Discontinuous Galerkin Method in Fluid Dynamics

Valentin Sonneville

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Université de Liège

Outline

- Motivations
- Brief timeline
- Transport equation
- Method formulation
- Stability and Accuracy
- Issues
- Prospects and conclusions

Motivations

When it comes to solve numerically a problem...

Challenges

- Complex geometries
- Steady – unsteady
- Multiphysics
- Many scales in space and time

Properties required

- Flexibility (geometry and problem types)
- Robustness
- Efficiency
- High Order precision

Brief timeline

- 1973 : Transport equation solved by Reed and Hill
- 1974 : First analysis (error) by LeSaint and Raviart
- …. nothing …
- 1986 : Analysis of the scalar hyperbolic equation by (Johnson and Pitkäranta)
- 1989 : RK for non-linear conservation laws (Cockburn, Shu)
- 1997-1998 : convection-diffusion problems by Bassi and Rebay, Cockburn and Shu...
- 1998 : Extenstion to Hamilton-Jacobi equations by Shu
- Last decade : Applications in various areas (Mechanics, Fluid dyamics, electromagnetism, plasma...). Mostly academic cases.

CFD : Transport equation

Illustration with a 1D linear partial differential equation in fluid dynamics.

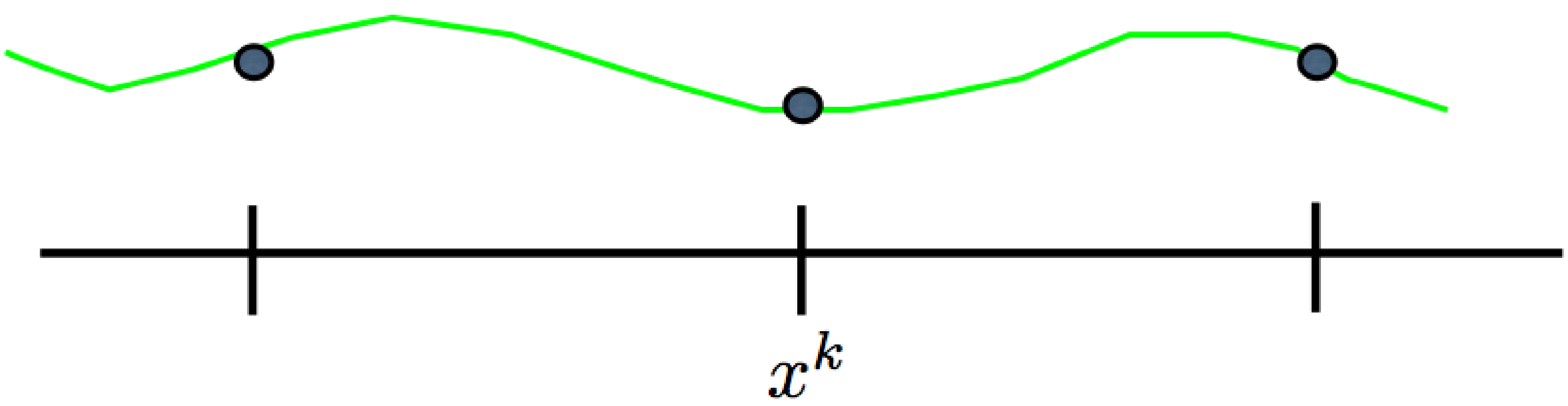
$$\frac{\partial}{\partial t} u(x,t) + c \frac{\partial}{\partial x} u(x,t) = g(x,t)$$

u(x,t) : unknown field
c : speed (here :constant and positive)
g(x,t) : source

Usual Methods :

- Finite Differences
- Finite Volumes
- Finite Elements

Finite Differences



Discretized form :

$$\frac{u_k^{i+1} - u_k^i}{\Delta t} + c\,\frac{u_k^i - u_{k-1}^i}{\Delta x} = g\left(x_k^i\right)$$

$$\Leftrightarrow \quad u_k^{i+1} = u_k^i + \frac{c\,\Delta t}{\Delta x}\left(u_{k-1}^i - u_k^i\right) + \Delta t\, g\left(x_k^i\right)$$

CFL : Stability if $|CFL| \leqslant 1$

Time : Euler Forward (explicit : simplicity)
Space : Euler Backward (physical reason : upwind c>0)

Finite Differences

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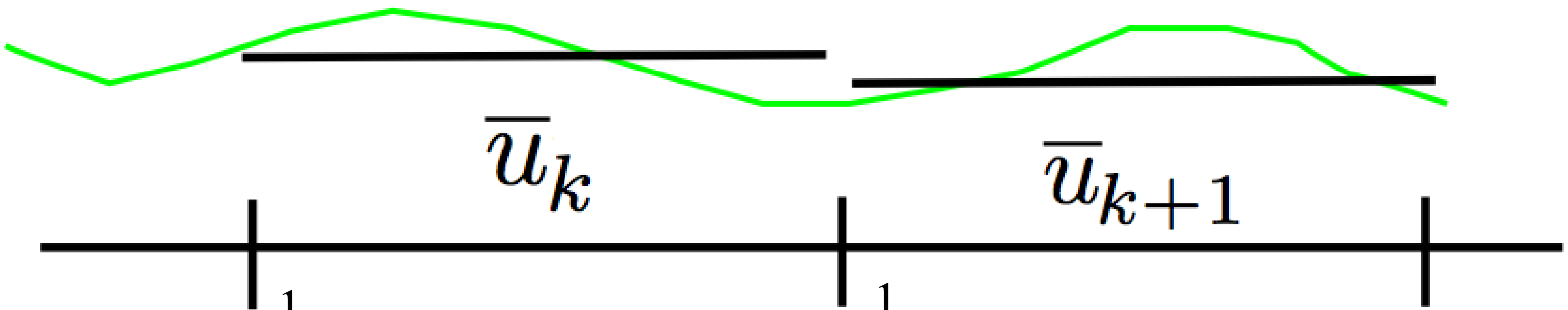
- Simple and fast implementation
- HO feasible but tedious
- Direction can be used
- BC can be imposed
- Explicit in time

-

- Simple local approximation, nothing between k's
- Only simple geometry
- Inherently 1D

Finite Volumes



$$k-\frac{1}{2} \qquad \Omega_k : [k-\frac{1}{2}, k+\frac{1}{2}] \qquad k+\frac{1}{2}$$

$$\int_{\Omega_k} (\partial_t u + c\,\partial_x u - g)\,dx = 0 \quad \& \quad \int_{\Omega_k} u\,dx = \bar{u}_k h_k \quad \forall k$$

Gauss theorem

$$h_k \frac{\bar{u}_k^{\,i+1} - \bar{u}_k^{\,i}}{\Delta t} + c\,[u^i]_{k-\frac{1}{2}}^{k+\frac{1}{2}} = \int_{\Omega_k} g^i\,dx \qquad CFL = \frac{c\,\Delta t}{h_k}$$

A numerical flux must be defined because
- u is unknown and discontinuous at the boundaries
- volumes are disconnected

→ “degree of freedom” to control the numerical behavior

Finite Volumes

+

- Robust
- Complex geometries
- Direction can be used
- Explicit in time

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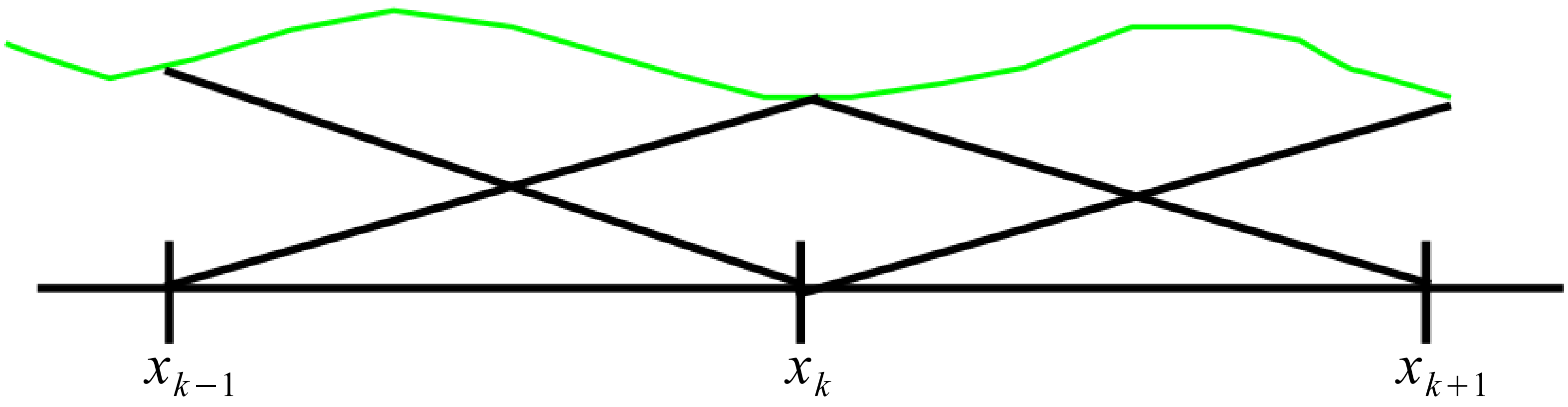
- Unable to achieve HO on general grid (flux formulation)
- Conservation law-based (elliptic problem ?)
- BC imposed weakly

Finite Elements



$$\int_{\Omega} (\partial_t u + c\,\partial_x u - g)\,u\,dx = 0 \quad \& \quad u(x,t) = \sum_k N_k(x)\,u_k(t)$$

$$\boldsymbol{M}\,\underline{\dot{u}} + c\,\boldsymbol{S}\,\underline{u} = \underline{g}$$

$$\boldsymbol{M}_{ij} = \int_{\Omega} N_i N_j\,dx$$

$$\boldsymbol{S}_{ij} = \int_{\Omega} N_i \frac{d N_j}{dx}\,dx$$

$$g_i = \int_{\Omega} g\;N_i\,dx$$

↓

-Even if Euler backward, inversion of **M** required
-Elements are connected by the shape functions N
→ implicit in time

Finite Elements

+

- HO easily handled
- Complex geometries
- Strong theory
- Approximate solution at any x (shape functions)
- BC can be imposed

-

- Implicit in Time
- Direction ?

Numerical Example

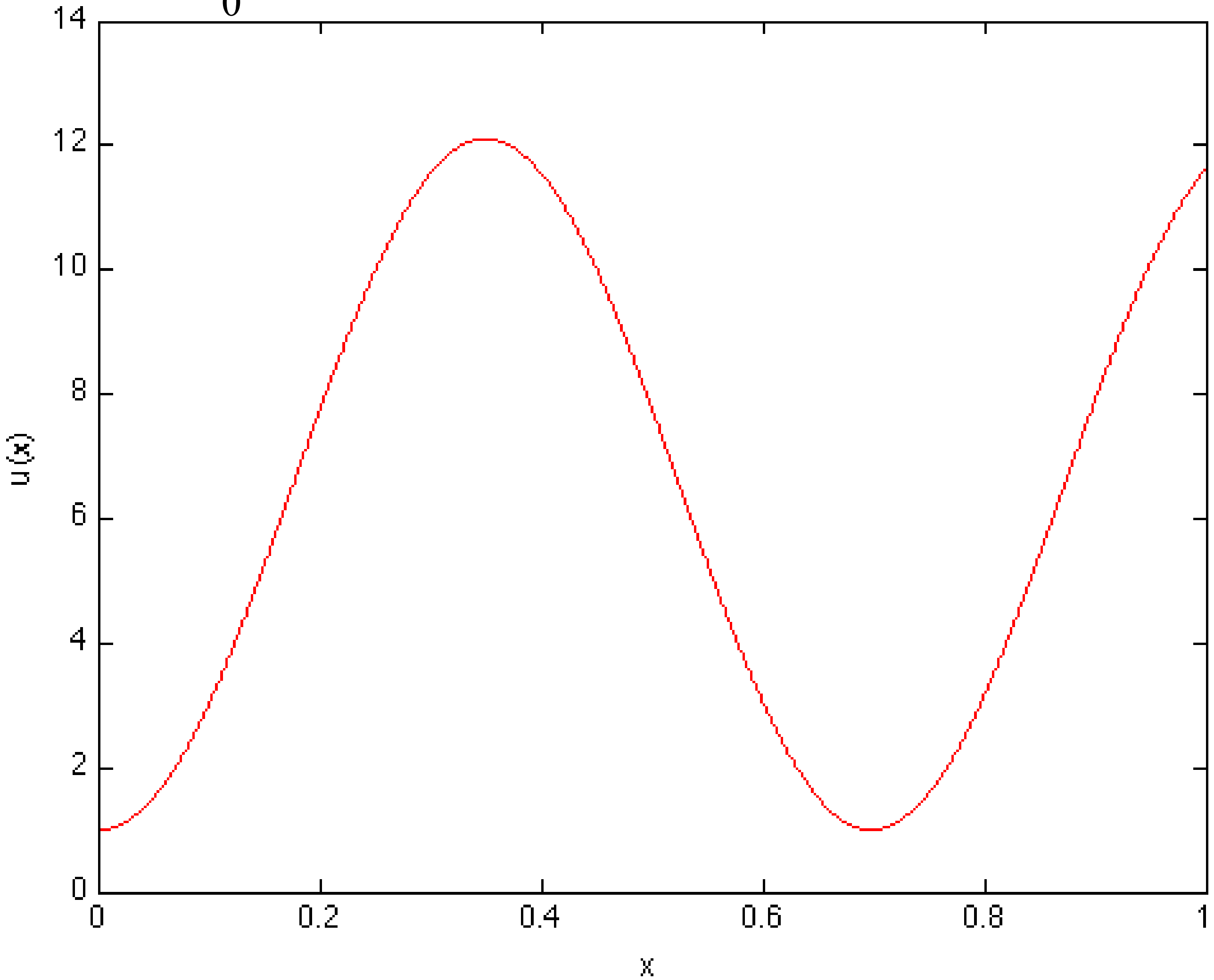
Transport equation with

- $c = 2$ 1/s
- Source : $g(x,t) = A \sin(ax)$, $A = 100$, $a = 9$
- BC : $u(0,t) = 1$
- IC : $u(x,0) = 1$
- CFL = 1 (FD and FV)
- Ω : [0,1], $dx = 0.1$
- T : [0,1] s, $dt = CFL*dx/c = 0.05$ s

Numerical example

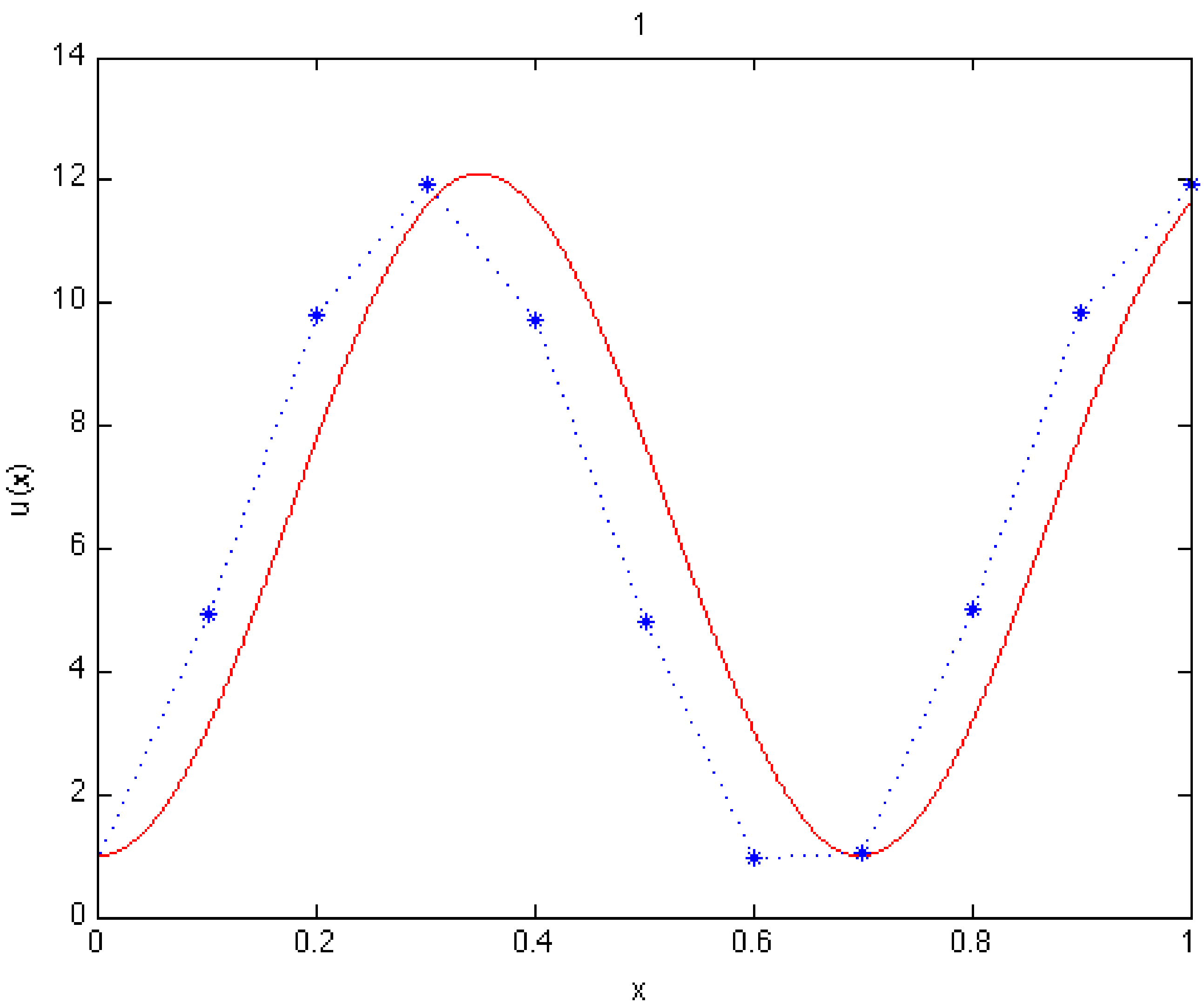
- Analytical stationary solution :

$$c\int_0^X \partial_x u\,dx = A\int_0^X \sin(ax)\,dx \iff u(X) = u(0) + \frac{A}{ac}\left(\cos(ax) - 1\right)$$



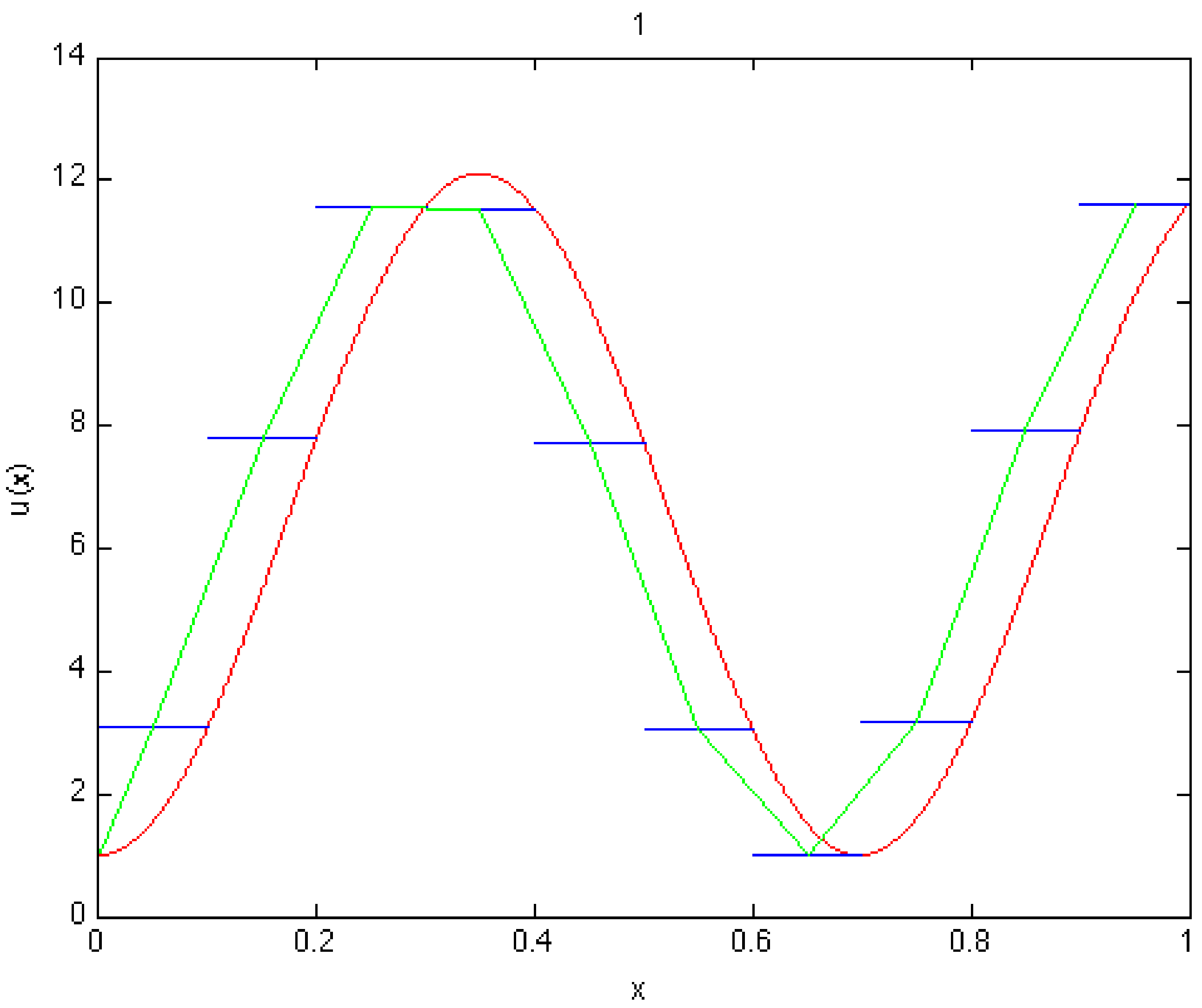
Numerical example

Finite differences method :



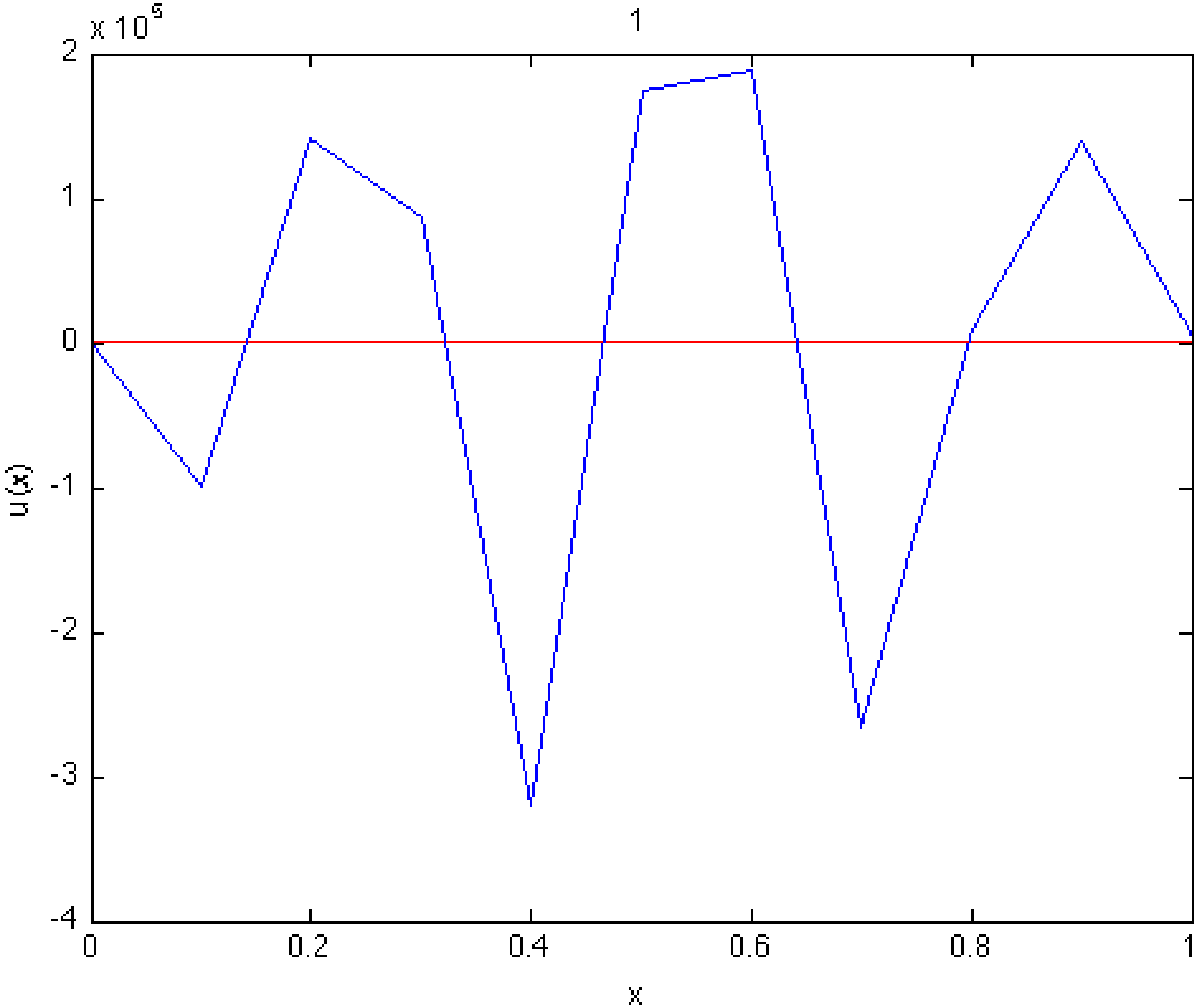
Numerical example

Finite volumes method :



Numerical example

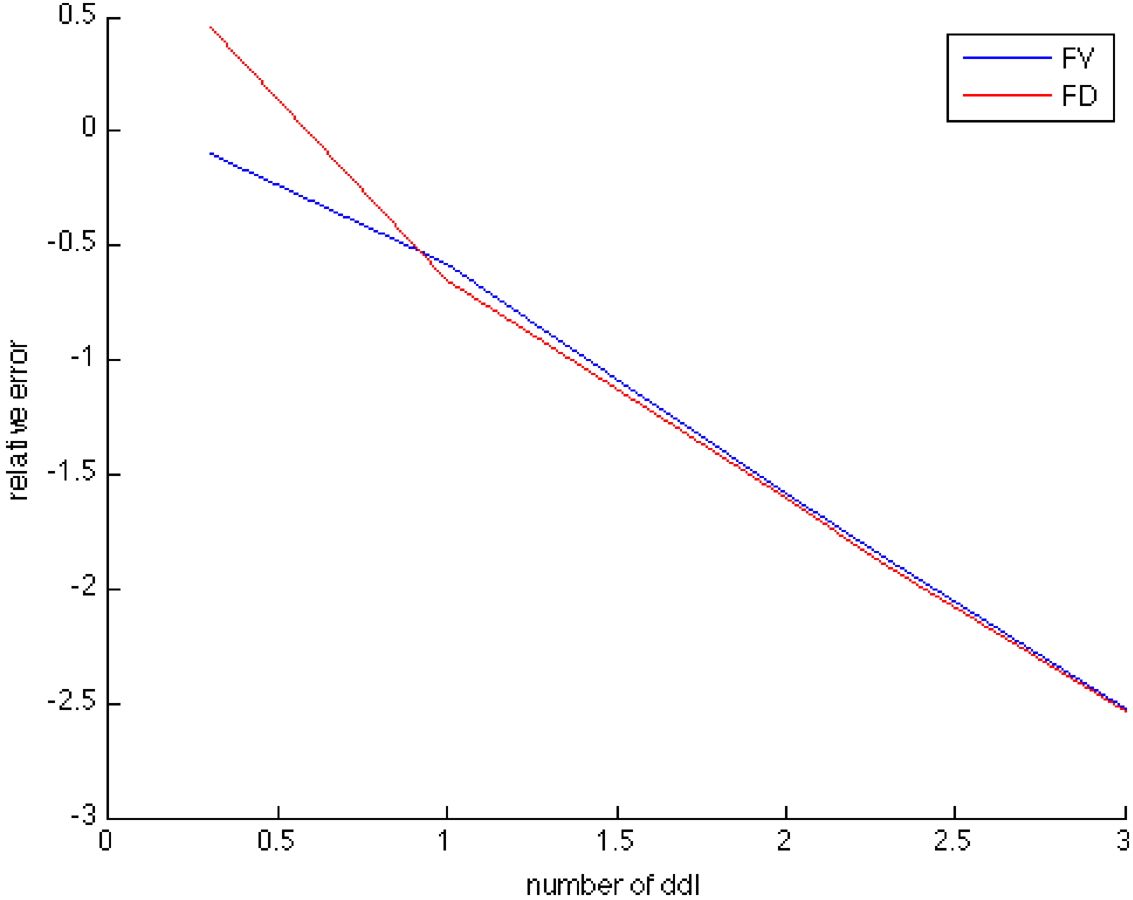
Finite elements method :



Numerical example

- FE : forget it (unconditionally unstable)
- FD and FV : relative error not that small.

and behaves as O(h), not really appealing...



DG method : first idea

We want :

- Stability : FV (FD)
- Flexibility : FE

Can we merge them to get something better ?

DG method : first idea

Starting from the local (Ω_k) Galerkin approach :

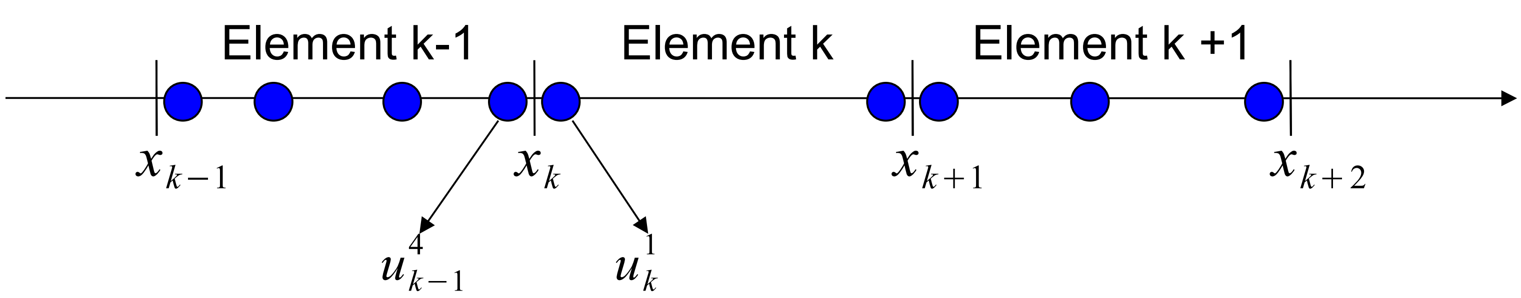
$$f(x,t)=cu$$

$$\int_{\Omega_k} \left(\partial_t u + \partial_x f - g\right) \underline{u}\, dx = 0$$

$$\Omega_k = [x_k, x_{k+1}]$$

$$u(x,t) = \sum_{k=1}^{K} u_k(x,t) = \sum_{k=1}^{K} N^i(x)\, u_k^i(t)$$

Ex :



DG method : first idea

Gauss integration to introduce a numerical flux :

$$\underbrace{\int_{\Omega_k} \partial_t(u) u\, dx}_{\text{FE}} + \underbrace{[\hat{f}\, u]_{x_k}^{x_{k+1}}}_{\text{FV}} - \underbrace{\int_{\Omega_k} cu\, \partial_x u\, dx}_{\approx \text{FE}} = \underbrace{\int_{\Omega_k} g\, u\, dx}_{\text{FE}}$$

$$\int_{\Omega_k} \partial_t u_k^i N^i N^j dx - c \int_{\Omega_k} N^j \partial_x u_k^i N^i dx$$

$$= \int_{\Omega_k} g\, N^i dx - \underbrace{[\hat{f}\, N^i]_{x_k}^{x_{k+1}}}_{\hat{\phi}_k^i}$$

In matrix form :

$$\boldsymbol{M}\, \underline{\dot{u}_k} - c\, \boldsymbol{S}^T \underline{u_k} = \underline{g} - \underline{\hat{\phi}}$$

→ the procedure to assemble the elements is the same as for FE

DG method : first idea

Essential points :

- u is discontinuous between elements, more than one node is defined at the interfaces between elements

→ numerical fluxes (which enable the connection between elements and the control of stability)

- The shape functions of an element are non zero only on the support of this element

→ **M** and **S** of the assembled system are block-diagonal matrices

DG method : Stability analysis

- How do we choose the flux ?

It must be consistent (tend to the real flux) and control the dissipation to insure the stability

- Analytical result from the original continuous equation :

$$\int_{\Omega} (\partial_t u + c\,\partial_x u)\, u\, dx = 0 \qquad \Omega = [x^l, x^r]$$

Appropriate BC : c>0 $\rightarrow u(x^l, t) = h(t)$

$$\Leftrightarrow \frac{1}{2}\frac{d}{dt}\|u\|_{\Omega}^{2} = -\frac{c}{2}\left(u^2(x^r) - u^2(x^l)\right)$$

Energy conservation when $u(x^r) = u(x^l)$

DG method : Stability analysis

- Analytical result from the modified equation :

$$\int_{\Omega_k} \partial_t(u)u\,dx - \int_{\Omega_k} c\,u\,\partial_x u\,dx = -[\hat{f}\,u]_{x_k}^{x_{k+1}}$$

$$\frac{1}{2}\frac{d}{dt}\|u_k\|_{\Omega_k}^2 = \frac{c}{2}\left(u_k^2(x_{k+1}) - u_k^2(x_k)\right) - [\hat{f}\,u_k]_{x_k}^{x_{k+1}}$$

- The scheme is bounded (stable) if

$$\frac{d}{dt}\|u\|_{\Omega}^2 = \sum_{k=1}^{K} \frac{d}{dt}\|u_k\|_{\Omega_k}^2 \leqslant 0$$

$$\Leftrightarrow \sum_{k=1}^{K} c\left(u_k^2(x_{k+1}) - u_k^2(x_k)\right) - 2[\hat{f}\,u_k]_{x_k}^{x_{k+1}} \leqslant 0$$

DG method : Stability analysis

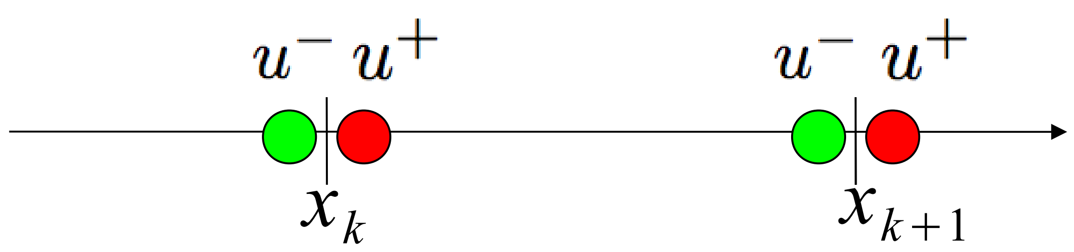
$$\Leftrightarrow c\left(u_k^2(x_{k+1}) - u_k^2(x_k)\right) - 2\left[\hat{f}\, u_k\right]_{x_k}^{x_{k+1}} \leqslant 0$$

- Must be verified at all the interfaces (because u is discontinuous, we can't sum up over the k's)
- Classical flux at the interface:

$$\hat{f} = c\{\{u\}\} + c\frac{1-\alpha}{2}[\![u]\!]$$

$$\{\{u\}\} = \frac{u^- + u^+}{2}$$

$$[\![u]\!] = \hat{\boldsymbol{n}}^- u^- + \hat{\boldsymbol{n}}^+ u^+$$



- α = 0 : upwind ; α = 1 : central flux

DG method : Stability analysis

- Contribution from the flux at each interface :

$$-c\frac{(1-\alpha)}{2}[\![u_k(x_{k+1})]\!]^2 \leqslant 0$$

- Sum over all elements yields

$$\frac{d}{dt}\|u\|_{\Omega}^2=\sum_{k=1}^{K}\frac{d}{dt}\|u_k\|_{\Omega_k}^2=-c(1-\alpha)\sum_{k=1}^{K}[\![u_k(x_{k+1})]\!]^2 \leqslant 0$$

- Stable if α ≤ 1 (rem : α = 1 is generally not stable)
- They are different ways to take the BC into account but they don't influence much the stability

DG method : BC

- As an hybrid method, there are several ways to deal with the boundary conditions.

In case of the example :

- FV-like :
  - Impose the flux : $\hat{f} = c\,x(l,t) = ch(t)$
  - Verify the interface condition (best) :

$$\hat{f} = -cu_1(x_{1,}t) + 2\mathrm{h}(t)$$

- FE-like :
  - Enforce the value of u at the boundary

→ modify a line in the matrices

DG method : Stability analysis

- We discussed stability w.r.t. discontinuities.

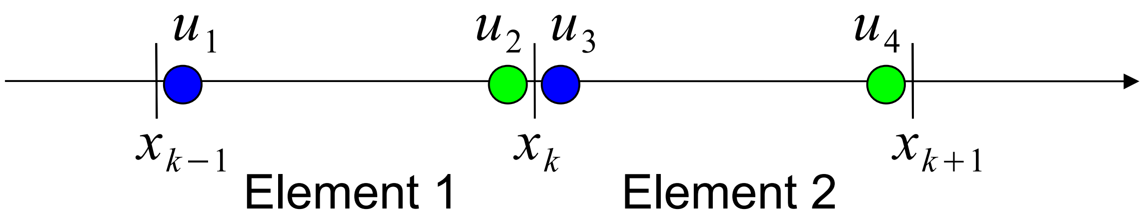
What about stability w.r.t. to time-integration ?

- A general formula is not available. It is not trivial and one must pay attention.

- Illustration with the eigenvalues of the amplification matrix for an Euler forward scheme in time with a full upwind numerical flux (α = 0) and linear shape functions.

$$\boldsymbol{M}\,\underline{\dot{u}} - c\,\boldsymbol{S}^T\,\underline{u} = c\,\boldsymbol{F}\,\underline{u}$$

DG method : Sensibility analysis



$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \quad \boldsymbol{M} = \begin{bmatrix} \boldsymbol{M_1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M_2} \end{bmatrix} \quad \boldsymbol{S} = \begin{bmatrix} \boldsymbol{S_1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{S_2} \end{bmatrix} \quad \boldsymbol{F} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\boldsymbol{M_k} = \frac{h_k}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \boldsymbol{S_k} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\underline{u}^{t+1} = \left(\boldsymbol{I} + \Delta t \, \boldsymbol{M}^{-1} \left(c \, \boldsymbol{S}^{T} + c \, \boldsymbol{F} \right) \right) \underline{u}^{t}$$

Eigenvalues :

$$\underline{\lambda} = \begin{matrix} 1 - 2 * CFL + 2\sqrt{2} * i * CFL \\ 1 - 2 * CFL - 2\sqrt{2} * i * CFL \\ 1 - 2 * CFL + 2\sqrt{2} * i * CFL \\ 1 - 2 * CFL - 2\sqrt{2} * i * CFL \end{matrix}$$

$$CFL = \frac{c \, \Delta t}{h_k}$$

$$|\lambda_i| \leqslant 1 \Leftrightarrow 0 \leqslant CFL \leqslant \frac{1}{3}$$

DG method : Stability analysis

- With an upwind flux in FV and FD, stability condition is only CFL ≤ 1.

The DG elements are less stable.

- However, no CFL with FEM, unconditionally unstable (eigenvalues of amplification matrix = 1)

- How might DG elements be interesting then ?

Accuracy ? Yes !

DG method : Accuracy

- The following result can be shown (regular grid) :

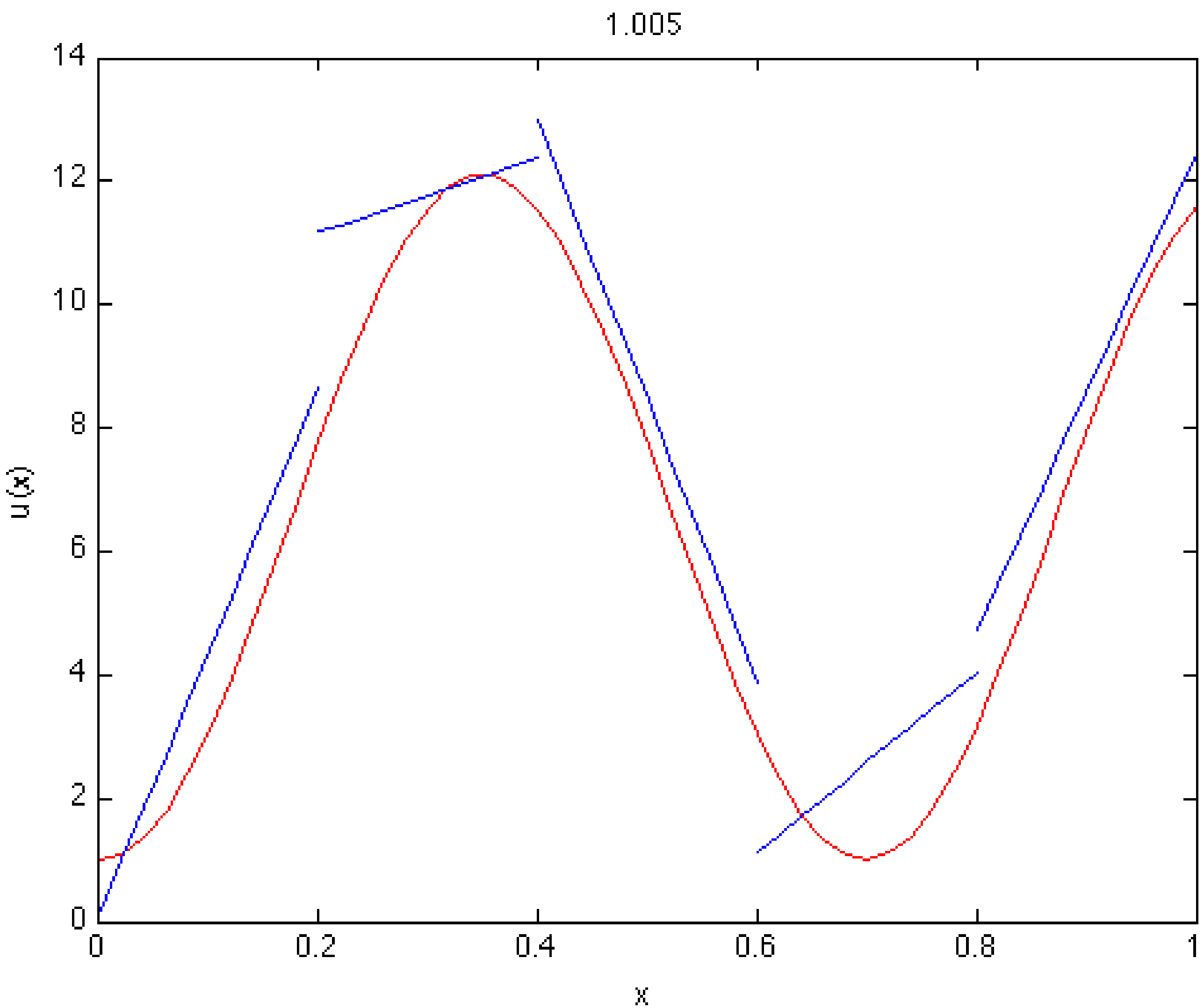
$$\|u - \sum_{k=1}^{K} u_k\| \leqslant C h_k^{N+1}$$

with N : order of the polynomial

- Actually at least N+1/2 and N+1 in smooth cases
- Higher than FD and FV !
- The higher the order, the better the approximation

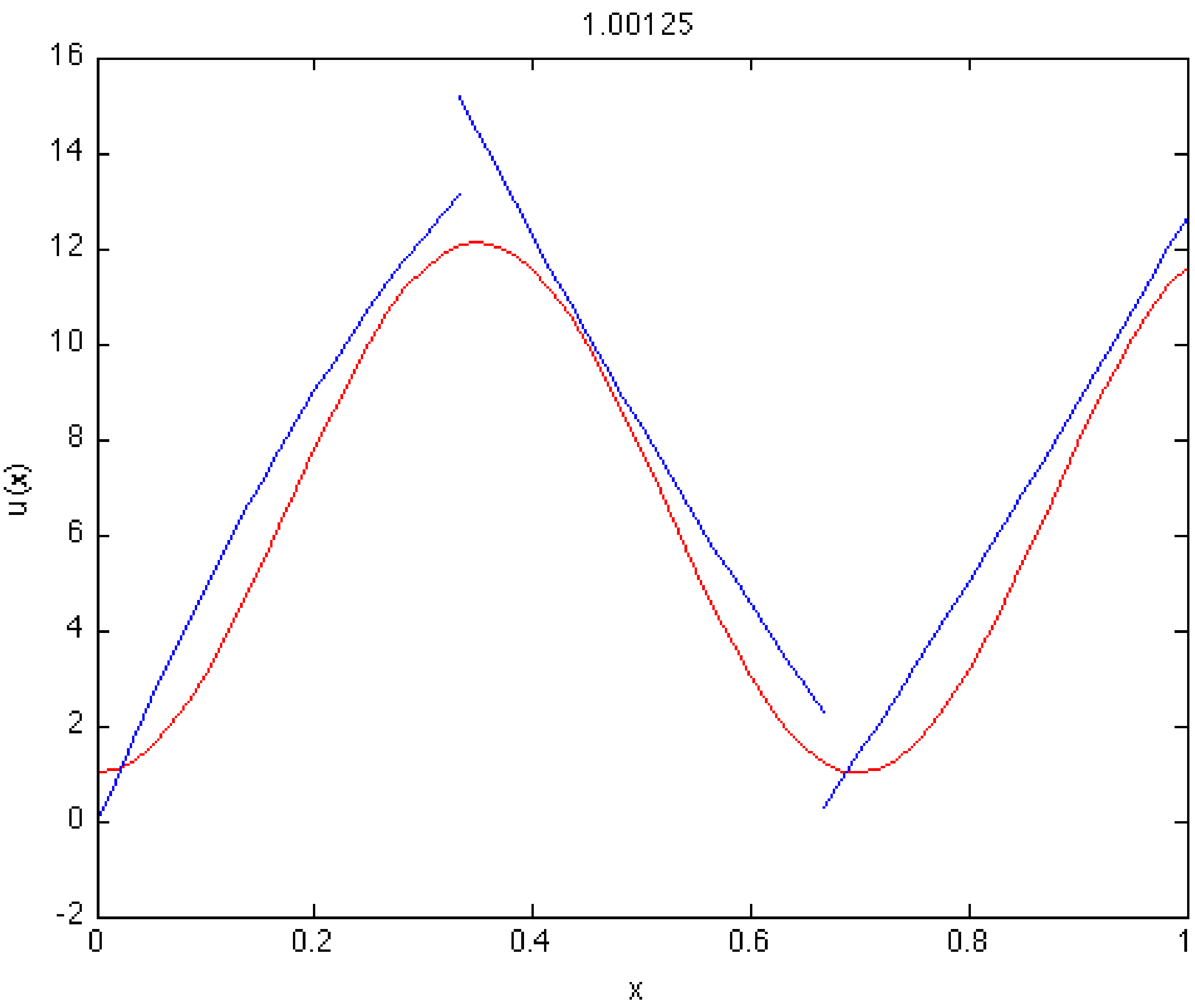
DG method : Accuracy

Back to the numerical example, order 1



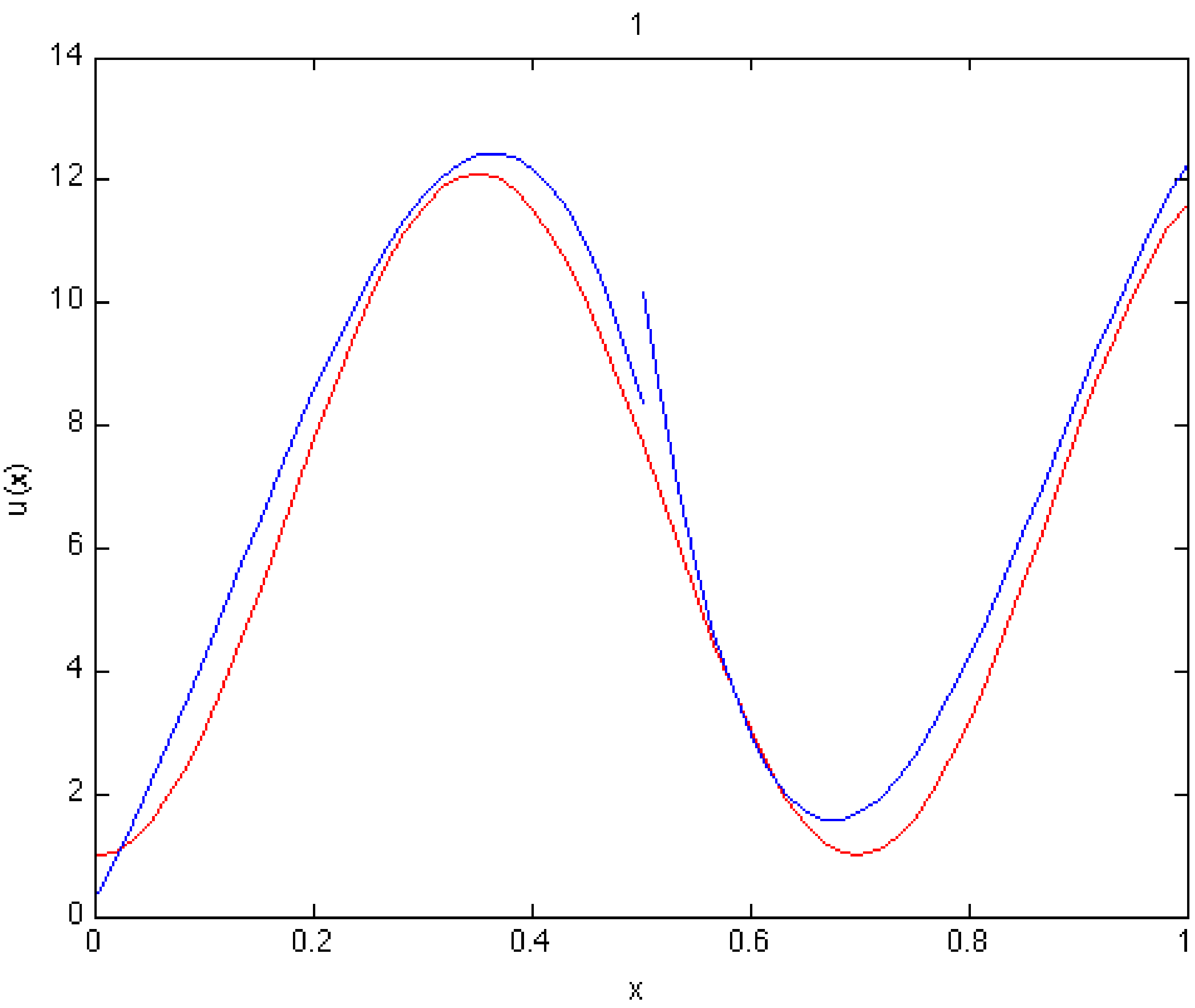
DG method : Accuracy

Back to the numerical example, order 2 (with same number of dof)



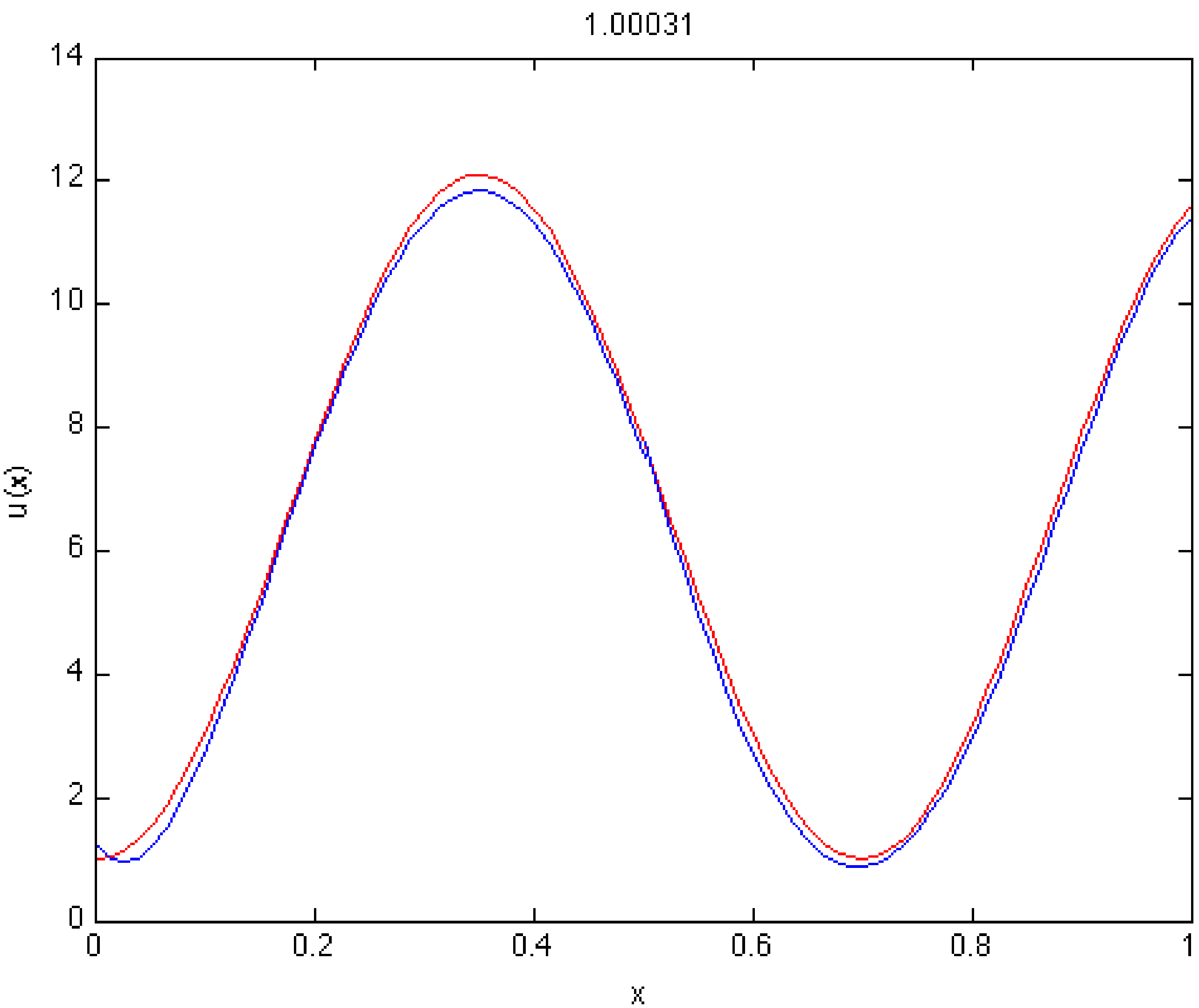
DG method : Accuracy

Back to the numerical example, order 3 (with same number of dof)



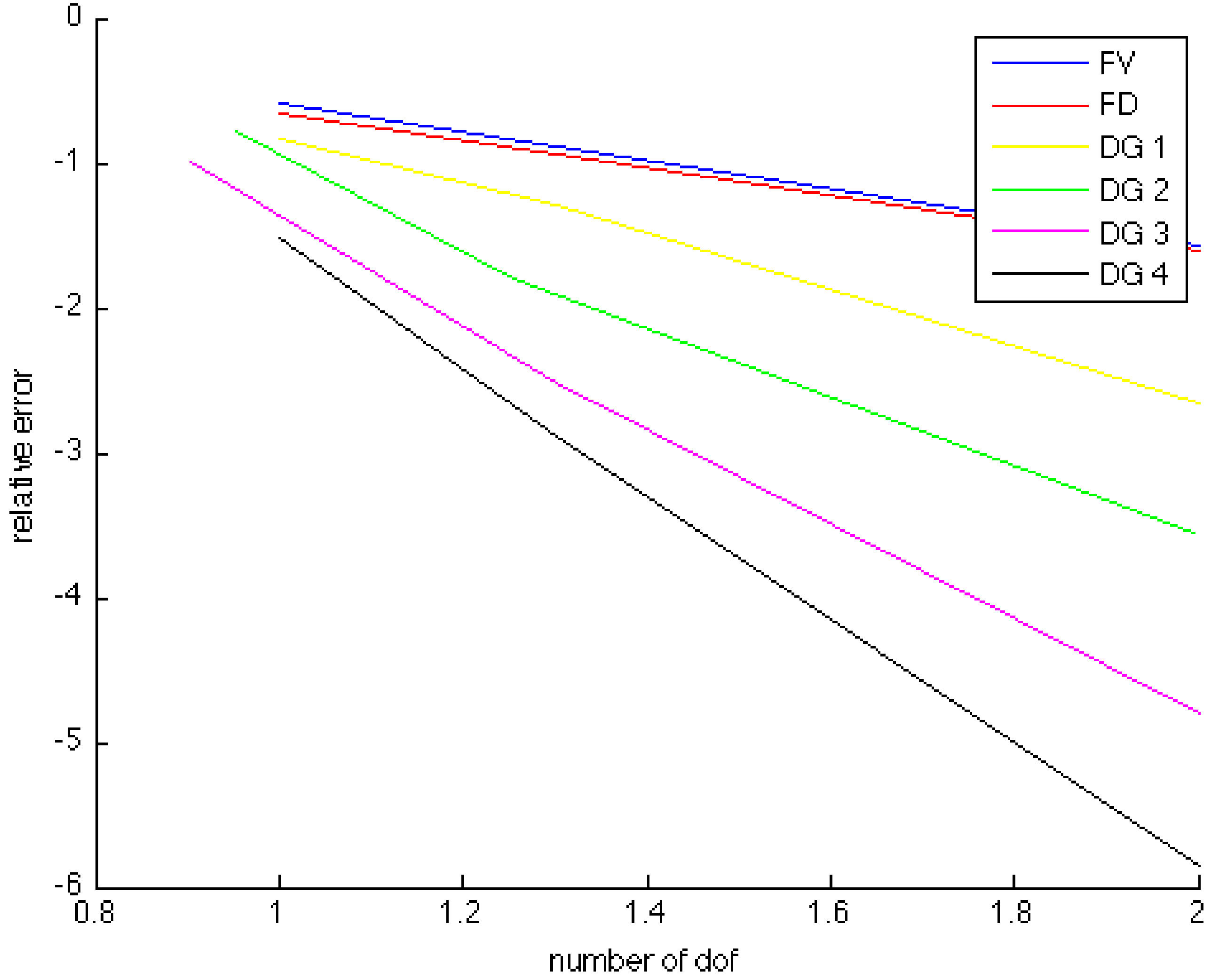
DG method : Accuracy

Back to the numerical example, order 4 (with same number of dof)



DG method : Accuracy

How does the relative error behave ? $\sim O\left(dx^{N+1}\right)$



DG method : Issues

<u>Formulation</u>

- For a given grid the number of node is doubled along the interfaces.

1D : just one node is added. But 2D and 3D...

DG method : Issues

Stability

- We saw that the numerical dissipation is proportional to the jump at interfaces. If the approximation is close to the actual continuous solution (high order or small elements), the jump is small and the scheme is less stable. One has to reduce CFL or use a higher order time integration scheme (RK).

- Improvement example : Time-step taming thanks to appropriate (complicated) mapping of the polynomials

DG method : Issues

Accuracy

- Lagrange polynomials of high order lead to large peaks at the boundaries (cfr Vandermonde matrix condition number).

- Solution : use a better basis for interpolation : Legendre polynomials (diagonal Vandermonde matrix).

DG method : Issues

<u>Discontinuities</u>

- Discontinuous approximation → easy to handle actual discontinuous solution ?
- No, the method requires slope limiters.
- Example : discontinuous initial condition

$$\partial_t u + c \partial_x u = 0$$

$$c > 0, \quad x \in [0,1], \quad x(0,t) = 0$$

$$u(x,0) = \begin{matrix} 1 & \frac{1}{20} \leqslant x \leqslant \frac{1}{10} \\ 0 & otherwise \end{matrix}$$

DG method : Issues

- Analytical solution :

method of characteristics $u(x,t)=u(x-ct)$

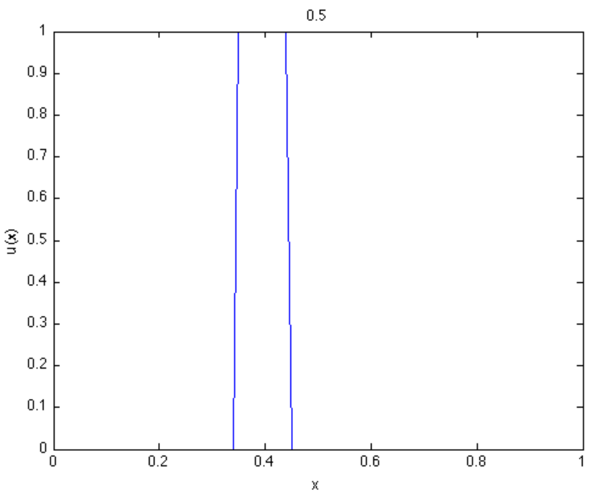
- if CFL = 1, no problem for FV and FD

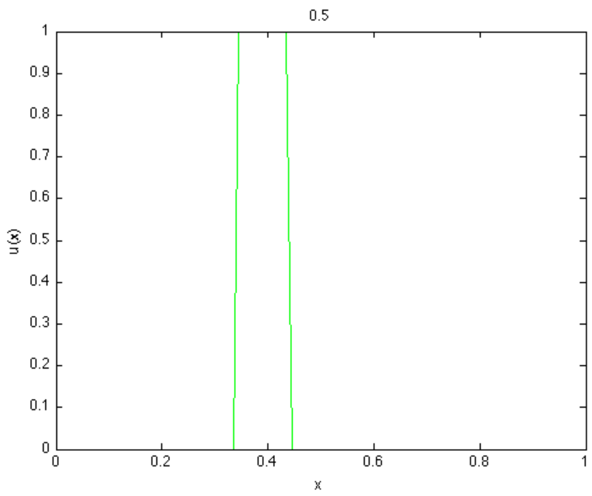
if CFL < 1, some dissipation

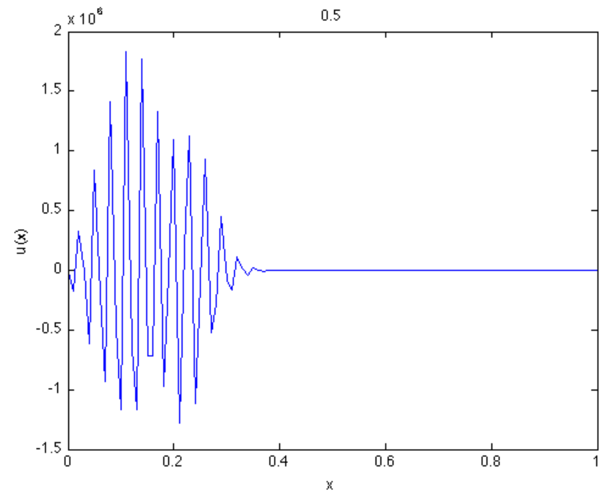
- FE unstable

DG method : Issues

- Finite Differences
- Finite Volumes
- Finite elements

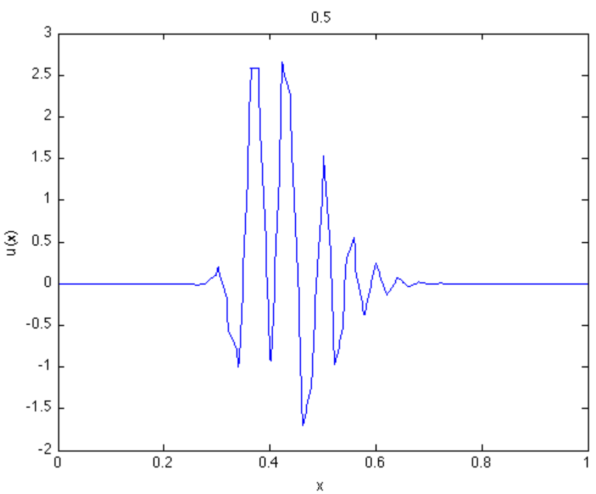


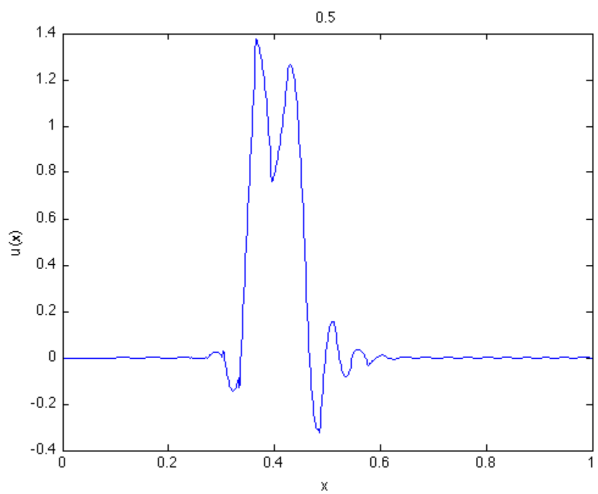


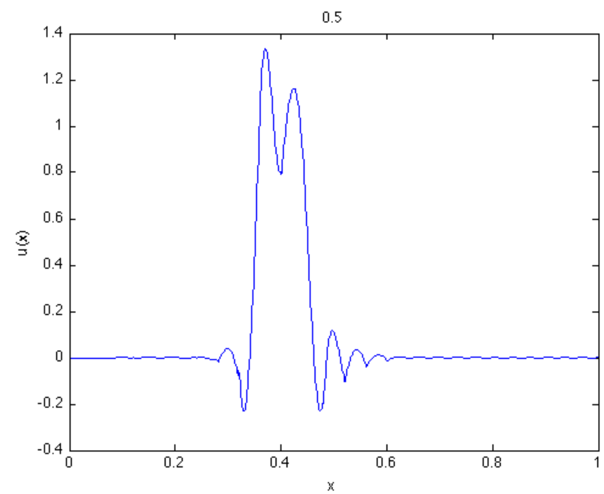


DG method : Issues

- Order 1
- Order 2
- Order 3







- ODG method unstable. Instabilities decreases with higher order polynomials but never vanish
- Solution : Slope limiters (the derivatives at the discontinuities introduce big flux that should be numerically bounded)
- Or put the discontinuities between the elements at the actual discontinuities (hard !)

Prospects

- Method essentially developed by mathematicians

→ solid mathematical background and development but not widely applied

- Applications in many areas
- Extensions have been made to be used for
  - General problems (elliptic, parabolic, hyperbolic)
  - Linear and non-linear problems

Conclusions

- Powerful numerical method
- Rely on well understood basics (flux : FV, shape functions : FE) but need some specific theory to be used properly
- Worth to be used when high accuracy is needed because stability conditions involve small time steps anyway
- Still in development, not yet spread in the industry

Main references

Discontinuous Galerkin methods, Bernardo Cockburn, Plenary lecture presented at the 80th Annual GAMM Conference, Augsburg, 25–28 March 2002, ZAMM Z. Angew. Math. Mech. 83, No. 11, 731 – 754 (2003) / DOI 10.1002/zamm.200310088

http-//www.cfm.brown.edu/people/jansh/resources/Publications/Lectures/

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