Alternative numerical methods in continuum mechanics
Summary of the Finite Element method

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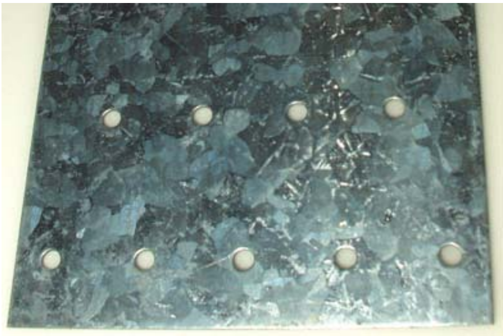
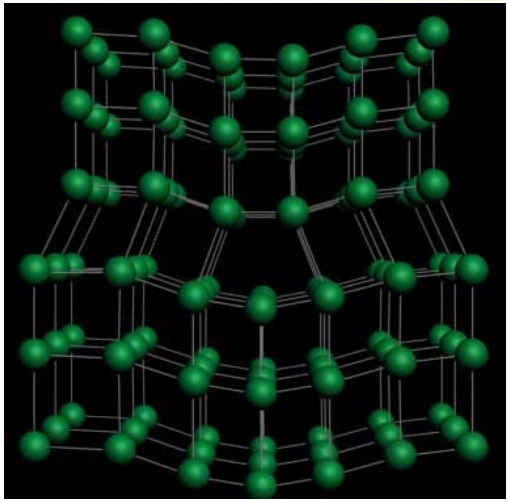
Continuum mechanics

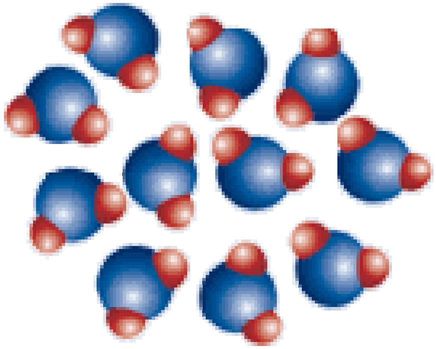
- Definition
  - Materials are modeled as a continuum
  - Matter
    - Is continuously distributed &
    - Fills the entire region of space the body occupies
- Consequences
  - The body can be continually sub-divided into infinitesimal elements
    - Kinematics and material behavior laws are deduced from these infinitesimal elements analysis
  - Kinematics and material behavior obey to
    - Constitutive equations
      - Elasticity
      - Elasto-plasticity
    - Conservation laws
      - Conservation of mass
      - Conservation of linear momentum
      - Conservation of angular momentum
      - Conservation of energy

- Idea
  - A real material is
    - Heterogeneous
      - Grains
      - Inclusions
    - Made of discontinuities
      - Cracks
      - Grain-boundaries
      - Plastic dislocations
    - Composed of molecules/atoms
      - Fluids, Solids
  - Instead of studying the motion of every atoms, continuum mechanics models these
    - Heterogeneities
    - Discontinuities
    - …

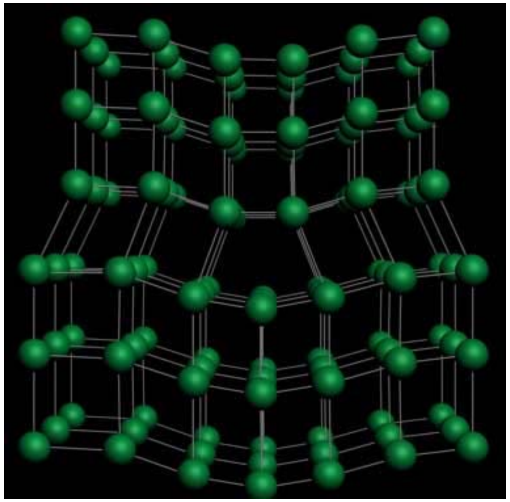
    at the macroscopic level through
    - Material laws
    - ….

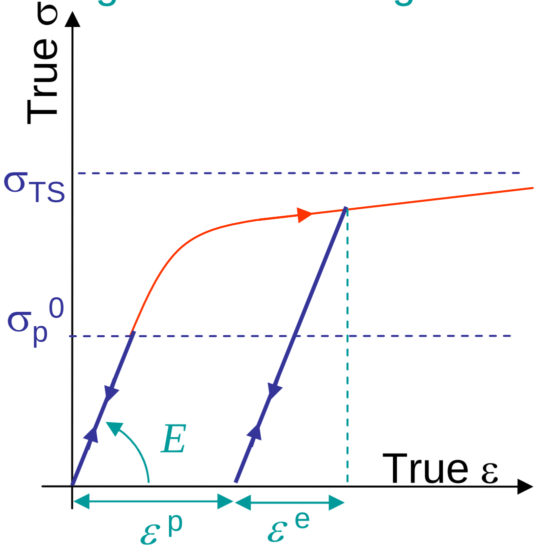




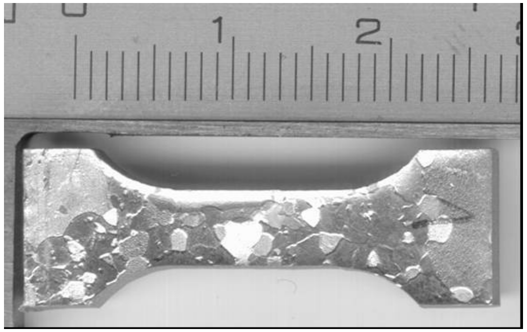


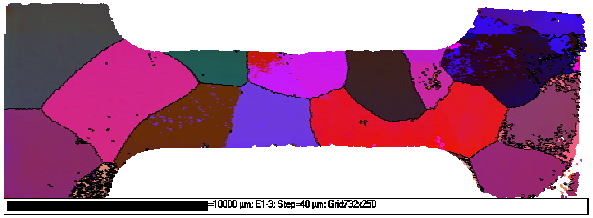
- Examples
  - Dislocation motions are modeled using an elasto-plastic material law
    - Grain sizes, inclusions, … are accounted for through the hardening law

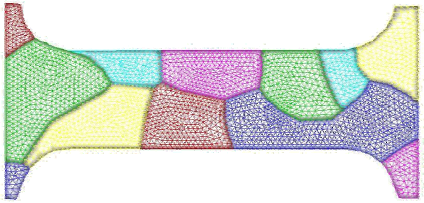




  - Each grain can also be modeled by continuum mechanics
    - A crystal plasticity model is used in each grain

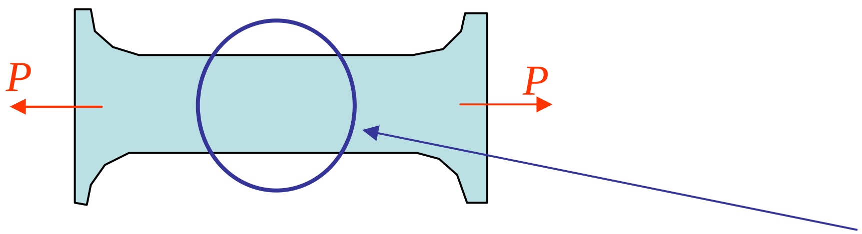




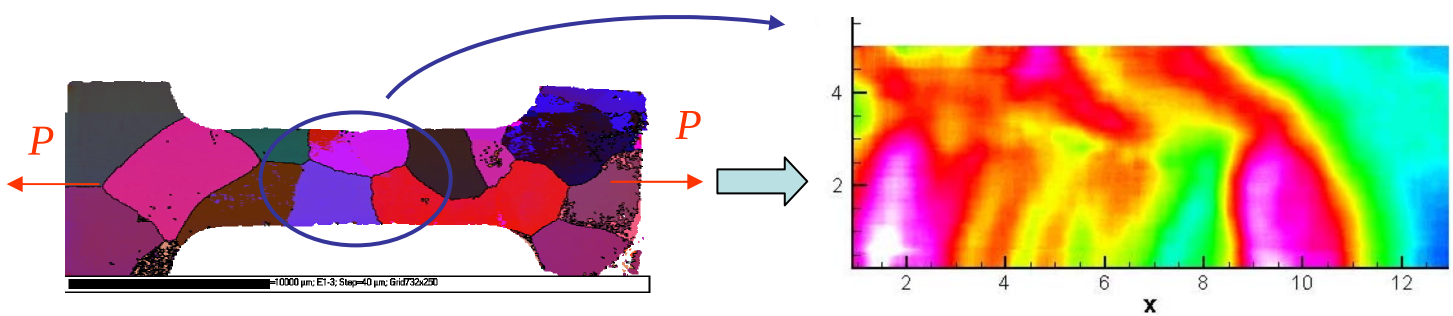


Continuum mechanics

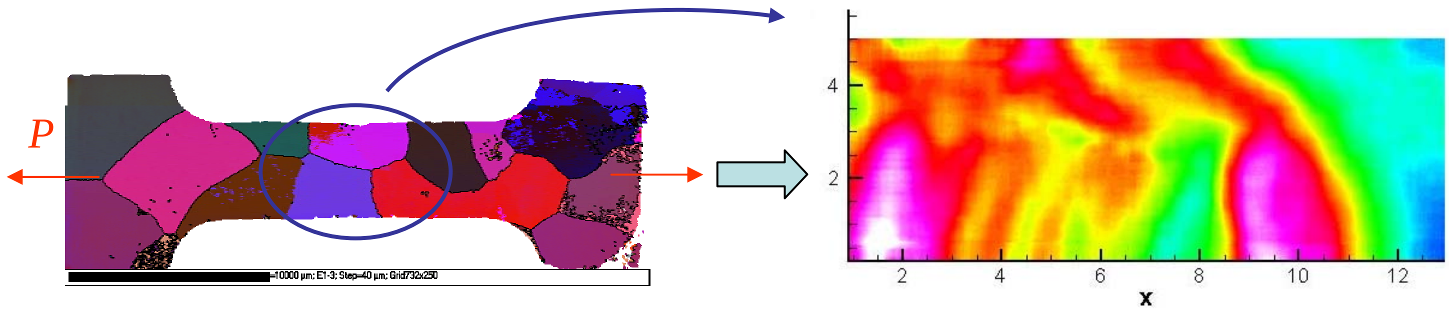
- Limitations
  - The model should be able to capture the physics
  - Example
    - Tensile test with an homogeneous elasto-plastic material



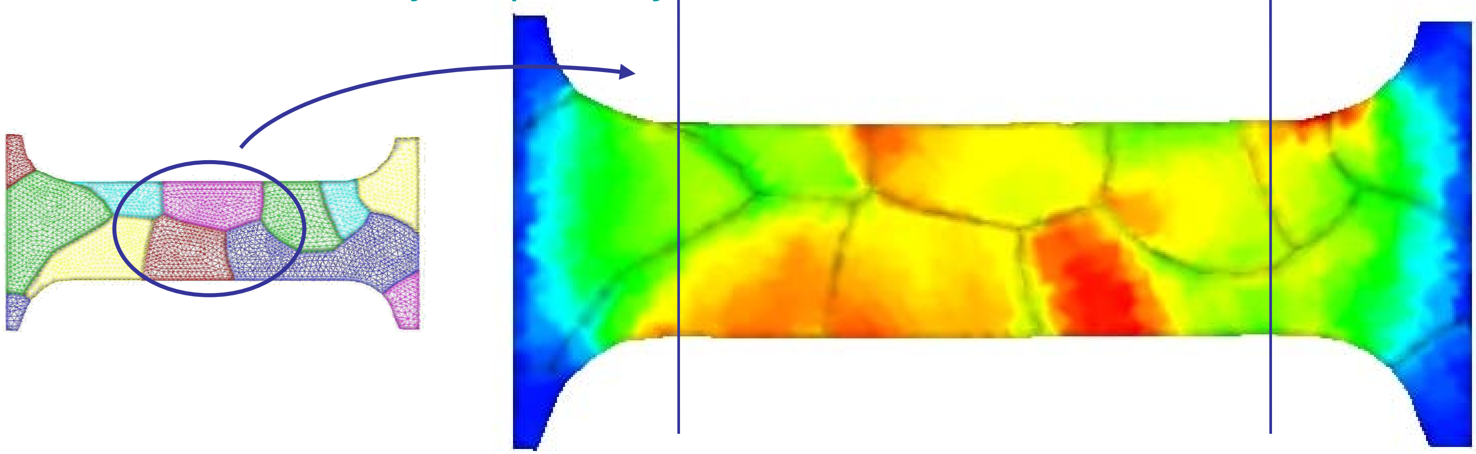
      - Deformations (plastic & elastic) will be uniform in the central zone
      - This can be a good model as long as grain size is small compared to the macroscopic characteristic length
    - Real structure with grain size comparable to the macroscopic length
      - Plastic deformations at the surface are not uniform



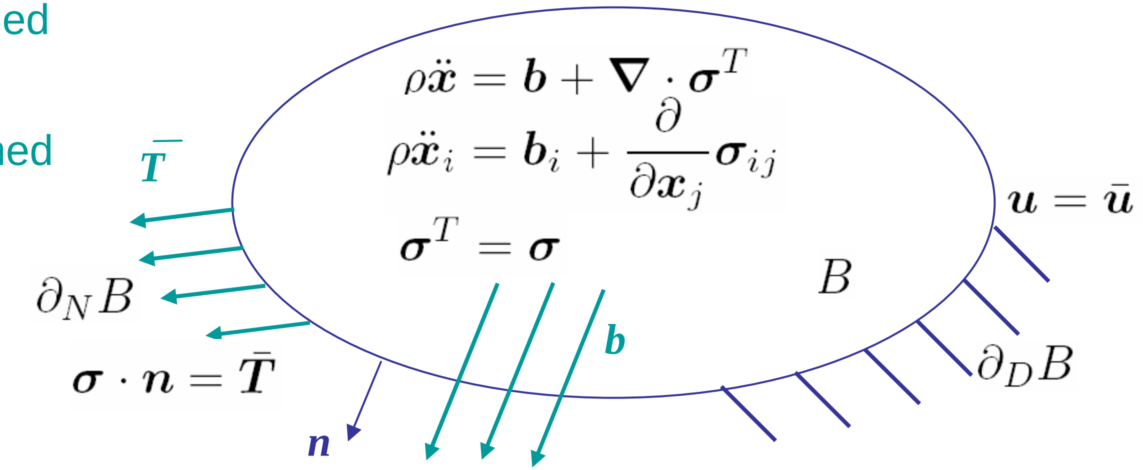
- Limitations (2)
  - Example (2)
    - Real structure with grains size comparable to the macroscopic length
      - Plastic deformations at the surface are not uniform



    - It does not mean continuum mechanics cannot be assumed, but the model should be enhanced to consider crystal plasticity

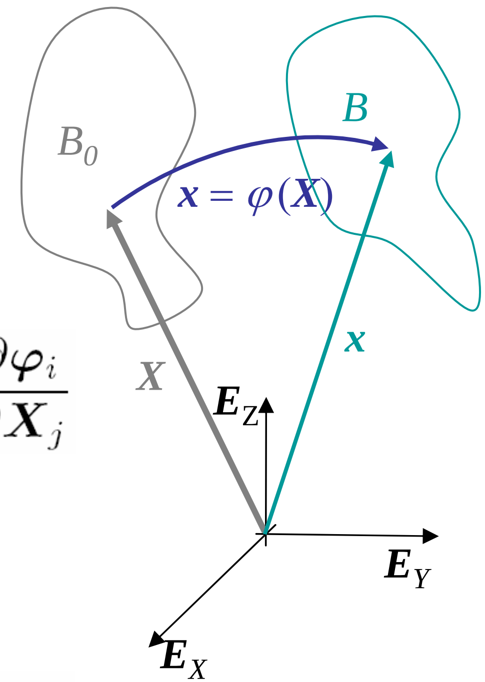


- Application to solid mechanics
  - Strong form of continuum mechanics
    - Equations that are satisfied
      - At every point $\boldsymbol{x}$ of body B in its deformed configuration
      - At every point $\boldsymbol{X}$ of body B_0 in its initial configuration



  - Static assumption
    - Linear momentum conservation $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^T + \boldsymbol{b} = 0 \ \forall\ \boldsymbol{X} \in B_0$
    - Angular momentum conservation $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \ \forall\ \boldsymbol{X} \in B_0$
    - Neumann BC on surface traction $\boldsymbol{\sigma}\cdot\boldsymbol{n} = \bar{\boldsymbol{T}} \ \forall\ \boldsymbol{X} \in \partial_N B_0$
    - Dirichlet BC on surface displacement $\boldsymbol{u} = \bar{\boldsymbol{u}} \ \forall\ \boldsymbol{X} \in \partial_D B_0$
  - Remark, B is an open manifold of boundary ∂B

- Deformations & strains
  - Deformation (or motion) mapping
    - Current position $\boldsymbol{x}$ of the material point $\boldsymbol{X}$ is obtained from a mapping $\boldsymbol{\varphi}$: $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}) \quad : \quad B_0 \rightarrow B$
  - Two-point (non symmetrical) deformation gradient
    - $\mathbf{F} = \nabla_0 \boldsymbol{\varphi} = \dfrac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{X}} \quad : \quad B_0 \rightarrow \mathrm{GL}_+(3, \mathbb{R})$ or $\mathbf{F}_{ij} = \dfrac{\partial \boldsymbol{\varphi}_i}{\partial \boldsymbol{X}_j}$
    - Where the Lie group $\mathrm{GL}_+(3, \mathbb{R})$
      - Is the smooth manifold in the Euclidean space, in which matrix can be inverted (as smooth)
      - + means the determinant is positive: $J = \det(\mathbf{F}) > 0$
    - $\mathbf{F}$ is non-symmetrical
  - Jacobian $J = \det(\mathbf{F}) > 0$
    - Corresponds to the change of (infinitesimal) volume
    - Using mass conservation leads to $\dfrac{dB}{dB_0} = J = \dfrac{\rho_0}{\rho}$ for any material point $\boldsymbol{X}$ in B_0

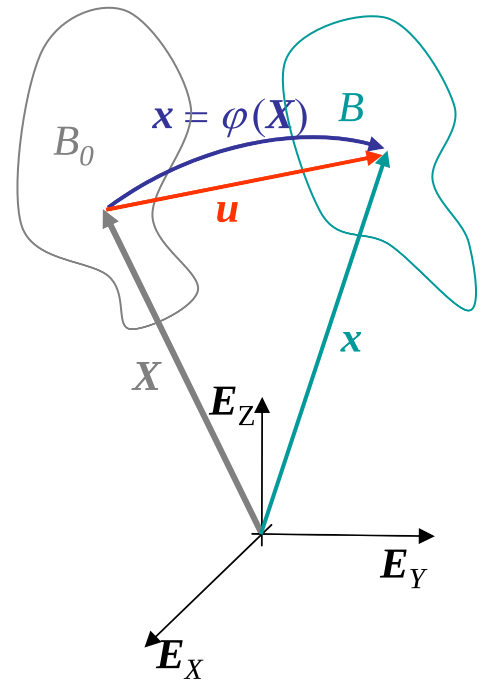


- Deformations & strains (2)
  - In terms of displacements
    - $\boldsymbol{\varphi}\left(\boldsymbol{X}\right) = \boldsymbol{X} + \boldsymbol{u}\left(\boldsymbol{X}\right) \quad : \quad B_0 \rightarrow B$
    - The deformation gradient is rewritten

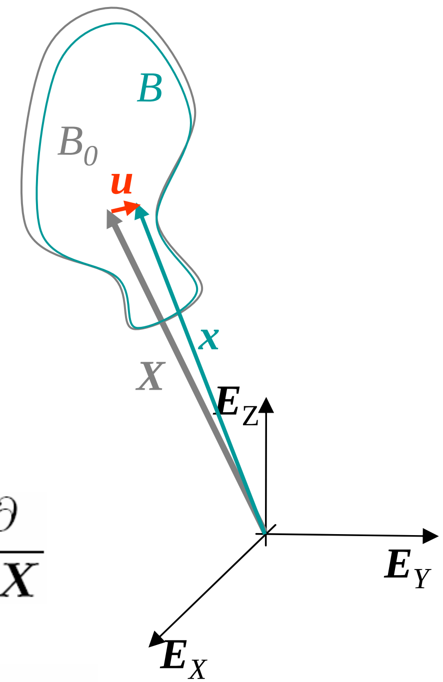
$$\mathbf{F} = \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{X}} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{X}} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = \mathbf{I} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}$$

    - The symmetrical part
      - Which corresponds to material deformations
      - Which removes rotation
      - Is obtained from the (symmetrical) right Cauchy tensor

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = \left(\mathbf{I} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)^T\right)\left(\mathbf{I} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)$$



- Small deformations
  - Small displacements (including rotations) assumption
    - Satisfied if $\|\boldsymbol{u}\| << |B|$
    - Implies
      - Integration can be performed on the current or on the initial configuration: $\int_B \simeq \int_{B_0}$
      - Differentiation can be performed with respect to the current or initial configuration: $\nabla = \frac{\partial}{\partial \boldsymbol{x}} \simeq \nabla_0 = \frac{\partial}{\partial \boldsymbol{X}}$
      - Definition of the small-deformation tensor
        - » From $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \left(\mathbf{I} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)^T\right)\left(\mathbf{I} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)$



$$\Longrightarrow \mathbf{C} \simeq \mathbf{I} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^T + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \Longrightarrow \boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \nabla\right) \simeq \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right)$$

- Other notations: $\boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{x}_i}\boldsymbol{u}_j + \frac{\partial}{\partial \boldsymbol{x}_j}\boldsymbol{u}_i\right)$ or again $\boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\boldsymbol{u}_{j,i} + \boldsymbol{u}_{i,j}\right)$

- Material law
  - We have the governing equations in the strong form,
  - What is still missing is the stress-strain relationship
  - Linear elasticity for small deformations
    - $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon}$ or $\sigma_{ij} = \mathcal{H}_{ijkl}\varepsilon_{kl}$ for any material point $\boldsymbol{X}$ in B_0

      with $\mathcal{H}_{ijkl} = \dfrac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \dfrac{E}{1+\nu}\left(\dfrac{1}{2}\delta_{ik}\delta_{jl} + \dfrac{1}{2}\delta_{il}\delta_{jk}\right)$

      $\lambda = K - 2\mu/3$ $\quad$ 2μ

    - Which can be inverted into $\boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma}$

      with $\mathcal{G}_{ijkl} = \dfrac{1+\nu}{E}\left(\dfrac{1}{2}\delta_{ik}\delta_{jl} + \dfrac{1}{2}\delta_{il}\delta_{jk}\right) - \dfrac{\nu}{E}\delta_{ij}\delta_{kl}$

    - An internal potential U can be defined at each material point $\boldsymbol{X}$ in B_0
      - $U = \dfrac{1}{2}\boldsymbol{\varepsilon} : \mathcal{H} : \boldsymbol{\varepsilon} \quad B_0 \to \mathbb{R}^+$
      - Stress tensor derives from the internal potential: $\boldsymbol{\sigma} = \partial_{\boldsymbol{\varepsilon}} U = \mathcal{H} : \boldsymbol{\varepsilon}$

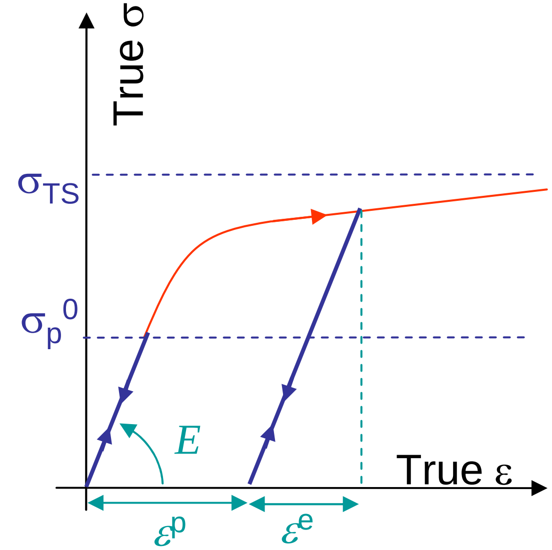
- Elasto-plasticity
  - Yield surface

$$f(\boldsymbol{\sigma}) \leq 0 \quad \begin{cases} f < 0\text{: elastic region} \\ f = 0\text{: plasticity} \end{cases}$$

  - Plastic flow
    - Assumption: deformations can be added

$$d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^{\mathrm{e}} + d\boldsymbol{\varepsilon}^{\mathrm{p}} \Longrightarrow d\boldsymbol{\sigma} = \mathcal{H} : d\boldsymbol{\varepsilon}^{\mathrm{e}}$$

    - Normal plastic flow $d\boldsymbol{\varepsilon}^{\mathrm{p}} = d\lambda \dfrac{\partial f}{\partial \boldsymbol{\sigma}}$

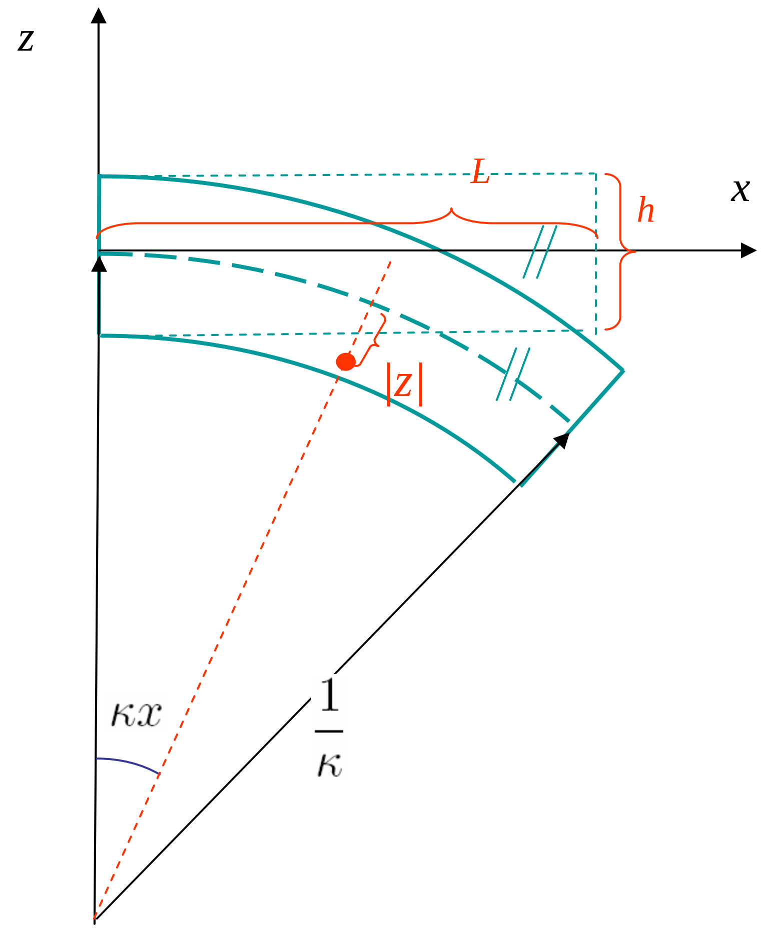


    - Von Mises surface with isotropic plastic flow (J2-plasticity)
      - Deviatoric part of the stress tensor $\mathbf{s} = \boldsymbol{\sigma} - \dfrac{\mathrm{tr}(\boldsymbol{\sigma})}{3}\mathbf{I}$
      - Yield surface $f = \sqrt{\dfrac{3}{2}\mathbf{s} : \mathbf{s}} - \sigma_p(\bar{\epsilon}^{\mathrm{p}}) \leq 0$
      - Normality: since $\dfrac{\partial\left(\boldsymbol{\sigma}_{ij} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\delta_{ij}\right)}{\partial \boldsymbol{\sigma}_{kl}} = \dfrac{1}{2}\delta_{ik}\delta_{jl} + \dfrac{1}{2}\delta_{il}\delta_{jk} - \dfrac{1}{3}\delta_{ij}\delta_{kl}$

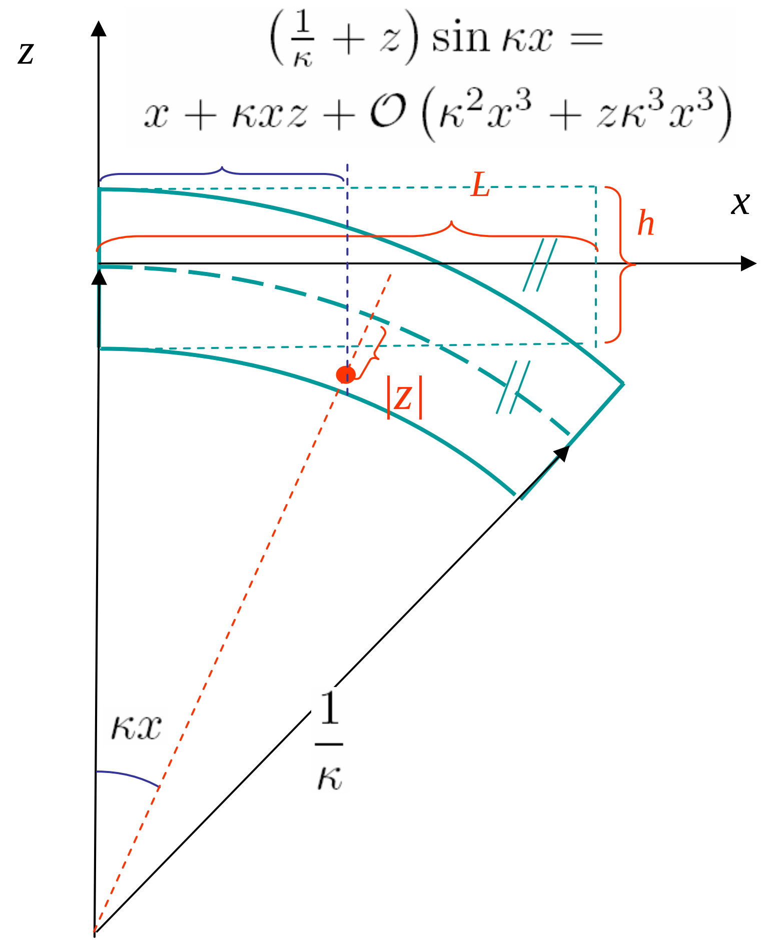
$$\Longrightarrow \frac{\partial f}{\partial \boldsymbol{\sigma}} = \sqrt{\frac{3}{2}}\frac{\mathbf{s}}{\sqrt{\mathbf{s} : \mathbf{s}}}$$

      - Then the plastic flow becomes $d\lambda = d\bar{\varepsilon}^{\mathrm{p}} = \sqrt{\dfrac{2}{3} d\boldsymbol{\varepsilon}^{\mathrm{p}} : d\boldsymbol{\varepsilon}^{\mathrm{p}}}$
  - Path dependency (incremental equations in *d*) $\Longrightarrow$ no internal energy

- 1-D pure bending
  - Assumptions
    - Symmetrical beam
    - Filled cross-section
    - Cross-section remains plane (Bernoulli or Kirchhoff-Love)
    - Only for thin structures (h/L << 1)
    - Limited bending: κL << 1
  - Curvature radius
    - $\kappa = -\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}$



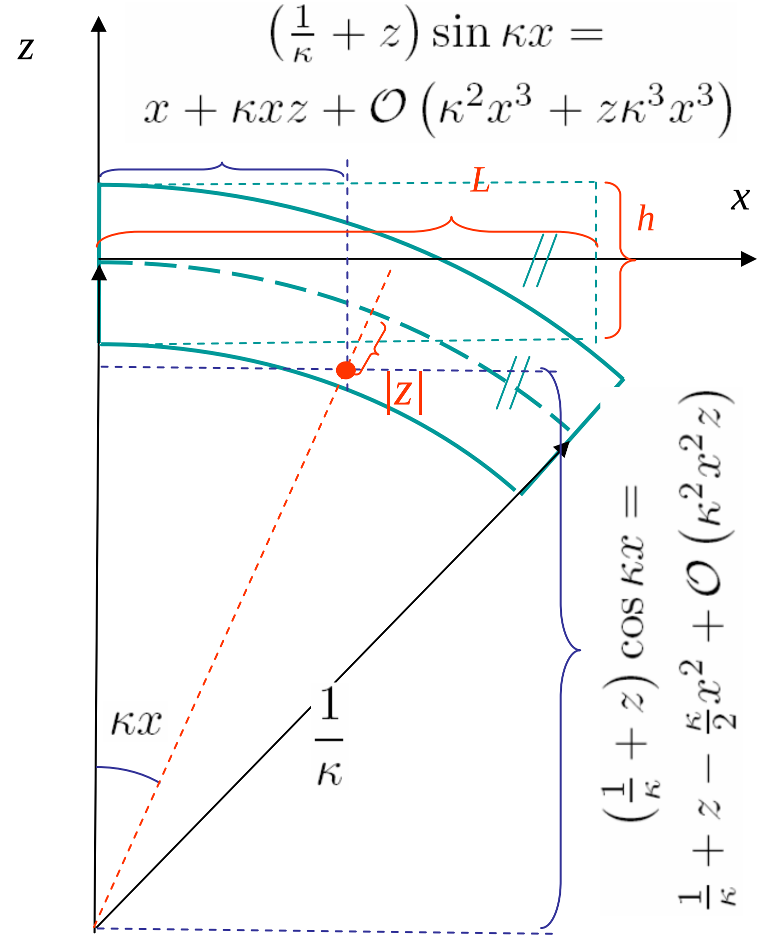
- 1-D pure bending (2)
  - Kinematics
    - $\boldsymbol{u}_x = \kappa x z$



- 1-D pure bending (3)
  - Kinematics (2)
    - $\boldsymbol{u}_x = \kappa x z$
    - $\boldsymbol{u}_z = -\frac{\kappa}{2}x^2 + ?$

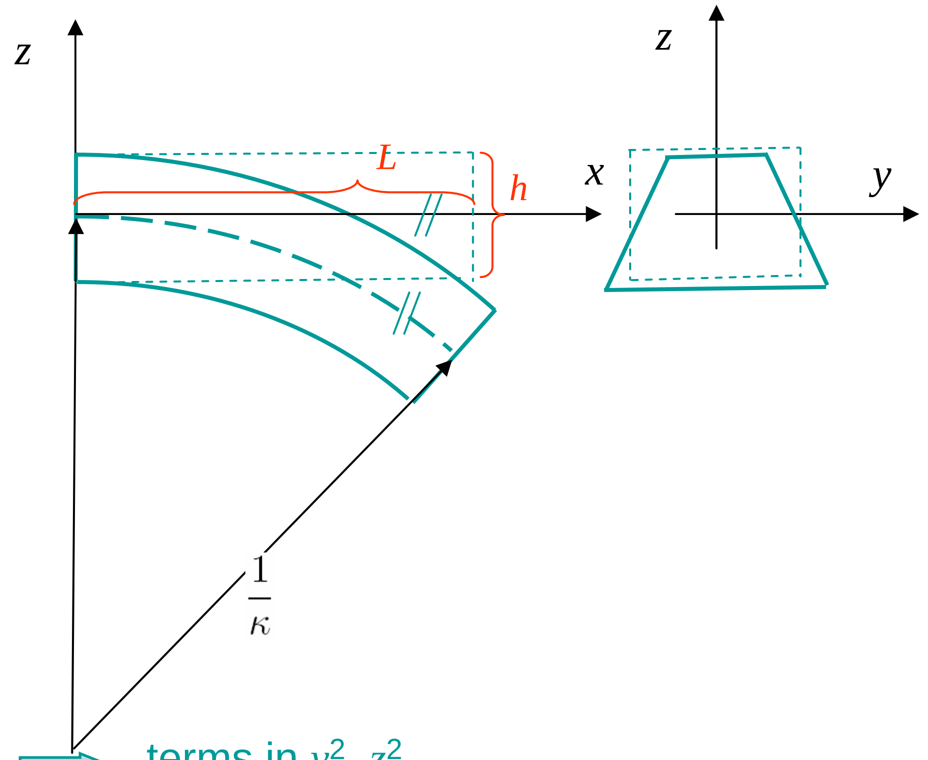
Section remains plane,
but the shape can change

$$\Rightarrow \begin{cases} \boldsymbol{u}_z = -\frac{\kappa}{2}x^2 + f\,(?) \\ \boldsymbol{u}_y = g\,(?) \end{cases}$$



- 1-D pure bending (4)
  - Kinematics (3)
    - $\boldsymbol{u}_x = \kappa x z$
    - $\boldsymbol{u}_z = -\frac{\kappa}{2}x^2 + f\,(?)$
    - $\boldsymbol{u}_y = g\,(?)$
    - *f* & *g* should
      - Involve quadratic terms
      - Be independent of *x*

      $\Longrightarrow$ terms in y^2, yz, z^2



    - *f(-y)* should be equal to *f(y)* $\Longrightarrow$ terms in y^2, z^2
    - *g(-y)* should be equal to *-g(y)* $\Longrightarrow$ terms in yz
    - No shearing $\Longrightarrow \boldsymbol{\varepsilon}_{yz} = \frac{1}{2}\left(2y\partial_{y^2}f + y\partial_{yz}g\right) = 0$ & $\boldsymbol{\varepsilon}_{xz} = 0$, $\boldsymbol{\varepsilon}_{xy} = 0$ (OK)
    - For linear elasticity there is a Poisson's effect
  - A solution satisfying these constraints $\left\{\begin{array}{l}\boldsymbol{u}_z = -\frac{\kappa}{2}\left[x^2 + \nu\left(\alpha z^2 - \beta y^2\right)\right] \\ \boldsymbol{u}_y = -\kappa\nu\beta y z\end{array}\right.$?

- 1-D pure bending (5)
  - Small deformations

$$\begin{cases} \boldsymbol{u}_x = \kappa x z \\ \boldsymbol{u}_y = -\kappa\nu\beta y z \\ \boldsymbol{u}_z = -\frac{\kappa}{2}\left[x^2 + \nu\left(\alpha z^2 - \beta y^2\right)\right] \end{cases} \Longrightarrow \begin{cases} \varepsilon_{xx} = \kappa z \\ \varepsilon_{yy} = -\kappa\beta\nu z \\ \varepsilon_{zz} = -\kappa\alpha\nu z \\ \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{xz} = 0 \end{cases}$$

  - For linear elasticity
    - $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon}$ with $\mathcal{H}_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{1+\nu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$

$$\Longrightarrow \begin{cases} \boldsymbol{\sigma}_{xx} = \frac{\kappa E z}{(1+\nu)(1-2\nu)}\left[1-\nu-(\alpha+\beta)\nu^2\right] \\ \boldsymbol{\sigma}_{yy} = \frac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1-\beta+\nu(\beta-\alpha)\right] \\ \boldsymbol{\sigma}_{zz} = \frac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1-\alpha+\nu(\alpha-\beta)\right] \end{cases}$$

    - Balance equation: $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = 0$

$\Longrightarrow \partial_z \boldsymbol{\sigma}_{zz} = 0 \Longrightarrow \alpha = \frac{1-\beta\nu}{1-\nu}$ or $\nu = 0$

- 1-D pure bending of beams
  - Small deformations & linear elasticity

$$\begin{cases} \varepsilon_{xx} = \kappa z \\ \varepsilon_{yy} = -\kappa\beta\nu z \\ \varepsilon_{zz} = -\kappa\alpha\nu z \\ \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{xz} = 0 \end{cases}$$

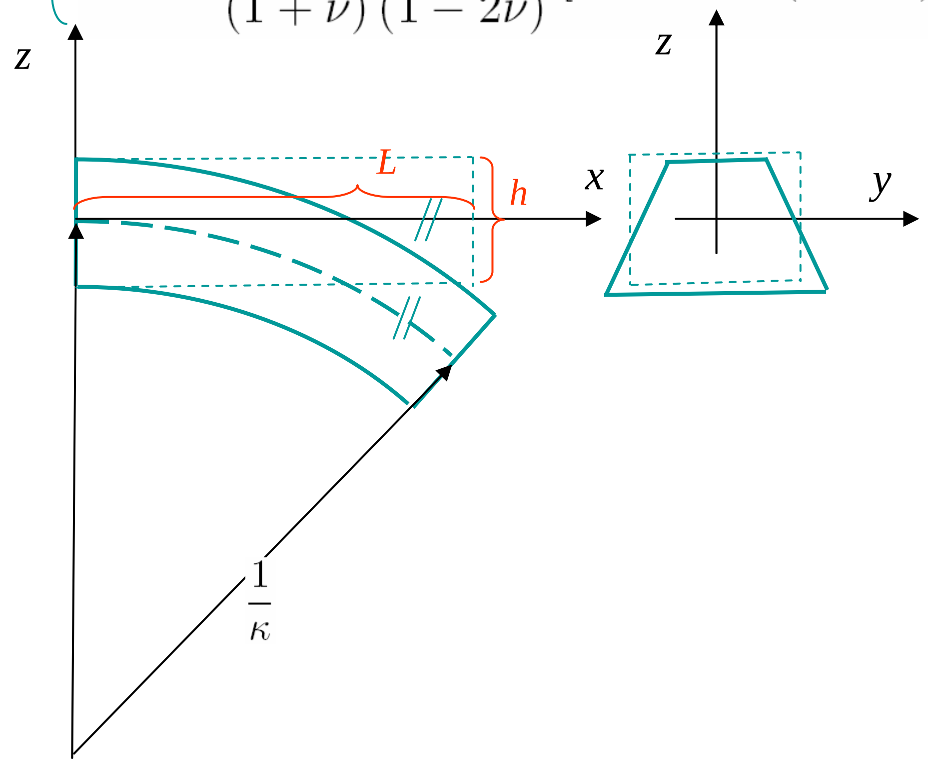
$$\begin{cases} \boldsymbol{\sigma}_{xx} = \frac{\kappa E z}{(1+\nu)(1-2\nu)}\left[1-\nu-(\alpha+\beta)\nu^2\right] \\ \boldsymbol{\sigma}_{yy} = \frac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1-\beta+\nu(\beta-\alpha)\right] \\ \boldsymbol{\sigma}_{zz} = \frac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1-\alpha+\nu(\alpha-\beta)\right] \end{cases}$$

  - Beam
    - Stress-free on all cross-section edges
      - $\boldsymbol{\sigma}_{yy} = \boldsymbol{\sigma}_{zz} = 0$

        $\Rightarrow \alpha = \beta = 1$
    - Balance equation

      $\alpha = \frac{1-\beta\nu}{1-\nu}$ satisfied

$$\Rightarrow \begin{cases} \boldsymbol{\sigma}_{xx} = \kappa E z \\ \boldsymbol{\sigma}_{yy} = \boldsymbol{\sigma}_{zz} = 0 \end{cases}$$



- 1-D pure bending of beams (2)
  - Equations

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \kappa E z \\ \boldsymbol{\sigma}_{yy} = \boldsymbol{\sigma}_{zz} = 0 \end{cases}$$

  - Momentum

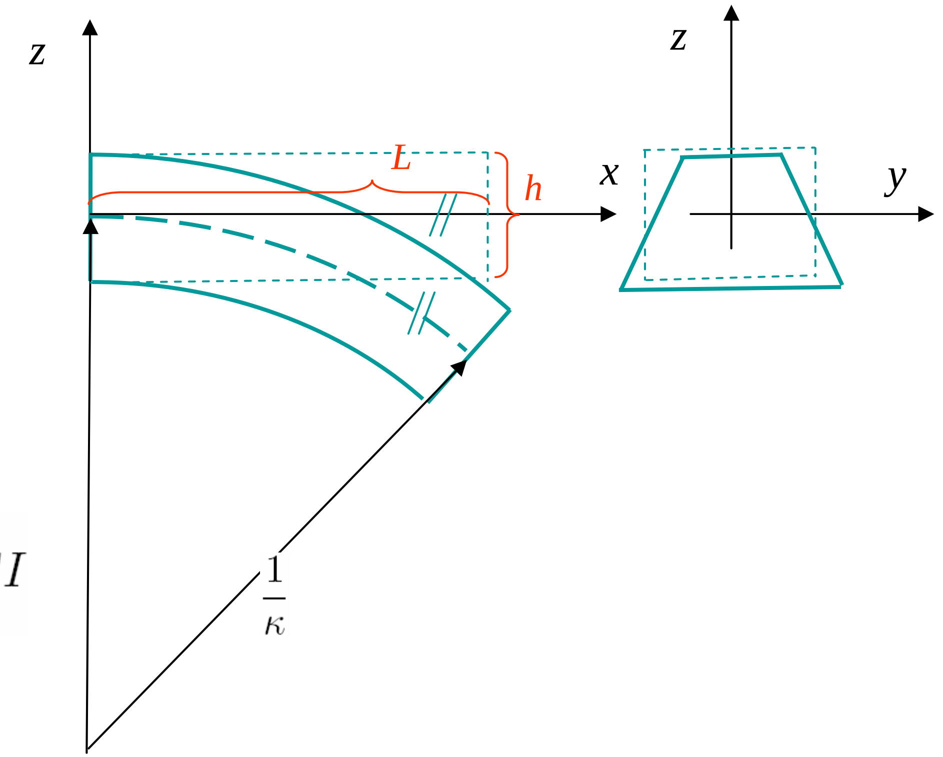
$$M_{xx} = \int_A \kappa E z^2 dy dz = \kappa E I$$

    - Inertia

$$I = \int_A z^2 dy dz$$

      - Rigorously we should call it I_{yy}

    - For a rectangular cross-section $I = \dfrac{bh^3}{12}$

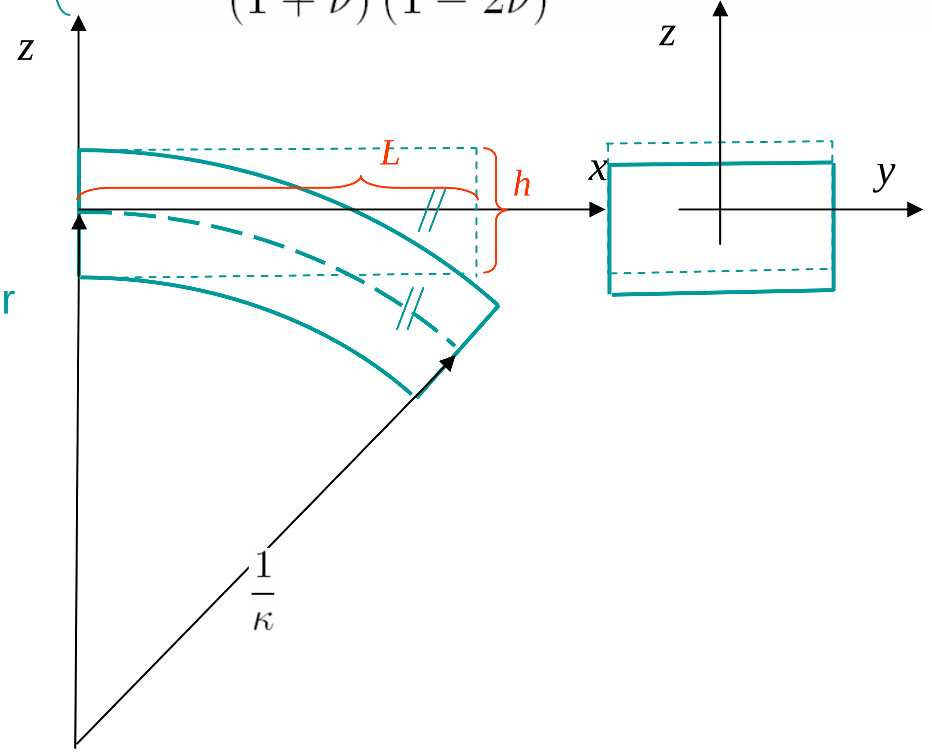


- 1-D pure bending of plates
  - Small deformations & linear elasticity

$$\begin{cases} \varepsilon_{xx} = \kappa z \\ \varepsilon_{yy} = -\kappa\beta\nu z \\ \varepsilon_{zz} = -\kappa\alpha\nu z \\ \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{xz} = 0 \end{cases}$$

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \frac{\kappa E z}{(1+\nu)(1-2\nu)}\left[1-\nu-(\alpha+\beta)\nu^2\right] \\ \boldsymbol{\sigma}_{yy} = \frac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1-\beta+\nu(\beta-\alpha)\right] \\ \boldsymbol{\sigma}_{zz} = \frac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1-\alpha+\nu(\alpha-\beta)\right] \end{cases}$$

  - Plate (plane - σ state)
    - No deformation along y
      ⟹ $\beta = 0$
    - Stress-free on upper and lower sides ⟹ $\boldsymbol{\sigma}_{zz} = 0$
      ⟹ $\alpha = 1 / (1-\nu)$
    - Balance equation
      $\alpha = \frac{1-\beta\nu}{1-\nu}$ satisfied

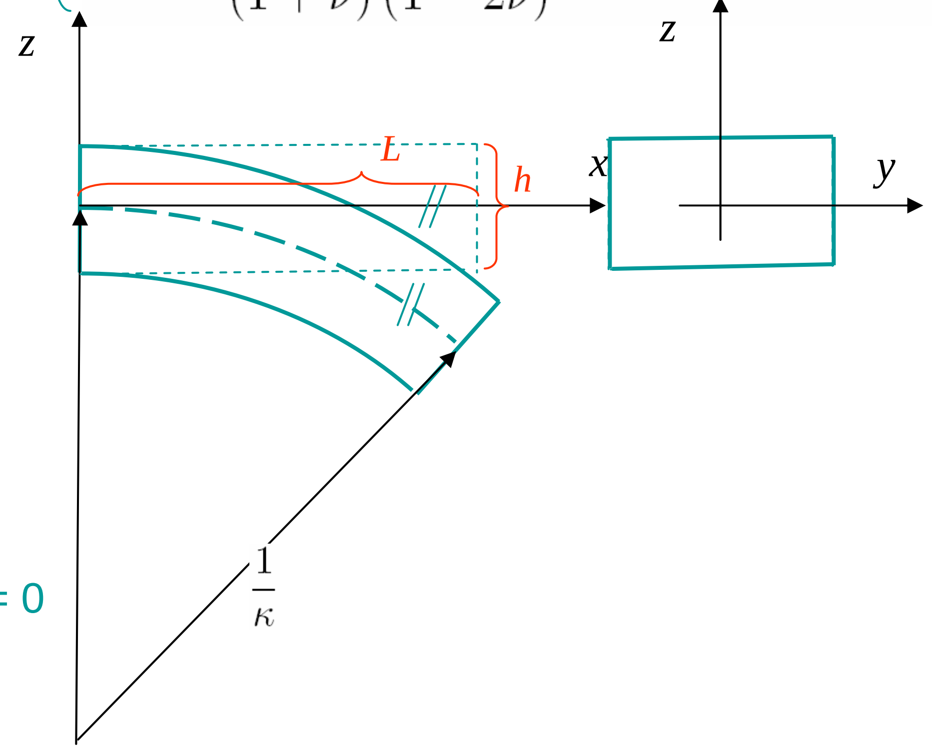


- 1-D pure bending of plates (2)
  - Small deformations & linear elasticity

$$\begin{cases} \varepsilon_{xx} = \kappa z \\ \varepsilon_{yy} = -\kappa\beta\nu z \\ \varepsilon_{zz} = -\kappa\alpha\nu z \\ \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{xz} = 0 \end{cases}$$

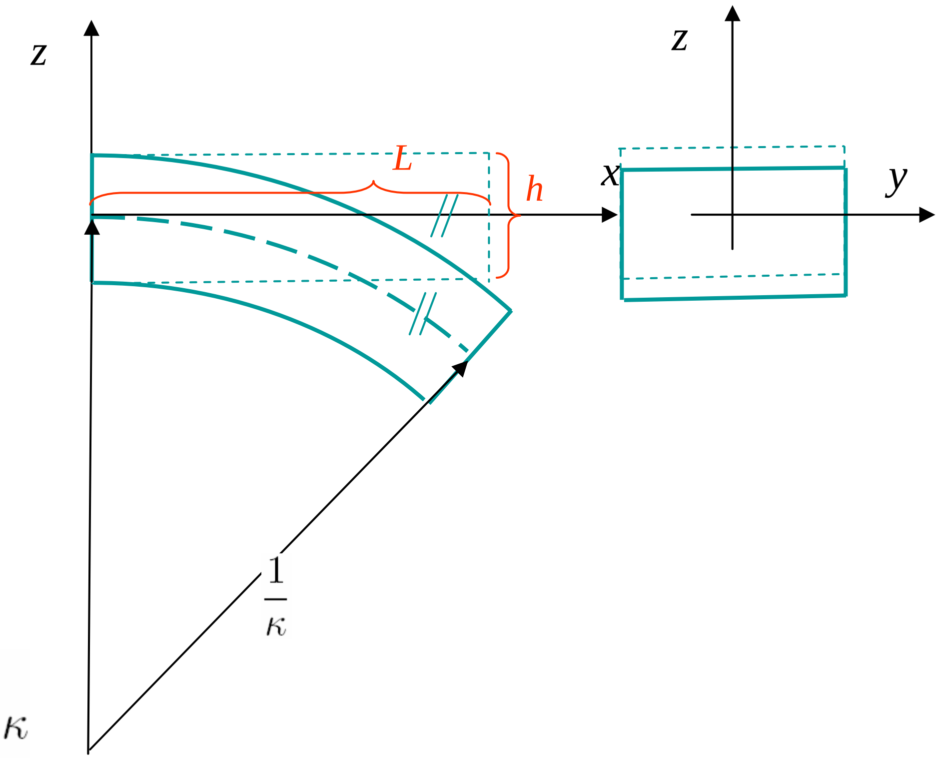
$$\begin{cases} \boldsymbol{\sigma}_{xx} = \dfrac{\kappa E z}{(1+\nu)(1-2\nu)}\left[1-\nu-(\alpha+\beta)\nu^2\right] \\ \boldsymbol{\sigma}_{yy} = \dfrac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1-\beta+\nu(\beta-\alpha)\right] \\ \boldsymbol{\sigma}_{zz} = \dfrac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1-\alpha+\nu(\alpha-\beta)\right] \end{cases}$$

  - Plate (plane - ε state)
    - No deformation along y
      ⟹ $\beta = 0$
    - No deformation along z
      ⟹ $\alpha = 0$
    - Balance equation
      $\alpha = \dfrac{1-\beta\nu}{1-\nu}$ NOT satisfied
    - This state actually requires $\nu = 0$



- 1-D pure bending of plates (3)
  - Back to plane - σ state
    - Equations

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \dfrac{\kappa E z}{1-\nu^2} \\ \boldsymbol{\sigma}_{zz} = 0 \\ \boldsymbol{\sigma}_{yy} = \dfrac{\nu \kappa E z}{1-\nu^2} \end{cases}$$



    - Momentum

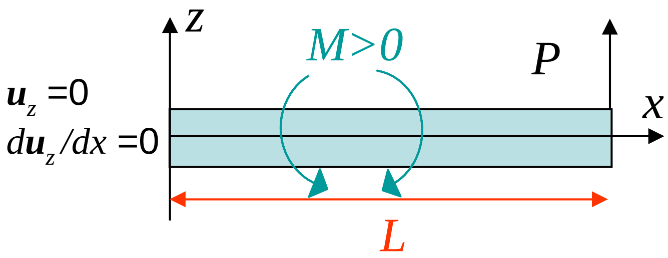
$$m_{xx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\kappa E}{1-\nu^2} z^2 dz = D\kappa$$

    - Flexural rigidity

$$D = \frac{E h^3}{12\left(1-\nu^2\right)}$$

- Strong form of pure-bending beam
  - Equations

$$\begin{cases} M_{xx} = \int_A \kappa E z^2 dy dz = \kappa E I \\ \kappa = -\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \quad \& \quad \boldsymbol{\sigma}_{xx} = \kappa E z \end{cases}$$



  - Concentrated load
    - For a uniform cross-section *hxb*: $I = \dfrac{bh^3}{12}$

$$P(L-x) = \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} EI \implies \boldsymbol{u}_z = \frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right)$$

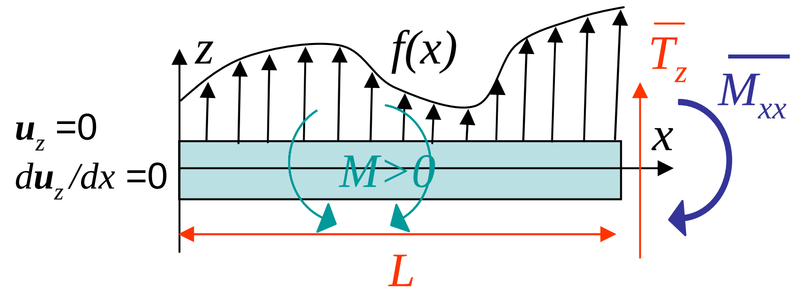
    - Stress

$$\boldsymbol{\sigma}_{xx}\big|_{z=-\frac{h}{2}} = -\kappa E \frac{h}{2} = \frac{Ph}{2I}(L-x) = \frac{6P}{bh^2}(L-x)$$

    - Shearing
      - There is a shearing $\boldsymbol{T}_z = P$: $\boldsymbol{T}_z = \dfrac{\partial M_{xx}}{\partial x} = \dfrac{\partial P(x-L)}{\partial x} = P$
      - Its effect on shearing stress can be neglected if h/L << as

$$\boldsymbol{\sigma}_{xy} = \mathcal{O}\left(\frac{P}{bh}\right) = \mathcal{O}\left(\frac{h}{L}\boldsymbol{\sigma}_{xx}\left(x=0,\ z=-\frac{h}{2}\right)\right)$$

- Strong form of pure-bending beam (2)



  - Equations

$$\begin{cases} M_{xx} = \int_A \kappa E z^2 dy dz = \kappa E I \\ \kappa = -\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \quad \& \quad \boldsymbol{\sigma}_{xx} = \kappa E z \end{cases}$$

  - Non-uniform loading
    - Internal energy variation

$$\delta E_{\text{int}} = \int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \boldsymbol{\varepsilon}_{xx} dA dx = \int_0^L \int_A E \kappa \delta \kappa z^2 dA dx =$$

$$\int_0^L \int_A E \delta \frac{\kappa^2}{2} z^2 dA dx = \delta \int_0^L \frac{M_{xx} \kappa}{2} dx$$

    - Work variation of external forces

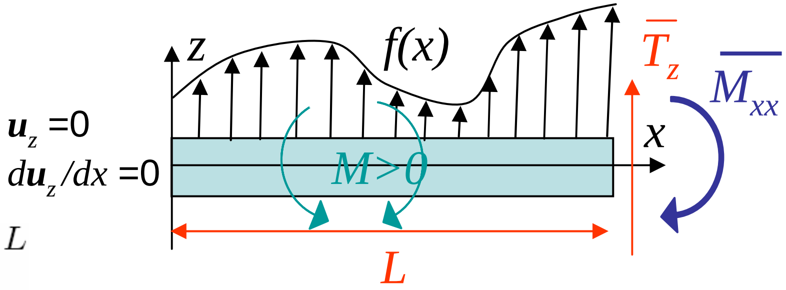
$$\delta W_{\text{ext}} = \int_0^L f(x) \delta \boldsymbol{u}_z dx + \bar{T}_z \delta \boldsymbol{u}_z \Big]_0^L - \bar{M}_{xx} \frac{\partial \delta \boldsymbol{u}_z}{\partial x} \Big|_0^L$$

$$\Rightarrow \int_0^L \frac{1}{2} E I \left(\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \right)^2 dx = \int_0^L f(x) \boldsymbol{u}_z dx + \bar{T}_z \boldsymbol{u}_z \Big]_0^L - \bar{M}_{xx} \frac{\partial \boldsymbol{u}_z}{\partial x} \Big|_0^L$$

- Strong form of pure-bending beam (3)
  - Energy conservation

$$\int_0^L \frac{1}{2} EI \left(\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)^2 dx =$$

$$\int_0^L f(x)\,\boldsymbol{u}_z dx + \bar{T}_z \boldsymbol{u}_z\Big]_0^L - \bar{M}_{xx} \frac{\partial \boldsymbol{u}_z}{\partial x}\Big|_0^L$$



- Integration by parts of the internal energy variation

$$\delta E_{\text{int}} = \int_0^L EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \frac{\partial^2 \delta \boldsymbol{u}_z}{\partial x^2} dx = \left[EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \frac{\partial \delta \boldsymbol{u}_z}{\partial x}\right]_0^L - \int_0^L \frac{\partial}{\partial x}\left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) \frac{\partial \delta \boldsymbol{u}_z}{\partial x} dx$$

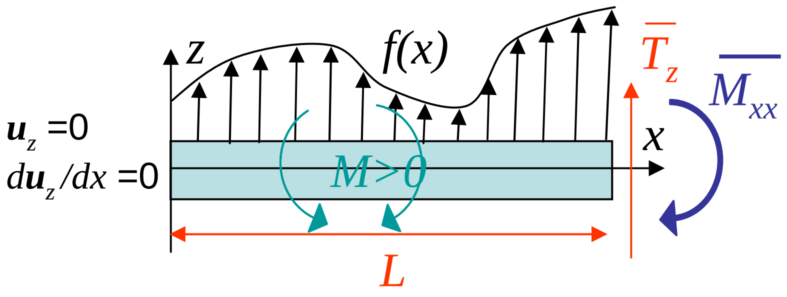
$$\delta E_{\text{int}} = \left[EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \frac{\partial \delta \boldsymbol{u}_z}{\partial x}\right]_0^L - \left[\frac{\partial}{\partial x}\left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) \delta \boldsymbol{u}_z\right]_0^L + \int_0^L \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) \delta \boldsymbol{u}_z dx$$

- Work variation of external forces

$$\delta W_{\text{ext}} = \int_0^L f(x)\,\delta \boldsymbol{u}_z dx + \bar{T}_z \delta \boldsymbol{u}_z\Big]_0^L - \bar{M}_{xx} \frac{\partial \delta \boldsymbol{u}_z}{\partial x}\Big]_0^L$$

- Elastic pure-bending beam (4)
  - Energy conservation (2)
    - As $\delta \boldsymbol{u}_z$ is arbitrary:

      ⟹ Euler-Bernoulli equations



- $$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f(x)$$ on [0, L] &

- $$\begin{cases} -\dfrac{\partial}{\partial x}\left(EI\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\Big|_{0,\,L} = \bar{T}_z\big|_{0,\,L} \\ -EI\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\Big|_{0,\,L} = \bar{M}_{xx}\big|_{0,\,L} \end{cases}$$

- Introduction to Hilbert space
  - When writing weak formulations of the continuum equations, solutions in usual C^n manifolds do not always exist
    - Because derivative is not always defined/continuous everywhere
    - However sometimes a numerical solution can be found
    - C^n manifolds are not the correct ones to be considered
  - Instead it is more convenient to consider manifolds where
    - Derivatives are understood in a weak sense: For non-differentiable functions, if they exist, the generalized derivatives are defined as
      - $f^{(k)}(x) \in \mathrm{L}^p([a,\, b]) \;:\; \int_a^b f \partial^k \phi dx = (-1)^k \int_a^b f^{(k)} \phi dx$
      - $\forall \phi \in C^\infty([a,\, b]) \;\&\; \phi(a) = \phi(b) = 0$
      - For differentiable functions, application of Greens' formula leads to classical derivatives
    - A norm can be defined
  - Sobolev spaces are the modern replacements for C^n manifolds
    - Hilbert spaces are particular cases of Sobolev spaces

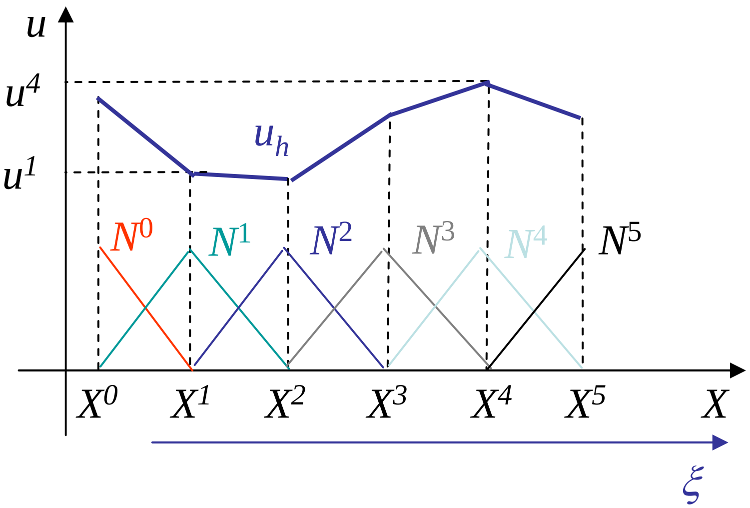
- Introduction to Hilbert space (2)
  - 1-D Sobolev space $W^{m,\,p}$
    - For a function *f(x)* : $f(x) \in \mathrm{W}^{m,\,p}([a,\ b])$ if the function and its generalized derivatives up to order m have a finite p-norm on $[a, b]$
    - p-norm is define as $\|g\|_{\mathrm{L}^p([a,\ b])} = \left(\int_a^b \sum_i |g_i|^p \, dx\right)^{\frac{1}{p}}$
      - For p = 2 & for tensors, we have the Euclidean norm and $\mathbf{H}^m = \mathbf{W}^{m,\,2}$
    - As the p-norms of the derivatives are finite, the Sobolev norm is also finite:
      - $\|f\|_{\mathrm{W}^{m,\,p}([a,\ b])} = \sum_{k=0}^{m} \left\| f^{(k)} \right\|_{\mathrm{L}^p([a,\ b])}$
  - Sobolev space and continuity
    - If a function belongs to $W^{1,p}$ then almost every line parallel to the coordinates is absolutely continuous

- 1-D exemple in Hilbert space
  - Considering the function *f(x)* on [-1, 1]

$$f(x) = 1 - |x| \quad \forall x \in [-1,\, 1]$$

    - Useful in FE methods
    - Derivative is not defined in [-1, 1]

$$f \notin C^1([-1,\, 1])$$



  - But we still want to use this function

- 1-D exemple in Hilbert space (2)
  - Derivative in a weak sense of $f\left(x\right)=1-|x|\quad \forall x\in\left[-1,\,1\right]$
    - Considering $\phi(x)\in C^{\infty}\left(\left[-1,\,1\right]\right)$ arbitrary, but with $\phi(-1)=\phi(1)=0$

$$\int_{-1}^{1}f(x)\phi'(x)dx=\int_{-1}^{0}f(x)\phi'(x)dx+\int_{0}^{1}f(x)\phi'dx$$

$$\int_{-1}^{1}f(x)\phi'(x)dx=\left.f\phi\right|_{-1}^{0^-}+\left.f\phi\right|_{0+}^{1}-\int_{-1}^{0}f'(x)\phi(x)dx-\int_{0}^{1}f'(x)\phi(x)dx$$

    - As
      - f is continuous at 0 (this is also the case for $\phi(x)\in C^{\infty}\left(\left[-1,\,1\right]\right)$) &
      - $\phi(-1)=\phi(1)=0$

$$\Longrightarrow\int_{-1}^{1}f(x)\phi'(x)dx=-\int_{-1}^{1}f'(x)\phi(x)dx$$

where f' is the usual derivative of f , and which is not defined at 0

    - So weak derivative $f^{(1)}(x)=f'(x)\ \ \forall\, x\in\left[-1,\,0\right[\cup\left]0,\,1\right]$

$$\Longrightarrow f^{(1)}\left(x\right)=\left\{\begin{array}{ll}1, & \text{if } -1\leq x<0\\ -1, & \text{if } 1\geq x>0\end{array}\right\}$$

- 1-D exemple in Hilbert space (3)
  - Function $f(x) = 1 - |x| \quad \forall x \in [-1,\ 1]$
  - Derivative in a weak sense $f^{(1)}(x) = \left\{ \begin{array}{ll} 1, & \text{if } -1 \leq x < 0 \\ -1, & \text{if } 1 \geq x > 0 \end{array} \right\}$
  - Norms
    - L^2 norm of the function

$$\|f(x)\|_{L^2([-1,\,1])} = \sqrt{\int_{-1}^{1} (1-|x|)^2\, dx}$$

$$= \sqrt{\frac{1}{3}(1+x)^3\Big|_{-1}^{0} - \frac{1}{3}(1-x)^3\Big|_{0}^{1}} = \sqrt{\frac{2}{3}}$$

    - L^2 norm of the weak derivative

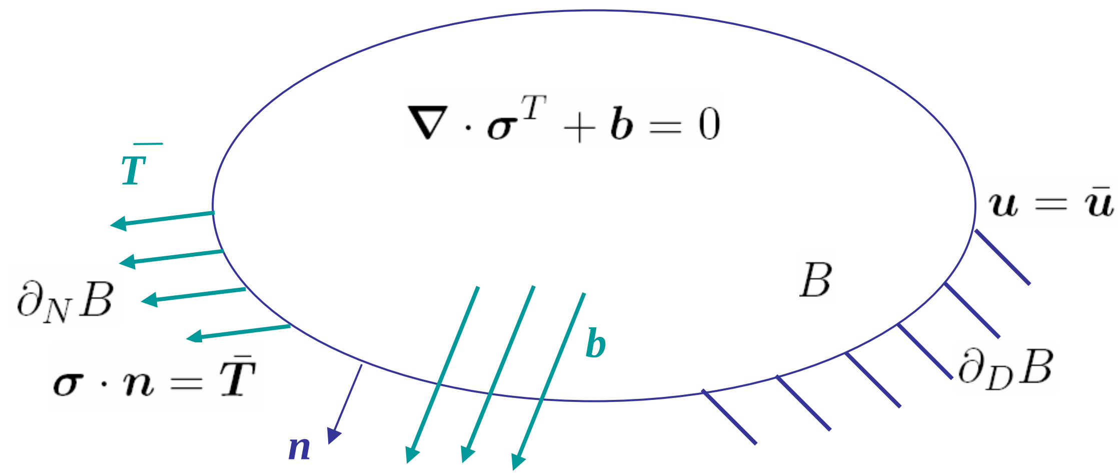
$$\left\| f(x)^{(1)} \right\|_{L^2([-1,\,1])} = \sqrt{\int_{-1}^{0} 1dx + \int_{0}^{1} 1dx} = \sqrt{2}$$

  - As these norms are finite
    - The function $f \in \mathrm{H}^1([-1,\ 1])$ but $f \notin C^1([-1,\ 1])$
    - With $\|f(x)\|_{\mathrm{H}^1([-1,\,1])} = \|f(x)\|_{L^2([-1,\,1])} + \left\| f(x)^{(1)} \right\|_{L^2([-1,\,1])} = \sqrt{\frac{2}{3}} + \sqrt{2}$

- Linear elasticity: Equations
  - Linear momentum equation becomes
    - $\nabla \cdot \left[\frac{1}{2}\mathcal{H} : (\nabla \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \nabla)\right] + \boldsymbol{b} = 0 \; \forall \boldsymbol{X} \in B_0$
  - Boundary conditions
    - $\boldsymbol{n} \cdot \left[\frac{1}{2}\mathcal{H} : (\nabla \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \nabla)\right] = \bar{\boldsymbol{T}} \;\; \forall \boldsymbol{X} \in \partial_N B_0$
    - $\boldsymbol{u} = \bar{\boldsymbol{u}} \;\; \forall \; \boldsymbol{X} \in \partial_D B_0$
  - It is assumed that
    - Boundary conditions &
    - Loadings

    are smooth enough, so
    - $\boldsymbol{u}^{\text{exact}}(\boldsymbol{X}) \in \mathbf{H}^2(B_0)$

    where $\boldsymbol{u}^{\text{exact}}$ is
    - The exact solution
    - Satisfying these equations (in the strong form)



Weak form of the continuum equations

- Linear elasticity: Virtual displacement
  - Let us defined an admissible virtual displacement (**not infinitesimal**)
    - Displacements are known on Dirichlet boundary
    - $\delta \boldsymbol{u} \in \mathbf{H}^1_c(B_0) = \{\delta \boldsymbol{u} \in \mathbf{H}^1(B_0) \ : \ \delta \boldsymbol{u}(\partial_D B_0) = 0\}$
  - Which multiplies the linear momentum equation
    - $\boldsymbol{\nabla} \cdot \left[\frac{1}{2}\mathcal{H} : (\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla})\right] + \boldsymbol{b} = 0 \ \forall \boldsymbol{X} \in B_0$

$$\Longrightarrow \boldsymbol{\nabla} \cdot \left[\frac{1}{2}\mathcal{H} : (\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla})\right] \cdot \delta \boldsymbol{u} + \boldsymbol{b} \cdot \delta \boldsymbol{u} = 0 \ \forall \boldsymbol{X} \in B_0 \quad \forall \delta \boldsymbol{u} \in \mathbf{H}^1_c(B_0)$$

  - As this equation is satisfied at any material point of B_0, we can integrate it
    - We choose the actual configuration B, as gradient are related to this one
    - But in small deformations: $B \sim B_0$ (we assume that for the following)
    - For large deformations, we could write down everything on B_0
    - $\int_B \boldsymbol{\nabla} \cdot \left[\frac{1}{2}\mathcal{H} : (\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla})\right] \cdot \delta \boldsymbol{u} dV + \int_B \boldsymbol{b} \cdot \delta \boldsymbol{u} dV = 0 \quad \forall \delta \boldsymbol{u} \in \mathbf{H}^1_c(B_0)$

- Linear elasticity: Volume integration
  - Integrating by parts, and using Gauss theorem:
    - $$\int_B \boldsymbol{\nabla}\cdot\left[\frac{1}{2}\mathcal{H}:(\boldsymbol{\nabla}\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\boldsymbol{\nabla})\right]\cdot\delta\boldsymbol{u}dV+\int_B \boldsymbol{b}\cdot\delta\boldsymbol{u}dV=0\ \ \forall\delta\boldsymbol{u}\in\mathbf{H}^1_c\left(B_0\right)$$

$$\Longrightarrow \int_B \boldsymbol{\nabla}\cdot\left\{\left[\tfrac{1}{2}\mathcal{H}:(\boldsymbol{\nabla}\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\boldsymbol{\nabla})\right]\cdot\delta\boldsymbol{u}\right\}dV-$$
$$\int_B\left[\tfrac{1}{2}\mathcal{H}:(\boldsymbol{\nabla}\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\boldsymbol{\nabla})\right]:\boldsymbol{\nabla}\otimes\delta\boldsymbol{u}dV+$$
$$\int_B \boldsymbol{b}\cdot\delta\boldsymbol{u}dV=0\ \ \forall\delta\boldsymbol{u}\in\mathbf{H}^1_c\left(B_0\right)$$

$$\Longrightarrow \int_{\partial B} \boldsymbol{n}\cdot\left\{\left[\tfrac{1}{2}\mathcal{H}:(\boldsymbol{\nabla}\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\boldsymbol{\nabla})\right]\cdot\delta\boldsymbol{u}\right\}dS-$$
$$\int_B\left[\tfrac{1}{2}\mathcal{H}:(\boldsymbol{\nabla}\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\boldsymbol{\nabla})\right]:\boldsymbol{\nabla}\otimes\delta\boldsymbol{u}dV+$$
$$\int_B \boldsymbol{b}\cdot\delta\boldsymbol{u}dV=0\ \ \forall\delta\boldsymbol{u}\in\mathbf{H}^1_c\left(B_0\right)$$

- Linear elasticity: Volume integration (2)
  - As
    - $\delta \boldsymbol{u} \left(\partial_D B_0\right) = 0$
    - $\boldsymbol{n} \cdot \left[\frac{1}{2}\mathcal{H} : (\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla})\right] = \bar{\boldsymbol{T}} \;\; \forall \boldsymbol{X} \in \partial_N B_0$
    - Hooke's tensor is symmetrical: $\mathcal{H}_{ijkl} = \mathcal{H}_{klij} = \mathcal{H}_{jikl} = \mathcal{H}_{ijlk}$
  - Equation can be simplified
    - $\int_{\partial B} \boldsymbol{n} \cdot \left\{\left[\frac{1}{2}\mathcal{H} : (\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla})\right] \cdot \delta \boldsymbol{u}\right\} dS - \int_B \left[\frac{1}{2}\mathcal{H} : (\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla})\right] : \boldsymbol{\nabla} \otimes \delta \boldsymbol{u} dV + \int_B \boldsymbol{b} \cdot \delta \boldsymbol{u} dV = 0 \;\; \forall \delta \boldsymbol{u} \in \mathbf{H}_c^1\left(B_0\right)$

$$\Longrightarrow \int_B \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \delta \boldsymbol{u} + \delta \boldsymbol{u} \otimes \boldsymbol{\nabla}\right) : \mathcal{H} : \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla}\right) dV = \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \delta \boldsymbol{u} dS + \int_B \boldsymbol{b} \cdot \delta \boldsymbol{u} dV = 0 \;\; \forall \delta \boldsymbol{u} \in \mathbf{H}_c^1\left(B_0\right)$$

- Linear elasticity: Weak formulation
  - Existence of solution
    - $\int_B \frac{1}{2}\left(\boldsymbol{\nabla}\otimes\delta\boldsymbol{u}+\delta\boldsymbol{u}\otimes\boldsymbol{\nabla}\right):\mathcal{H}:\frac{1}{2}\left(\boldsymbol{\nabla}\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\boldsymbol{\nabla}\right)dV = \int_{\partial_N B}\bar{\boldsymbol{T}}\cdot\delta\boldsymbol{u}dS+\int_B \boldsymbol{b}\cdot\delta\boldsymbol{u}dV=0 \;\; \forall\delta\boldsymbol{u}\in\mathbf{H}^1_c\left(B_0\right)$
    - It can be shown that
      - A solution in $C^2(B_0)\cap C^1(\overline{B_0})$ does not always exist BUT
      - For adequate boundary conditions (such as $\bar{\boldsymbol{u}}\left(\boldsymbol{X}\right)\in\mathbf{H}^1$), a solution $\boldsymbol{u}\left(\boldsymbol{X}\right)\in\mathbf{H}^1\left(B_0\right)$ always exists*
      - This explains why we are looking for a solution (with virtual displacements defined) in the Sobolev spaces instead of looking in usual C spaces
  - Weak form is stated as
    - Finding $\boldsymbol{u}\left(\boldsymbol{X}\right)\in\mathbf{H}^1\left(B_0\right)$
    - Such that $\int_B \frac{1}{2}\left(\boldsymbol{\nabla}\otimes\delta\boldsymbol{u}+\delta\boldsymbol{u}\otimes\boldsymbol{\nabla}\right):\mathcal{H}:\frac{1}{2}\left(\boldsymbol{\nabla}\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\boldsymbol{\nabla}\right)dV = \int_{\partial_N B}\bar{\boldsymbol{T}}\cdot\delta\boldsymbol{u}dS+\int_B \boldsymbol{b}\cdot\delta\boldsymbol{u}dV=0 \;\; \forall\delta\boldsymbol{u}\in\mathbf{H}^1_c\left(B_0\right)$

*Finite elements: theory, fast solvers, and applications in solid mechanics, Dietrich Braess, Cambridge Press, 2001, ISBN 0521 011957

- Linear elasticity: Bilinear form
  - Bilinear form
    - Let us define the bilinear form $\mathbf{H}^1(B_0)\times\mathbf{H}^1(B_0)\rightarrow\mathbb{R}$:

$$a(\boldsymbol{u},\,\boldsymbol{v})=\int_B\frac{1}{2}\left(\boldsymbol{\nabla}\otimes\boldsymbol{v}+\boldsymbol{v}\otimes\boldsymbol{\nabla}\right):\mathcal{H}:\frac{1}{2}\left(\boldsymbol{\nabla}\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\boldsymbol{\nabla}\right)dV$$

      - Which is symmetrical $a(\boldsymbol{u},\,\boldsymbol{v})=a(\boldsymbol{v},\,\boldsymbol{u})$ &
      - Positive $a(\boldsymbol{u},\,\boldsymbol{u})>0\ \ \forall\boldsymbol{u}\in\mathbf{H}^1(B_0),\ \boldsymbol{u}\neq 0$
    - Let us define the linear form $\mathbf{H}^1(B_0)\rightarrow\mathbb{R}$:

$$b(\boldsymbol{v})=\int_{\partial_N B}\bar{\boldsymbol{T}}\cdot\boldsymbol{v}\,dS+\int_B\boldsymbol{b}\cdot\boldsymbol{v}\,dV$$

  - Weak form of the problem can be stated as finding $\boldsymbol{u}(\boldsymbol{X})\in\mathbf{H}^1(B_0)$ such that $a(\boldsymbol{u},\,\delta\boldsymbol{u})=b(\delta\boldsymbol{u})\ \ \forall\delta\boldsymbol{u}\in\mathbf{H}^1_c(B_0)\subset\mathbf{H}^1(B_0)$

- Linear elasticity: Weak form & consistency
  - Weak form of the problem can be stated as finding $\boldsymbol{u}(\boldsymbol{X}) \in \mathbf{H}^1(B_0)$ such that $a(\boldsymbol{u},\, \delta\boldsymbol{u}) = b(\delta\boldsymbol{u}) \quad \forall \delta\boldsymbol{u} \in \mathbf{H}^1_c(B_0) \subset \mathbf{H}^1(B_0)$
  - Remarks
    - This weak form is written $\forall \delta\boldsymbol{u} \in \mathbf{H}^1_c(B_0) \subset \mathbf{H}^1(B_0)$
      - As $\delta\boldsymbol{u}$ is arbitrary in $\mathbf{H}^1{}_c$, it can be shown* that
        - » If the exact solution $\boldsymbol{u}^{\text{exact}}$ of the strong form exists in $C^2(B_0) \cap C^1(B_0)$
        - » The solution $\boldsymbol{u}(\boldsymbol{X}) \in \mathbf{H}^1(B_0)$ of the weak form corresponds to the exact solution $\boldsymbol{u}^{\text{exact}}$
      - Reciprocally, the exact solution satisfies the weak form
        - » Directly obtained by integration by parts
    - For a general weak formulation
      - A solution of a weak form is sought in a particularized subspace of $\mathbf{H}^1$ for $\delta\boldsymbol{u}$ arbitrary in this subspace (see finite-element method)
      - In that case, the solution of the weak form does not verify the strong form at each material point $\boldsymbol{X}$ of B_0
      - This solution verifies the strong form equations on average

*Finite elements: theory, fast solvers, and applications in solid mechanics, Dietrich Braess, Cambridge Press, 2001, ISBN 0521 011957 – chapter 3, boundary-value problems

- Linear elasticity: One-field functional
  - Weak form of the problem can be stated as finding $\boldsymbol{u}(\boldsymbol{X}) \in \mathbf{H}^1(B_0)$ such that $a(\boldsymbol{u},\, \delta\boldsymbol{u}) = b(\delta\boldsymbol{u}) \quad \forall \delta\boldsymbol{u} \in \mathbf{H}_c^1(B_0) \subset \mathbf{H}^1(B_0)$
  - In Hilbert spaces, the directional Gâteaux derivative can be used:
    - For a functional $I(\boldsymbol{u}) = \int_B f(\boldsymbol{u})\, dV$
    - The Gâteaux derivative
      - In the direction ***v*-*u***
      - $\forall \boldsymbol{u},\ \boldsymbol{v} \in \mathbf{H}^1(B_0)$ (***v*-*u*** is not necessarily infinitesimal)
      - Reads

$$I'(\boldsymbol{u};\, \boldsymbol{v}-\boldsymbol{u}) = \int_B f'(\boldsymbol{u}) \cdot (\boldsymbol{v}-\boldsymbol{u})\, dV = \lim_{\epsilon \longrightarrow 0} \frac{I(\boldsymbol{u}+\epsilon(\boldsymbol{v}-\boldsymbol{u})) - I(\boldsymbol{u})}{\epsilon}$$

- Linear elasticity: One-field functional (2)
  - In linear elasticity the stress tensor derives from an internal potential
    - $U = \frac{1}{2}\boldsymbol{\varepsilon} : \mathcal{H} : \boldsymbol{\varepsilon} \quad B_0 \rightarrow \mathbb{R}^+$
    - As $B \sim B_0$ we can write

$$\begin{aligned} a(\boldsymbol{u}, \delta\boldsymbol{u}) &= \int_B \frac{1}{2}(\boldsymbol{\nabla} \otimes \delta\boldsymbol{u} + \delta\boldsymbol{u} \otimes \boldsymbol{\nabla}) : \mathcal{H} : \frac{1}{2}(\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla})\, dV \\ &= \int_B \delta\boldsymbol{\varepsilon} : \mathcal{H} : \boldsymbol{\varepsilon} dV = \int_B \partial_{\boldsymbol{\varepsilon}} U(\boldsymbol{\varepsilon}) : \left[\left(\frac{1}{2}\boldsymbol{\nabla} \otimes + \frac{1}{2} \otimes \boldsymbol{\nabla}\right)\delta\boldsymbol{u}\right] dV \end{aligned}$$

    - Internal energy of the body is defined by $E_{\text{int}} = \int_B U(\boldsymbol{\varepsilon})\, dV$
    - The bilinear term is the directional derivative of the internal energy with respect to the displacements $a(\boldsymbol{u}, \delta\boldsymbol{u}) = E'_{\text{int}}(\boldsymbol{u};\, \delta\boldsymbol{u})$, with $\delta\boldsymbol{u} = \boldsymbol{v}\text{-}\boldsymbol{u}$ (not infinitesimal)
  - Similarly
    - $b(\delta\boldsymbol{u}) = \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \delta\boldsymbol{u}\, dS + \int_B \boldsymbol{b} \cdot \delta\boldsymbol{u}\, dV = W'_{\text{ext}}(\boldsymbol{u};\, \delta\boldsymbol{u})$
    - With the work of the external forces $W_{\text{ext}}(\boldsymbol{u}) = \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \boldsymbol{u}\, dS + \int_B \boldsymbol{b} \cdot \boldsymbol{u}\, dV$

- Linear elasticity: One-field functional (3)
  - Starting from $\begin{cases} a(\boldsymbol{u},\, \delta\boldsymbol{u}) = E'_{\text{int}}(\boldsymbol{u};\, \delta\boldsymbol{u}) \\ b(\delta\boldsymbol{u}) = W'_{\text{ext}}(\boldsymbol{u};\, \delta\boldsymbol{u}) \end{cases}$
  - We can define the one-field functional $I(\boldsymbol{u})$: $\mathbf{H}^1(B_0) \to \mathbb{R}$

$$I(\boldsymbol{u}) = E_{\text{int}}(\boldsymbol{u}) - W_{\text{ext}}(\boldsymbol{u}) = \int_B U(\boldsymbol{\varepsilon}(\boldsymbol{u}))\, dV - \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \boldsymbol{u}\, dS - \int_B \boldsymbol{b} \cdot \boldsymbol{u}\, dV$$

$$\Longrightarrow I'(\boldsymbol{u};\, \delta\boldsymbol{u}) = a(\boldsymbol{u},\, \delta\boldsymbol{u}) - b(\delta\boldsymbol{u})$$

  - Functional extremum
    - As $\boldsymbol{u}$ is known on Dirichlet boundary, $\boldsymbol{v}$-$\boldsymbol{u}$ = 0 on $\partial_D B$ $\Longrightarrow$ $\delta\boldsymbol{u} \in \mathbf{H}^1_c(B_0)$
    - Gâteaux derivatives are valid $\forall \boldsymbol{u},\ \boldsymbol{v} \in \mathbf{H}^1(B_0)$

      $\Longrightarrow$ in expression $I'(\boldsymbol{u};\, \delta\boldsymbol{u}) = a(\boldsymbol{u},\, \delta\boldsymbol{u}) - b(\delta\boldsymbol{u})$, $\delta\boldsymbol{u}$ is arbitrary in $\mathbf{H}^1_c(B_0)$
    - At the extremum of the functional $a(\boldsymbol{u},\, \delta\boldsymbol{u}) = b(\delta\boldsymbol{u})$
    - So the functional extremum corresponds to $\boldsymbol{u}(\boldsymbol{X}) \in \mathbf{H}^1(B_0)$

      such that $a(\boldsymbol{u},\, \delta\boldsymbol{u}) = b(\delta\boldsymbol{u}) \quad \forall \delta\boldsymbol{u} \in \mathbf{H}^1_c(B_0) \subset \mathbf{H}^1(B_0)$
  - The functional extremum corresponds to the solution of the weak form
    - Which corresponds to the exact solution $\boldsymbol{u}^{\text{exact}}$ of the strong form in $C^2(B_0) \cap C^1(B_0)$, if it exists

- Linear elasticity: Two-field functional
  - Weak form of the problem can be stated as finding $\boldsymbol{u}(\boldsymbol{X}) \in \mathbf{H}^1(B_0)$ such that $a(\boldsymbol{u},\, \delta\boldsymbol{u}) = b(\delta\boldsymbol{u}) \quad \forall \delta\boldsymbol{u} \in \mathbf{H}^1_c(B_0) \subset \mathbf{H}^1(B_0)$
  - The solution corresponds to the extremum
    - Of the two-field functional $I(\boldsymbol{u},\, \boldsymbol{\sigma}) \;:\; \mathbf{H}^1(B_0) \times \mathbf{H}^0(B_0) \rightarrow \mathbb{R}$

$$I(\boldsymbol{u},\, \boldsymbol{\sigma}) = \int_B \left[\boldsymbol{\sigma} : \frac{1}{2}(\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla}) - U(\boldsymbol{\sigma})\right] dV - \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \boldsymbol{u}\, dS - \int_{\partial_D B} [\boldsymbol{u} - \bar{\boldsymbol{u}}] \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}\, dS - \int_B \boldsymbol{b} \cdot \boldsymbol{u}\, dV$$

    - With the internal potential $U(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma} : \mathcal{H}^{-1} : \boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\sigma} : \mathcal{G} : \boldsymbol{\sigma}$

Weak form of the continuum equations

- Linear elasticity: Two-field functional (2)
  - Functional $I(\boldsymbol{u},\ \boldsymbol{\sigma}) = \int_B \left[\boldsymbol{\sigma} : \frac{1}{2}(\boldsymbol{\nabla}\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\boldsymbol{\nabla}) - U(\boldsymbol{\sigma})\right] dV - \int_{\partial_N B} \bar{\boldsymbol{T}}\cdot\boldsymbol{u}\, dS - \int_{\partial_D B} [\boldsymbol{u}-\bar{\boldsymbol{u}}]\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\, dS - \int_B \boldsymbol{b}\cdot\boldsymbol{u}\, dV$
  - Directional derivative with respect to σ
    - $I'(\boldsymbol{u},\ \boldsymbol{\sigma};\ \delta\boldsymbol{\sigma}) = \int_B \left[\frac{1}{2}(\boldsymbol{\nabla}\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\boldsymbol{\nabla}) - \partial_{\boldsymbol{\sigma}} U(\boldsymbol{\sigma})\right] \delta\boldsymbol{\sigma} dV - \int_{\partial_D B} [\boldsymbol{u}-\bar{\boldsymbol{u}}]\cdot\delta\boldsymbol{\sigma}\cdot\boldsymbol{n}\, dS$
    - As on Neumann boundary σ is known, this equation holds for
      $\forall\delta\boldsymbol{\sigma} \in \mathbf{H}^0_s(B_0) \ = \ \left\{\delta\boldsymbol{\sigma} \in \mathbf{H}^0(B_0) \ : \ \delta\boldsymbol{\sigma}(\partial_N B_0) = 0\right\}$
    - As $\delta\sigma$ is arbitrary in B_0 and on $\partial_D B$, equaling the derivative to zero leads to
      - $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on $\partial_D B$
      - $\partial_{\boldsymbol{\sigma}} U(\boldsymbol{\sigma}) = \mathcal{H}^{-1} : \boldsymbol{\sigma} = \frac{1}{2}(\boldsymbol{\nabla}\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\boldsymbol{\nabla})$ in B
    - Extremum of the functional satisfies the material behavior and Dirichlet BC

- Linear elasticity: Two-field functional (3)
  - Functional $$I(\boldsymbol{u},\ \boldsymbol{\sigma}) = \int_B \left[\boldsymbol{\sigma} : \frac{1}{2}(\boldsymbol{\nabla}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{\nabla}) - U(\boldsymbol{\sigma})\right] dV - \int_{\partial_N B} \bar{\boldsymbol{T}}\cdot\boldsymbol{u}\, dS - \int_{\partial_D B} [\boldsymbol{u} - \bar{\boldsymbol{u}}]\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\, dS - \int_B \boldsymbol{b}\cdot\boldsymbol{u}\, dV$$
  - Directional derivative with respect to $\delta\boldsymbol{u}$
    - $$I'(\boldsymbol{u},\ \boldsymbol{\sigma};\ \delta\boldsymbol{u}) = \int_B \left[\boldsymbol{\sigma} : \frac{1}{2}(\boldsymbol{\nabla}\otimes\delta\boldsymbol{u} + \delta\boldsymbol{u}\otimes\boldsymbol{\nabla})\right] dV - \int_{\partial_N B} \bar{\boldsymbol{T}}\cdot\delta\boldsymbol{u}\, dS - \int_{\partial_D B} \delta\boldsymbol{u}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\, dS - \int_B \boldsymbol{b}\cdot\delta\boldsymbol{u}\, dV$$
    - $\delta\boldsymbol{u} \in \mathbf{H}^1_c(B_0) = \left\{\delta\boldsymbol{u} \in \mathbf{H}^1(B_0)\ :\ \delta\boldsymbol{u}(\partial_D B_0) = 0\right\}$
    - Applying integration by parts and Gauss theorem

$$I'(\boldsymbol{u},\ \boldsymbol{\sigma};\ \delta\boldsymbol{u}) = \int_{\partial_N B} \delta\boldsymbol{u}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n} dV - \int_B \left[\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^T\right]\cdot\delta\boldsymbol{u}\, dV - \int_{\partial_N B} \bar{\boldsymbol{T}}\cdot\delta\boldsymbol{u}\, dV - \int_B \boldsymbol{b}\cdot\delta\boldsymbol{u}\, dV$$

    - As $\delta\boldsymbol{u}$ is arbitrary in B and on $\partial_N B$, equaling the derivative to zero leads to
      - $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^T + \boldsymbol{b} = 0$ in B
      - $\boldsymbol{\sigma}\cdot\boldsymbol{n} = \bar{\boldsymbol{T}}$ on $\partial_N B$

- Linear elasticity: Two-field functional (4)
  - Extremum of functional $I(\boldsymbol{u},\ \boldsymbol{\sigma})\ :\ \mathbf{H}^1(B_0) \times \mathbf{H}^0(B_0) \rightarrow \mathbb{R}$

$$I(\boldsymbol{u},\ \boldsymbol{\sigma}) = \int_B \left[\boldsymbol{\sigma} : \frac{1}{2}(\boldsymbol{\nabla}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{\nabla}) - U(\boldsymbol{\sigma})\right] dV - \int_{\partial_N B} \bar{\boldsymbol{T}}\cdot\boldsymbol{u}\, dS - \int_{\partial_D B} [\boldsymbol{u} - \bar{\boldsymbol{u}}]\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\, dS - \int_B \boldsymbol{b}\cdot\boldsymbol{u}\, dV$$

    - Satisfies
      - $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^T + \boldsymbol{b} = 0$ in B
      - $\partial_{\boldsymbol{\sigma}} U(\boldsymbol{\sigma}) = \mathcal{H}^{-1} : \boldsymbol{\sigma} = \frac{1}{2}(\boldsymbol{\nabla}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{\nabla})$ in B
      - $\boldsymbol{\sigma}\cdot\boldsymbol{n} = \bar{\boldsymbol{T}}$ on $\partial_N B$
      - $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on $\partial_D B$
    - So the extremum satisfies the equations
      - Weak form is the stationary point of the functional in $\mathbf{H}^1(B_0) \times \mathbf{H}^0(B_0)$
      - Corresponds to the exact solution $\boldsymbol{u}^{\text{exact}}$ of the strong form in $C^2(B_0) \cap C^1(B_0)$, if it exists

- Linear elasticity: Three-field functional
  - Weak form of the problem can be stated as finding $\boldsymbol{u}\left(\boldsymbol{X}\right) \in \mathbf{H}^1\left(B_0\right)$ such that $a\left(\boldsymbol{u},\, \delta\boldsymbol{u}\right) = b\left(\delta\boldsymbol{u}\right) \quad \forall \delta\boldsymbol{u} \in \mathbf{H}^1_c\left(B_0\right) \subset \mathbf{H}^1\left(B_0\right)$
  - The solution corresponds to the extremum
    - Of the three-field functional $I\left(\boldsymbol{u},\, \boldsymbol{\sigma},\, \boldsymbol{\varepsilon}\right) \;:\; \mathbf{H}^1\left(B_0\right) \times \mathbf{H}^0\left(B_0\right) \times \mathbf{H}^0\left(B_0\right) \rightarrow \mathbb{R}$

$$I\left(\boldsymbol{u},\, \boldsymbol{\sigma},\, \boldsymbol{\varepsilon}\right) = \int_B \left\{ U\left(\boldsymbol{\varepsilon}\right) + \boldsymbol{\sigma} : \left[\frac{1}{2}\left(\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla}\right) - \boldsymbol{\varepsilon}\right]\right\} dV - \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \boldsymbol{u}\, dS - \int_{\partial_D B} \left[\boldsymbol{u} - \bar{\boldsymbol{u}}\right] \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}\, dS - \int_B \boldsymbol{b} \cdot \boldsymbol{u}\, dV$$

    - With the internal potential $U\left(\boldsymbol{\varepsilon}\right) = \frac{1}{2}\boldsymbol{\varepsilon} : \mathcal{H} : \boldsymbol{\varepsilon}$

- Linear elasticity: Three-field functional (2)
  - Functional $I(\boldsymbol{u},\ \boldsymbol{\sigma},\ \boldsymbol{\varepsilon}) = \int_B \left\{ U(\boldsymbol{\varepsilon}) + \boldsymbol{\sigma} : \left[\frac{1}{2}(\boldsymbol{\nabla}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{\nabla}) - \boldsymbol{\varepsilon} \right] \right\} dV - \int_{\partial_N B} \bar{\boldsymbol{T}}\cdot\boldsymbol{u}\, dS - \int_{\partial_D B} [\boldsymbol{u} - \bar{\boldsymbol{u}}]\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\, dS - \int_B \boldsymbol{b}\cdot\boldsymbol{u}\, dV$
    - Directional derivative with respect to σ

$$I'(\boldsymbol{u},\ \boldsymbol{\sigma},\ \boldsymbol{\varepsilon};\ \delta\boldsymbol{\sigma}) = \int_B \delta\boldsymbol{\sigma} : \left[\frac{1}{2}(\boldsymbol{\nabla}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{\nabla}) - \boldsymbol{\varepsilon} \right] dV - \int_{\partial_D B} [\boldsymbol{u} - \bar{\boldsymbol{u}}]\cdot\delta\boldsymbol{\sigma}\cdot\boldsymbol{n}\, dS$$

    - As on Neumann boundary σ is known, this equation holds for

$$\forall\delta\boldsymbol{\sigma} \in \mathbf{H}^0_s(B_0) \ = \ \left\{ \delta\boldsymbol{\sigma} \in \mathbf{H}^0(B_0) \ :\ \delta\boldsymbol{\sigma}(\partial_N B_0) = 0 \right\}$$

    - As $\delta\sigma$ is arbitrary in B_0 and on $\partial_D B$, equaling the derivative to zero leads to
      - $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on $\partial_D B$
      - $\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{\nabla}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{\nabla})$ in B
    - Extremum of the functional satisfies the compatibility and Dirichlet BC

- Linear elasticity: Three-field functional (3)
  - Functional $I(\boldsymbol{u},\ \boldsymbol{\sigma},\ \boldsymbol{\varepsilon}) = \int_B \left\{U(\boldsymbol{\varepsilon}) + \boldsymbol{\sigma} : \left[\frac{1}{2}(\boldsymbol{\nabla}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{\nabla}) - \boldsymbol{\varepsilon}\right]\right\} dV - \int_{\partial_N B} \bar{\boldsymbol{T}}\cdot\boldsymbol{u}\, dS - \int_{\partial_D B} [\boldsymbol{u} - \bar{\boldsymbol{u}}]\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\, dS - \int_B \boldsymbol{b}\cdot\boldsymbol{u}\, dV$
    - Directional derivative with respect to $\boldsymbol{\varepsilon}$

$$I'(\boldsymbol{u},\ \boldsymbol{\sigma},\ \boldsymbol{\varepsilon};\ \delta\boldsymbol{\varepsilon}) = \int_B [\partial_{\boldsymbol{\varepsilon}} U(\boldsymbol{\varepsilon}) - \boldsymbol{\sigma}] : \delta\boldsymbol{\varepsilon}\ dV$$

    - As $\delta\boldsymbol{\varepsilon}$ is arbitrary in B_0, equaling the derivative to zero leads to

$$\boldsymbol{\sigma} = \partial_{\boldsymbol{\varepsilon}} U(\boldsymbol{\varepsilon}) = \mathcal{H} : \boldsymbol{\varepsilon} \quad \text{in } B$$

    - As on Neumann boundary $\boldsymbol{\sigma}$, and so $\boldsymbol{\varepsilon}$, are known validity is ensured

$$\forall\delta\boldsymbol{\varepsilon} \in \mathbf{H}^0_s(B_0) = \left\{\delta\boldsymbol{\varepsilon} \in \mathbf{H}^0(B_0) : \delta\boldsymbol{\varepsilon}(\partial_N B_0) = 0\right\}$$

    - Extremum of the functional satisfies the material behavior

- Linear elasticity: Three-field functional (4)
  - Functional $$I(\boldsymbol{u},\ \boldsymbol{\sigma},\ \boldsymbol{\varepsilon}) = \int_B \left\{ U(\boldsymbol{\varepsilon}) + \boldsymbol{\sigma} : \left[\frac{1}{2} (\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla}) - \boldsymbol{\varepsilon} \right] \right\} dV - \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \boldsymbol{u}\, dS - \int_{\partial_D B} [\boldsymbol{u} - \bar{\boldsymbol{u}}] \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}\, dS - \int_B \boldsymbol{b} \cdot \boldsymbol{u}\, dV$$
    - Directional derivative with respect to $\boldsymbol{u}$
$$I'(\boldsymbol{u},\ \boldsymbol{\sigma},\ \boldsymbol{\varepsilon};\ \delta\boldsymbol{u}) = \int_B \boldsymbol{\sigma} : \frac{1}{2} (\boldsymbol{\nabla} \otimes \delta\boldsymbol{u} + \delta\boldsymbol{u} \otimes \boldsymbol{\nabla})\, dV - \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \delta\boldsymbol{u}\, dS - \int_B \boldsymbol{b} \cdot \delta\boldsymbol{u}\, dV - \int_{\partial_D B} \delta\boldsymbol{u} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}\, dS$$
    - $\delta\boldsymbol{u} \in \mathbf{H}^1_c(B_0) \ = \ \left\{ \delta\boldsymbol{u} \in \mathbf{H}^1(B_0)\ :\ \delta\boldsymbol{u}(\partial_D B_0) = 0 \right\}$
    - Applying integration by parts and Gauss theorem
$$I'(\boldsymbol{u},\ \boldsymbol{\sigma},\ \boldsymbol{\varepsilon};\ \delta\boldsymbol{u}) = -\int_B \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T \cdot \delta\boldsymbol{u}\, dV + \int_{\partial_N B} \delta\boldsymbol{u} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}\, dV - \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \delta\boldsymbol{u}\, dS - \int_B \boldsymbol{b} \cdot \delta\boldsymbol{u}\, dV$$

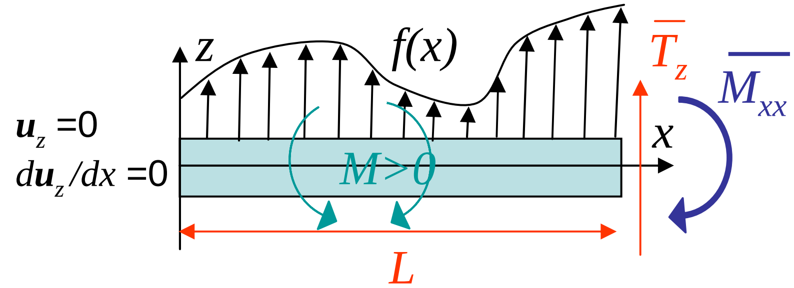
- Linear elasticity: Three-field functional (5)
  - Functional $$I(\boldsymbol{u},\ \boldsymbol{\sigma},\ \boldsymbol{\varepsilon}) = \int_B \left\{ U(\boldsymbol{\varepsilon}) + \boldsymbol{\sigma} : \left[\frac{1}{2} (\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla}) - \boldsymbol{\varepsilon} \right] \right\} dV - \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \boldsymbol{u}\, dS - \int_{\partial_D B} [\boldsymbol{u} - \bar{\boldsymbol{u}}] \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}\, dS - \int_B \boldsymbol{b} \cdot \boldsymbol{u}\, dV$$
    - Directional derivative with respect to $\boldsymbol{u}$, integration by parts and Gauss theorem
$$I'(\boldsymbol{u},\ \boldsymbol{\sigma},\ \boldsymbol{\varepsilon};\ \delta\boldsymbol{u}) = -\int_B \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^T\cdot\delta\boldsymbol{u}\, dV + \int_{\partial_N B} \delta\boldsymbol{u}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\, dV - \int_{\partial_N B} \bar{\boldsymbol{T}}\cdot\delta\boldsymbol{u}\, dS - \int_B \boldsymbol{b}\cdot\delta\boldsymbol{u}\, dV$$
    - As $\delta\boldsymbol{u}$ is arbitrary in B_0 and on $\partial_N B$, equaling the derivative to zero leads to
      - $\boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{T}}$ on $\partial_N B$
      - $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T + \boldsymbol{b} = 0$ in B

- Linear elasticity: Three-field functional (6)
  - Extremum of functional $I\left(\boldsymbol{u},\ \boldsymbol{\sigma},\ \boldsymbol{\varepsilon}\right)\ :\ \mathbf{H}^1\left(B_0\right)\times\mathbf{H}^0\left(B_0\right)\times\mathbf{H}^0\left(B_0\right)\rightarrow\mathbb{R}$

$$I\left(\boldsymbol{u},\ \boldsymbol{\sigma},\ \boldsymbol{\varepsilon}\right)=\int_B\left\{U\left(\boldsymbol{\varepsilon}\right)+\boldsymbol{\sigma}:\left[\frac{1}{2}\left(\boldsymbol{\nabla}\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\boldsymbol{\nabla}\right)-\boldsymbol{\varepsilon}\right]\right\}dV-$$
$$\int_{\partial_N B}\bar{\boldsymbol{T}}\cdot\boldsymbol{u}\,dS-\int_{\partial_D B}\left[\boldsymbol{u}-\bar{\boldsymbol{u}}\right]\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\,dS-\int_B\boldsymbol{b}\cdot\boldsymbol{u}\,dV$$

- Satisfies
  - $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^T+\boldsymbol{b}=0$ in B
  - $\boldsymbol{\sigma}=\partial_{\boldsymbol{\varepsilon}}U\left(\boldsymbol{\varepsilon}\right)=\mathcal{H}:\boldsymbol{\varepsilon}$ in B
  - $\boldsymbol{\varepsilon}=\frac{1}{2}\left(\boldsymbol{\nabla}\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\boldsymbol{\nabla}\right)$ in B
  - $\boldsymbol{\sigma}\cdot\boldsymbol{n}=\bar{\boldsymbol{T}}$ on $\partial_N B$
  - $\boldsymbol{u}=\bar{\boldsymbol{u}}$ on $\partial_D B$
- So the extremum satisfies the equations
  - Weak form is the stationary point of the functional in $\mathbf{H}^1\left(B_0\right)\times\mathbf{H}^0\left(B_0\right)\times\mathbf{H}^0\left(B_0\right)$
  - Corresponds to the exact solution $\boldsymbol{u}^{\text{exact}}$ of the strong form in $C^2(B_0)\cap C^1(B_0)$, if it exists

- Euler-Bernoulli beam equations



  – $-\dfrac{\partial^2}{\partial x^2}\left(EI\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f(x)$ for x in $]0\ L[$

  – Boundary conditions

    - $-\dfrac{\partial}{\partial x}\left(EI\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\Big|_{0,\ L} = \bar{T}_z|_{0,\ L}$ or $\boldsymbol{u}_z|_{0,\ L} = \bar{\boldsymbol{u}}_z|_{0,\ L}$
    - $-EI\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\Big|_{0,\ L} = \bar{M}_{xx}|_{0,\ L}$ or $\partial_x \boldsymbol{u}_z|_{0,\ L} = \bar{\theta}|_{0,\ L}$

    - Let us define the boundaries such that

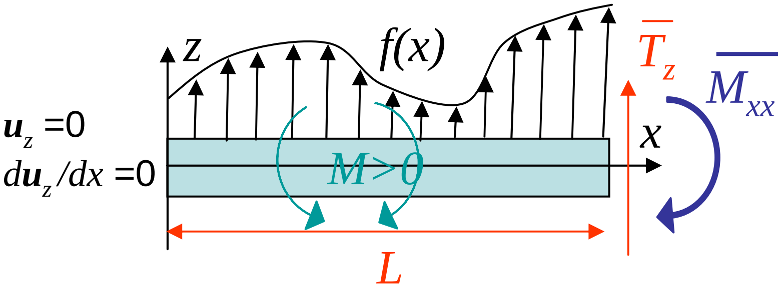
      – Displacement constrained on $\partial_U L$ & vertical load known on $\partial_N L = \partial L \setminus \partial_U L$

      – Rotation constrained on $\partial_T L$ & momentum known on $\partial_M L = \partial L \setminus \partial_T L$

  – Exact solution can be found as $\boldsymbol{u}_z^{\text{exact}} \in \mathrm{H}^4\left(]0\ L[\right)$

Weak form of the continuum equations for beams

- Weak form
  - For x in $]0\ L[$



$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f(x)$$

- Multiply the equation by

$$\delta \boldsymbol{u}_z \in \mathrm{H}_c^2(]0\ L[) = \left\{\delta \boldsymbol{u}_z \in \mathrm{H}^2(]0\ L[)\ :\ \delta \boldsymbol{u}_z|_{\partial_U L} = \partial_x \delta \boldsymbol{u}_z|_{\partial_T L} = 0\right\}$$

and integrate the product on the beam length

$$\Longrightarrow \int_0^L \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\delta \boldsymbol{u}_z\, dx - \int_0^L f(x)\,\delta \boldsymbol{u}_z\, dx = 0\quad \forall \delta \boldsymbol{u}_z \in \mathrm{H}_c^2(]0\ L[)$$

- Integrations by parts

$$\Longrightarrow \int_0^L EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\frac{\partial^2 \delta \boldsymbol{u}_z}{\partial x^2}\, dx + \frac{\partial}{\partial x}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\delta \boldsymbol{u}_z\Big|_0^L - \left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\frac{\partial \delta \boldsymbol{u}_z}{\partial x}\Big|_0^L - \int_0^L f(x)\,\delta \boldsymbol{u}_z\, dx = 0\quad \forall \delta \boldsymbol{u}_z \in \mathrm{H}_c^2(]0\ L[)$$

- Weak form (2)
  - Starting from

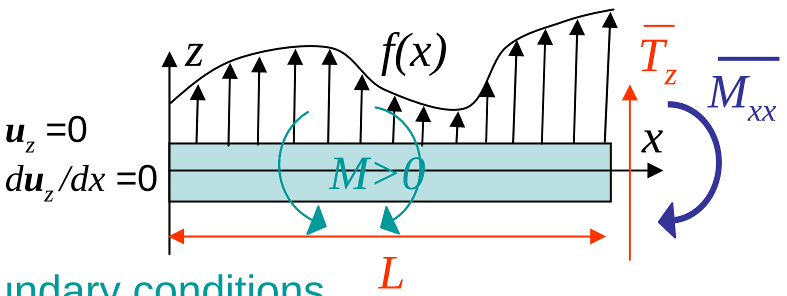
$$\int_0^L EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \frac{\partial^2 \delta \boldsymbol{u}_z}{\partial x^2} dx + \frac{\partial}{\partial x}\left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) \delta \boldsymbol{u}_z \Big|_0^L - \left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) \frac{\partial \delta \boldsymbol{u}_z}{\partial x} \Big|_0^L - \int_0^L f(x)\, \delta \boldsymbol{u}_z\, dx = 0 \quad \forall \delta \boldsymbol{u}_z \in \mathrm{H}_c^2(]0\ L[)$$

$$\Longrightarrow \int_0^L EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \frac{\partial^2 \delta \boldsymbol{u}_z}{\partial x^2} dx = \int_0^L f(x)\, \delta \boldsymbol{u}_z\, dx - \frac{\partial}{\partial x}\left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) n_x \delta \boldsymbol{u}_z \Big|_{\partial_N L} + \left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) \frac{\partial \delta \boldsymbol{u}_z}{\partial x} n_x \Big|_{\partial_M L} \quad \forall \delta \boldsymbol{u}_z \in \mathrm{H}_c^2(]0\ L[)$$

- With

$$\begin{cases} -\dfrac{\partial}{\partial x}\left(EI \dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = \bar{T}_z & \text{on } \partial_N B \\ -EI \dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2} = \bar{M}_{xx} & \text{on } \partial_M B \end{cases}$$

- Weak form (3)
  - Combining previous expressions leads to the weak form statement:
    - Finding $\boldsymbol{u}_z \in \mathrm{H}^2\left(]0\ L[\right)$ - and not in H^4 - such that
    - $$\int_0^L EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \frac{\partial^2 \delta \boldsymbol{u}_z}{\partial x^2}\, dx = \int_0^L f\left(x\right) \delta \boldsymbol{u}_z\, dx + n_x \bar{T}_z \delta \boldsymbol{u}_z \big|_{\partial_N L} + n_x \bar{M}_{xx} \frac{\partial\left(-\delta \boldsymbol{u}_z\right)}{\partial x}\bigg|_{\partial_M L} \quad \forall \delta \boldsymbol{u}_z \in \mathrm{H}_c^2\left(]0\ L[\right)$$
  - Remarks
    - For small deflections $\theta_y = -\dfrac{\partial \boldsymbol{u}_z}{\partial x}$



    - For sufficiently smooth loadings and boundary conditions
      - A solution in $\mathrm{H}^2(]0\ L[)$ can be found
      - A solution in $C^4(]0\ L[)$ cannot always by found

- Bilinear form
  - Let us define the bilinear form:

$$a\left(\boldsymbol{u}_z,\, \boldsymbol{v}_z\right) = \int_0^L EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \frac{\partial^2 \boldsymbol{v}_z}{\partial x^2}\, dx \quad : \mathrm{H}^2\left(]0\ L[\right) \times \mathrm{H}^2\left(]0\ L[\right) \to \mathbb{R}$$

    - Which is symmetrical $a\left(\boldsymbol{u}_z,\, \boldsymbol{v}_z\right) = a\left(\boldsymbol{v}_z,\, \boldsymbol{u}_z\right)$ &
    - Positive $a\left(\boldsymbol{u}_z,\, \boldsymbol{u}_z\right) > 0\ \ \forall \boldsymbol{u}_z \in \mathrm{H}^2\left(]0\ L[\right),\ \boldsymbol{u}_z \neq 0$
    - Let us define the linear form $\mathrm{H}^2\left(]0\ L[\right) \to \mathbb{R}$:

$$b\left(\boldsymbol{v}_z\right) = \int_0^L f\left(x\right) \boldsymbol{v}_z\, dx + n_x \bar{T}_z \boldsymbol{v}_z \Big|_{\partial_N L} + n_x \bar{M}_{xx} \frac{\partial\left(-\boldsymbol{v}_z\right)}{\partial x}\Bigg|_{\partial_M L}$$

  - Weak form of the problem can be stated as finding $\boldsymbol{u}_z \in \mathrm{H}^2\left(]0\ L[\right)$

    such that $a\left(\boldsymbol{u}_z,\, \delta\boldsymbol{u}_z\right) = b\left(\delta\boldsymbol{u}_z\right)\ \ \forall \delta\boldsymbol{u}_z \in \mathrm{H}^2_c\left(]0\ L[\right) \subset \mathrm{H}^2\left(]0\ L[\right)$

- One-field functional
  - Solution $\boldsymbol{u}_z \in \mathrm{H}^2(]0\ L[)$ of the bilinear form :

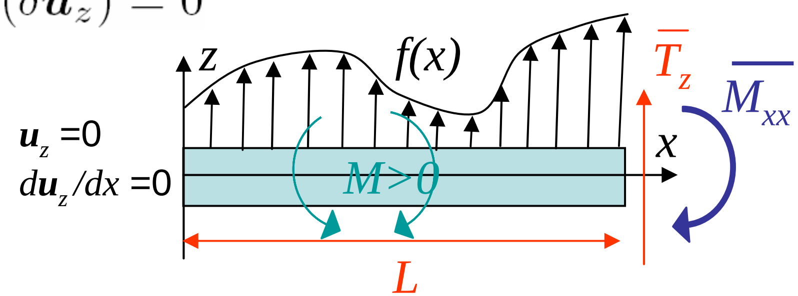
$$a(\boldsymbol{u}_z,\ \delta\boldsymbol{u}_z) = b(\delta\boldsymbol{u}_z) \quad \forall \delta\boldsymbol{u}_z \in \mathrm{H}^2_c(]0\ L[) \subset \mathrm{H}^2(]0\ L[)$$

- With
$$\begin{cases} a(\boldsymbol{u}_z,\ \boldsymbol{v}_z) = \int_0^L EI \dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\dfrac{\partial^2 \boldsymbol{v}_z}{\partial x^2}\,dx \\ b(\boldsymbol{v}_z) = \int_0^L f(x)\,\boldsymbol{v}_z\,dx + n_x \bar{T}_z \boldsymbol{v}_z\big|_{\partial_N L} + n_x \bar{M}_{xx} \dfrac{\partial(-\boldsymbol{v}_z)}{\partial x}\Big|_{\partial_M L} \end{cases}$$

  - Is the extremum of the one-field functional

$$I(\boldsymbol{u}_z) = \int_0^L \frac{EI}{2}\left(\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)^2 dx - \int_0^L f(x)\,\boldsymbol{u}_z\,dx - n_x \bar{T}_z \boldsymbol{u}_z\big|_{\partial_N L} - n_x \bar{M}_{xx}\frac{\partial(-\boldsymbol{u}_z)}{\partial x}\Big|_{\partial_M L} \quad : \mathrm{H}^2(]0\ L[) \to \mathbb{R}$$

- Indeed $I'(\boldsymbol{u}_z;\ \delta\boldsymbol{u}_z) = a(\boldsymbol{u}_z,\ \delta\boldsymbol{u}_z) - b(\delta\boldsymbol{u}_z) = 0$
- As $\boldsymbol{u}_z$ is known on $\partial_N L$ & $\partial_x \boldsymbol{u}_z$ on $\partial_M L$
  - $\delta\boldsymbol{u}_z$ belongs to $\mathrm{H}_c{}^2$



Weak form of the continuum equations for beams

- Two-field functional
  - Functional $I\left(\boldsymbol{u}_z,\, M_{xx}\right) \quad : \mathrm{H}^2\left(]0\; L[\right) \times \mathrm{H}^0\left(]0\; L[\right) \rightarrow \mathbb{R}$

$$I\left(\boldsymbol{u}_z,\, M_{xx}\right) = \int_0^L \left[M_{xx} \frac{\partial^2\left(-\boldsymbol{u}_z\right)}{\partial x^2} - U\left(M_{xx}\right)\right] dx - \int_0^L f\left(x\right) \boldsymbol{u}_z \, dx -$$
$$n_x \bar{T}_z \boldsymbol{u}_z\big|_{\partial_N L} - n_x \bar{M}_{xx} \frac{\partial\left(-\boldsymbol{u}_z\right)}{\partial x}\bigg|_{\partial_M L} +$$
$$n_x \partial_x M_{xx}\left(\boldsymbol{u}_z - \bar{\boldsymbol{u}}_z\right)\big|_{\partial_U L} - n_x M_{xx}\left[\frac{\partial\left(-\boldsymbol{u}_z\right)}{\partial x} - \bar{\theta}_y\right]\bigg|_{\partial_T L}$$

  - With the internal energy $U\left(M_{xx}\right) = \dfrac{M_{xx}^2}{2EI}$
    - Bending moment $M_{xx} = EI\kappa = EI \dfrac{\partial^2\left(-\boldsymbol{u}_z\right)}{\partial x^2}$ (obtained at the extremum)
  - Solution of the weak form corresponds to the stationary point of the functional
    - See annex I

Weak form of the continuum equations for beams

- Three-field functional
  - Functional $I\left(\boldsymbol{u}_z,\, M_{xx},\, \kappa\right) \quad : \mathrm{H}^2\left(]0\ L[\right) \times \mathrm{H}^0\left(]0\ L[\right) \times \mathrm{H}^0\left(]0\ L[\right) \rightarrow \mathbb{R}$

$$I\left(\boldsymbol{u}_z,\, M_{xx},\, \kappa\right) = \int_0^L \left[U\left(\kappa\right) - M_{xx}\left(\kappa + \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\right] dx - \int_0^L f\left(x\right) \boldsymbol{u}_z\, dx -$$

$$n_x \bar{T}_z \boldsymbol{u}_z\big|_{\partial_N L} - n_x \bar{M}_{xx} \frac{\partial\left(-\boldsymbol{u}_z\right)}{\partial x}\bigg|_{\partial_M L} +$$

$$n_x \partial_x M_{xx}\left(\boldsymbol{u}_z - \bar{\boldsymbol{u}}_z\right)\big|_{\partial_U L} - n_x M_{xx}\left[\frac{\partial\left(-\boldsymbol{u}_z\right)}{\partial x} - \bar{\theta}_y\right]\bigg|_{\partial_T L}$$

    - With the internal energy $U\left(\kappa\right) = \frac{EI}{2}\kappa^2$

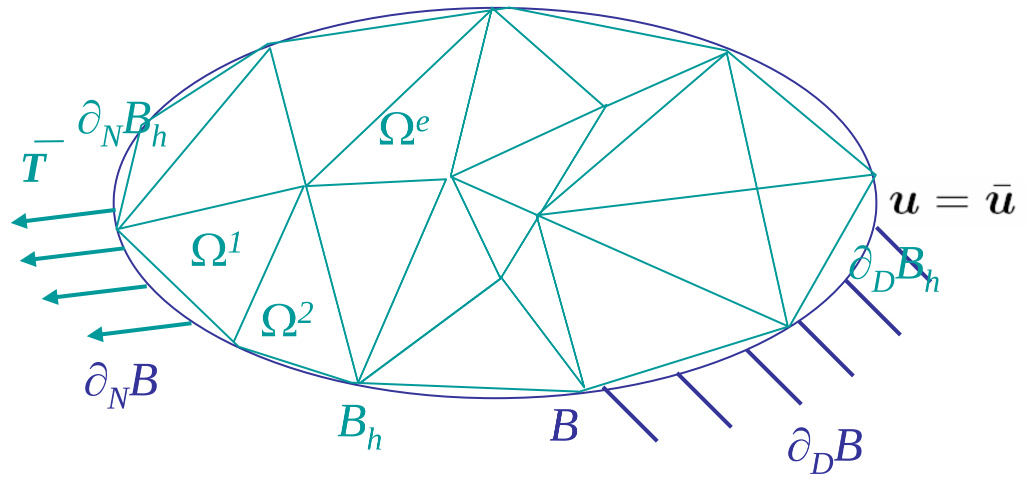
  - Solution of the weak form corresponds to the stationary point of the functional
    - See annex II

Finite element discretization of the weak form

- General weak form of linear elasticity
  - Weak form of the problem can be stated as finding $\boldsymbol{u}\left(\boldsymbol{X}\right) \in \mathbf{H}^1\left(B_0\right)$ such that $a\left(\boldsymbol{u},\, \delta\boldsymbol{u}\right) = b\left(\delta\boldsymbol{u}\right) \quad \forall \delta\boldsymbol{u} \in \mathbf{H}^1_c\left(B_0\right) \subset \mathbf{H}^1\left(B_0\right)$
    - With $\begin{cases} a\left(\boldsymbol{u},\, \boldsymbol{v}\right) = \int_B \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \boldsymbol{\nabla}\right) : \mathcal{H} : \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla}\right) dV \\ b\left(\boldsymbol{v}\right) = \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \boldsymbol{v}\, dS + \int_B \boldsymbol{b} \cdot \boldsymbol{v}\, dV \end{cases}$
- Finite-element method
  - Instead of seeking $\boldsymbol{u}\left(\boldsymbol{X}\right) \in \mathbf{H}^1\left(B_0\right)$, $\forall \delta\boldsymbol{u} \in \mathbf{H}^1_c\left(B_0\right)$
  - We are particularizing
    - The solution $\boldsymbol{u} \rightarrow \boldsymbol{u}_h$: test functions
    - The virtual displacements $\delta\boldsymbol{u} \rightarrow \delta\boldsymbol{u}_h$: trial functions

    in a manifold which is
    - A polynomial approximation
    - The same for test and trial functions: Galerkin method
    - Built on an approximation B_h of the body B: The finite-element discretization

- Finite-element discretization
  - Approximation of the body
    - Reference configuration $$B_{0h} = \bigcup_{e=1}^{E} \bar{\Omega}_0^e$$
    - Similar in configuration B

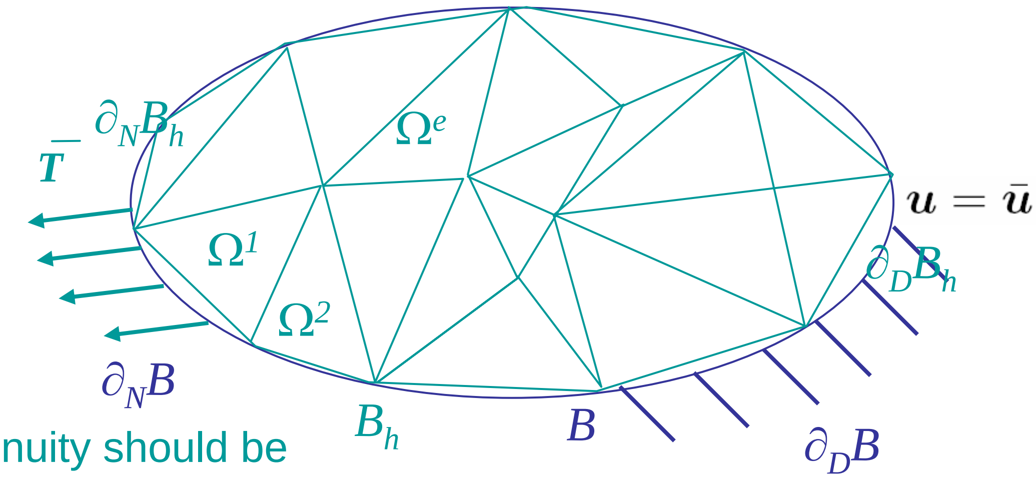


  - With
    - Interior of one element Ω_0^e with $\Omega_0^e \cap_{\forall e' \neq e} \Omega_0^{e'} = 0$
    - Boundary of one element $\partial\Omega_0^e$
    - $\bar{\Omega}_0^e = \Omega_0^e \cap \partial\Omega_0^e$
    - Dirichlet boundary of an element (can be empty): $\partial_D\Omega_0^e = \partial\Omega_0^e \cap \partial_D B_{0h} = 0$
    - Neumann boundary of an element (can be empty): $\partial_N\Omega_0^e = \partial\Omega_0^e \cap \partial_N B_{0h} = 0$
  - Characteristic size
    - Size of an element $h_e = \dfrac{\Omega_0^e}{\partial\Omega_0^e}$
    - Size of the mesh $h_{\max} = \max_e \left(h_e = \dfrac{\Omega_0^e}{\partial\Omega_0^e} \right)$

- Polynomial approximation
  - The approximation should be
    - In $\mathbf{H}^1(B_{0h})$ on the whole body (due to the weak form statement)
      - Meaning (absolute) continuity should be ensured
      - Meaning (absolute) continuity of the derivative is not always ensured
    - A polynomial approximation $\mathbb{P}^k(\Omega_0^e)$ of degree **up to** k on each element
  - Eventually
    - Test functions $\boldsymbol{u}_h \in X_h^k$

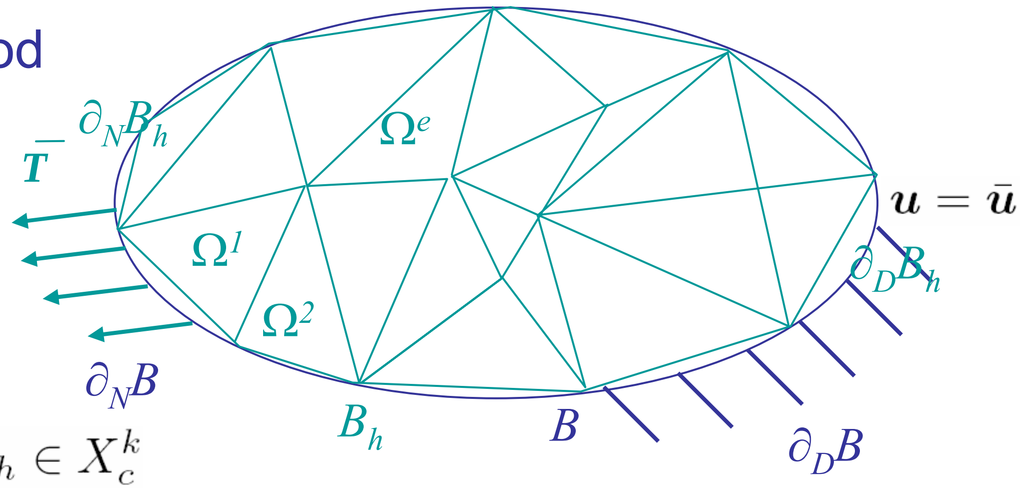
      with $X_h^k = \left\{ \boldsymbol{u}_h \in \mathbf{H}^1(B_{0h}) \ : \ \boldsymbol{u}_h|_{\Omega_0^e} \in \mathbb{P}^k(\Omega_0^e) \ \forall \Omega_0^e \in B_{0h} \right\} \subset \mathbf{H}^1(B_{0h})$
    - Trial functions $\delta\boldsymbol{u}_h \in X_c^k$

      with $X_c^k = \left\{ \delta\boldsymbol{u}_h \in X_h^k \ : \ \delta\boldsymbol{u}_h|_{\partial_D \Omega_0^e} = 0 \ \forall \Omega_0^e \in B_{0h} \right\} \subset \mathbf{H}_c^1(B_{0h})$



- New weak form of the FE method
  - Using the new manifolds
    - The problem can be restated as finding $\boldsymbol{u}_h \in X_h^k$ such that

$$a\left(\boldsymbol{u}_h,\, \delta\boldsymbol{u}_h\right) = b\left(\delta\boldsymbol{u}_h\right) \quad \forall \delta\boldsymbol{u}_h \in X_c^k$$



    - With

$$\begin{cases} a\left(\boldsymbol{u}_h,\, \boldsymbol{v}_h\right) = \sum_e \int_{\Omega^e} \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \boldsymbol{v}_h + \boldsymbol{v}_h \otimes \boldsymbol{\nabla}\right) : \mathcal{H} : \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \boldsymbol{u}_h + \boldsymbol{u}_h \otimes \boldsymbol{\nabla}\right) dV \\ b\left(\boldsymbol{v}_h\right) = \sum_e \int_{\partial_N \Omega^e} \bar{\boldsymbol{T}} \cdot \boldsymbol{v}_h \, dS + \sum_e \int_{\Omega^e} \boldsymbol{b} \cdot \boldsymbol{v}_h \, dV \end{cases}$$

  - Remark
    - Solution of the strong form: $\boldsymbol{u}^{\text{exact}}$ in $C^2(B_0) \cap C^1(B_0)$, if it exists
    - Solution of the general weak form: $\boldsymbol{u}\left(\boldsymbol{X}\right) \in \mathbf{H}^1\left(B_0\right)$
      - Corresponds to $\boldsymbol{u}^{\text{exact}}$ in $C^2(B_0) \cap C^1(B_0)$, if it exists
    - Solution of the FE-approximation: $\boldsymbol{u}_h\left(\boldsymbol{x}\right) \in X_h^k$
      - Verifies the strong form only on average

Finite element discretization of the weak form

- Does the FE weak form converge toward the exact solution?
  - First mandatory property: consistency
    - The exact solution of the problem $\boldsymbol{u}^{\text{exact}}$ in $C^2(B_0) \cap C^1(B_0)$, which satisfies the strong form, should also satisfy $a\left(\boldsymbol{u}_h,\, \delta \boldsymbol{u}_h\right) = b\left(\delta \boldsymbol{u}_h\right) \quad \forall \delta \boldsymbol{u}_h \in X_c^k$
    - Proof
      - $$a\left(\boldsymbol{u}^{\text{exact}},\, \delta \boldsymbol{u}_h\right) = \sum_e \int_{\Omega^e} \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \delta \boldsymbol{u}_h + \delta \boldsymbol{u}_h \otimes \boldsymbol{\nabla}\right) : \mathcal{H} : \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \boldsymbol{u}^{\text{exact}} + \boldsymbol{u}^{\text{exact}} \otimes \boldsymbol{\nabla}\right) dV$$
      - As for the exact solution $\mathcal{H} : \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \boldsymbol{u}^{\text{exact}} + \boldsymbol{u}^{\text{exact}} \otimes \boldsymbol{\nabla}\right) = \boldsymbol{\sigma}$

$$\Longrightarrow \quad a\left(\boldsymbol{u}^{\text{exact}},\, \delta \boldsymbol{u}_h\right) = \int_{B_h} \boldsymbol{\nabla} \otimes \delta \boldsymbol{u}_h : \boldsymbol{\sigma}^T dV$$

$$= -\int_{B_h} \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T \cdot \delta \boldsymbol{u}_h + \int_{\partial B_h} \delta \boldsymbol{u}_h \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} dS$$

      - As $\delta \boldsymbol{u}_h \in X_c^k$, $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T + \boldsymbol{b} = 0 \ \forall\, \boldsymbol{X} \in B_0$ & $\boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{T}} \ \forall\, \boldsymbol{X} \in \partial_N B_0$

$$\Longrightarrow \quad a\left(\boldsymbol{u}^{\text{exact}},\, \delta \boldsymbol{u}_h\right) = \int_{B_h} \boldsymbol{b} \cdot \delta \boldsymbol{u}_h + \int_{\partial_N B_h} \delta \boldsymbol{u}_h \cdot \bar{\boldsymbol{T}} dS = b\left(\delta \boldsymbol{u}_h\right)$$

- Does the FE weak form converge toward the exact solution (2)?
  - Second mandatory property: stability
    - The deformation energy should be bounded by (half) the work of external forces
    - For a conservative formulation the equality should be obtained
    - Proof
      - Let us assume constrained displacements (on $\partial_D B$) equal to zero, so $\boldsymbol{u}_h \in X_c^k$
      - The energy norm (twice the internal energy) : $\mathbf{H}_c^1(B_{0h}) \rightarrow \mathbb{R}^+$

        is defined as $|||\boldsymbol{u}|||^2 = \sum_e \left\| \sqrt{\mathcal{H}} : \boldsymbol{\nabla} \otimes \boldsymbol{u} \right\|^2_{\mathbf{L}^2(\Omega^e)}$
        - » This is a norm as it is equal to zero only if $\boldsymbol{u}$=0 on B_{0h} (in $\mathrm{H}_c^1(B_{0h})$)
      - Considering $\boldsymbol{u}_h \in X_c^k \subset \mathbf{H}_c^1(B_{0h})$ a particular choice for $\delta\boldsymbol{u}_h \in X_c^k$

$$W_{\text{ext}} = \int_{B_h} \boldsymbol{b} \cdot \boldsymbol{u}_h + \int_{\partial_N B_h} \boldsymbol{u}_h \cdot \bar{\boldsymbol{T}} dS = b(\boldsymbol{u}_h)$$

      - The bilinear form allows to write

$$W_{\text{ext}} = b(\boldsymbol{u}_h) = a(\boldsymbol{u}_h, \boldsymbol{u}_h) = |||\boldsymbol{u}_h|||^2$$

Finite element discretization of the weak form

- Does the FE weak form converge toward the exact solution (3)?
  - Third property: what is the convergence rate toward the exact solution?
    - Some preliminary results
      - Energy norm $|||\boldsymbol{u}|||^2 = \sum_e \left\| \sqrt{\mathcal{H}} : \boldsymbol{\nabla} \otimes \boldsymbol{u} \right\|^2_{\mathbf{L}^2(\Omega^e)}$: $\mathbf{H}^1_c\left(B_{0h}\right) \rightarrow \mathbb{R}^+$
      - Upper bound of the bilinear form in $\mathbf{H}^1_c \times \mathbf{H}^1_c \rightarrow \mathbb{R}$

$$|a\left(\boldsymbol{u},\, \boldsymbol{v}\right)| = \left| \sum_e \int_{\Omega^e} \frac{1}{2}\left(\boldsymbol{\nabla}\otimes\boldsymbol{v} + \boldsymbol{v}\otimes\boldsymbol{\nabla}\right) : \mathcal{H} : \frac{1}{2}\left(\boldsymbol{\nabla}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{\nabla}\right) dV \right|$$

$$\Longrightarrow |a\left(\boldsymbol{u},\, \boldsymbol{v}\right)| \leq \sum_e \left| \int_{\Omega^e} \frac{1}{2}\left(\boldsymbol{\nabla}\otimes\boldsymbol{v} + \boldsymbol{v}\otimes\boldsymbol{\nabla}\right) : \mathcal{H} : \frac{1}{2}\left(\boldsymbol{\nabla}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{\nabla}\right) dV \right|$$

$$\Longrightarrow |a\left(\boldsymbol{u},\, \boldsymbol{v}\right)| \leq \sum_e \left\| \sqrt{\mathcal{H}} : \boldsymbol{\nabla}\otimes\boldsymbol{u} \right\|_{\mathbf{L}^2(\Omega^e)} \left\| \sqrt{\mathcal{H}} : \boldsymbol{\nabla}\otimes\boldsymbol{v} \right\|_{\mathbf{L}^2(\Omega^e)}$$

» Using the Cauchy-Swartz inequality

$$|a\left(\boldsymbol{u},\, \boldsymbol{v}\right)| \leq \sqrt{\sum_e \left\| \sqrt{\mathcal{H}} : \boldsymbol{\nabla}\otimes\boldsymbol{u} \right\|^2_{\mathbf{L}^2(\Omega^e)} \sum_{e'} \left\| \sqrt{\mathcal{H}} : \boldsymbol{\nabla}\otimes\boldsymbol{v} \right\|^2_{\mathbf{L}^2(\Omega^{e'})}}$$

$$\Longrightarrow |a\left(\boldsymbol{u},\, \boldsymbol{v}\right)| \leq |||\boldsymbol{u}|||\; |||\boldsymbol{v}|||$$

Finite element discretization of the weak form

- Does the FE weak form converge toward the exact solution (4)?
  - Third property: what is the convergence rate toward the exact solution (2)?
    - Some preliminary results (2)
      - Orthogonality relation
        - » By linearity: $a\left(\boldsymbol{u}_h - \boldsymbol{u}^{\text{exact}},\, \boldsymbol{v}_h\right) = a\left(\boldsymbol{u}_h,\, \boldsymbol{v}_h\right) - a\left(\boldsymbol{u}^{\text{exact}},\, \boldsymbol{v}_h\right)$
        - » Using consistency & weak form statement leads to

$$a\left(\boldsymbol{u}_h - \boldsymbol{u}^{\text{exact}},\, \boldsymbol{v}_h\right) = b\left(\boldsymbol{v}_h\right) - b\left(\boldsymbol{v}_h\right) = 0 \;\; \forall \boldsymbol{u}_h,\, \boldsymbol{v}_h \in X_c^k$$

      - Interpolation $\boldsymbol{u}^k \in X_c^k$ of the exact solution $\boldsymbol{u}^{\text{exact}}\left(\boldsymbol{X}\right) \in \mathbf{H}_c^2\left(B_0\right)$ in the FE representation is defined such that

$$\int_{B_h} \left(\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k\right) \cdot \boldsymbol{v}\, dV = 0 \;\; \forall \boldsymbol{v} \in X_c^k$$

      - Interpolation theory: for $\boldsymbol{u}^k \in \mathbb{P}^k\left(\Omega_0^e\right)$ interpolating $\boldsymbol{u}^{\text{exact}} \in \mathbf{H}^{k+1}\left(\Omega_0^e\right)$ it can be shown* that $\left\|\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k\right\|_{\mathbf{H}^q\left(\Omega_0^e\right)} \leq C h_e^{k+1-q} \left|\boldsymbol{u}^{\text{exact}}\right|_{\mathbf{H}^{k+1}\left(\Omega_0^e\right)}$ with C independent on the element size h_e and $\forall\, 0 \leq q \leq k+1$

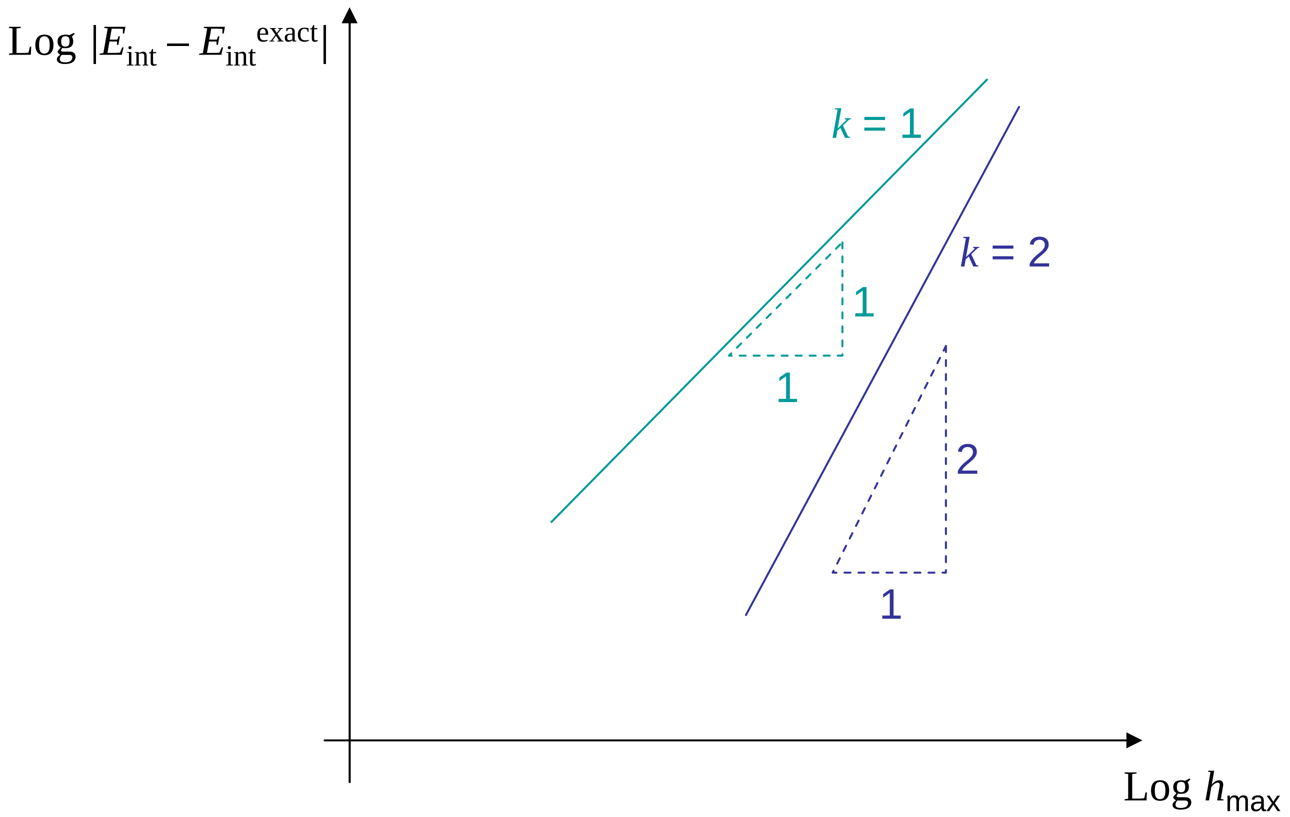
*Ciarlet PG. The Finite Element Method for Elliptic Problems. North-Holland: Amsterdam, 1978, ISBN 0 4448 5028 7

- Does the FE weak form converge toward the exact solution (5)?
  - Third property: what is the convergence rate toward the exact solution (3)?
    - Some preliminary results: Summary
      - Energy norm $|||\boldsymbol{u}|||^2 = \sum_e \left\| \sqrt{\mathcal{H}} : \boldsymbol{\nabla} \otimes \boldsymbol{u} \right\|^2_{\mathbf{L}^2(\Omega^e)}$: $\mathbf{H}^1_c(B_{0h}) \rightarrow \mathbb{R}^+$
      - Upper bound of the bilinear form in $\mathbf{H}^1_c \times \mathbf{H}^1_c \rightarrow \mathbb{R}$: $|a(\boldsymbol{u},\, \boldsymbol{v})| \leq |||\boldsymbol{u}|||\; |||\boldsymbol{v}|||$
      - Orthogonality $a\left(\boldsymbol{u}_h - \boldsymbol{u}^{\text{exact}},\, \boldsymbol{v}_h\right) = 0 \;\; \forall \boldsymbol{u}_h,\, \boldsymbol{v}_h \in X^k_c$
      - Interpolation theory: for $\boldsymbol{u}^k \in \mathbb{P}^k(\Omega^e_0)$ interpolating $\boldsymbol{u}^{\text{exact}} \in \mathbf{H}^{k+1}(\Omega^e_0)$

        one has $\left\| \boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k \right\|_{\mathbf{H}^q(\Omega^e_0)} \leq C h_e^{k+1-q} \left| \boldsymbol{u}^{\text{exact}} \right|_{\mathbf{H}^{k+1}(\Omega^e_0)}$, $q \leq k+1$
    - If the method converges
      - The interpolated error $\boldsymbol{e}^k = \boldsymbol{u}_h - \boldsymbol{u}^k$ should converge toward zero
      - With an optimal rate with the mesh size
    - Analyzing $|||\boldsymbol{e}^k|||^2 = a\left(\boldsymbol{u}_h - \boldsymbol{u}^k,\, \boldsymbol{u}_h - \boldsymbol{u}^k\right)$ & using the preliminary results
      - Annex III: $\Longrightarrow$ $|||\boldsymbol{e}^k||| \leq C h^k_{\max} \left| \boldsymbol{u}^{\text{exact}} \right|_{\mathbf{H}^{k+1}(B_h)}$

- Does the FE weak form converge toward the exact solution (6)?
  - Third property: what is the convergence rate toward the exact solution (4)?
    - Convergence rate in the energy norm
      - With respect to the mesh size
      - Is equal to the polynomial order

$$\left|\left|\left|\boldsymbol{e}^k\right|\right|\right| \leq C h_{\max}^k \left|\boldsymbol{u}^{\text{exact}}\right|_{\mathbf{H}^{k+1}(B_h)}$$



Finite element discretization of the weak form

- Does the FE weak form converge toward the exact solution (7)?
  - Fourth property: what is the convergence rate of the displacement?
    - For the problem under consideration
      - $a\left(\boldsymbol{u}_h,\, \delta\boldsymbol{u}_h\right) = b\left(\delta\boldsymbol{u}_h\right) \quad \forall \delta\boldsymbol{u}_h \in X_c^k$
      - With $b\left(\boldsymbol{v}_h\right) = \sum_e \int_{\partial_N \Omega^e} \bar{\boldsymbol{T}} \cdot \boldsymbol{v}_h \, dS + \sum_e \int_{\Omega^e} \boldsymbol{b} \cdot \boldsymbol{v}_h \, dV$
    - Let us consider a dual problem governed by loadings ($\bar{\boldsymbol{T}}^d$, $\boldsymbol{b}^d$), with
      - $b^d\left(\boldsymbol{v}_h\right) = \sum_e \int_{\partial_N \Omega^e} \bar{\boldsymbol{T}}^d \cdot \boldsymbol{v}_h \, dS + \sum_e \int_{\Omega^e} \boldsymbol{b}^d \cdot \boldsymbol{v}_h \, dV$
      - With $\boldsymbol{u}_h^d$ the FE solution satisfying $a\left(\boldsymbol{u}_h^d,\, \delta\boldsymbol{u}_h\right) = b^d\left(\delta\boldsymbol{u}_h\right) \quad \forall \delta\boldsymbol{u}_h \in X_c^k$
      - With $\boldsymbol{u}^{d,\,k}$ the interpolation in X_c^k of the exact solution $\boldsymbol{u}^{d,\,\mathrm{exact}} \in \mathbf{H}_c^2\left(B_0\right)$

        of the dual problem
    - Let us consider the error of the initial problem: $\boldsymbol{e} = \boldsymbol{u}_h - \boldsymbol{u}^{\mathrm{exact}}$
      - $\boldsymbol{e}$ is a possible particular choice as virtual displacement

$$\Longrightarrow b^d\left(\boldsymbol{e}\right) = a\left(\boldsymbol{u}^{d,\,\mathrm{exact}},\, \boldsymbol{e}\right)$$

- Does the FE weak form converge toward the exact solution (8)?
  - Fourth property: what is the convergence rate of the displacement (2)?
    - Starting from $b^d\left(\boldsymbol{e}\right) = a\left(\boldsymbol{u}^{d,\,\mathrm{exact}},\,\boldsymbol{e}\right)$
      - Particularize the loading of the dual problem $\boldsymbol{b}^d = \boldsymbol{e}$ & $\bar{\boldsymbol{T}} = 0$

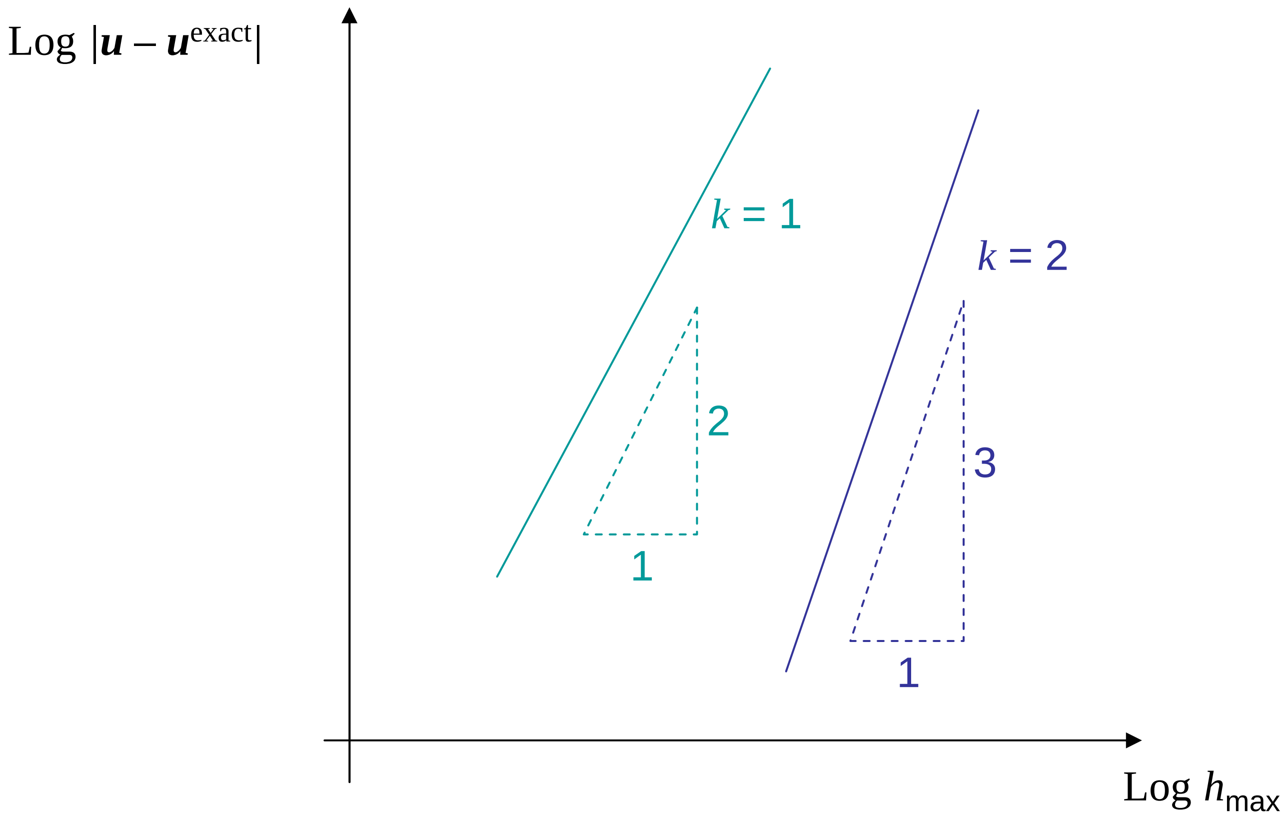
$$b^d\left(\boldsymbol{v}_h\right) = \sum_e \int_{\Omega^e} \boldsymbol{e}\cdot\boldsymbol{v}_h\,dV \Longrightarrow b^d\left(\boldsymbol{e}\right) = \left\|\boldsymbol{e}\right\|^2_{\mathbf{L}^2(B_{0h})}$$

      - Developing $b^d\left(\boldsymbol{e}\right)$, annex IV

$$\Longrightarrow \left\|\boldsymbol{e}\right\|_{\mathbf{L}^2(B_{0h})} \leq C h^{k+1}_{\max}\left|\boldsymbol{u}^{\mathrm{exact}}\right|_{\mathbf{H}^{k+1}(B_h)}$$

- Does the FE weak form converge toward the exact solution (9)?
  - Fourth property: what is the convergence rate of the displacement (3)?
    - Convergence rate in the L^2-norm
      - With respect to the mesh size
      - Is equal to the polynomial order+1

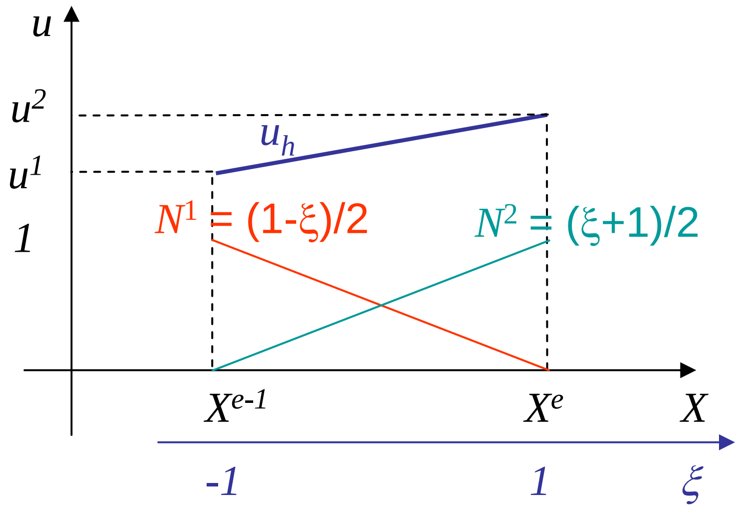
$$\|\boldsymbol{e}\|_{\mathbf{L}^2(B_{0h})} \leq C h_{\max}^{k+1} \left|\boldsymbol{u}^{\text{exact}}\right|_{\mathbf{H}^{k+1}(B_h)}$$



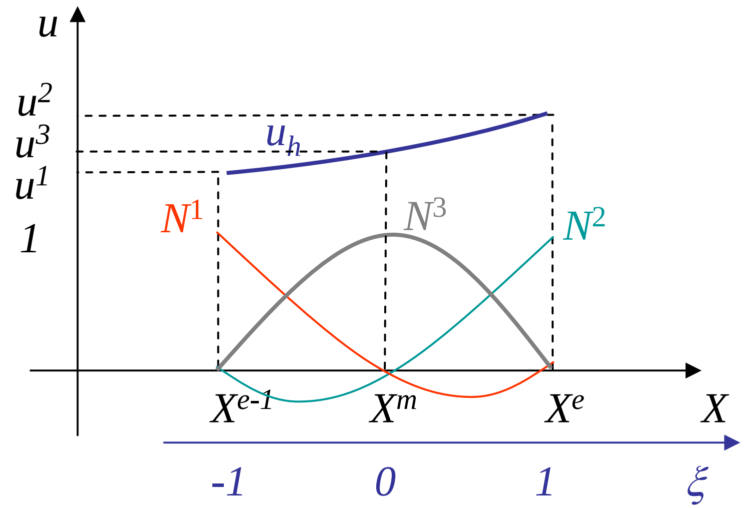
- Shape functions
  - In order to define
    - The test functions $\boldsymbol{u}_h \in X_h^k$
      with $X_h^k = \left\{ \boldsymbol{u}_h \in \mathbf{H}^1(B_{0h}) \ : \ \boldsymbol{u}_h|_{\Omega_0^e} \in \mathbb{P}^k(\Omega_0^e) \ \forall \Omega_0^e \in B_{0h} \right\} \subset \mathbf{H}^1(B_{0h})$
    - The trial functions $\delta\boldsymbol{u}_h \in X_c^k$
      with $X_c^k = \left\{ \delta\boldsymbol{u}_h \in X_h^k \ : \ \delta\boldsymbol{u}_h|_{\partial_D \Omega_0^e} = 0 \ \forall \Omega_0^e \in B_{0h} \right\} \subset \mathbf{H}_c^1(B_{0h})$
  - Polynomial shape functions $N^a(\xi)$ are defined, with
    - a the node number
    - ξ the coordinates in the element basis
  - On one element Ω^e
    - $\boldsymbol{u}_h^e(\boldsymbol{X}) = \sum_a^{n^e} N^a(\boldsymbol{\xi})\, \boldsymbol{u}^a$ & $\delta\boldsymbol{u}_h^e(\boldsymbol{X}) = \sum_a^{n^e} N^a(\boldsymbol{\xi})\, \delta\boldsymbol{u}^a$ for $\boldsymbol{X}$ in Ω^e
    - With n^e the number of nodes of the element
    - With $\boldsymbol{u}^a$ the nodal displacements at node a
      - For adequate shape functions satisfying $N^a(\boldsymbol{\xi}^b) = \delta_{ab}$
      - Where ξ^b are the coordinates of node b

- 1-D shape functions
  - On one element from $X^{e\text{-}1}$ to X^e
    - $u_h^e(X) = \sum_a^{n^e} N^a(\xi)\, u^a$ for X in $[X^{e\text{-}1}, X^e]$

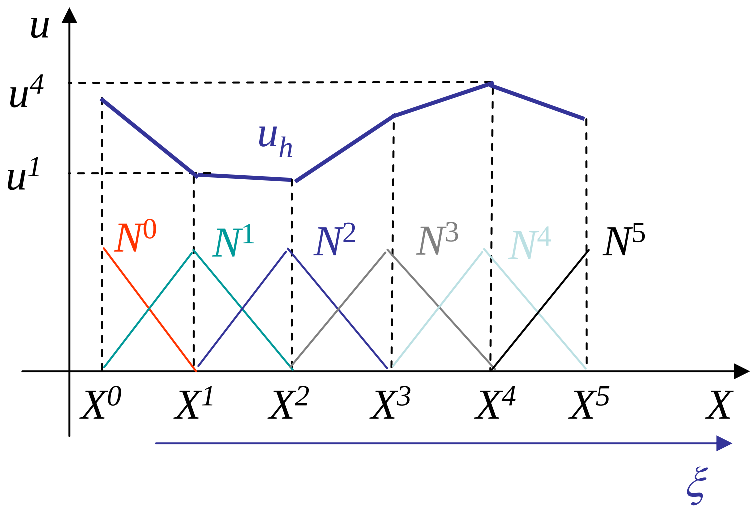
Linear approximation



Quadratic approximation



- Shape functions on the mesh
  - On Body B
    - $\boldsymbol{u}_h(\boldsymbol{X}) = \sum_a^n N^a(\boldsymbol{\xi})\,\boldsymbol{u}^a$ & $\delta\boldsymbol{u}_h(\boldsymbol{X}) = \sum_a^n N^a(\boldsymbol{\xi})\,\delta\boldsymbol{u}^a$ for $\boldsymbol{X}$ in B
    - With n the number of nodes of the mesh
    - With $\boldsymbol{u}^a$ the nodal displacements at node a
      - For adequate shape functions satisfying $N^a(\boldsymbol{\xi}^b) = \delta_{ab}$
      - Where ξ^b are the coordinates of node b
  - 1-D linear example



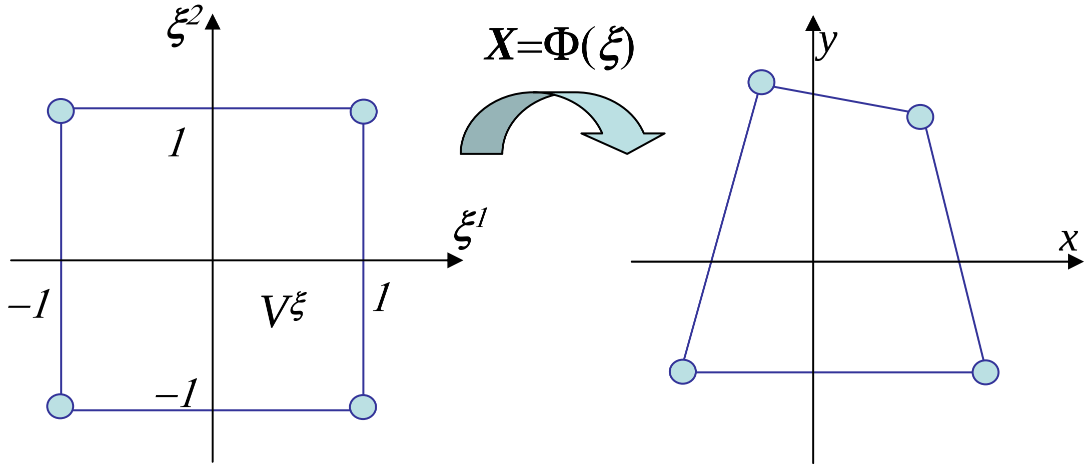
- Finite-element equations
  - FE formulation of the problem can be stated as finding $\boldsymbol{u}_h \in X_h^k$ such that $a\left(\boldsymbol{u}_h,\, \delta\boldsymbol{u}_h\right) = b\left(\delta\boldsymbol{u}_h\right) \quad \forall \delta\boldsymbol{u}_h \in X_c^k$,with
    - $$\begin{cases} a\left(\boldsymbol{u}_h,\, \boldsymbol{v}_h\right) = \sum_e \int_{\Omega^e} \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \boldsymbol{v}_h + \boldsymbol{v}_h \otimes \boldsymbol{\nabla}\right) : \mathcal{H} : \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \boldsymbol{u}_h + \boldsymbol{u}_h \otimes \boldsymbol{\nabla}\right) dV \\ b\left(\boldsymbol{v}_h\right) = \sum_e \int_{\partial_N \Omega^e} \bar{\boldsymbol{T}} \cdot \boldsymbol{v}_h \, dS + \sum_e \int_{\Omega^e} \boldsymbol{b} \cdot \boldsymbol{v}_h \, dV \end{cases}$$
  - Using $\boldsymbol{u}_h\left(\boldsymbol{X}\right) = \sum_a^n N^a\left(\boldsymbol{\xi}\right)\boldsymbol{u}^a$ & $\delta\boldsymbol{u}_h\left(\boldsymbol{X}\right) = \sum_a^n N^a\left(\boldsymbol{\xi}\right)\delta\boldsymbol{u}^a$ for $\boldsymbol{X}$ in B
  - This is restated as finding $\boldsymbol{u}^a$ in $\mathbb{R}^{3n}$ such that
    - $a\left(\boldsymbol{u}^a,\, \delta\boldsymbol{u}^b\right) = b\left(\delta\boldsymbol{u}^b\right) \quad \forall \delta\boldsymbol{u}^b \in \mathbb{R}^{3n} \ : \ \delta\boldsymbol{u}^b\big|_{\partial_D B_{0h}} = 0$
    - With $$\begin{cases} a\left(\boldsymbol{u}^a,\, \delta\boldsymbol{u}^b\right) = \left\{\sum_e \int_{\Omega^e} \boldsymbol{\nabla} N^b\left(\boldsymbol{\xi}\right) \cdot \mathcal{H} \cdot \boldsymbol{\nabla} N^a dV\right\} : \boldsymbol{u}^a \otimes \delta\boldsymbol{u}^b \\ b\left(\delta\boldsymbol{u}^b\right) = \left\{\sum_e \int_{\partial_N \Omega^e} \bar{\boldsymbol{T}} N^b\left(\boldsymbol{\xi}\right)\, dS + \sum_e \int_{\Omega^e} \boldsymbol{b} N^b\left(\boldsymbol{\xi}\right)\, dV\right\} \cdot \delta\boldsymbol{u}^b \end{cases}$$

Implementation of the finite element discretization

- Finite-element equations (2)
  - FE formulation of the problem can be stated as finding $\boldsymbol{u}^a$ in $\mathbb{R}^{3n}$ such that
    - $a\left(\boldsymbol{u}^a,\, \delta\boldsymbol{u}^b\right) = b\left(\delta\boldsymbol{u}^b\right) \quad \forall \delta\boldsymbol{u}^b \in \mathbb{R}^{3n} \;:\; \delta\boldsymbol{u}^b\big|_{\partial_D B_{0h}} = 0$
    - With
$$\begin{cases} a\left(\boldsymbol{u}^a,\, \delta\boldsymbol{u}^b\right) = \left\{\sum_e \int_{\Omega^e} \boldsymbol{\nabla} N^b\left(\boldsymbol{\xi}\right) \cdot \mathcal{H} \cdot \boldsymbol{\nabla} N^a dV\right\} : \boldsymbol{u}^a \otimes \delta\boldsymbol{u}^b \\ \text{(using symmetrical properties of } \mathcal{H}\text{)} \\ b\left(\delta\boldsymbol{u}^b\right) = \left\{\sum_e \int_{\partial_N \Omega^e} \bar{\boldsymbol{T}} N^b\left(\boldsymbol{\xi}\right)\, dS + \sum_e \int_{\Omega^e} \boldsymbol{b} N^b\left(\boldsymbol{\xi}\right)\, dV\right\} \cdot \delta\boldsymbol{u}^b \end{cases}$$
  - This can be reformulated
    - Using $a\left(\boldsymbol{u}^a,\, \delta\boldsymbol{u}^b\right) = \sum_e \mathbf{K}^{e,\,ab} : \boldsymbol{u}^a \otimes \delta\boldsymbol{u}^b$ & $b\left(\delta\boldsymbol{u}^b\right) = \sum_e \boldsymbol{f}_{\text{ext}}^{e,\,b} \cdot \delta\boldsymbol{u}^b$
    - As $\delta\boldsymbol{u}_b$ is arbitrary, except for the n' values constrained on $\partial_D B$, the problem is finding $\boldsymbol{u}^a$ in $\mathbb{R}^{3n-n'}$ such that $\sum_e \mathbf{K}^{e,\,ab} \boldsymbol{u}^a = \sum_e \boldsymbol{f}_{\text{ext}}^{e,\,b}$
    - Remarks:
      - This corresponds to solving a system of $3n$-n' equations with $3n$-n' unknowns
      - n' should be large enough so the system is not singular

Implementation of the finite element discretization

- Elementary stiffness matrix
  - For one element, the second-order tensor related to nodes a and b reads
    - $\mathbf{K}^{e,\,ab} = \int_{\Omega^e} \boldsymbol{\nabla} N^b (\boldsymbol{\xi}) \cdot \mathcal{H} \cdot \boldsymbol{\nabla} N^a dV$
  - Curvilinear coordinates
    - Element and shape functions are defined in the ξ-space



    - Changing frame using mapping $\boldsymbol{X} = \boldsymbol{\Phi}(\boldsymbol{\xi})$ of Jacobian determinant J

$$\Rightarrow \begin{cases} \boldsymbol{\nabla} N^a = \dfrac{\partial N^a(\boldsymbol{\xi})}{\partial \boldsymbol{X}} = \dfrac{\partial N^a(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \cdot \dfrac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{X}} = \boldsymbol{\nabla}_{\boldsymbol{\xi}} N^a \cdot \left(\boldsymbol{\nabla}_{\boldsymbol{\xi}} \otimes \boldsymbol{\Phi}\right)^{-1} \\ dV = \left|\boldsymbol{\nabla}_{\boldsymbol{\xi}} \otimes \boldsymbol{\Phi}\right| dV^{\xi} = J dV^{\xi} \end{cases}$$

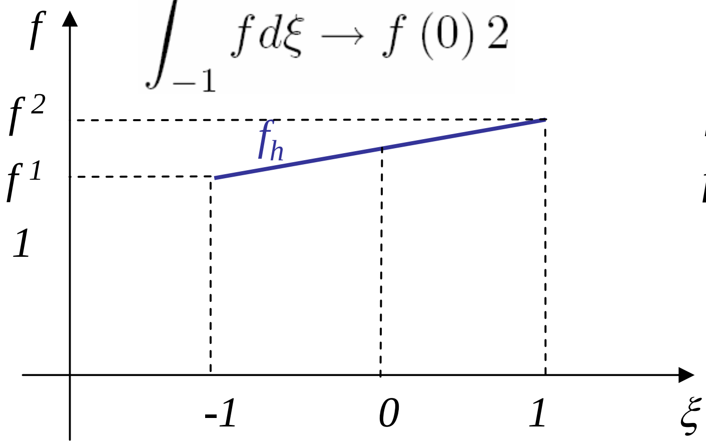
- Elementary stiffness matrix (2)
  - Elementary second-order tensor related to nodes a and b
    - $\mathbf{K}^{e,\,ab} = \int_{\Omega^e} \boldsymbol{\nabla} N^b\left(\boldsymbol{\xi}\right) \cdot \mathcal{H} \cdot \boldsymbol{\nabla} N^a dV$
    - Curvilinear coordinates $\boldsymbol{X} = \boldsymbol{\Phi}\left(\boldsymbol{\xi}\right)$

$$\Rightarrow \left\{ \begin{array}{l} \boldsymbol{\nabla} N^a = \dfrac{\partial N^a\left(\boldsymbol{\xi}\right)}{\partial \boldsymbol{X}} = \dfrac{\partial N^a\left(\boldsymbol{\xi}\right)}{\partial \boldsymbol{\xi}} \cdot \dfrac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{X}} = \boldsymbol{\nabla}_{\boldsymbol{\xi}} N^a \cdot \left(\boldsymbol{\nabla}_{\boldsymbol{\xi}} \otimes \boldsymbol{\Phi}\right)^{-1} \\ dV = \left|\boldsymbol{\nabla}_{\boldsymbol{\xi}} \otimes \boldsymbol{\Phi}\right| dV^{\xi} = J dV^{\xi} \end{array} \right.$$

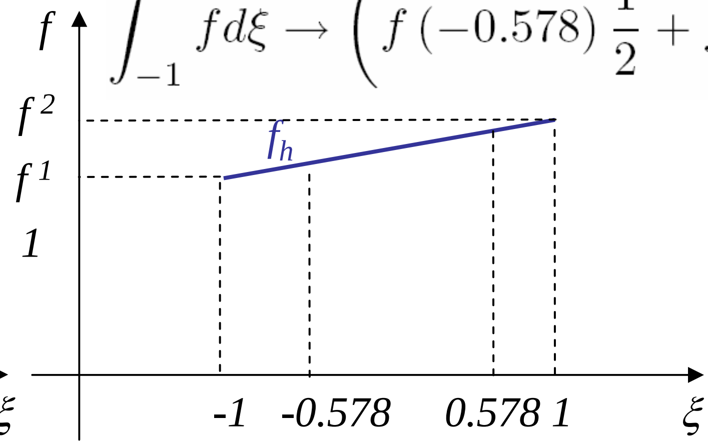
    - $\mathbf{K}^{e,\,ab} = \int_{V^{\xi}} \left(\boldsymbol{\nabla}_{\boldsymbol{\xi}} N^b \cdot \left(\boldsymbol{\nabla}_{\boldsymbol{\xi}} \otimes \boldsymbol{\Phi}\right)^{-1}\right) \cdot \mathcal{H} \cdot \left(\boldsymbol{\nabla}_{\boldsymbol{\xi}} N^a \cdot \left(\boldsymbol{\nabla}_{\boldsymbol{\xi}} \otimes \boldsymbol{\Phi}\right)^{-1}\right) J dV^{\xi}$
  - Integrating this term is not always possible in closed form
    - Complex element shapes
    - For non-linear elements, the expression is more complex
  - This integration is performed considering evaluation points
    - Gauss points
    - Lobatto
    - ……..

- Gauss integration
  - 1-D example

$$\int_{-1}^{1} f d\xi \rightarrow f(0)\, 2$$



$$\int_{-1}^{1} f d\xi \rightarrow \left(f(-0.578)\, \frac{1}{2} + f(0.578)\, \frac{1}{2} \right) 2$$

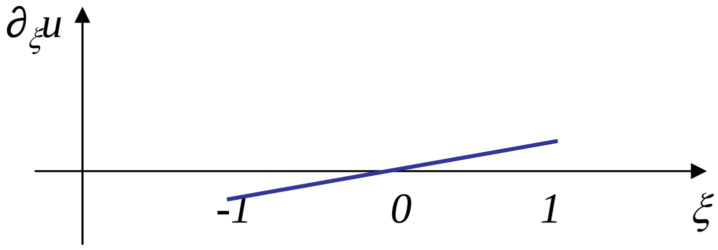


- Depending on the element shape, dimension, polynomial approximation, … there is an optimal number of Gauss points n_{pg} to capture the field

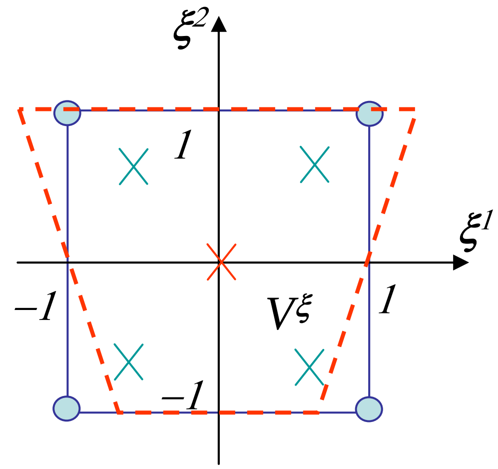
$$\int_{V^\xi} f dV^\xi \rightarrow \sum_{n=0}^{n_{pg}} f(\xi^n)\, w^n V^\xi$$

  - ξ^n is the location of the n^{th} Gauss point
  - w^n is the weight of the n^{th} Gauss point
- What is the optimal number?
  - Too many: computational cost
  - Not enough: hourglass modes

- Gauss integration (2)
  - Hourglass modes
    - Correspond to deformation modes leading to a zero-internal energy
    - 1D-example
      - Assume
        - » Linear strain approximation
        - » Antisymmetrical deformation
      - For one Gauss-point the stiffness matrix will be equal to zero
      - So two Gauss points are required



    - 2D-example
      - Assume linear square element
      - With a single Gauss-point integration
      - For some deformation modes
        - » Deformation gradient at the center is zero
        - » Zero internal energy (zero stiffness)
      - At least 4 Gauss points are required
        - » Or hourglass control

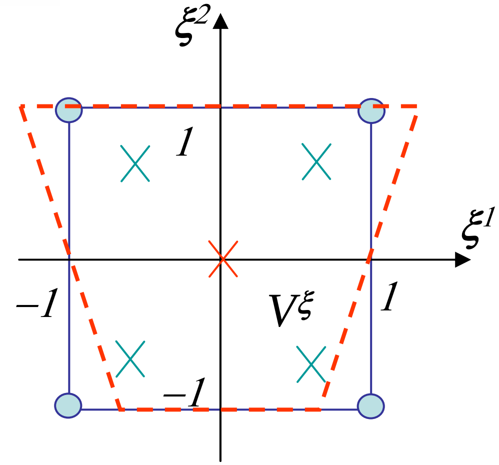


- Gauss integration (3)
  - Locking
    - Elementary stiffness matrix reads

$$\mathbf{K}^{e,\,ab} = \sum_{n=0}^{n_{pg}} \left(\boldsymbol{\nabla}_{\boldsymbol{\xi}} N^b\left(\boldsymbol{\xi}^n\right) \cdot \left(\boldsymbol{\nabla}_{\boldsymbol{\xi}} \otimes \boldsymbol{\Phi}\right)^{-1}\left(\boldsymbol{\xi}^n\right)\right) \cdot \mathcal{H} \cdot \left(\boldsymbol{\nabla}_{\boldsymbol{\xi}} N^a\left(\boldsymbol{\xi}^n\right) \cdot \left(\boldsymbol{\nabla}_{\boldsymbol{\xi}} \otimes \boldsymbol{\Phi}\right)^{-1}\left(\boldsymbol{\xi}^n\right)\right) J\left(\boldsymbol{\xi}^n\right) w^n V^{\xi}$$

    - 2D-example
      - Assume linear square element
      - At least 4 Gauss points are required to avoid hourglass modes
      - If a constrain is added to the system
        - » Incompressibility (rubber, plasticity)

        there are more equations than unknowns and the solution of the system is zero deformation



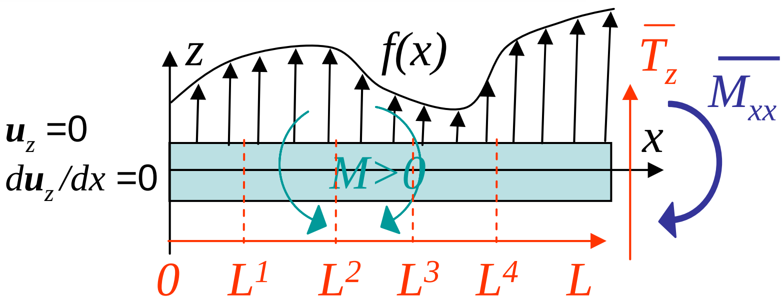
    - Solutions
      - Linear element with 1 Gauss point & hourglass control
      - Linear element with Selected Reduced Integration
      - Higher polynomial approximation
      - Internal degrees of freedom (Enhanced Assumed Strain elements)

- FE weak form for beams
  - The problem is finding $\boldsymbol{u}_z \in \mathrm{H}^2\left(]0\ L[\right)$ such that

$$a\left(\boldsymbol{u}_z,\, \delta\boldsymbol{u}_z\right) = b\left(\delta\boldsymbol{u}_z\right) \quad \forall \delta\boldsymbol{u}_z \in \mathrm{H}^2_c\left(]0\ L[\right) \subset \mathrm{H}^2\left(]0\ L[\right)$$

    - With
$$\begin{cases} a\left(\boldsymbol{u}_z,\, \boldsymbol{v}_z\right) = \displaystyle\int_0^L EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\frac{\partial^2 \boldsymbol{v}_z}{\partial x^2}\, dx \\ b\left(\boldsymbol{v}_z\right) = \displaystyle\int_0^L f\left(x\right) \boldsymbol{v}_z\, dx + n_x \bar{T}_z \boldsymbol{v}_z\Big|_{\partial_N L} + n_x \bar{M}_{xx} \frac{\partial\left(-\boldsymbol{v}_z\right)}{\partial x}\Bigg|_{\partial_M L} \end{cases}$$

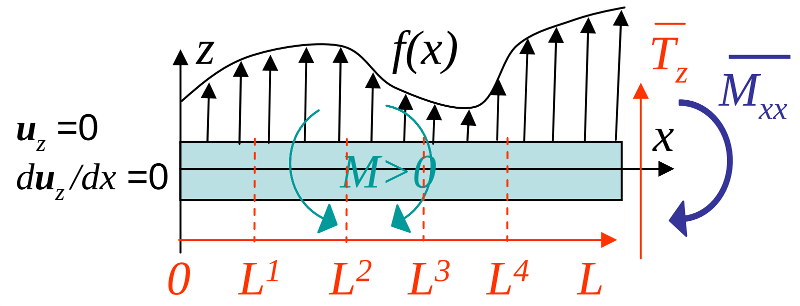
  - Finite element approximation
    - Let us try to write a displacement FE formulation, with



$$\boldsymbol{u}_{zh} \in U_h^k = \left\{ \boldsymbol{u}_{zh} \in \mathbf{H}^2\left(]0\ L[\right) \ : \right.$$
$$\left. \boldsymbol{u}_{zh}\big|_{]L^{e-1}\ L^e[} \in \mathbb{P}^k\left(]L^{e-1}\ L^e[\right)\ \forall]L^{e-1}\ L^e[\right\} \subset \mathbf{H}^2\left(]0\ L[\right)$$

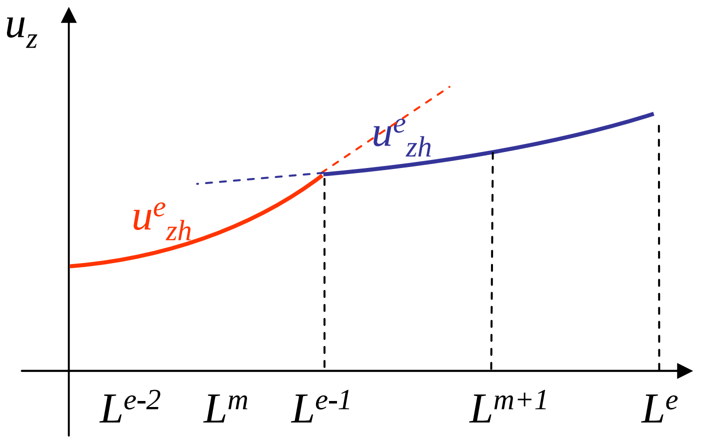
    - Is it possible?

- FE weak form for beams (2)
  - Finite element approximation (2)
    - Let us try to write a displacement FE formulation, with



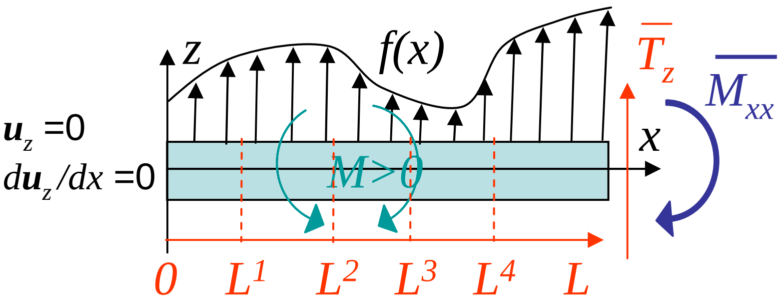
$$\boldsymbol{u}_{zh} \in U_h^k = \Big\{ \boldsymbol{u}_{zh} \in \mathbf{H}^2\left(]0\ L[\right)\ :$$
$$\boldsymbol{u}_{zh}\big|_{]L^{e-1}\ L^e[} \in \mathbb{P}^k\left(]L^{e-1}\ L^e[\right)\ \forall]L^{e-1}\ L^e[\Big\} \subset \mathbf{H}^2\left(]0\ L[\right)$$

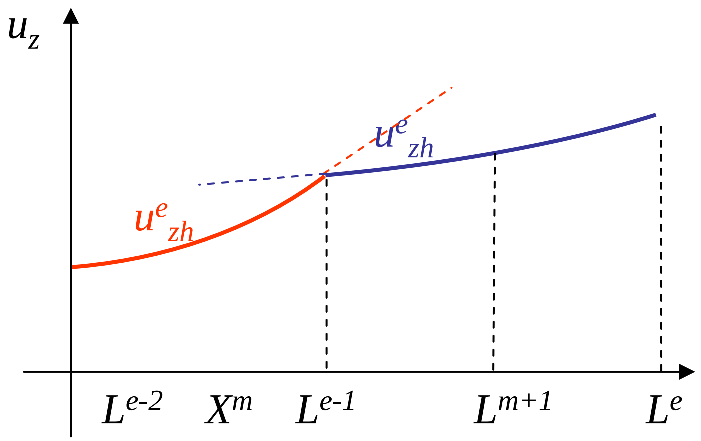
    - The problem is that, even for quadratic shape functions in each element, the $\mathbf{H}^2$ condition is not ensured at inter-element boundaries



    - Solution?

- FE weak form for beams (3)
  - Solutions
    - Special shape functions $C^1]0\ L[$
      - Shell in 3D ?
    - Considering formulation with displacement & rotation degrees of freedom
      - Requires shearing
      - More degrees of freedom





Limitations of the FE methods

- High order equations
  - C^1 is difficult to enforce strongly
  - Solutions
    - Discontinuous Galerkin methods
    - Meshless methods
- Mesh compatibility
  - Problem of mesh definition: crack propagation, at material boundaries
  - Solutions
    - XFEM
    - Smooth Particles Hydrodynamics
- Mesh deformation
  - For large deformations
  - Solution
    - Smooth Particles Hydrodynamics
- …

- References
  - Books

    - Finite elements: theory, fast solvers, and applications in solid mechanics, Dietrich Braess, Cambridge Press, 2001, ISBN 0521 011957
    - The Finite Element Method: Its Basis and Fundamentals, O.C. Zienkiewicz, R.L. Taylor and J.Z. Zhu, Elsevier Butterworth-Heinemann, 2005, ISBN 0 7506 6320 0
    - The Finite Element Method for Elliptic Problems, P.G. Ciarlet. North-Holland: Amsterdam, 1978, ISBN 0 4448 5028 7

- Two-field functional for beams
  - Extremum with respect to M_{xx}
    - $$I'\left(\boldsymbol{u}_z,\ M_{xx};\ \delta M_{xx}\right) = \int_0^L \delta M_{xx}\left[\frac{\partial^2\left(-\boldsymbol{u}_z\right)}{\partial x^2} - \partial_{M_{xx}} U\left(M_{xx}\right)\right] dx - n_x \delta M_{xx}\left[\frac{\partial\left(-\boldsymbol{u}_z\right)}{\partial x} - \bar{\theta}_y\right]\Bigg|_{\partial_T L} + n_x \partial_x \delta M_{xx}\left(\boldsymbol{u}_z - \bar{\boldsymbol{u}}_z\right)\big|_{\partial_U L}$$
    - $$\forall \delta M_{xx} \in \mathrm{H}^0_M\left(]0\ L[\right) = \left\{\delta M_{xx} \in \mathrm{H}^0\left(]0\ L[\right)\ :\ \delta M_{xx}|_{\partial_M L} = 0\right\}$$
    - Due to the arbitrary nature of δM_{xx}
      - $\frac{\partial^2\left(-\boldsymbol{u}_z\right)}{\partial x^2} = \partial_{M_{xx}} U\left(M_{xx},\ T_z\right) = \frac{M_{xx}}{EI}$ on $]0\ L[$, satisfying bending law
      - $\frac{\partial\left(-\boldsymbol{u}_z\right)}{\partial x} = \bar{\theta}_y$ on $\partial_T L$, satisfying constrained rotations
      - $\boldsymbol{u}_z = \bar{\boldsymbol{u}}_z$ on $\partial_U L$, satisfying constrained displacements

- Two-field functional for beams (2)
  - Extremum with respect to $\boldsymbol{u}_z$
    - $$I'\left(\boldsymbol{u}_z,\, M_{xx};\, \delta\boldsymbol{u}_z\right) = \int_0^L M_{xx}\frac{\partial^2\left(-\delta\boldsymbol{u}_z\right)}{\partial x^2}\, dx - \int_0^L f\left(x\right)\delta\boldsymbol{u}_z\, dx - n_x\bar{T}_z\delta\boldsymbol{u}_z\big|_{\partial_N L} - n_x\bar{M}_{xx}\frac{\partial\left(-\delta\boldsymbol{u}_z\right)}{\partial x}\bigg|_{\partial_M L} - n_x M_{xx}\frac{\partial\left(-\delta\boldsymbol{u}_z\right)}{\partial x}\bigg|_{\partial_T L} + n_x\partial_x M_{xx}\delta\boldsymbol{u}_z\big|_{\partial_U L}$$
    - $$\delta\boldsymbol{u}_z \in \mathrm{H}_c^2\left(\left]0\ L\right[\right) = \left\{\delta\boldsymbol{u}_z \in \mathrm{H}^2\left(\left]0\ L\right[\right)\ :\ \delta\boldsymbol{u}_z\big|_{\partial_U L} = \partial_x\delta\boldsymbol{u}_z\big|_{\partial_T L} = 0\right\}$$
    - Integration by parts

$$I'\left(\boldsymbol{u}_z,\, M_{xx};\, \delta\boldsymbol{u}_z\right) = \int_0^L -\frac{\partial^2 M_{xx}}{\partial x^2}\delta\boldsymbol{u}_z\, dx - \int_0^L f\left(x\right)\delta\boldsymbol{u}_z\, dx - n_x\left[\bar{T}_z - \frac{\partial M_{xx}}{\partial x}\right]\delta\boldsymbol{u}_z\bigg|_{\partial_N L} - n_x\left[\bar{M}_{xx} - M_{xx}\right]\frac{\partial\left(-\delta\boldsymbol{u}_z\right)}{\partial x}\bigg|_{\partial_M L} = 0$$

- Two-field functional for beams (3)
  - Extremum with respect to $\boldsymbol{u}_z$ (2)
    - $$I'\left(\boldsymbol{u}_z,\, M_{xx};\, \delta\boldsymbol{u}_z\right) = \int_0^L -\frac{\partial^2 M_{xx}}{\partial x^2}\delta\boldsymbol{u}_z\, dx - \int_0^L f\left(x\right)\delta\boldsymbol{u}_z\, dx - n_x\left[\bar{T}_z - \frac{\partial M_{xx}}{\partial x}\right]\delta\boldsymbol{u}_z\Big|_{\partial_N L} - n_x\left[\bar{M}_{xx} - M_{xx}\right]\frac{\partial\left(-\delta\boldsymbol{u}_z\right)}{\partial x}\Big|_{\partial_M L} = 0$$
    - With $\dfrac{\partial^2\left(-\boldsymbol{u}_z\right)}{\partial x^2} = \partial_{M_{xx}} U\left(M_{xx},\, T_z\right) = \dfrac{M_{xx}}{EI}$
    - Due to the arbitrary nature of $\delta\boldsymbol{u}_z$
      - $\dfrac{\partial^2}{\partial x^2}\left(EI\dfrac{\partial^2\boldsymbol{u}_z}{\partial x^2}\right) = f\left(x\right)$ on $]0\ L[$, satisfying linear momentum
      - $-\dfrac{\partial}{\partial x}\left(EI\dfrac{\partial^2\boldsymbol{u}_z}{\partial x^2}\right) = \bar{T}_z$ on $\partial_N L$, satisfying shear loading BC
      - $-EI\dfrac{\partial^2\boldsymbol{u}_z}{\partial x^2} = \bar{M}_{xx}$ on $\partial_M L$, satisfying momentum loading BC

- Three-field functional for beams
  - Extremum with respect to M_{xx}
    - $$I'\left(\boldsymbol{u}_z,\ M_{xx},\ \kappa;\ \delta M_{xx}\right) = \int_0^L -\delta M_{xx}\left(\kappa + \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\,dx - n_x \delta M_{xx}\left[\frac{\partial\left(-\boldsymbol{u}_z\right)}{\partial x} - \bar{\theta}_y\right]\Bigg|_{\partial_T L} + n_x \partial_x \delta M_{xx}\left(\boldsymbol{u}_z - \bar{\boldsymbol{u}}_z\right)\big|_{\partial_U L} = 0$$
    - $$\forall \delta M_{xx} \in \mathrm{H}^0_M\left(]0\ L[\right) \ = \ \left\{\delta M_{xx} \in \mathrm{H}^0\left(]0\ L[\right)\ :\ \delta M_{xx}|_{\partial_M L} = 0\right\}$$
    - Due to the arbitrary nature of δM_{xx}
      - $\kappa = -\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2}$ on $]0\ L[$, satisfying bending compatibility
      - $\bar{\theta}_y = \dfrac{\partial\left(-\boldsymbol{u}_z\right)}{\partial x}$ on $\partial_T L$, satisfying constrained rotations
      - $\boldsymbol{u}_z = \bar{\boldsymbol{u}}_z$ on $\partial_N L$, satisfying constrained displacements

- Three-field functional for beams (2)
  - Extremum with respect to κ
    - $I'(\boldsymbol{u}_z,\, M_{xx},\, \kappa;\, \delta\kappa) = \int_0^L [\partial_\kappa U(\kappa) - M_{xx}]\, \delta\kappa\, dx = 0$
    - $\forall \delta\kappa \in \mathrm{H}^0_M(]0\ L[) \ = \ \left\{\delta\kappa \in \mathrm{H}^0(]0\ L[) \ : \ \delta\kappa|_{\partial_M L} = 0\right\}$
    - Due to the arbitrary nature of $\delta\kappa$
      - $M_{xx} = \partial_\kappa U(\kappa) = EI\kappa$ on]0 L[, satisfying bending law

- Three-field functional for beams (3)
  - Extremum with respect to $\boldsymbol{u}_z$
    - $$I'\left(\boldsymbol{u}_z,\, M_{xx},\, \kappa;\, \delta\boldsymbol{u}_z\right) = \int_0^L \left[-M_{xx}\frac{\partial^2 \delta\boldsymbol{u}_z}{\partial x^2}\right]\, dx - \int_0^L f\left(x\right)\delta\boldsymbol{u}_z\, dx - n_x\bar{T}_z\delta\boldsymbol{u}_z\big|_{\partial_N L} - n_x\bar{M}_{xx}\frac{\partial\left(-\delta\boldsymbol{u}_z\right)}{\partial x}\bigg|_{\partial_M L} + n_x\partial_x M_{xx}\delta\boldsymbol{u}_z|_{\partial_U L} - n_x M_{xx}\frac{\partial\left(-\delta\boldsymbol{u}_z\right)}{\partial x}\bigg|_{\partial_T L}$$
    - $\delta\boldsymbol{u}_z \in \mathrm{H}^2_c\left(]0\ L[\right) = \left\{\delta\boldsymbol{u}_z \in \mathrm{H}^2\left(]0\ L[\right)\ :\ \delta\boldsymbol{u}_z|_{\partial_U L} = \partial_x\delta\boldsymbol{u}_z|_{\partial_T L} = 0\right\}$
    - Integration by parts

$$I'\left(\boldsymbol{u}_z,\, M_{xx},\, \kappa;\, \delta\boldsymbol{u}_z\right) = \int_0^L \left[-\frac{\partial^2 M_{xx}}{\partial x^2}\right]\delta\boldsymbol{u}_z\, dx - \int_0^L f\left(x\right)\delta\boldsymbol{u}_z\, dx - n_x\left(\bar{T}_z + \frac{\partial M_{xx}}{\partial x}\right)\delta\boldsymbol{u}_z\bigg|_{\partial_N L} - n_x\left(\bar{M}_{xx} - M_{xx}\right)\frac{\partial\left(-\delta\boldsymbol{u}_z\right)}{\partial x}\bigg|_{\partial_M L} = 0$$

- Three-field functional for beams (4)
  - Extremum with respect to $\boldsymbol{u}_z$ (2)
    - $$I'\left(\boldsymbol{u}_z,\, M_{xx},\, \kappa;\, \delta\boldsymbol{u}_z\right) = \int_0^L \left[-\frac{\partial^2 M_{xx}}{\partial x^2}\right]\delta\boldsymbol{u}_z\, dx - \int_0^L f(x)\,\delta\boldsymbol{u}_z\, dx - n_x\left(\bar{T}_z + \frac{\partial M_{xx}}{\partial x}\right)\delta\boldsymbol{u}_z\Big|_{\partial_N L} - n_x\left(\bar{M}_{xx} - M_{xx}\right)\frac{\partial\left(-\delta\boldsymbol{u}_z\right)}{\partial x}\Big|_{\partial_M L} = 0$$
    - With
      - $M_{xx} = \partial_\kappa U(\kappa) = EI\kappa$
      - $\kappa = -\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2}$
    - Due to the arbitrary nature of $\delta\boldsymbol{u}_z$
      - $\dfrac{\partial^2}{\partial x^2}\left(EI\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f(x)$ on $]0\ L[$, satisfying linear momentum
      - $-\dfrac{\partial}{\partial x}\left(EI\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = \bar{T}_z$ on $\partial_N L$, satisfying shear loading BC
      - $-EI\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2} = \bar{M}_{xx}$ on $\partial_M L$, satisfying momentum loading BC

- Convergence rate in the energy norm
  - Starting from $\left|\left|\left|\boldsymbol{e}^k\right|\right|\right|^2 = a\left(\boldsymbol{u}_h - \boldsymbol{u}^k,\, \boldsymbol{u}_h - \boldsymbol{u}^k\right)$
    - Using linearity of a

$$\Longrightarrow \left|\left|\left|\boldsymbol{e}^k\right|\right|\right|^2 = a\left(\boldsymbol{u}_h - \boldsymbol{u}^{\text{exact}},\, \boldsymbol{u}_h - \boldsymbol{u}^k\right) + a\left(\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k,\, \boldsymbol{u}_h - \boldsymbol{u}^k\right)$$

    - Using orthogonality relation ($a\left(\boldsymbol{u}_h - \boldsymbol{u}^{\text{exact}},\, \boldsymbol{v}_h\right) = 0 \;\; \forall \boldsymbol{u}_h,\, \boldsymbol{v}_h \in X_c^k$)

$$\Longrightarrow \left|\left|\left|\boldsymbol{e}^k\right|\right|\right|^2 = a\left(\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k,\, \boldsymbol{u}_h - \boldsymbol{u}^k\right)$$

    - Using upper bound ($\left|a\left(\boldsymbol{u},\, \boldsymbol{v}\right)\right| \leq |||\boldsymbol{u}|||\;|||\boldsymbol{v}|||$)

$$\Longrightarrow \left|\left|\left|\boldsymbol{e}^k\right|\right|\right|^2 \leq \left|\left|\left|\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k\right|\right|\right| \;\left|\left|\left|\boldsymbol{u}_h - \boldsymbol{u}^k\right|\right|\right| = \left|\left|\left|\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k\right|\right|\right| \;\left|\left|\left|\boldsymbol{e}^k\right|\right|\right|$$

$$\Longrightarrow \left|\left|\left|\boldsymbol{e}^k\right|\right|\right|^2 \leq \left|\left|\left|\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k\right|\right|\right|^2 = \sum_e \left\|\sqrt{\mathcal{H}} : \boldsymbol{\nabla} \otimes \left(\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k\right)\right\|^2_{\mathbf{L}^2(\Omega^e)}$$

$$\Longrightarrow \left|\left|\left|\boldsymbol{e}^k\right|\right|\right|^2 \leq |\mathcal{H}| \sum_e \left\|\boldsymbol{\nabla} \otimes \left(\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k\right)\right\|^2_{\mathbf{L}^2(\Omega^e)}$$

- Convergence rate in the energy norm (2)
  - Starting from $\left|\left|\left|\boldsymbol{e}^k\right|\right|\right|^2 \leq |\mathcal{H}| \sum_e \left\|\boldsymbol{\nabla} \otimes \left(\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k\right)\right\|^2_{\mathbf{L}^2(\Omega^e)}$
    - Using Sobolov norm definition ($\|f\|_{\mathrm{W}^{m,\,p}([a,\,b])} = \sum_{k=0}^{m} \left\|f^{(k)}\right\|_{\mathrm{L}^p([a,\,b])}$)

      $$\Longrightarrow \left|\left|\left|\boldsymbol{e}^k\right|\right|\right|^2 \leq |\mathcal{H}| \sum_e \left\|\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k\right\|^2_{\mathbf{H}^1(\Omega^e)}$$

    - As $\boldsymbol{u}^k \in \mathbb{P}^k\left(\Omega_0^e\right)$, assuming $\boldsymbol{u}^{\text{exact}} \in \mathbf{H}^{k+1}\left(\Omega_0^e\right)$ using the interpolation theory ($\left\|\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k\right\|_{\mathbf{H}^q\left(\Omega_0^e\right)} \leq C h_e^{k+1-q} \left|\boldsymbol{u}^{\text{exact}}\right|_{\mathbf{H}^{k+1}\left(\Omega_0^e\right)}$), with $q = 1$

      $$\Longrightarrow \left|\left|\left|\boldsymbol{e}^k\right|\right|\right|^2 \leq C \sum_e h_e^{2k} \left|\boldsymbol{u}^{\text{exact}}\right|^2_{\mathbf{H}^{k+1}(\Omega^e)}$$

      $$\Longrightarrow \left|\left|\left|\boldsymbol{e}^k\right|\right|\right| \leq C h_{\max}^k \left|\boldsymbol{u}^{\text{exact}}\right|_{\mathbf{H}^{k+1}(B_h)}$$

    - Using similar argumentation for $\boldsymbol{e} = \boldsymbol{u}_h - \boldsymbol{u}^{\text{exact}} \in \mathbf{H}_c^2\left(B_0\right)$

      $$\Longrightarrow |||\boldsymbol{e}||| \leq C h_{\max}^k \left|\boldsymbol{u}^{\text{exact}}\right|_{\mathbf{H}^{k+1}(B_h)}$$

    - As $\left|\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k\right| \leq |\boldsymbol{e}| + \left|\boldsymbol{e}^k\right|$, using the 2 error estimates

      $$\Longrightarrow \left|\left|\left|\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k\right|\right|\right| \leq C h_{\max}^k \left|\boldsymbol{u}^{\text{exact}}\right|_{\mathbf{H}^{k+1}(B_h)}$$

- Convergence rate in the L^2-norm
  - Starting from $b^d\left(\boldsymbol{e}\right) = a\left(\boldsymbol{u}^{d,\,\text{exact}},\, \boldsymbol{e}\right) = a\left(\boldsymbol{u}^{d,\,\text{exact}} - \boldsymbol{u}^{d,\,k},\, \boldsymbol{e}\right) + a\left(\boldsymbol{u}^{d,\,k},\, \boldsymbol{e}\right)$
    - As a is symmetrical

$$\Longrightarrow b^d\left(\boldsymbol{e}\right) = a\left(\boldsymbol{u}^{d,\,\text{exact}} - \boldsymbol{u}^{d,\,k},\, \boldsymbol{e}\right) + a\left(\boldsymbol{u}_h - \boldsymbol{u}^{\text{exact}},\, \boldsymbol{u}^{d,\,k}\right)$$

    - Using orthogonality relation ($a\left(\boldsymbol{u}_h - \boldsymbol{u}^{\text{exact}},\, \boldsymbol{v}_h\right) = 0 \;\; \forall \boldsymbol{u}_h,\; \boldsymbol{v}_h \in X_c^k$)

$$\Longrightarrow b^d\left(\boldsymbol{e}\right) = a\left(\boldsymbol{u}^{d,\,\text{exact}} - \boldsymbol{u}^{d,\,k},\, \boldsymbol{e}\right) = a\left(\boldsymbol{u}^{d,\,\text{exact}} - \boldsymbol{u}^{d,\,k},\, \boldsymbol{u}_h - \boldsymbol{u}^{\text{exact}}\right)$$

$$\Longrightarrow b^d\left(\boldsymbol{e}\right) = a\left(\boldsymbol{u}^{d,\,\text{exact}} - \boldsymbol{u}^{d,\,k},\, \boldsymbol{u}_h - \boldsymbol{u}^{k}\right) + a\left(\boldsymbol{u}^{d,\,\text{exact}} - \boldsymbol{u}^{d,\,k},\, \boldsymbol{u}^{k} - \boldsymbol{u}^{\text{exact}}\right)$$

    - Let us particularize the loading of the dual problem $\boldsymbol{b}^d = \boldsymbol{e}$ & $\bar{\boldsymbol{T}} = 0$

$$b^d\left(\boldsymbol{v}_h\right) = \sum_e \int_{\Omega^e} \boldsymbol{e} \cdot \boldsymbol{v}_h \, dV \Longrightarrow b^d\left(\boldsymbol{e}\right) = \|\boldsymbol{e}\|^2_{\mathbf{L}^2(B_{0h})}$$

$$\Longrightarrow \|\boldsymbol{e}\|^2_{\mathbf{L}^2(B_{0h})} = a\left(\boldsymbol{u}^{d,\,\text{exact}} - \boldsymbol{u}^{d,\,k},\, \boldsymbol{u}_h - \boldsymbol{u}^{k}\right) + a\left(\boldsymbol{u}^{d,\,\text{exact}} - \boldsymbol{u}^{d,\,k},\, \boldsymbol{u}^{k} - \boldsymbol{u}^{\text{exact}}\right)$$

    - Using upper bound ($\left|a\left(\boldsymbol{u},\, \boldsymbol{v}\right)\right| \leq |||\boldsymbol{u}|||\; |||\boldsymbol{v}|||$)

$$\Longrightarrow \|\boldsymbol{e}\|^2_{\mathbf{L}^2(B_{0h})} \leq \left|\left|\left|\boldsymbol{u}^{d,\,\text{exact}} - \boldsymbol{u}^{d,\,k}\right|\right|\right| \; \left|\left|\left|\boldsymbol{e}^{k}\right|\right|\right| + \left|\left|\left|\boldsymbol{u}^{d,\,\text{exact}} - \boldsymbol{u}^{d,\,k}\right|\right|\right| \; \left|\left|\left|\boldsymbol{u}^{k} - \boldsymbol{u}^{\text{exact}}\right|\right|\right|$$

- Convergence rate in the L²-norm (2)
  - Starting from $\|\boldsymbol{e}\|^2_{\mathbf{L}^2(B_{0h})} \leq \left|\left|\left|\boldsymbol{u}^{d,\,\text{exact}} - \boldsymbol{u}^{d,\,k}\right|\right|\right| \left[\left|\left|\left|\boldsymbol{e}^k\right|\right|\right| + \left|\left|\left|\boldsymbol{u}^k - \boldsymbol{u}^{\text{exact}}\right|\right|\right|\right]$
    - Assuming the problem is elliptic with
      - $\mathbf{A}\cdot\boldsymbol{u} = \boldsymbol{b}$ in B_0

        with $\mathbf{A} \;:\; C^{\infty}(B_0) \to \mathbf{H}^{p-2m}(B_0)$ the elliptic operator
        - » m=1 in elasticity
        - » m=2 for beams
      - $\partial^i \boldsymbol{u} = 0$ on ∂B_0 $\forall\, 0 \leq i \leq m-1$
      - If the exact solution $\boldsymbol{u} \in \mathbf{H}^{2m}(B_0)$, then*

$$\|\boldsymbol{u}\|_{\mathbf{H}^p} \leq C^p \|\mathbf{A}\cdot\boldsymbol{u}\|_{\mathbf{H}^{p-2m}(B_0)} \quad \forall\, p \geq 2m$$

    - Using m=1, p=2, as $\boldsymbol{b}^d = \boldsymbol{e}$ this theorem applied to the dual problem leads to

$$\left\|\boldsymbol{u}^{d,\,\text{exact}}\right\|_{\mathbf{H}^2(B_{0h})} \leq C\left\|\boldsymbol{b}^d\right\|_{\mathbf{H}^0(B_{0h})} = C\|\boldsymbol{e}\|_{\mathbf{L}^2(B_{0h})}$$

    - Using $\left|\left|\left|\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k\right|\right|\right| \leq C h^k_{\max}\left|\boldsymbol{u}^{\text{exact}}\right|_{\mathbf{H}^{k+1}(B_h)}$ for the dual problem

$$\Longrightarrow \left|\left|\left|\boldsymbol{u}^{d,\,\text{exact}} - \boldsymbol{u}^{d,\,k}\right|\right|\right| \leq C' h_{\max}\left\|\boldsymbol{u}^{d,\,\text{exact}}\right\|_{\mathbf{H}^2(B_{0h})} \leq C h_{\max}\|\boldsymbol{e}\|_{\mathbf{L}^2(B_{h0})}$$

*J. Lions, E. Magenes, Problèmes aux limites non homogènes, Dunod, Paris, France, 1968.

- Convergence rate in the L²-norm (3)
  - Starting from $\|\boldsymbol{e}\|^2_{\mathbf{L}^2(B_{0h})} \leq \left|\left|\left|\boldsymbol{u}^{d,\,\text{exact}} - \boldsymbol{u}^{d,\,k}\right|\right|\right| \left[\left|\left|\left|\boldsymbol{e}^k\right|\right|\right| + \left|\left|\left|\boldsymbol{u}^k - \boldsymbol{u}^{\text{exact}}\right|\right|\right|\right]$
    - As $\left|\left|\left|\boldsymbol{u}^{d,\,\text{exact}} - \boldsymbol{u}^{d,\,k}\right|\right|\right| \leq C' h_{\max} \left\|\boldsymbol{u}^{d,\,\text{exact}}\right\|_{\mathbf{H}^2(B_{0h})} \leq C h_{\max} \|\boldsymbol{e}\|_{\mathbf{L}^2(B_{h0})}$

$$\Longrightarrow \|\boldsymbol{e}\|_{\mathbf{L}^2(B_{0h})} \leq C h_{\max} \left[\left|\left|\left|\boldsymbol{e}^k\right|\right|\right| + \left|\left|\left|\boldsymbol{u}^k - \boldsymbol{u}^{\text{exact}}\right|\right|\right|\right]$$

    - Using $\begin{cases} \left|\left|\left|\boldsymbol{e}^k\right|\right|\right| \leq C h^k_{\max} \left|\boldsymbol{u}^{\text{exact}}\right|_{\mathbf{H}^{k+1}(B_h)} \\ \left|\left|\left|\boldsymbol{u}^{\text{exact}} - \boldsymbol{u}^k\right|\right|\right| \leq C h^k_{\max} \left|\boldsymbol{u}^{\text{exact}}\right|_{\mathbf{H}^{k+1}(B_h)} \end{cases}$

$$\Longrightarrow \|\boldsymbol{e}\|_{\mathbf{L}^2(B_{0h})} \leq C h^{k+1}_{\max} \left|\boldsymbol{u}^{\text{exact}}\right|_{\mathbf{H}^{k+1}(B_h)}$$

*J. Lions, E. Magenes, Problèmes aux limites non homogènes, Dunod, Paris, France, 1968.