MEAN-FIELD HOMOGENIZATION

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Outline

- Introduction
- Homogenization of isothermal elastic composites
- Homogenization of thermo-elastic composites
- Homogenization of viscoelastic composites
- Conclusion
- Bibliography
- Appendix

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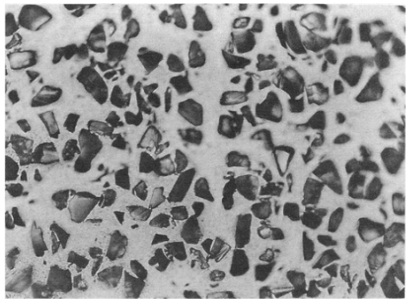
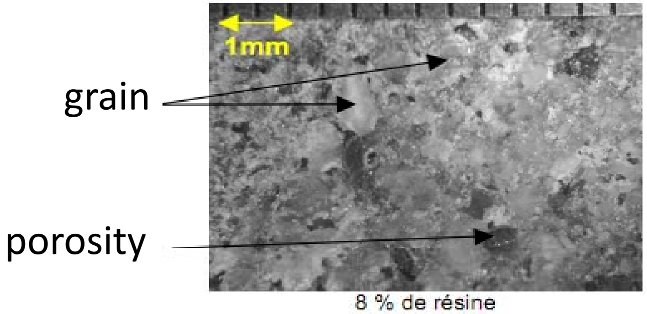
Objectives

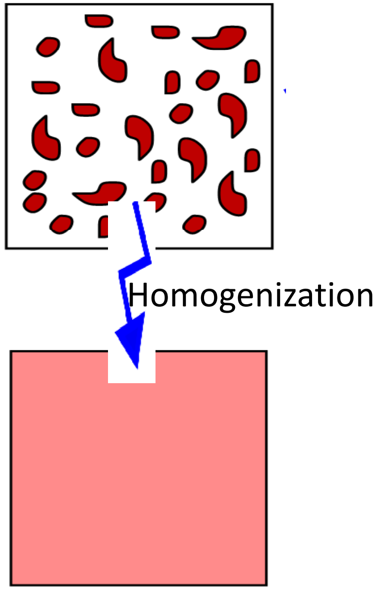
Resin mortars



Resin mortars



Heterogeneous milieu



Effective milieu

<u>Homogenization Methods:</u>

- Finite Element
- Asymptotic Homogenization
- Generalized Method of Cells
- Fast Fourrier Transform
- Mean-Field Homogenization
- …

Basis of MFH Method

Key idea: *Based on assumptions of the interaction laws between the different phases (which define the homogenization scheme)→give a macroscopic response as well as basic information on the state of deformation within the phases*

- Very efficient on a computational point of view.
- Some models have a good accuracy in the linear elastic regime for both inclusion-reinforced materials and poly-crystals.
- Extension of MFH schemes to rate independent and dependent elasto-plastic behaviors is the main challenge

Basis of MFH Method *(2)*

Limitation:

- unable to predict any strain or stress localization.
- cannot take into account clustering, percolation and size effects.

Complex geometries	fair
Ease of discretization	good
Accuracy macro response	fair
Accuracy microfields	weak
Computational cost	good
Nonlinear behaviors	fair

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Results of Eshelby and Hill

Figure (a)

- Homogeneous linear elastic material of stiffness C in which an embedded ellipsoidal inclusion made of the same material undergoes an eigenstrain ε^*

- Results of Eshelby: the resulting strain field inside the inclusion (ε_1) is uniform.

$$\varepsilon_1 = S : \varepsilon^*$$

S: *Eshelby's tensor* → *See Appendix A for more details*

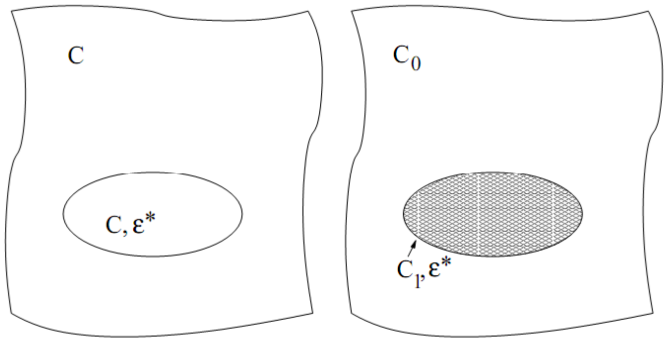
- The homogeneous stress field σ_1 inside the inclusion:

$$\sigma_1 = C : \varepsilon_1 + \tau \ , \quad \tau = -C : \varepsilon^*$$

τ: *polarization tensor*

$$\varepsilon_1 = -P : \tau \ , \qquad P = S : C^{-1}$$

P: *Hill's tensor*



(a) *(b)*

Results of Eshelby and Hill *(2)*

Figure (b)

- The single inclusion of stiffness C_1 undergoing an eigenstrain embedded in a matrix with a different stiffness $(C_0 = C_1 - \Delta C)$

$$\sigma_0 = C_0 : \varepsilon_0 \qquad \textit{In the matrix}$$

$$\sigma_1 = C_1 : \varepsilon_1 + \tau_1 \qquad \textit{In the inclusion}$$

$$= C_0 : \varepsilon_1 + \tau_0$$

where $$\tau_0 = \Delta C : \varepsilon_1 + \tau_1$$

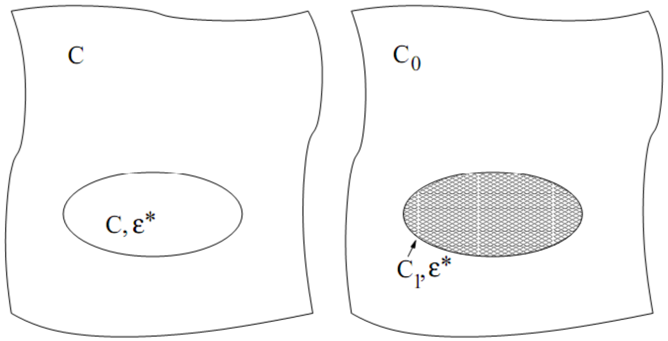
- Hypothesis of Hill: ε_1 is uniform in the inclusions :

$$\varepsilon_1 = -P(C_0) : \tau_0$$

$$-\tau_1 = \left(C^* + C_1\right) : \varepsilon_1$$

$$C^* = P^{-1} : C_0$$

C^**: Hill's tensor* $\sigma_1 = -C^* : \varepsilon_1$



(a) *(b)*

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Expressions of localization tensors

- Concentration tensors enable to link a local property to the corresponding macroscopic one

$$\bar{\sigma}(\bar{\varepsilon}) = \langle \sigma(\varepsilon(x)) \rangle = \langle \sigma(D^{\varepsilon}(x) : \bar{\varepsilon}) \rangle$$

D^{ε}: *Strain concentration tensor* $\varepsilon(x) = D^{\varepsilon}(x) : \bar{\varepsilon}$

- Each point of the *RVE* belong to phase *r* whose behaviors is given by a linear elastic relation as:

$$\sigma(x) = C_r : \varepsilon(x)$$

- Macroscopic relation:

$$\bar{\sigma} = C^{eff} : \bar{\varepsilon}$$

C^{eff}: *effective tensor*

- The macroscopic stress computed from the volume average of the stress microfield becomes:

$$\bar{\sigma} = \langle \sigma(x) \rangle = \langle C_r : \varepsilon(x) \rangle = \langle C_r : D^{\varepsilon}(x) \rangle : \bar{\varepsilon} = \bar{C} : \bar{\varepsilon}$$

⟶ *Hypothesis on* D^{ε} *give a* $\bar{C}$ *which is eigher an estimation or a bound of the effective stiffness* C^{eff}

Various homogenization schemes

- Expressions of concentration tensors A^{ε} and B^{ε}

$$\langle \varepsilon \rangle_{\omega_1} = B^{\varepsilon} : \langle \varepsilon \rangle_{\omega_0}$$

$$\langle \varepsilon \rangle_{\omega_1} = A^{\varepsilon} : \overline{\varepsilon}$$

- Relation between the concentration tensors

$$A^{\varepsilon} = B^{\varepsilon} : \left(v_1 B^{\varepsilon} + (1 - v_1) I \right)^{-1}$$

$$B^{\varepsilon} = (1 - v_1) A^{\varepsilon} : \left(I - v_1 A^{\varepsilon} \right)^{-1}$$

I : *fourth-order symmetric identity tensor*

- Macroscopic stiffness:

$$\begin{aligned} \overline{C} &= \left[v_1 C_1 : B^{\varepsilon} + (1 - v_1) C_0 \right] : \left[v_1 B^{\varepsilon} + (1 - v_1) I \right]^{-1}, \\ &= \left[v_1 \left(C^* + C_1 \right)^{-1} + (1 - v_1) \left(C^* + C_0 \right)^{-1} \right]^{-1} - C^* \\ &= C_0 + v_1 \left(C_1 - C_0 \right) : A^{\varepsilon} \end{aligned}$$

→ *Hypothesis on the strain concentration tensor defines an homogenization scheme so that the composite behavior can be predicted.*

Various homogenization schemes *(2)*

Dilute solution

- Each inclusion is considered as swimming in a infinite milieu having the properties of matrix and there isn't interaction among the inclusions.
- only apply to heterogeneous materials containing the low volume fraction of inclusions.

Mori-Tanaka model

- Taking into account interactions between the inclusions.
- fit for the heterogeneous materials with the moderate volume fraction 25-30%.

Self-consistent model

- Assuming that each inclusion is isolated and embedded in a fictitious homogeneous matrix possessing the unknown macroscopic stiffness that is being searched.
- generally giving good predictions for poly-cristals but is less satisfying in the case of two-phase composites.

Various homogenization schemes *(3)*

Bound of Voigt and Reuss

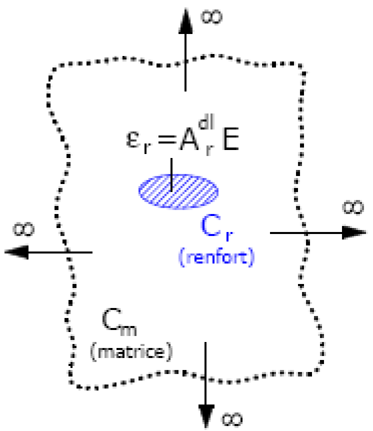
- **Bound of Voigt**
  - Assuming a uniform strain within the RVE
  - The Voigt model gives good predictions in the longitudinal direction of materials reinforced by long fiber.
- **Bound of Reuss model**
  - Assuming a uniform stress within the RVE

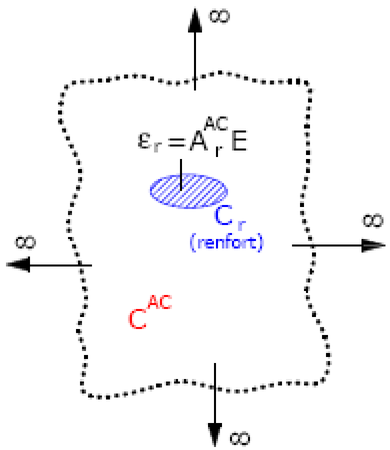
Bound of Hashin-Shtrikman-Walpole

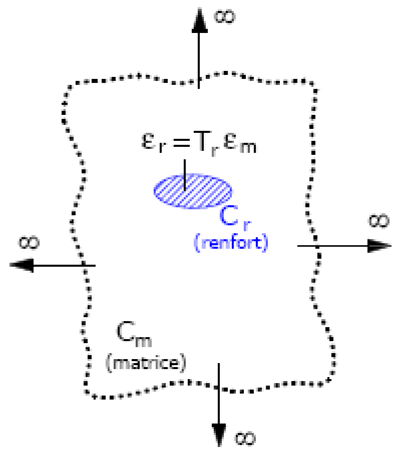
- Similar as Self-Consistent model but equivalent homogeneous material surrounding the inclusions is replaced by a material of comparison. Depending on the material of comparison that is more or less rigid, we have the inferior or superior boundary of composite.

Various homogenization schemes *(4)*

Dilute	**Self-consistent**	**Mori-Tanaka**







$$\mathbf{A}_r^{dl} = \left[\mathbf{I} + \mathbf{S}_r \mathbf{C}_m^{-1} \left(\mathbf{C}_r - \mathbf{C}_m \right) \right]^{-1}$$

$$\mathbf{A}_r^{AC} = \left[\mathbf{I} + \mathbf{S}_r \left(\mathbf{C}^{AC} \right)^{-1} \left(\mathbf{C}_r - \mathbf{C}^{AC} \right) \right]^{-1}$$

$$\mathbf{A}_{ph}^{MT} = \mathbf{A}_{ph}^{dl} \left[\sum_{ph=1}^{N} f_{ph} \mathbf{A}_{ph}^{dl} \right]^{-1}$$

$$\mathbf{C}^{dl} = \mathbf{C}_m + \sum_{r=2}^{N} f_r \left(\mathbf{C}_r - \mathbf{C}_m \right) \mathbf{A}_r^{dl}$$

$$\mathbf{C}^{AC} = \mathbf{C}_m + \sum_{r=2}^{N} f_r \left(\mathbf{C}_r - \mathbf{C}_m \right) \mathbf{A}_r^{AC}$$

$$\mathbf{C}^{MT} = \sum_{ph=1}^{N} f_{ph} \mathbf{C}_{ph} \mathbf{A}_{ph}^{MT} = \sum_{ph=1}^{N} f_{ph} \mathbf{C}_{ph} \mathbf{A}_{ph}^{dl} \left[\sum_{ph=1}^{N} f_{ph} \mathbf{A}_{ph}^{dl} \right]^{-1}$$

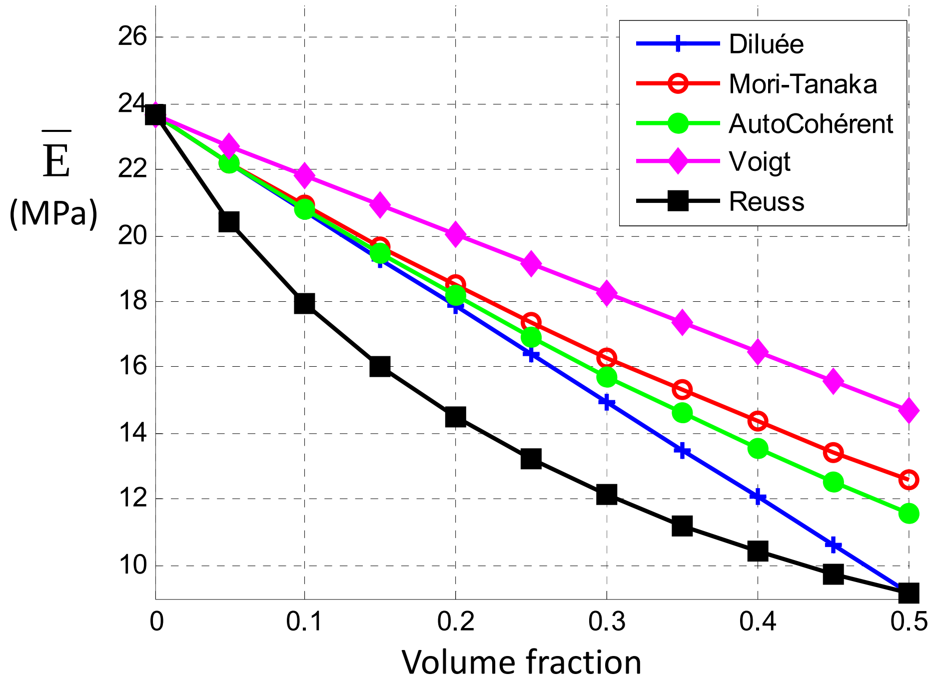
Various homogenization schemes *(5)*

Voigt	**Reuss**	**H.S.W max**	**H.S.W min**
$\underline{\underline{A}}_{ph} = \underline{\underline{I}}$	$\underline{\underline{B}}_{ph} = \underline{\underline{I}}$	$\underline{\underline{T}}_{ph}^{max} = \left[\underline{\underline{C}}_{ph} + \underline{\underline{C}}_{ph}^{max}\left(\underline{\underline{S}}_{r}^{-1} - \underline{\underline{I}}\right)\right]^{-1} \underline{\underline{C}}_{ph}^{max}\underline{\underline{S}}_{r}^{-1}$ $\rightarrow \underline{\underline{A}}_{ph}^{HSW\,max} = \underline{\underline{T}}_{ph}^{max}\left(\sum_{ph=1}^{N} f_{ph}\underline{\underline{T}}_{ph}^{max}\right)^{-1}$	$\underline{\underline{T}}_{ph}^{min} = \left[\underline{\underline{C}}_{ph} + \underline{\underline{C}}_{ph}^{min}\left(\underline{\underline{S}}_{r}^{-1} - \underline{\underline{I}}\right)\right]^{-1} \underline{\underline{C}}_{ph}^{min}\underline{\underline{S}}_{r}^{-1}$ $\rightarrow \underline{\underline{A}}_{ph}^{HSW\,min} = \underline{\underline{T}}_{ph}^{min}\left(\sum_{ph=1}^{N} f_{ph}\underline{\underline{T}}_{ph}^{min}\right)^{-1}$
$\underline{\underline{C}}^{Voigt} = \sum_{ph=1}^{N} f_{ph}\underline{\underline{C}}_{ph}$	$\underline{\underline{S}}^{Reuss} = \sum_{ph=1}^{N} f_{ph}\underline{\underline{C}}_{ph}^{-1}$	$\underline{\underline{C}}^{HSW\,max} = \sum_{ph=1}^{N} f_{ph}\underline{\underline{C}}_{ph}\underline{\underline{A}}_{ph}^{HSW\,max}$	$\underline{\underline{C}}^{HSW\,min} = \sum_{ph=1}^{N} f_{ph}\underline{\underline{C}}_{ph}\underline{\underline{A}}_{ph}^{HSW\,min}$

Various homogenization schemes *(6)*

Application: Reinforces less rigid than matrix

Lightweight concrete	Young's modulus (MPa)	Poisson's ratio
Matrix	23630	0.20
Granules (spheroid)	5679	0.15



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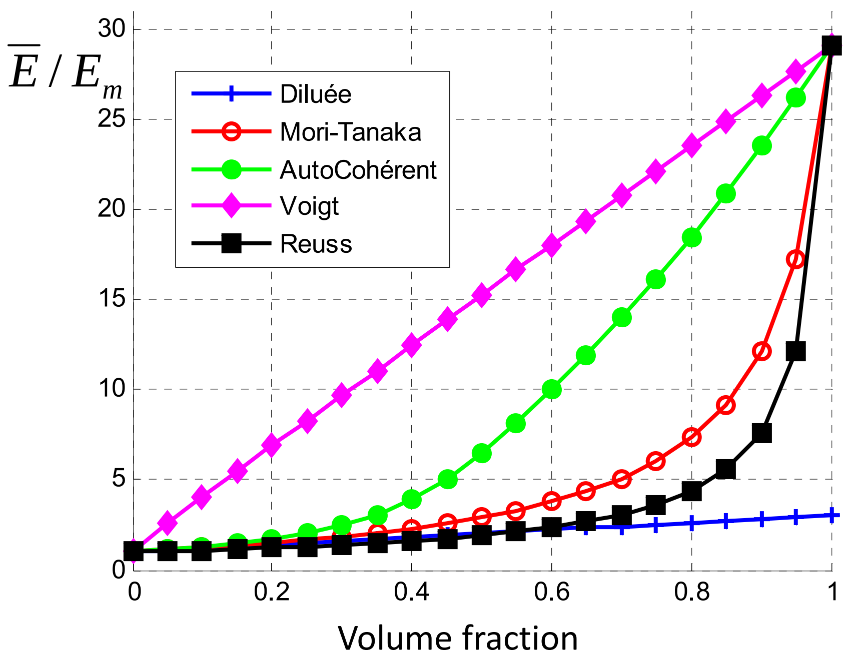
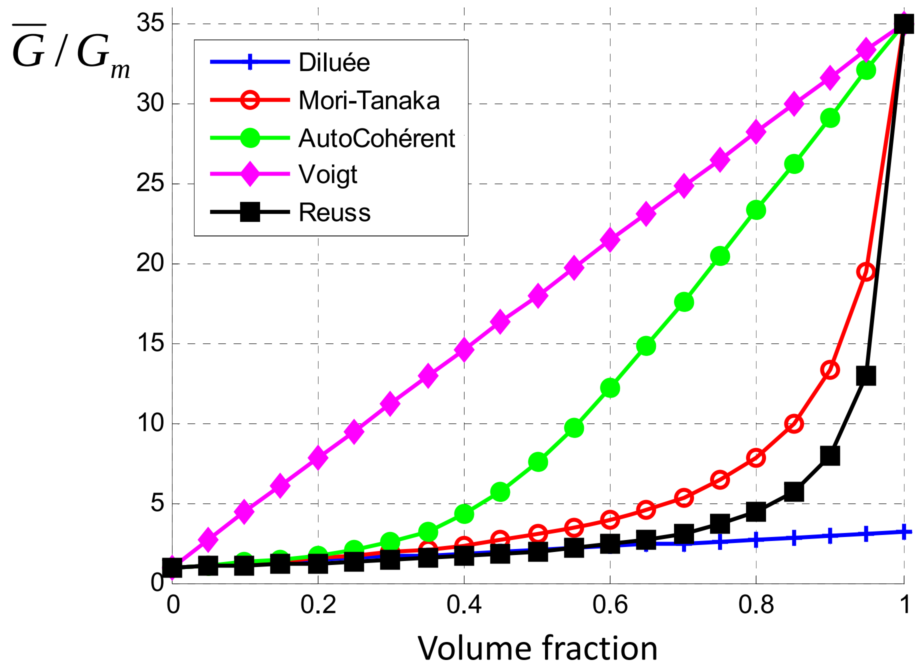
Various homogenization schemes *(7)*

Application: Reinforces more rigid than matrix

Concrete of resin	Young's modulus (GPa)	Poisson's ratio
Resin of epoxy	2.518	0.413
Standard sand (spheroid)	73.08	0.172





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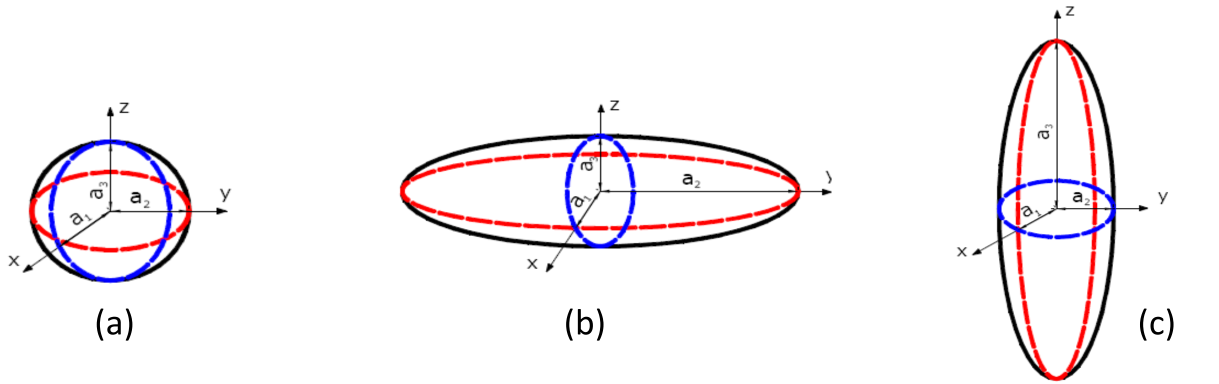
Influence of inclusion's geometry



(a) (b) (c)

		Case (a)	Case (b)	Case (c)
Macroscopic stiffness tensor	**Eshelby tensor**	Isotropic		
	Dilute solution	Isotropic	Transversal Isotropic	Transversal Isotropic
	Self-consistent	Isotropic		
	Mori-Tanaka	Isotropic	Transversal Isotropic	Transversal Isotropic

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Influence of inclusion's geometry

		Sphère (a)	*Sphéroïde aplati aux pôles* (b)	*Sphéroïde allongé* (c)
Tenseurs d'Eshelby		$\begin{bmatrix} 0.5805 & 0.1210 & 0.1210 & 0 & 0 & 0 \\ 0.1210 & 0.5805 & 0.1210 & 0 & 0 & 0 \\ 0.1210 & 0.1210 & 0.5805 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2198 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2198 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2198 \end{bmatrix}$ → ISOTROPE	$\begin{bmatrix} 0.2501 & 0.0577 & 0.0049 & 0 & 0 & 0 \\ 0.0577 & 0.2501 & 0.0049 & 0 & 0 & 0 \\ 0.4343 & 0.4343 & 0.9133 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3485 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3485 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0982 \end{bmatrix}$	$\begin{bmatrix} 0.6763 & 0.1360 & 0.2574 & 0 & 0 & 0 \\ 0.1360 & 0.6763 & 0.2574 & 0 & 0 & 0 \\ 0.0195 & 0.0195 & 0.2207 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2206 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2206 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2702 \end{bmatrix}$
Tenseurs des rigidités équivalents	Solution diluée	$\begin{bmatrix} 12.0934 & 7.2713 & 7.2713 & 0 & 0 & 0 \\ 7.2713 & 12.0934 & 7.2713 & 0 & 0 & 0 \\ 7.2713 & 7.2713 & 12.0934 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.4110 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.4110 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.4110 \end{bmatrix}$ → ISOTROPE → Module d'Young équivalent = 6.6328 GPa ; Coefficient de Poisson équivalent = 0.3754	$\begin{bmatrix} 15.6140 & 7.6055 & 7.0723 & 0 & 0 & 0 \\ 7.6055 & 15.6140 & 7.0723 & 0 & 0 & 0 \\ 7.0723 & 7.0723 & 10.6468 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.6722 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.6722 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.1043 \end{bmatrix}$ → Isotrope transverse : *5 coefficients indépendants*	$\begin{bmatrix} 11.4969 & 7.2126 & 7.2131 & 0 & 0 & 0 \\ 7.2126 & 11.4969 & 7.2131 & 0 & 0 & 0 \\ 7.2132 & 7.2132 & 16.4743 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.4056 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.4056 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.1421 \end{bmatrix}$ → Isotrope transverse : *5 coefficients indépendants*
	Solution Auto-cohérente	$\begin{bmatrix} 46.0667 & 6.6215 & 6.6215 & 0 & 0 & 0 \\ 6.6215 & 46.0667 & 6.6215 & 0 & 0 & 0 \\ 6.6215 & 6.6215 & 46.0667 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20.7226 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20.7226 & 0 \\ 0 & 0 & 0 & 0 & 0 & 20.7226 \end{bmatrix}$ → ISOTROPE → Module d'Young équivalent = 46.4633 GPa ; Coefficient de Poisson équivalent = 0.1208	$\begin{bmatrix} 57.0979 & 9.6110 & 6.9825 & 0 & 0 & 0 \\ 9.6110 & 57.0979 & 6.9825 & 0 & 0 & 0 \\ -3.2711 & -3.2711 & 31.3194 & 0 & 0 & 0 \\ 0 & 0 & 0 & 13.7740 & 0 & 0 \\ 0 & 0 & 0 & 0 & 13.7740 & 0 \\ 0 & 0 & 0 & 0 & 0 & 23.7435 \end{bmatrix}$	$\begin{bmatrix} 43.1243 & 5.7276 & 3.4261 & 0 & 0 & 0 \\ 5.7276 & 43.1243 & 3.4261 & 0 & 0 & 0 \\ 9.4316 & 9.4316 & 57.5616 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20.6947 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20.6947 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16.6984 \end{bmatrix}$
	Solution de Mori-Tanaka	$\begin{bmatrix} 26.5701 & 12.6370 & 12.6370 & 0 & 0 & 0 \\ 12.6370 & 26.5701 & 12.6370 & 0 & 0 & 0 \\ 12.6370 & 12.6370 & 26.5701 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6.9666 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6.9666 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.9666 \end{bmatrix}$ → ISOTROPE → Module d'Young équivalent = 16.4240 GPa ; Coefficient de Poisson équivalent = 0.3224	$\begin{bmatrix} 34.9660 & 12.1666 & 11.6611 & 0 & 0 & 0 \\ 12.1666 & 34.9660 & 11.6611 & 0 & 0 & 0 \\ 11.6611 & 11.6611 & 22.6606 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5.1131 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.1131 & 0 \\ 0 & 0 & 0 & 0 & 0 & 11.3666 \end{bmatrix}$ → Isotrope transverse : *5 coefficients indépendants*	$\begin{bmatrix} 25.0090 & 12.6561 & 11.7644 & 0 & 0 & 0 \\ 12.6561 & 25.0090 & 11.7644 & 0 & 0 & 0 \\ 11.7644 & 11.7644 & 36.2075 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6.9497 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6.9497 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.0754 \end{bmatrix}$ → Isotrope transverse : *5 coefficients indépendants*

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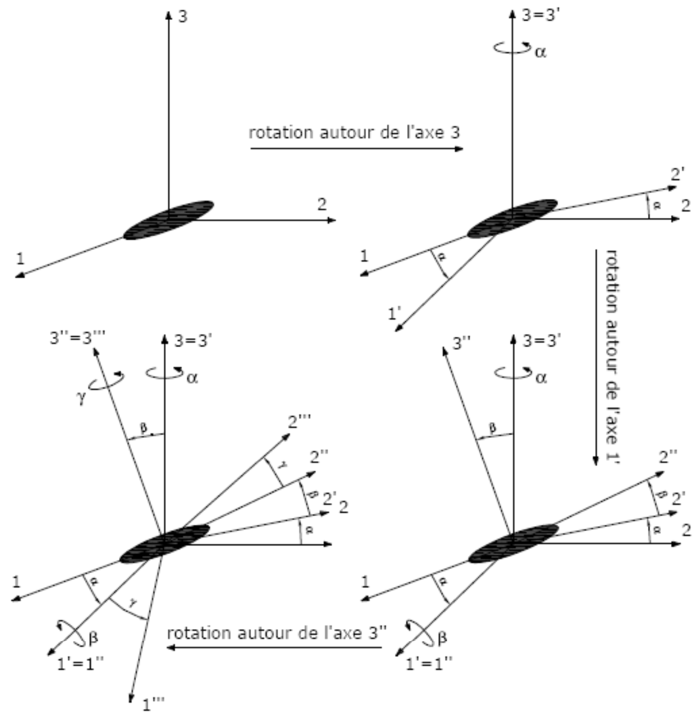
Orientation of inclusions

- 3 Euler angles (α, β, γ) describing the diverse orientations of inclusions.



- Transformation tensor Qij(α, β, γ):

$$Q_{ij}(\alpha,\beta,\gamma)=\begin{bmatrix} \cos\alpha\cos\gamma-\sin\alpha\cos\beta\sin\gamma & \sin\alpha\cos\gamma+\cos\alpha\cos\beta\sin\gamma & \sin\gamma\sin\beta \\ -\cos\alpha\sin\gamma-\sin\alpha\cos\beta\cos\gamma & -\sin\alpha\sin\gamma+\cos\alpha\cos\beta\cos\gamma & \sin\beta\cos\gamma \\ \sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta \end{bmatrix}$$

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Orientation of inclusions *(2)*

2 Methods:

➢ **1st Method:** consider a finite number of families of inclusions, each of which is oriented in the directions well determined.

$$\underline{\underline{\underline{C}}}^{MT}_{Nf} = \underline{\underline{\underline{C}}}_m + \sum_{i=1}^{Nf} \frac{f_r}{Nf} \overline{\left(\left(\underline{\underline{\underline{C}}}_r - \underline{\underline{\underline{C}}}_m \right) \underline{\underline{\underline{A}}}^{dl}_r \right)}_i \left(f_m I + \sum_{i=1}^{Nf} \frac{f_r}{Nf} \overline{\left(\underline{\underline{\underline{A}}}^{dl}_r \right)}_i \right)^{-1}$$

Nf :*the number of families of inclusions oriented in a given direction*

$$\overline{X}_{ijkl} = Q_{ip} Q_{jq} Q_{kr} Q_{ls} X_{pqrs}$$

Orientation of inclusions *(3)*

2 Methods:

➢ **2nd Method:** couple the ODF function to present the divers orientations of inclusions.

$$\underline{\underline{C}}^{MT}_{ODF} = \underline{\underline{C}}_m + f_r \left\langle \left(\underline{\underline{C}}_r - \underline{\underline{C}}_m \right) \underline{\underline{A}}^{dl}_r \right\rangle \left(f_m I + f_r \left\langle \underline{\underline{A}}^{dl}_r \right\rangle \right)^{-1}$$

(case of the biphasic composite)

with $$\left\langle X_{ijkl} \right\rangle = \frac{\int_{-\pi}^{\pi}\int_{0}^{\pi}\int_{0}^{\pi/2} \overline{X}_{ijkl}(\alpha,\beta,\gamma) ODF(\alpha,\gamma)\sin(\beta)\, d\alpha\, d\beta\, d\gamma}{\int_{-\pi}^{\pi}\int_{0}^{\pi}\int_{0}^{\pi/2} ODF(\alpha,\gamma)\sin(\beta)\, d\alpha\, d\beta\, d\gamma}$$

where $\overline{X}_{ijkl} = Q_{ip} Q_{jq} Q_{kr} Q_{ls} X_{pqrs}$ *and* $ODF(\alpha,\gamma) = \exp\left(-s_1 \alpha^2\right)\exp\left(-s_2 \gamma^2\right)$

➡ *See Appendix B for more details*

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Constitutive equations

Homogeneous linear thermo-elastic material of elastic stiffness C and thermal expansion α :

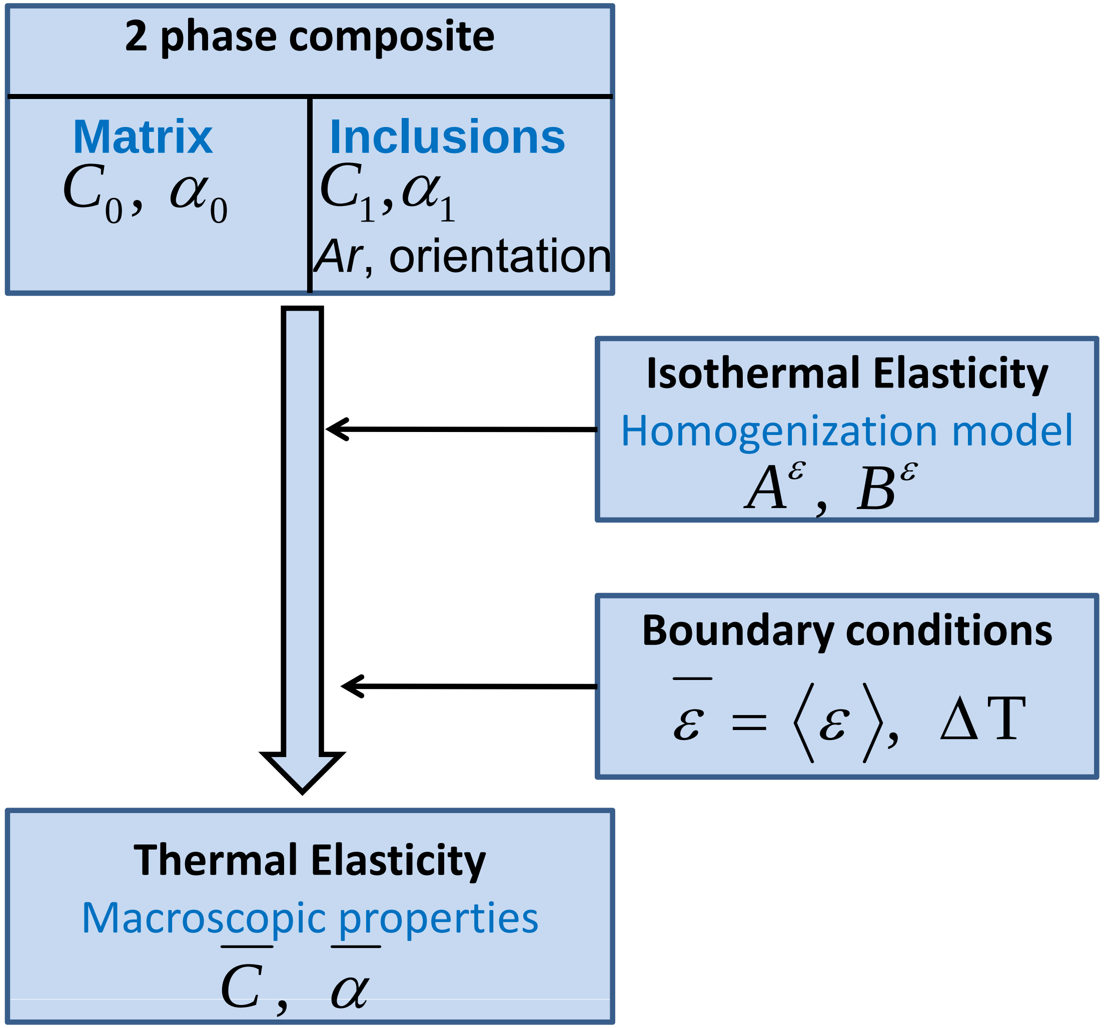
$$\sigma(x) = C : \left(\varepsilon(x) - \varepsilon^{th}(x)\right) \;, \qquad \varepsilon^{th}(x) = \alpha \Delta T(x)$$

$$= C : \varepsilon(x) + \beta \Delta T(x) \;, \qquad \beta = -C : \alpha$$

or $\quad \varepsilon(x) = C^{-1} : \sigma(x) + \varepsilon^{th}(x)$

Homogenization technique

Problem:



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Homogenization technique *(2)*

First step: The composite is subjected $\varepsilon^{s_1} = \overline{\varepsilon}$ and $\Delta T^{s_1} = 0$

→classical isothermal transformation

The per phase strain averages:

$$\left\langle \varepsilon^{s_1} \right\rangle_{w_1} = A^{\varepsilon} : \overline{\varepsilon} , \quad v_1 \left\langle \varepsilon^{s_1} \right\rangle_{w_1} + (1 - v_1) \left\langle \varepsilon^{s_1} \right\rangle_{w_0} = \overline{\varepsilon}$$

The stress averages:

$$\left\langle \sigma^{s_1} \right\rangle_{w_1} = C_1 : \left\langle \varepsilon^{s_1} \right\rangle_{w_1} , \quad \left\langle \sigma^{s_1} \right\rangle_{w0} = C_0 : \left\langle \varepsilon^{s_1} \right\rangle_{w_0}$$

Second step: The RVE witnesses $\Delta T^{s2} = \Delta T$

and a macroscopic strain increment $\Delta \overline{\varepsilon}^{s_2}$

The uniform strain and stress increments are obtained:

$$\Delta \sigma^{s2} = C_0 : \Delta \overline{\varepsilon}^{s2} + \beta_0 \Delta T = C_1 : \Delta \overline{s}^{s_2} + \beta_1 \Delta T$$

$$\Delta \overline{\varepsilon}^{s2} = -\left(C_1 - C_0\right)^{-1} : \left(\beta_1 - \beta_0\right) \Delta T$$

Homogenization technique *(3)*

Third step: The composite is subjected to linear boundary displacements corresponding to a macroscopic total strain increment $\Delta\overline{\varepsilon}^{s_3}$ and $\Delta T^{s_3} = 0$

$$\left\langle \varepsilon^{s_3} \right\rangle_{w_1} = A^{\varepsilon} : \Delta\overline{\varepsilon}^{s_3}, v_1 \left\langle \Delta\varepsilon^{s_3} \right\rangle_{w_1} + (1 - v_1)\left\langle \Delta\varepsilon^{s_3} \right\rangle_{w_0} = \Delta\overline{\varepsilon}^{s_3}$$

$$\left\langle \Delta\sigma^{s_3} \right\rangle_{w_1} = C_1 : \left\langle \Delta\varepsilon^{s_3} \right\rangle_{w_1}, \qquad \left\langle \Delta\sigma^{s_3} \right\rangle_{w_0} = C_0 : \left\langle \Delta\varepsilon^{s_3} \right\rangle_{w_0}$$

Superposition:

$$\left\langle \varepsilon \right\rangle_{w_1} = A^{\varepsilon} : \left(\overline{\varepsilon} + \Delta\overline{\varepsilon}^{s_3}\right) + \Delta\overline{\varepsilon}^{s_2}$$

$$v_1 \left\langle \varepsilon \right\rangle_{w_1} + (1 - v_1)\left\langle \varepsilon \right\rangle_{w_0} = \overline{\varepsilon} + \Delta\overline{\varepsilon}^{s_2} + \Delta\overline{\varepsilon}^{s_3}$$

$$\longrightarrow \quad \Delta\overline{\varepsilon}^{s_3} = -\Delta\overline{\varepsilon}^{s_2}$$

and

$$\left\langle \sigma \right\rangle = \Delta\sigma^{s_2} + (1 - v_1)\left(\left\langle \sigma^{s_1} \right\rangle_{w_0} + \left\langle \Delta\sigma^{s_1} \right\rangle_{w_0} \right) + v_1 \left(\left\langle \sigma^{s_1} \right\rangle_{w_1} + \left\langle \Delta\sigma^{s_3} \right\rangle_{w_1} \right)$$

Homogenization technique *(4)*

Final expressions:

$$\langle \varepsilon \rangle_{\omega_1} = A^{\varepsilon} : \overline{\varepsilon} + a^{\varepsilon} \Delta T$$
$$a^{\varepsilon} = \left(A^{\varepsilon} - I\right) : \left(C_1 - C_0\right)^{-1} : \left(\beta_1 - \beta_0\right) \qquad (*)$$
$$v_1 \langle \varepsilon \rangle_{w_1} + \left(1 - v_1\right)\langle \varepsilon \rangle_{w_0} = \overline{\varepsilon}$$

Macroscopic thermo-elastic response:

$$\langle \sigma \rangle = \overline{C} : \overline{\varepsilon} + \overline{\beta} \Delta T$$
$$\overline{\beta} = \left(1 - v_1\right)\beta_0 + v_1 \beta_1 + v_1 \left(C_1 - C_0\right) : a^{\varepsilon}$$

$\overline{C}$ is given by the isothermal expression, a^{ε} by (*) and $\overline{\alpha} = -\overline{C}^{-1} : \overline{\beta}$

Special case: When $C_0 = C_1 = C$

step 1 is unchanged

steps 2 and 3 are replaced by $\Delta T^{s_2} = \Delta T$

and $\Delta \overline{\varepsilon}^{s_2} = 0$

Homogenization technique *(5)*

Application: Short glass fiber reinforced composite

Inclusion: E_1 =72.5 GPa, $\nu_1 = 0.2,\ \alpha_1 = 4.9\text{x}10^{-6} K^{-1}$

$A_r = 35.58,\ \text{v}_1 = 8\%$

Matrix: E_0 =1.57 GPa, $\nu_0 = 0.335,\ \alpha_0 = 108.3\text{x}10^{-6} K^{-1}$

Model	E_L [GPa]	E_T [GPa]	$\bar{\alpha}_L$ [10^{-6}K^{-1}]	$\bar{\alpha}_T$ [10^{-6}K^{-1}]
Experiment [65]	5.99		27.7 ± 1.7	121 ± 1
M-T	6.097	1.929	30.1	120.6
M-T($A_r \times 1.25$)	6.401	1.932	28.9	121.0
Interpol. model	6.136	1.938	30.0	120.1
Interpol. model ($A_r \times 1.25$)	6.432	1.941	28.8	120.5
FE [65]			29.3 ± 0.1	119 ± 0.1
Takao-Taya/aggregate [65]			32.0	120
Tandon-Weng/laminate [65]			29.4	119
McCullough [65]	< 5.89		31.4	121

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Constitutive equations

The constitutive model of Homogeneous viscoelastic material:

$$\sigma(x,t) = G(t) : \varepsilon(\mathrm{x},0) + \int_0^t G(t-\tau) : \dot{\varepsilon}(\mathrm{x},\tau)\,d\tau,$$

$$= G(t) : \varepsilon(\mathrm{x},0) + \dot{\varepsilon}(\mathrm{x}) \otimes \mathrm{G},$$

where $\varepsilon(\mathrm{x},0) = \lim_{t \to 0^+} \varepsilon(\mathrm{x},\mathrm{t})$

Or equivalently: $\varepsilon(\mathrm{x},\mathrm{t}) = J(t) : \sigma(\mathrm{x},0) + \int_0^t J(t-\tau) : \dot{\sigma}(\mathrm{x},\tau)\,d\tau.$

$$= J(t) : \sigma(\mathrm{x},0) + \dot{\sigma}(\mathrm{x}) \otimes \mathrm{J},$$

where $\sigma(\mathrm{x},0) = \lim_{t \to 0^+} \sigma(\mathrm{x},\mathrm{t})$

G is the fourth order relaxation tensor $\overline{\sigma}^*(s) = \overline{G}^*(s) : \overline{\varepsilon}^*(\mathrm{s})$

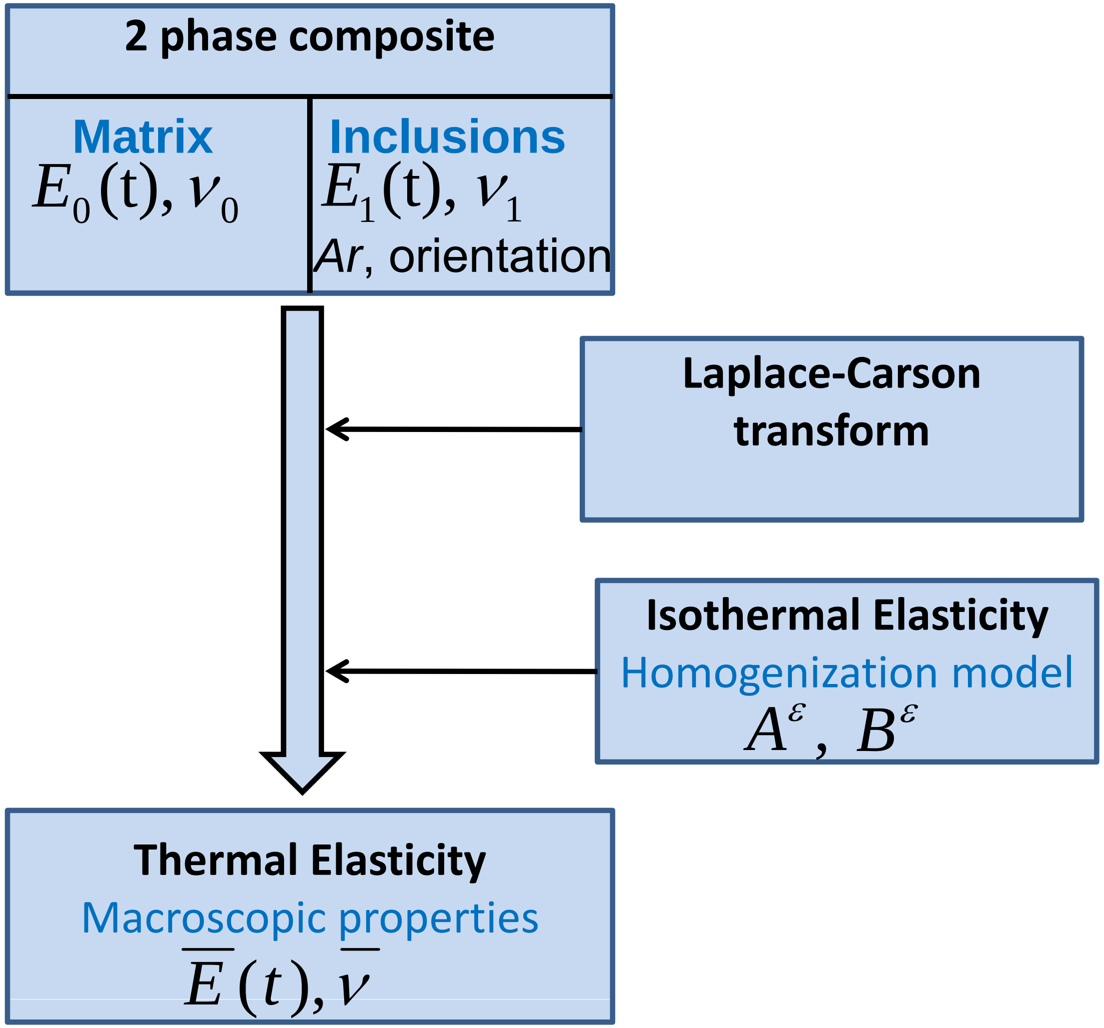
J is the fourth order creep tensor $\overline{\varepsilon}^*(\mathrm{s}) = \overline{J}^*(s) : \overline{\sigma}^*(\mathrm{s})$

Using the Laplace-Carson transform: $\overline{G}^*(s) = [\overline{J}^*(s)]^{-1}$

*f** is the laplace-carson transform of function *f* and *s* is the laplace variable

Homogenization technique

Problem:



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Homogenization technique *(2)*

Applying the homogenization schemes valid in linear isothermal elasticity in order to obtain the homogenized modulus $\overline{G}^*(s)$ or $\overline{J}^*(s)$

$$\overline{\sigma}^*(s) = \overline{G}^*(s) : \overline{\varepsilon}^*(s)$$

Or equivalently $\overline{\varepsilon}^*(s) = \overline{J}^*(s) : \overline{\sigma}^*(s)$

with $\overline{G}^*(s) = [\overline{J}^*(s)]^{-1}$

Using a numerical inversion of the Laplace-Carson transform, the macroscopic response is calculated as:

$$\overline{\sigma}(t) = \overline{G}(t) : \overline{\varepsilon}(0) + \int_0^t \overline{G}(t-\tau) : \dot{\overline{\varepsilon}}(\tau) d\tau ,$$

$$\overline{\varepsilon}(t) = \overline{J}(t) : \overline{\sigma}(0) + \int_0^t \overline{J}(t-\tau) : \dot{\overline{\sigma}}(\tau) d\tau .$$

See Appendix C for more details

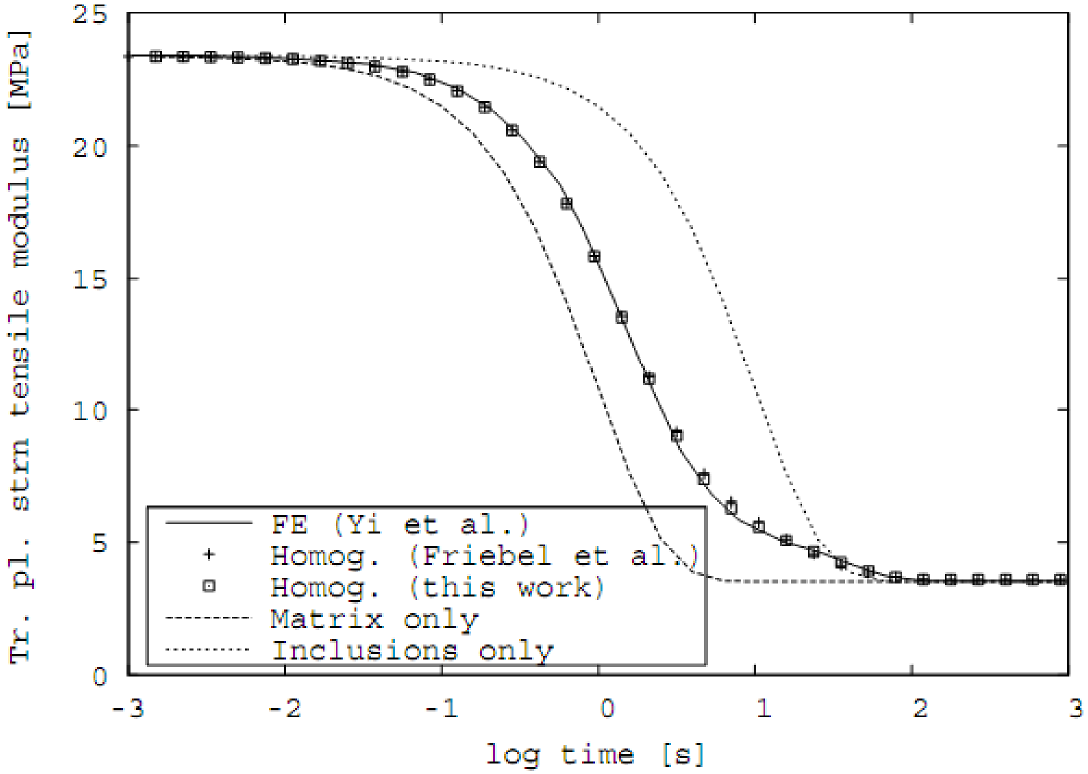
Homogenization technique *(3)*

Application: linear viscoelastic material defined by Prony series

Inclusion (long fibers): $E_1 = 3+17e^{-t/10}$, $\nu_1 = 0.38$

$v_1 = 50\%$

Matrix: $E_0 = 3+17e^{-t}$, $\nu_0 = 0.38$



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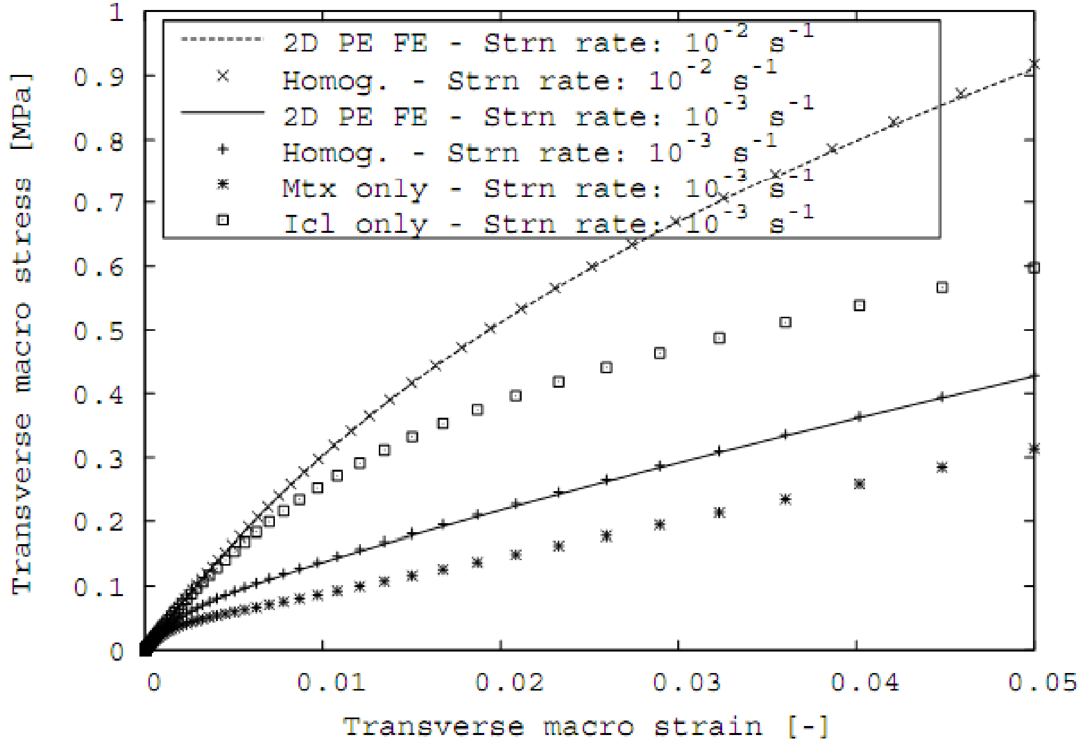
Homogenization technique *(4)*

Application: linear viscoelastic material defined by Prony series

Inclusion (long fibers): E_1 =3+17e-$^{t/10}$, $\nu_1 = 0.38$

$$v_1 = 50\%$$

Matrix: E_0 =3+17e^{-t}, $\nu_0 = 0.38$



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Conclusion

✓ Major mean-field homogenization schemes for two-phase isothermal linear elastic composites

✓ General method allows to formulate the thermo-elastic version of any homogenization model defined by its isothermal strain concentration tensors

✓ The problem of visco-elastic composite is examined

➡ The predictions have been extensively validated against experimental data or FE results for numerous composite systems

➡ These homogenization schemes are used as sub-problems in the homogenization of nonlinear materials

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Appendix A: Eshelby's tensor

➢The Eshelby's tensor for inclusions of arbitrary geometry embedded in an isotropic matrix characterized by μ and ν:

$$\varepsilon_{ijkl} = \frac{1}{8\pi(1-\nu)}\left[\delta_{ij}\delta_{kl}(2\nu I_i + J_{ik}) + (\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})\{(1-\nu)(I_k + I_l) + J_{ij}\}\right]$$

$$I_i = I(\lambda) = 2\pi a_1 a_2 a_3 \int_{\lambda}^{\infty} \frac{(A_i + s)^{-1}}{\Delta s} ds$$

$$J_{ij} = J_{ij}(\lambda) A_i I_{ij}(\lambda) - I_i(\lambda)$$

$$I_{ij} = I_{ij}(\lambda) = 2\pi a_1 a_2 a_3 \int_{\lambda}^{\infty} \frac{(A_i + s)^{-1}(A_j + s)^{-1}}{\Delta s} ds$$

$$\Delta s = \sqrt{(A_1 + s)(A_2 + s)(A_3 + s)}$$

$$A_i = a_i^2$$

Appendix A: Eshelby's tensor *(2)*

➢The Eshelby's tensor for spheroids of aspect ratio Ar embedded in an isotropic matrix.

$$
\begin{aligned}
S_{1111} &= \frac{1}{2(1-\nu)}\left[2(1-\nu)(1-g)+g-Ar^2\frac{3g-2}{Ar^2-1}\right],\\
S_{2222}=S_{3333} &= \frac{1}{4(1-\nu)}\left[2(2-\nu)g-\frac{1}{2}-(Ar^2-\frac{1}{4})\frac{3g-2}{Ar^2-1}\right],\\
S_{1122}=S_{1133} &= \frac{1}{4(1-\nu)}\left[4\nu(1-g)-g+Ar^2\frac{3g-2}{Ar^2-1}\right],\\
S_{2233}=S_{3322} &= \frac{1}{4(1-\nu)}\left[-(1-2\nu)g+\frac{1}{2}-\frac{1}{4}\frac{3g-2}{Ar^2-1}\right],\\
S_{2211}=S_{3311} &= \frac{1}{4(1-\nu)}\left[-(1-2\nu)g+Ar^2\frac{3g-2}{Ar^2-1}\right],\\
S_{1212}=S_{1313} &= \frac{1}{4(1-\nu)}\left[(1-\nu)(2-g)-g+Ar^2\frac{3g-2}{Ar^2-1}\right],\\
S_{2323} &= \frac{S_{2222}-S_{1122}}{2},
\end{aligned}
$$

where
$$
\begin{aligned}
g &= \frac{Ar}{(Ar^2-1)^{3/2}}\left[Ar(Ar^2-1)^{1/2}-cos^{-1}Ar\right] \text{ for } 0<Ar<1,\\
g &= \frac{Ar}{(1-Ar^2)^{3/2}}\left[cosh^{-1}Ar-Ar(1-Ar^2)^{1/2}\right] \text{ for } 1<Ar<\infty. \text{(A.2)}
\end{aligned}
$$

Appendix A: Eshelby's tensor *(3)*

- The Eshelby's tensor for case of spherical inclusions Ar=1:

$$S_{1111} = S_{2222} = S_{3333} = \frac{7-5\nu}{15(1-\nu)},$$

$$S_{1122} = S_{1133} = S_{2233} = \frac{5\nu-1}{15(1-\nu)},$$

$$S_{1212} = S_{1313} = S_{2323} = \frac{4-5\nu}{15(1-\nu)}.$$

Appendix B: ODF function

- The different values of two parameters of ODF function:

Aligné	Axisymétrique	Aléatoire
$s_1 = s_2 = \infty$	$s_1 = k, \quad s_2 = \infty$	$s_1 = s_2 = 0$
$ODF(\alpha,\gamma) = \delta(\alpha - 0)\delta(\gamma - 0)$	$ODF(\alpha,\gamma) = \exp\left(-k\alpha^2\right)\delta(\gamma - 0)$	$ODF(\alpha,\gamma) = 1$

Appendix C: Laplace-Carson transform

- Laplace-Carson transform:

$$[\mathcal{L}(f)]_{(s)} = f^*(s) = s\int_0^{\infty} f(t)e^{-st}dt.$$

- Laplace-Carson inversion:

$$f(t) = A + Bt + \sum_{k=1}^{k=M} b_k \underbrace{\left(1 - e^{-t/\theta_k}\right)}_{\text{Basis functions}}.$$

$$f^*(s) = A + \frac{B}{s} + \sum_{k=1}^{k=M} b_k \underbrace{\frac{1}{1+s\theta_k}}_{\text{Transf. basis funct.}}$$

$$A = \lim_{s\to+\infty} f^*(s) = \lim_{t\to 0} f(t),$$

$$B = \lim_{s\to 0} sf^*(s) = \lim_{t\to+\infty} \frac{f(t)}{t}.$$

$$f^*(s_l) = A + \frac{B}{s_l} + \sum_{k=1}^{k=M} b_k \frac{1}{1+s_l\theta_k},$$

$f(t)$	$f^*(s)$	Conditions
a	a	$s > 0$
at	a/s	$s > 0$
e^{at}	$\frac{s}{s-a}$	$s > a$
$cos(\omega t)$	$\frac{s^2}{s^2+\omega^2}$	-
$\delta(t-c)$	se^{-cs}	-
$H(t-c)$	e^{-cs}	-
$af(t)+bg(t)$	$af^*(s)+bg^*(s)$	-
$A \otimes B$	$\frac{1}{s}A^*(s) : B^*(s)$	-
$A \odot B$	$A^*(s) : B^*(s)$	-

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