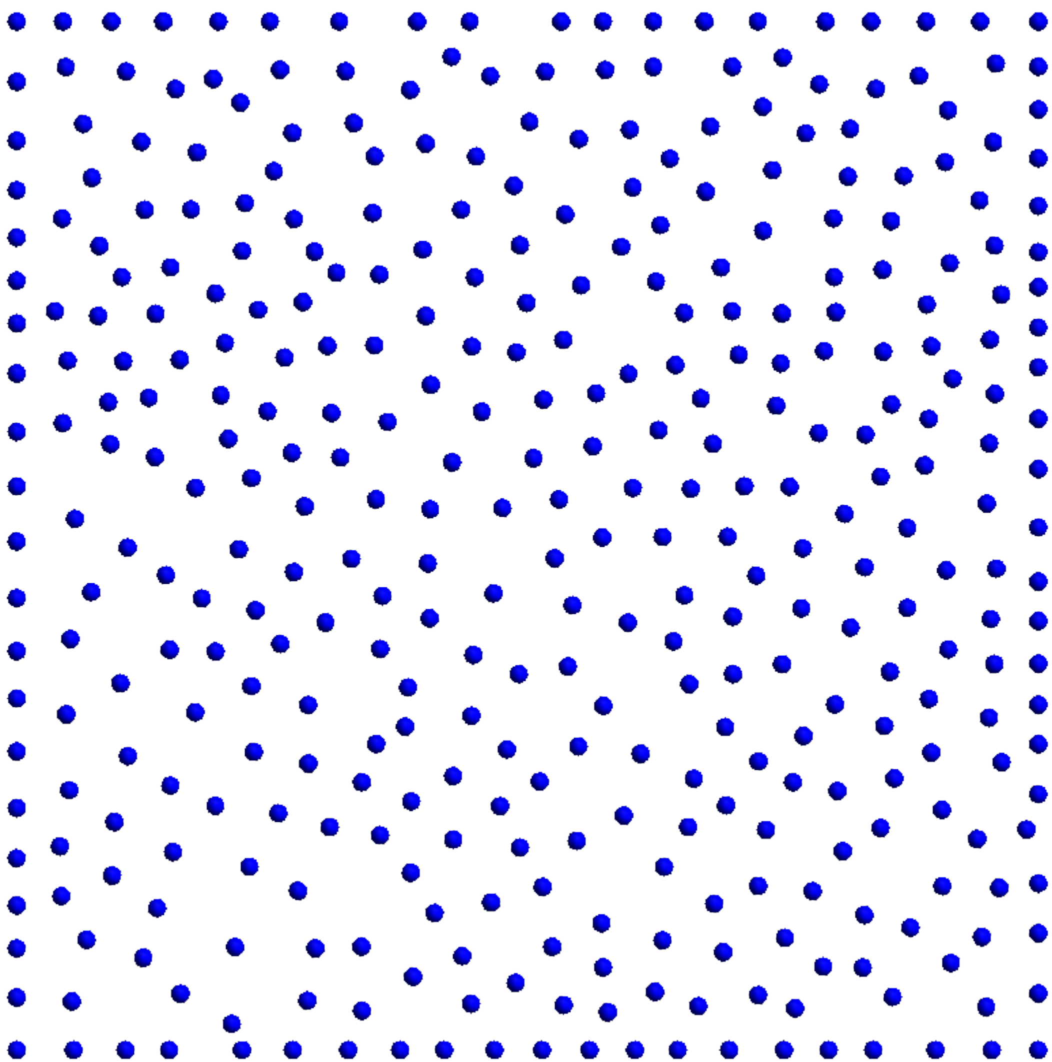
Projection of 3D nodes



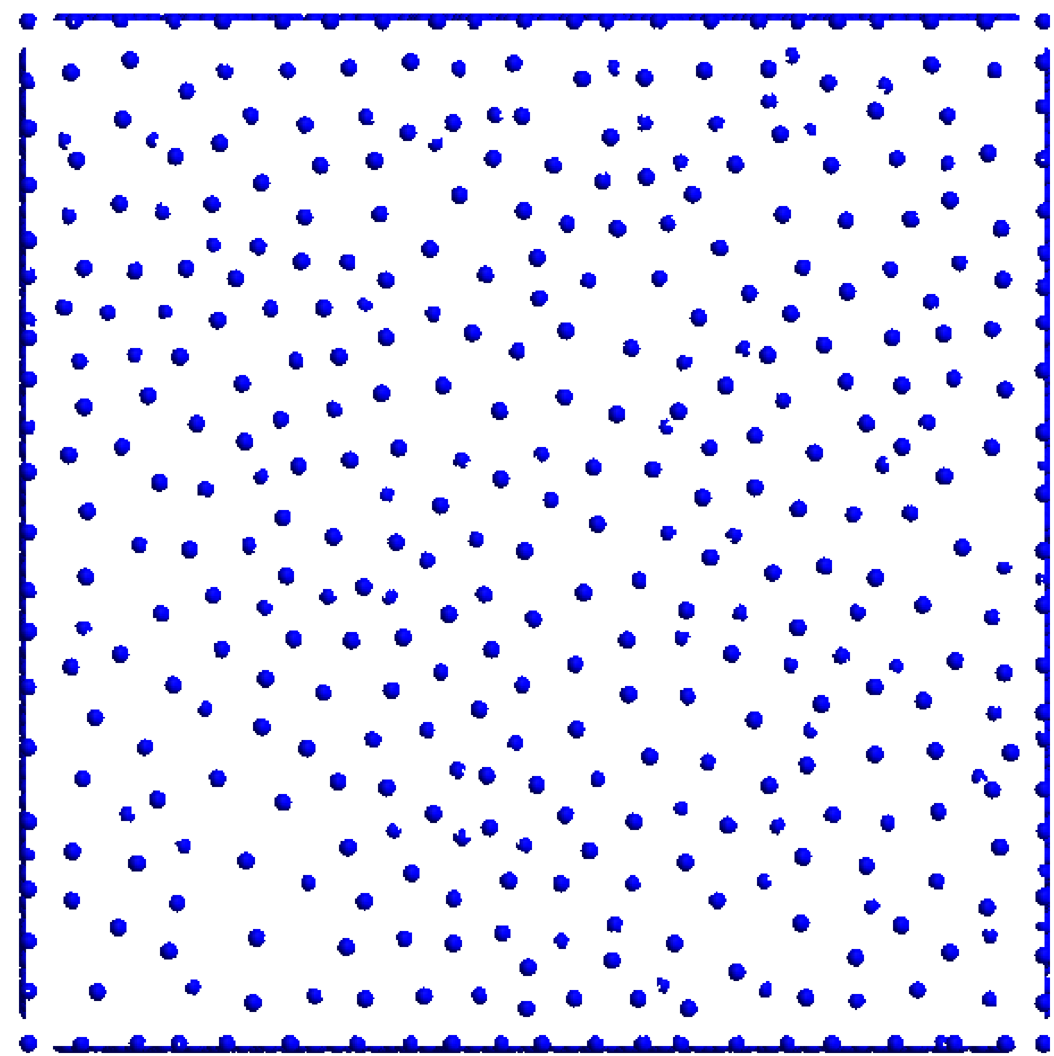
# Extended Finite Elements

## Dirichlet boundary conditions

Result of the decimation



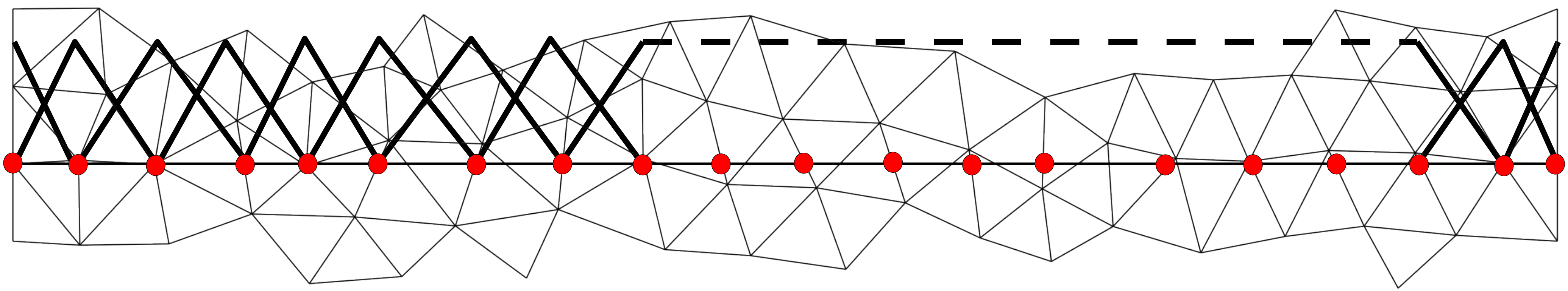
Projection of 3D nodes



# Extended Finite Elements

## Dirichlet boundary conditions

- How to build shape functions from this ?
  - Directly on the interface ?



  - Works...

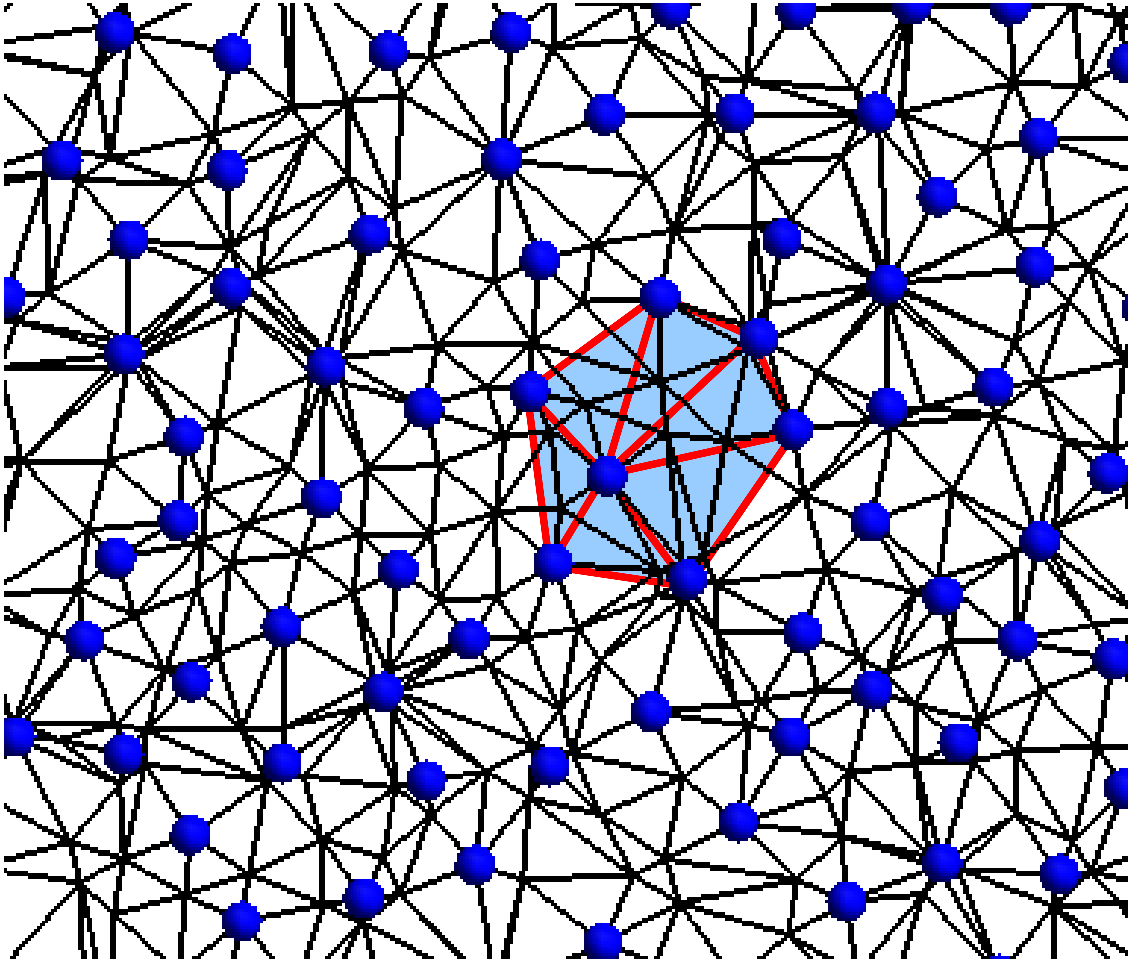
    … only in 2D !!!

## Dirichlet boundary conditions

- In 3D : one would have to build a triangulation of the set of nodes V

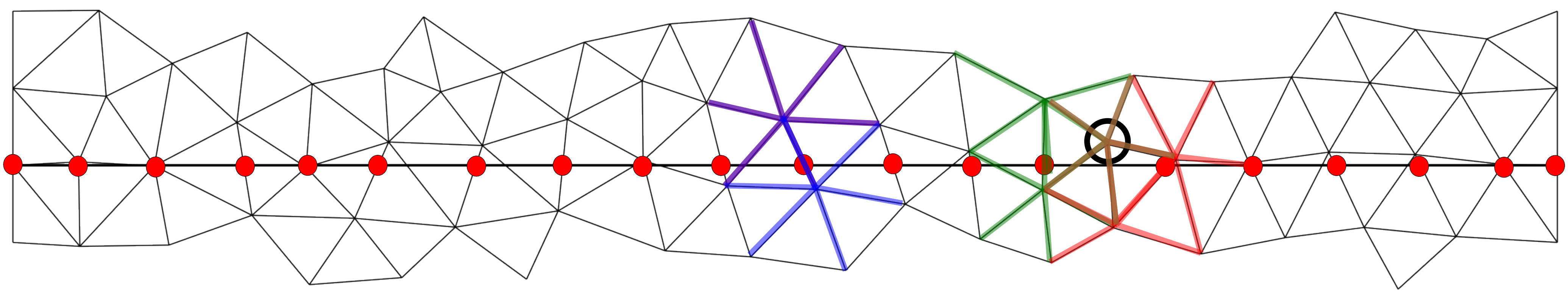
What about :

- Curvy interfaces
- Discrepancy (non conformity) btw. triangulations
  - Integration problems
- So we must find a better way in 3D...



## Dirichlet boundary conditions

- Another solution
  - Lets take the trace of volume shape functions – but there are too many !
    - One will combine SFs. (linear combinations) for each V-node



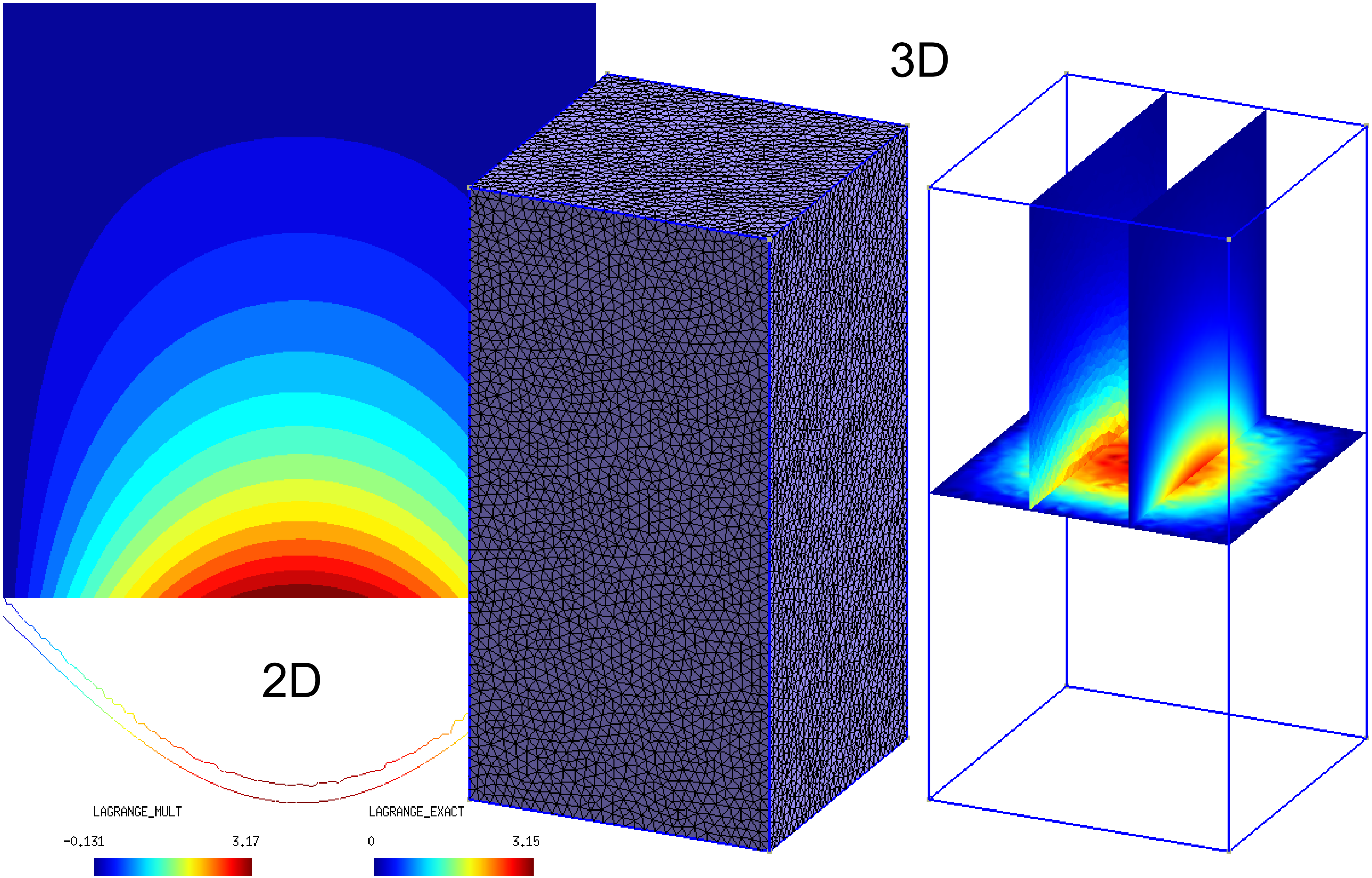
    - At some places, a volume SF may be linked to more than one V-node.
    - There is room for freedom : 100% with the green, or 100% with the red or whatever combination such that the sum is 100% (to keep “partition of unity”)

## Dirichlet boundary conditions

- Advantages of using trace shape function for Lagrange multipliers
  - Easy integration
  - Compact shape functions
  - Partition of unity on the interface
  - Same algorithm in 3D and 2D
  - Good numerical results ? See what’s follow !

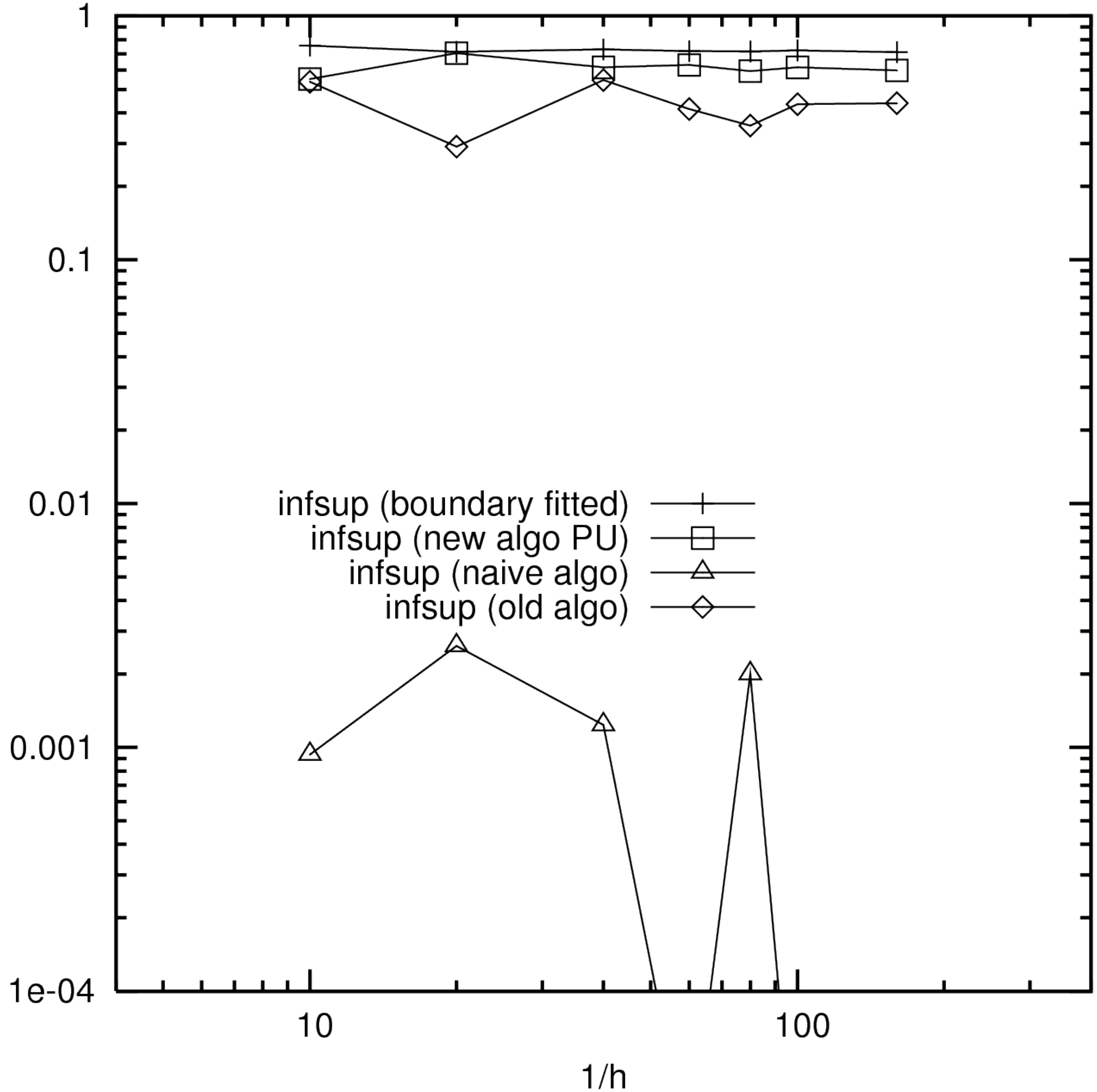
# Extended Finite Elements

## Dirichlet boundary conditions

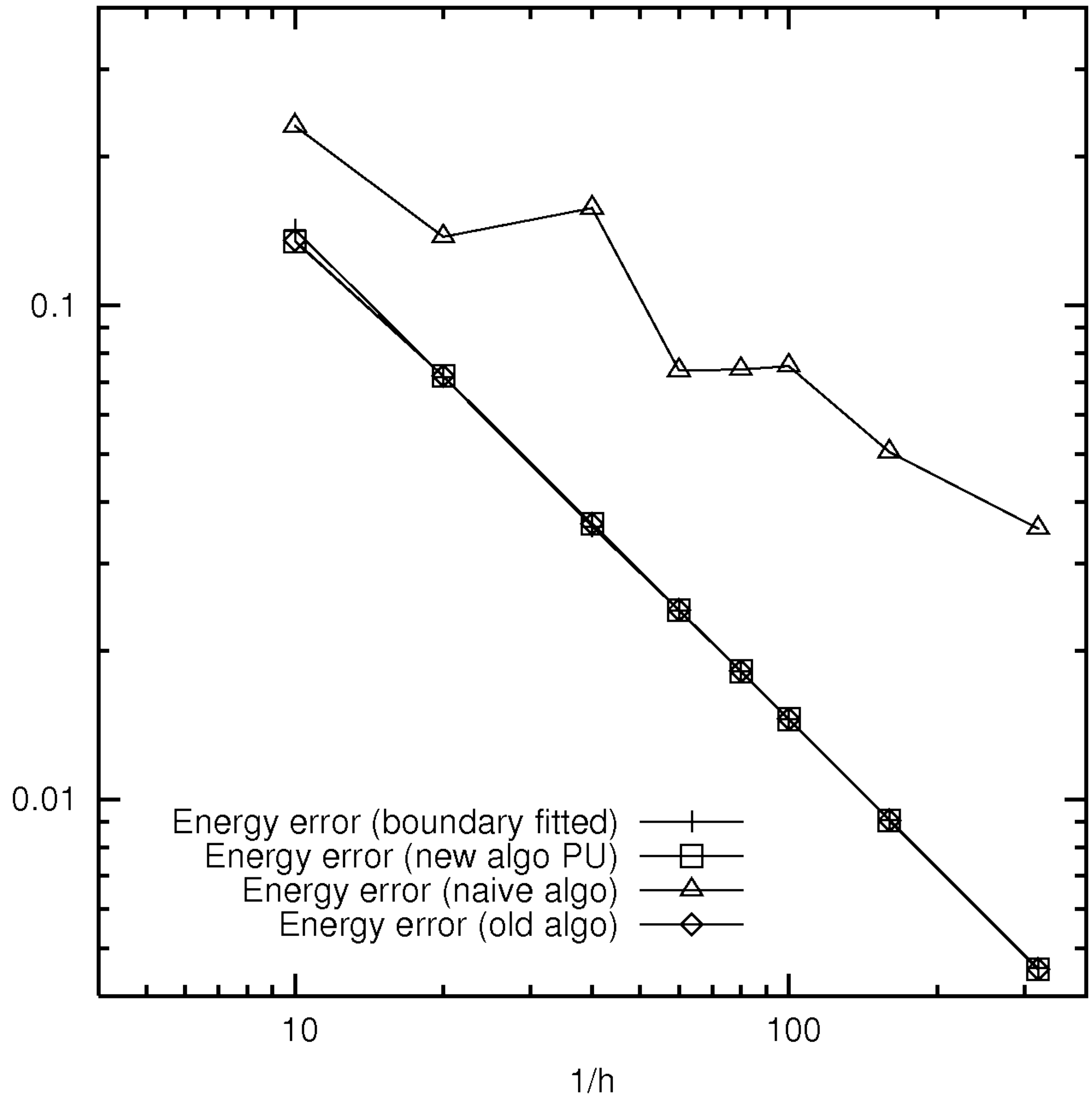


# Extended Finite Elements

## Dirichlet boundary conditions

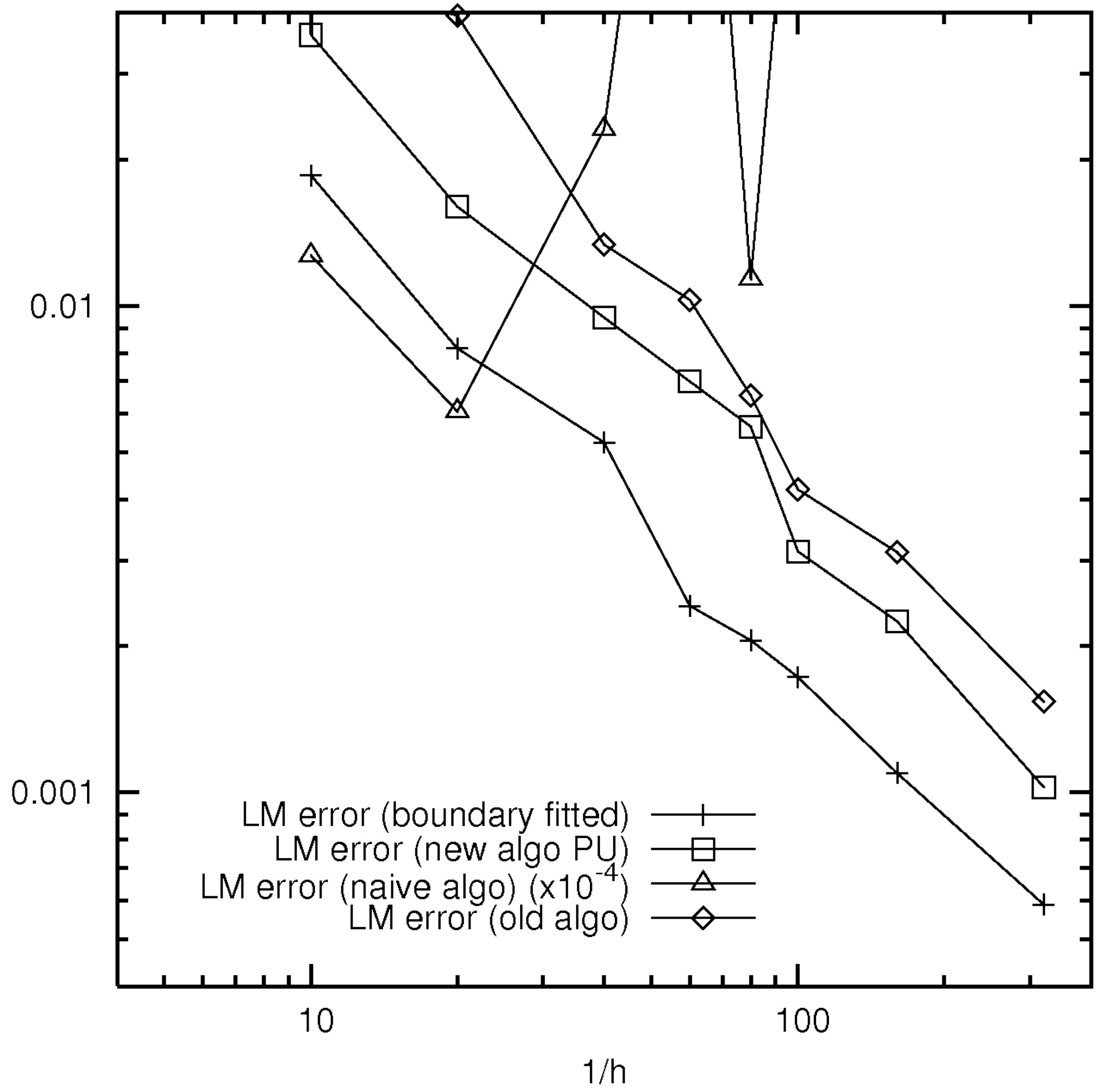


# Dirichlet boundary conditions



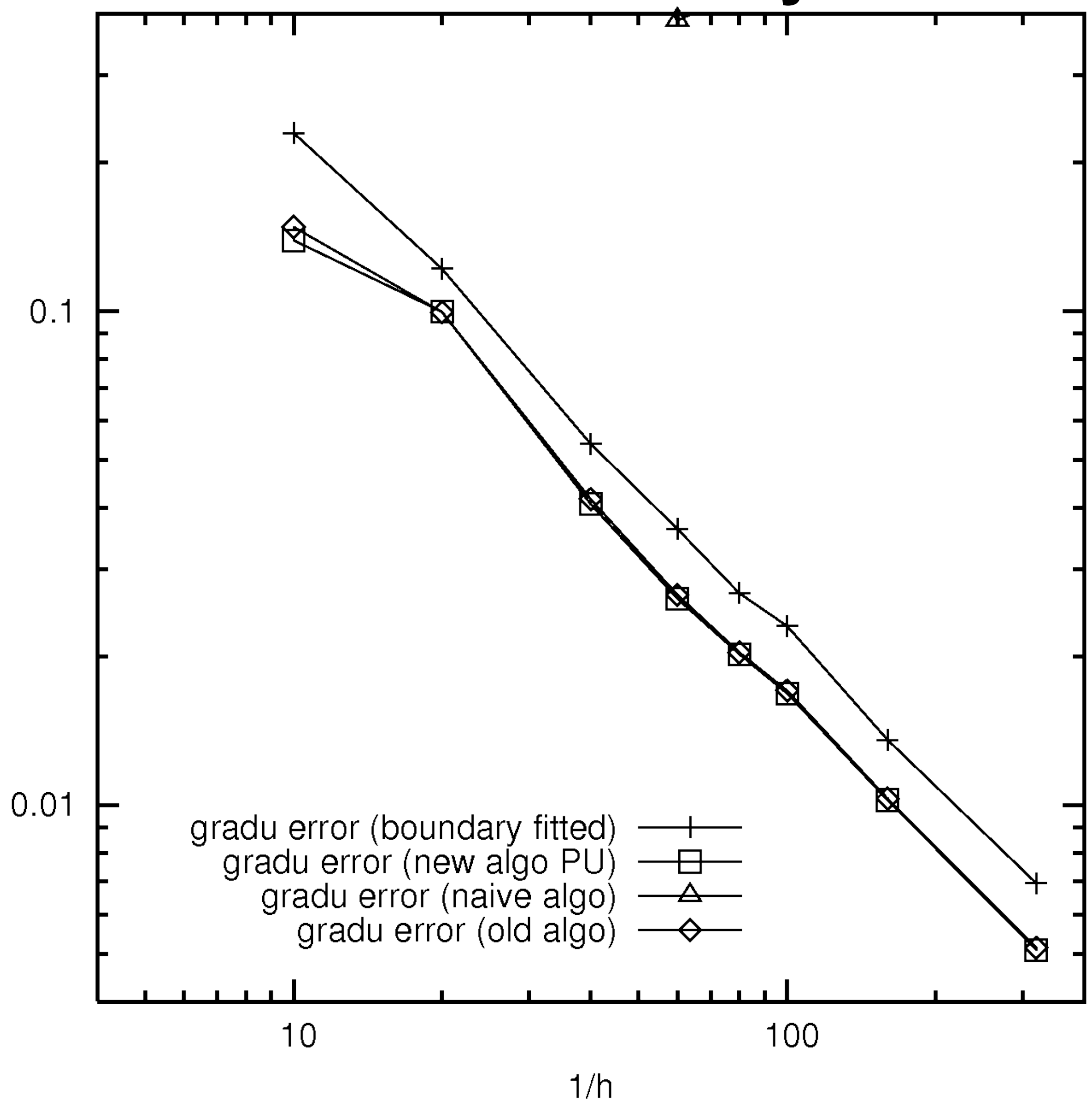
# Extended Finite Elements

## Dirichlet boundary conditions



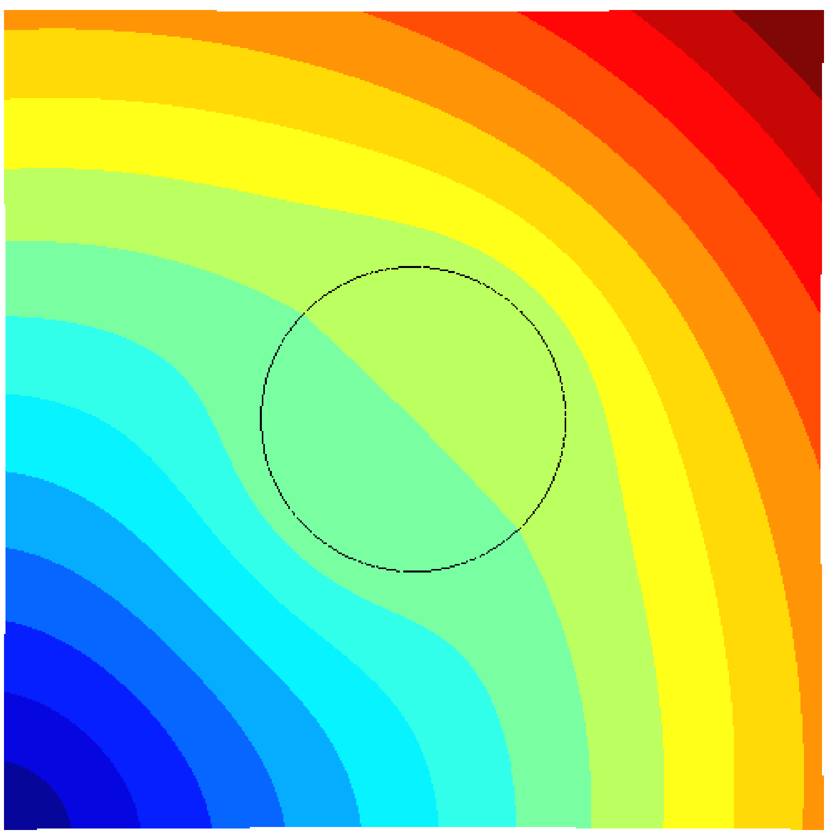
# Extended Finite Elements

## Dirichlet boundary conditions

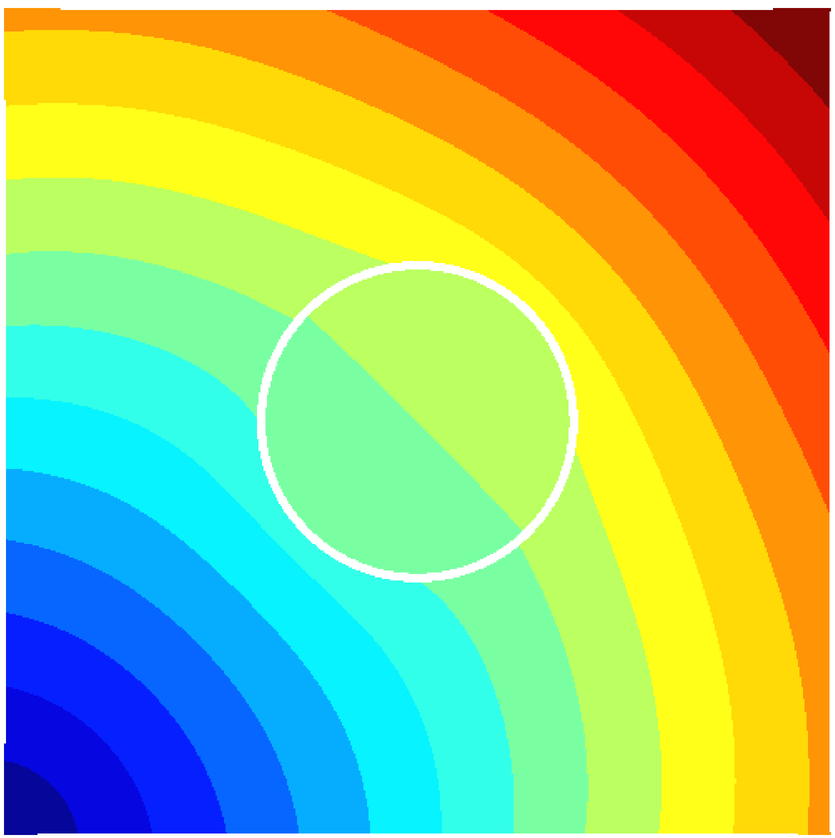


# Extended Finite Elements

## Dirichlet boundary conditions



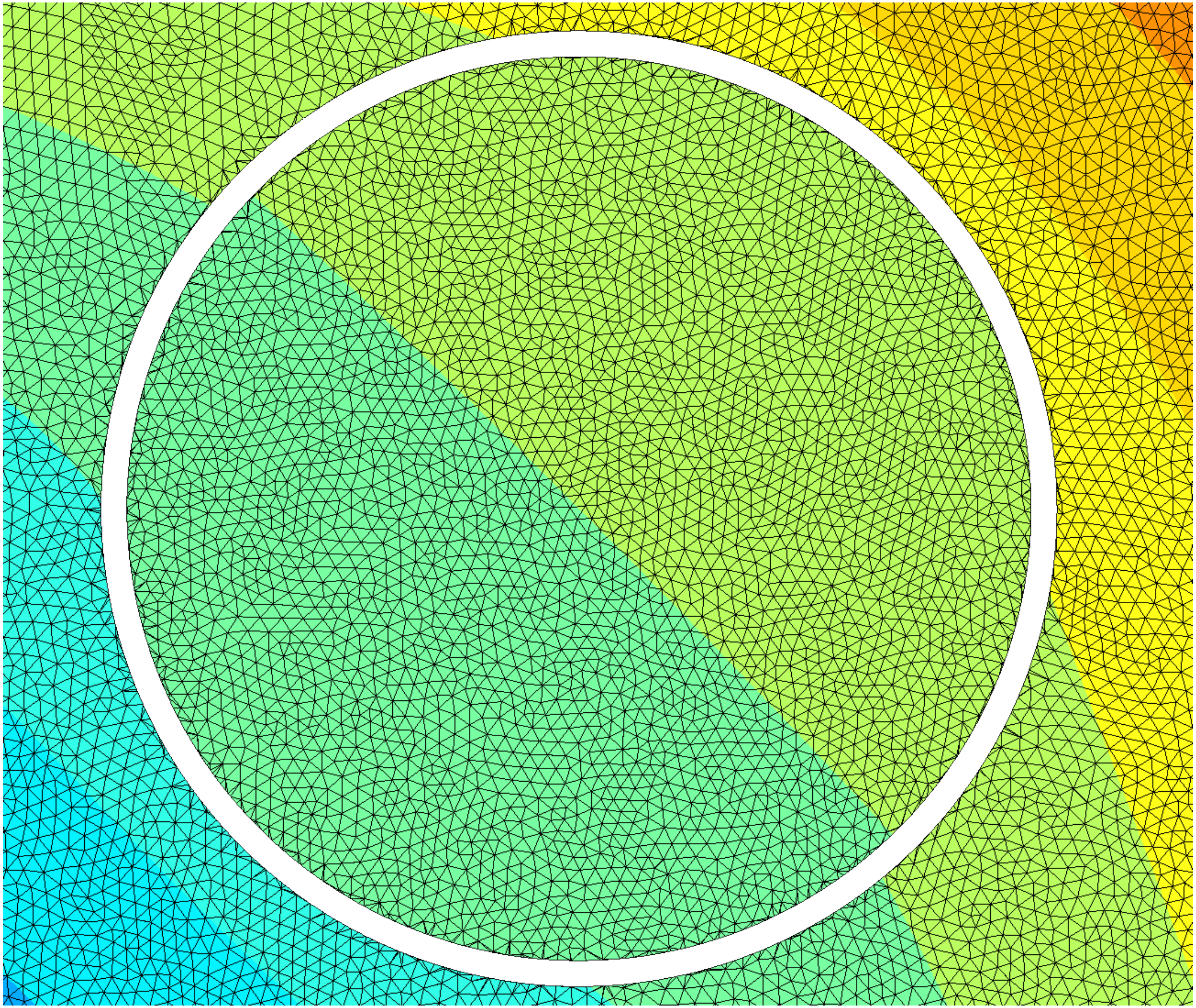
Composites : perfect glueing



Imperfect glueing

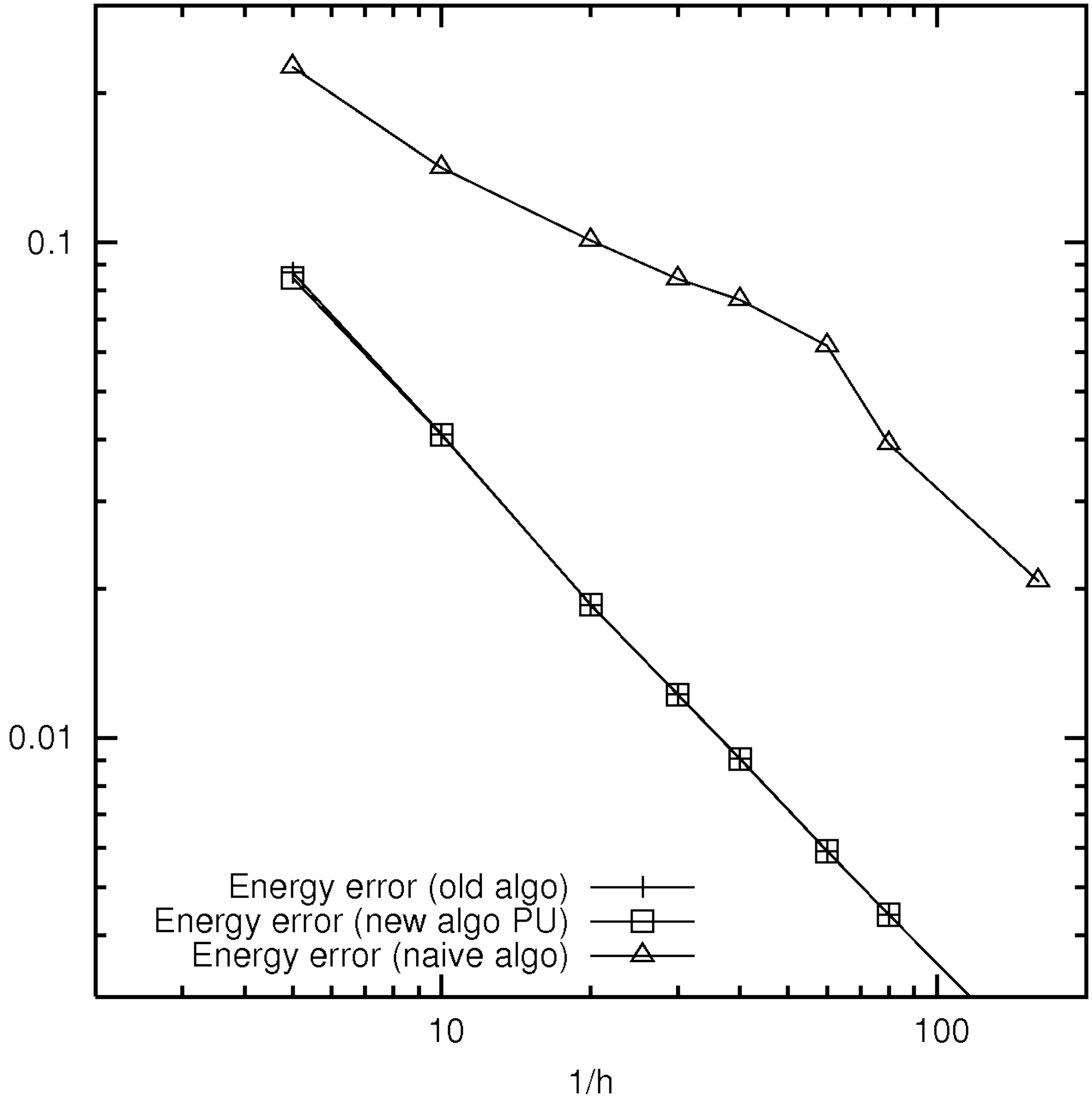
# Extended Finite Elements

## Dirichlet boundary conditions



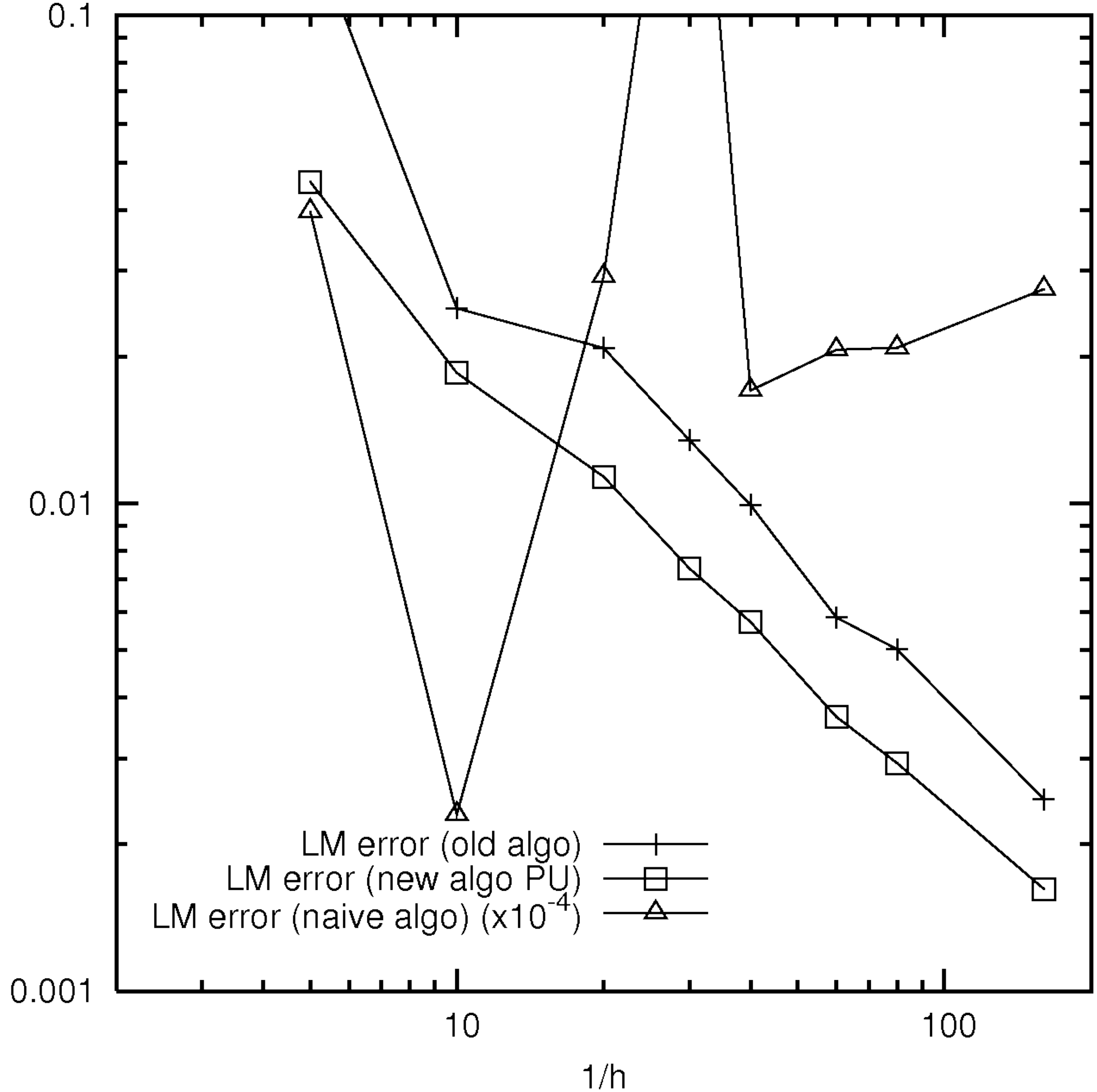
# Extended Finite Elements

## Dirichlet boundary conditions

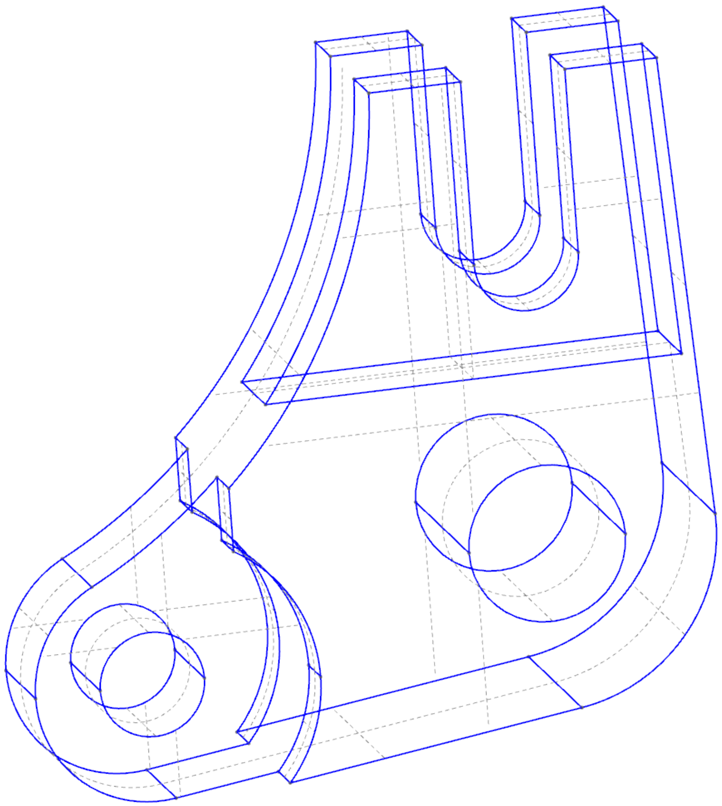


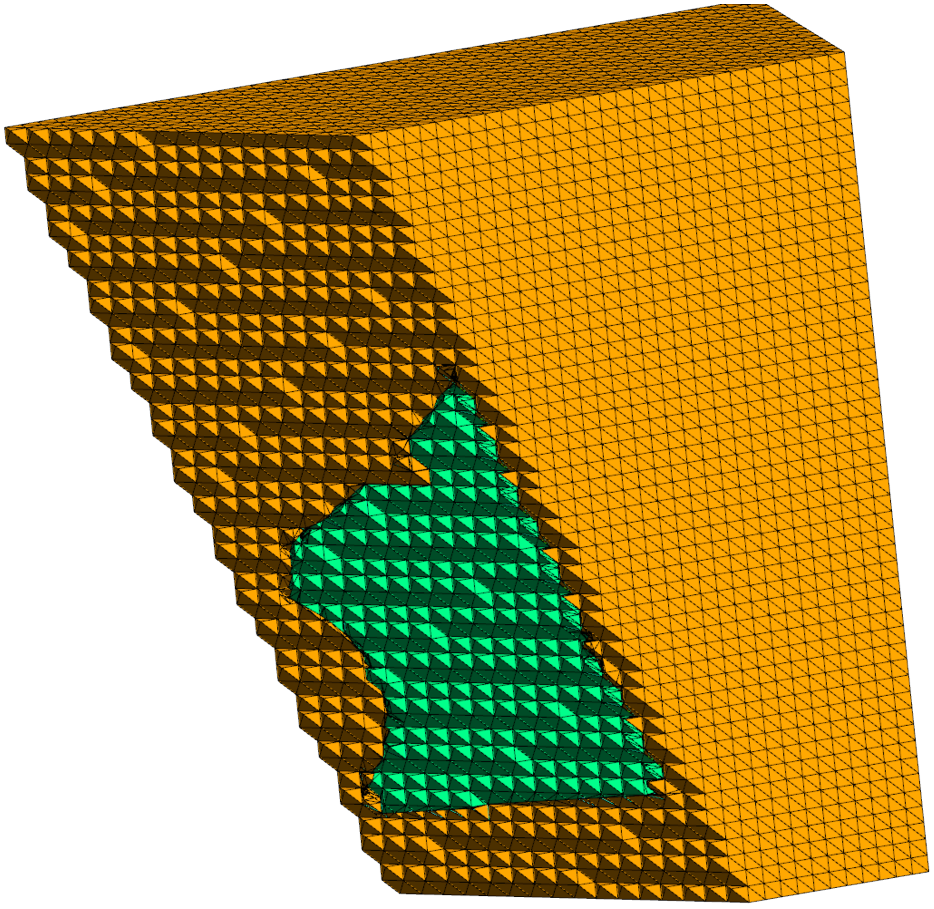
# Extended Finite Elements

## Dirichlet boundary conditions

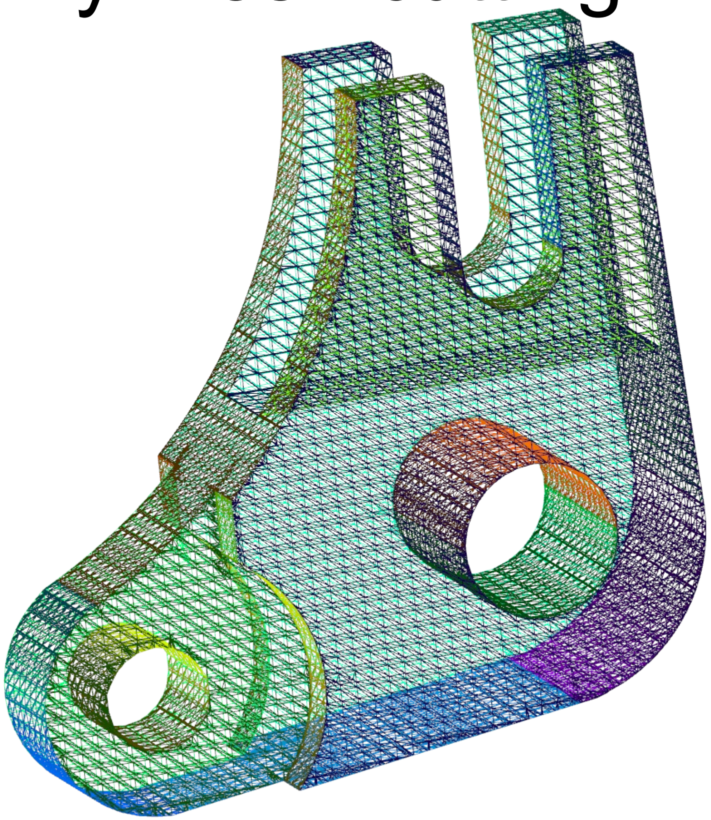


- From a traditional CAD (B-rep) representation ...





- … To an implicit representation and F.E. computation (here, no mesh generation steps, only mesh cutting … )



# Extended Finite Elements

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# Extended Finite Elements

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Nota :

IJNME = International journal for numerical methods in engineering (Wiley)
CMAME = Computer methods in applied mechanics and engineering (Elsevier)
FEAD = Finite element in analysis and design (Elsevier)
JCP = Journal of computational physics (Elsevier)