THE STRESS ANALYSIS OF CRACKS HANDBOOK

THIRD EDITION

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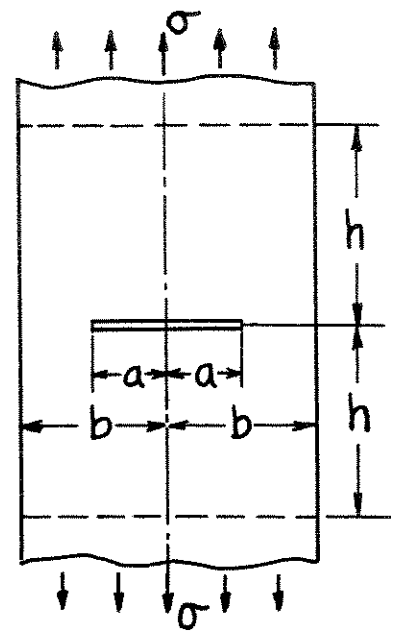
THE CENTER CRACKED TEST SPECIMEN

A. Stress Intensity Factor

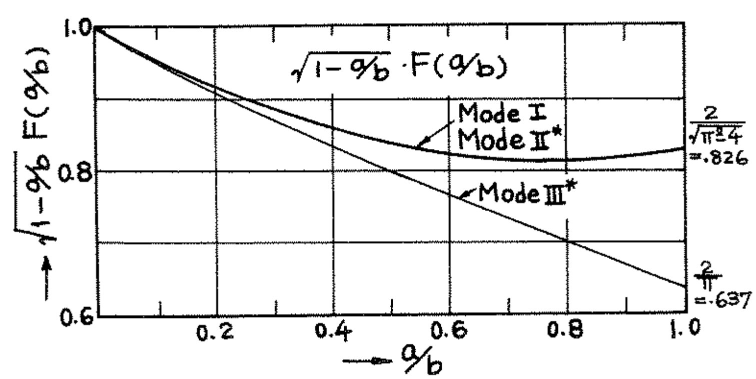
$$K_I = \sigma\sqrt{\pi a}\, F(a/b)$$



Numerical Values of $F(a/b)$

(Isida 1962, 1965a, b, 1973)

Isida's 36-term power series of $(a/b)^2$ (Laurent series expansion of complex stress potential, 1973) gives practically exact values of $F(a/b)$ up to $a/b = 0.9$. Numerical values of $F(a/b)$ are shown in the following graph and table.



a/b	$F(a/b)$
0.0	1.0000
0.1	1.0060
0.2	1.0246
0.3	1.0577
0.4	1.1094
0.5	1.1867
0.6	1.3033
0.7	1.4882
0.8	1.8160
0.9	2.5776
1.0	$\frac{2}{\sqrt{\pi^2-4}} \Big/ \sqrt{1-a/b}$ **

**Exact Limit (Koiter 1965b)

*See Note 2

Empirical Formulas

a. Accuracy
b. Method of derivation, reference

$$F(a/b) = \sqrt{\frac{2b}{\pi a}\tan\frac{\pi a}{2b}}$$

a. Better than 5% for $a/b \leq 0.5$
b. Approximation by periodic crack solution (**Irwin 1957**)

$$F(a/b) = 1 + 0.128(a/b) - 0.288(a/b)^2 + 1.525(a/b)^3$$

a. 0.5% for $a/b \leq 0.7$
b. Least squares fitting to Isida's results (**Brown 1966**)

$$F(a/b) = \sqrt{\sec\frac{\pi a}{2b}}$$

a. 0.3% for $a/b \leq 0.7$, 1% at $a/b = 0.8$
b. Guess based on Isida's results (**Feddersen 1966**)

$$F(a/b) = \frac{1 - 0.5(a/b) + 0.326(a/b)^2}{\sqrt{1 - a/b}}$$

a. 1% for any a/b
b. Asymptotic approximation (**Koiter 1965b**)

$$F(a/b) = \frac{1 - 0.5(a/b) + 0.370(a/b)^2 - 0.044(a/b)^3}{\sqrt{1 - a/b}}$$

a. 0.3% for any a/b
b. Modification of Koiter's formula (**Tada 1973**)

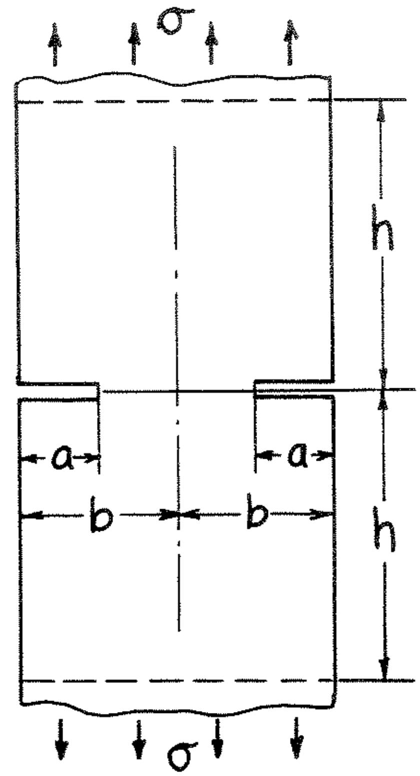
$$F(a/b) = \left\{1 - 0.025(a/b)^2 + 0.06(a/b)^4\right\}\sqrt{\sec\frac{\pi a}{2b}}$$

a. 0.1% for any a/b
b. Modification of Feddersen's formula (**Tada 1973**)

THE DOUBLE EDGE NOTCH TEST SPECIMEN

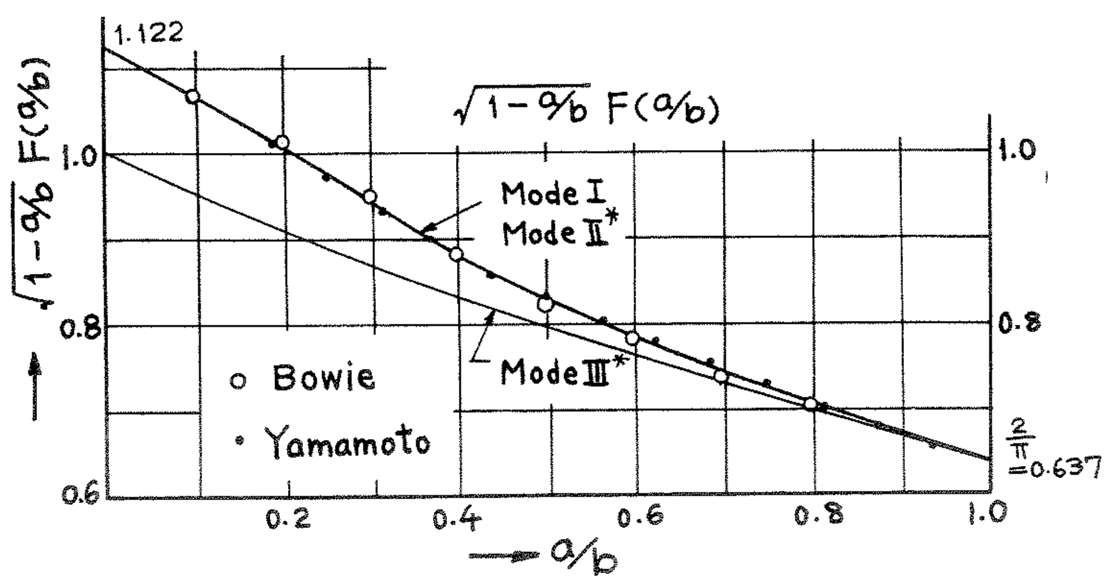
A. Stress Intensity Factor

$$K_I = \sigma\sqrt{\pi a}\, F(a/b)$$



Numerical Values of $F(a/b)$

Bowie's results ($h/b = 3.0$, mapping function method) have 1% accuracy and Yamamoto's results ($h/b = 2.75$, finite element method) have 0.5% accuracy for $0.2 < a/b < 0.9$ (**Bowie 1964a; Yamamoto 1972**).



*See Note 2

(See also **pages 2.32, 2.33, 2.34, 11.5, 15.1** etc., for corrections and various effects.)

Empirical Formulas

a. Accuracy
b. Method of derivation, reference

$$F(a/b) = \sqrt{\frac{2b}{\pi a}\tan\frac{\pi a}{2b}}$$

a. Better than 5% for $a/b > 0.4$
b. Approximation by periodic crack solution (**Irwin 1957**)

$$F(a/b) = 1.12 + 0.203(a/b) - 1.197(a/b)^2 + 1.930(a/b)^3$$

a. Better than 2% for $a/b \leq 0.7$
b. Least squares fitting to Bowie's results (**Brown 1966**)

$$F(a/b) = \frac{1.122 - 0.561(a/b) - 0.015(a/b)^2 + 0.091(a/b)^3}{\sqrt{1 - a/b}}$$

a. Better than 2% for any a/b
b. Asymptotic approximation (**Benthem 1972**)

$$F(a/b) = \left(1 + 0.122\cos^4\frac{\pi a}{2b}\right)\sqrt{\frac{2b}{\pi a}\tan\frac{\pi a}{2b}}$$

a. 0.5% for any a/b
b. Modification of Irwin's interpolation formula (**Tada 1973**)

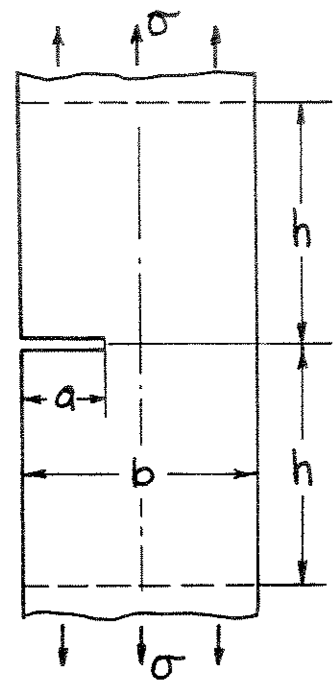
$$F(a/b) = \frac{1.122 - 0.561(a/b) - 0.205(a/b)^2 + 0.471(a/b)^3 - 0.190(a/b)^4}{\sqrt{1 - a/b}}$$

a. 0.5% for any a/b
b. Modification of Benthem's formula (**Tada 1973**)

THE SINGLE EDGE NOTCH TEST SPECIMEN

A. Stress Intensity Factor

$$K_I = \sigma\sqrt{\pi a}\, F(a/b)$$

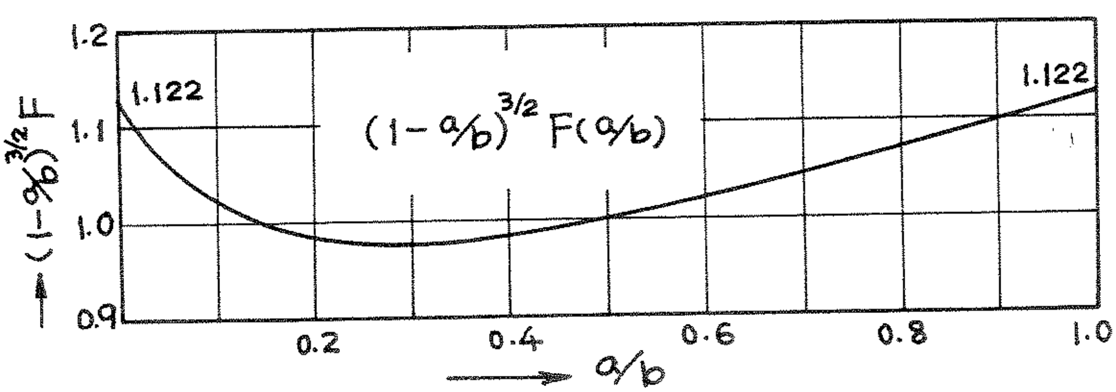


Numerical Values of $F(a/b)$

The curve in the following figure was drawn based on the results having better than 0.5% accuracy.

Methods and References

a. Boundary Collocation Method ($h/b > 0.8$): **Gross 1964**
b. Mapping Function Method ($h/b = 1.53$): **Bowie 1965**
c. Green's Function Method ($h/b > 1.5$): **Emery 1969, 1972**
d. Weight Function Method: **Bueckner 1970, 1971**
e. Asymptotic Approximation: **Benthem 1972**
f. Finite Element Method ($h/b = 2.75$, 1.0): **Yamamoto 1972**



NOTE: 1. Load is applied along the centerline of the strip at the crack location (or uniform pressure on crack surfaces).
2. The effect of h/b is practically negligible for $h/b \geq 1.0$.

(See also **pages 2.13, 2.16, 2.27 to 2.31** etc., for various corrections and effects.)

Empirical Formulas
a. Accuracy
b. Method, reference

$$F(a/b) = 1.122 - 0.231(a/b) + 10.550(a/b)^2 - 21.710(a/b)^3 + 30.382(a/b)^4$$

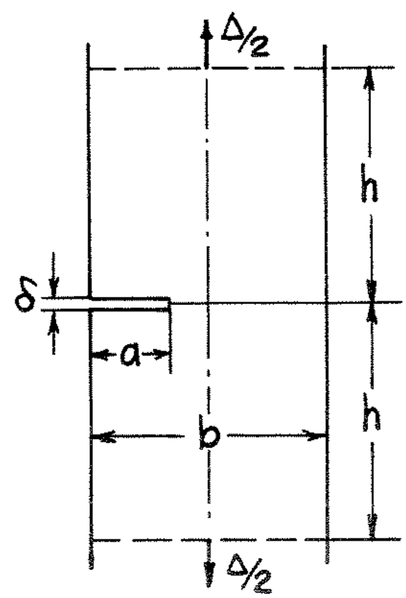
a. 0.5% for $a/b \leq 0.6$
b. Least squares fitting (**Gross 1964; Brown 1966**)

$$F(a/b) = 0.265(1 - a/b)^4 + \frac{0.857 + 0.265\, a/b}{(1 - a/b)^{3/2}}$$

a. Better than 1% for $a/b < 0.2$, 0.5% for $a/b \geq 0.2$
b. **Tada 1973**

$$F(a/b) = \sqrt{\frac{2b}{\pi a}\tan\frac{\pi a}{2b}} \cdot \frac{0.752 + 2.02(a/b) + 0.37\left(1 - \sin\frac{\pi a}{2b}\right)^3}{\cos\frac{\pi a}{2b}}$$

a. Better than 0.5% for any a/b
b. **Tada 1973**



B. Displacements

Crack Opening at Edge

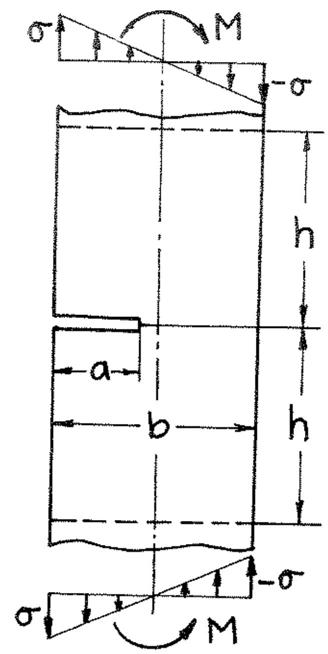
$$\delta = \frac{4\sigma a}{E'} V_1(a/b)$$

Gross' results (**Gross 1967**, Boundary Collocation Method) are expected to have 0.5% accuracy for $0.2 \leq a/b < 0.7$. An empirical formula with 1% accuracy for any a/b is (**Tada 1973**)

$$V_1(a/b) = \frac{1.46 + 3.42\left(1 - \cos\frac{\pi a}{2b}\right)}{\left(\cos\frac{\pi a}{2b}\right)^2}$$

THE PURE BENDING SPECIMEN

A. Stress Intensity Factor

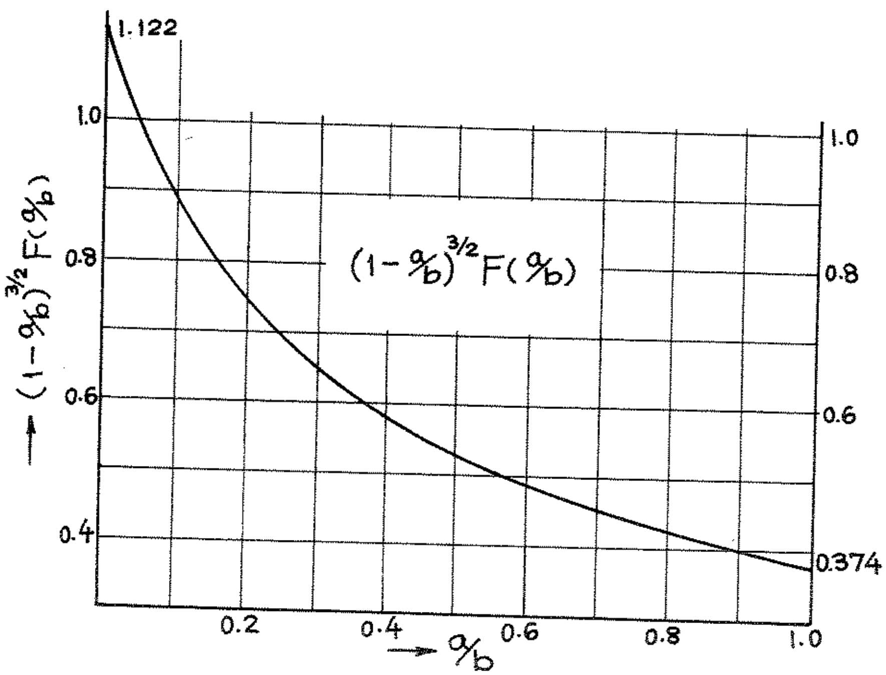


$$\sigma = \frac{6M}{b^2}$$

$$K_I = \sigma\sqrt{\pi a}\, F(a/b)$$

Numerical Values of $F(a/b)$

The curve in the following figure was drawn based on the results having better than 0.5% accuracy. Also used for four-point bending.



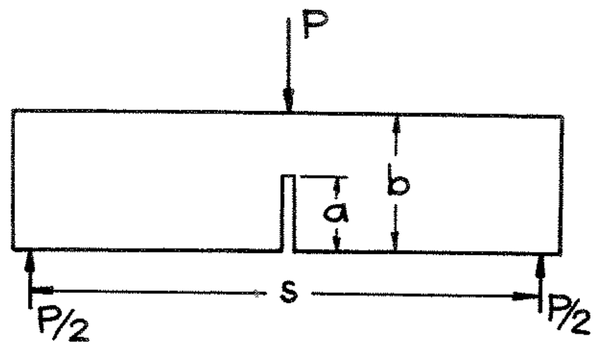
Methods and References

1. Singular Integral Equation, **Bueckner 1960**
2. Boundary Collocation Method $(h/b \geq 2)$, **Gross 1965a**
3. Weight Function Method, **Bueckner 1970, 1971**
4. Green's Function Method $(h/b \geq 1.5)$, **Emery 1969**
5. Asymptotic Approximation, **Benthem 1972**

THE THREE-POINT BEND TEST SPECIMEN

A. Stress Intensity Factor

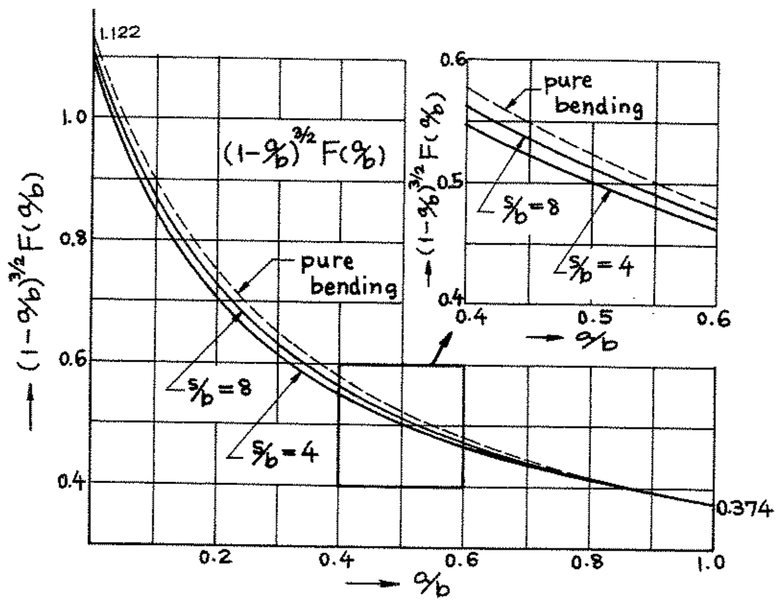
$$\sigma = \frac{6M}{b^2}\left(M = \frac{Ps}{4}\right)$$



$$K_I = \sigma\sqrt{\pi a}F(a/b)$$

Numerical Values of $F(a/b)$

The curves in the following figure have 1% accuracy.



Methods and References

1. Boundary Collocation Method $\left(s/b = 4, 8\right)$ **(Gross 1965b)**
2. Green's Function Method $\left(s/b = 3, 8\right)$ **(Emery 1969)**

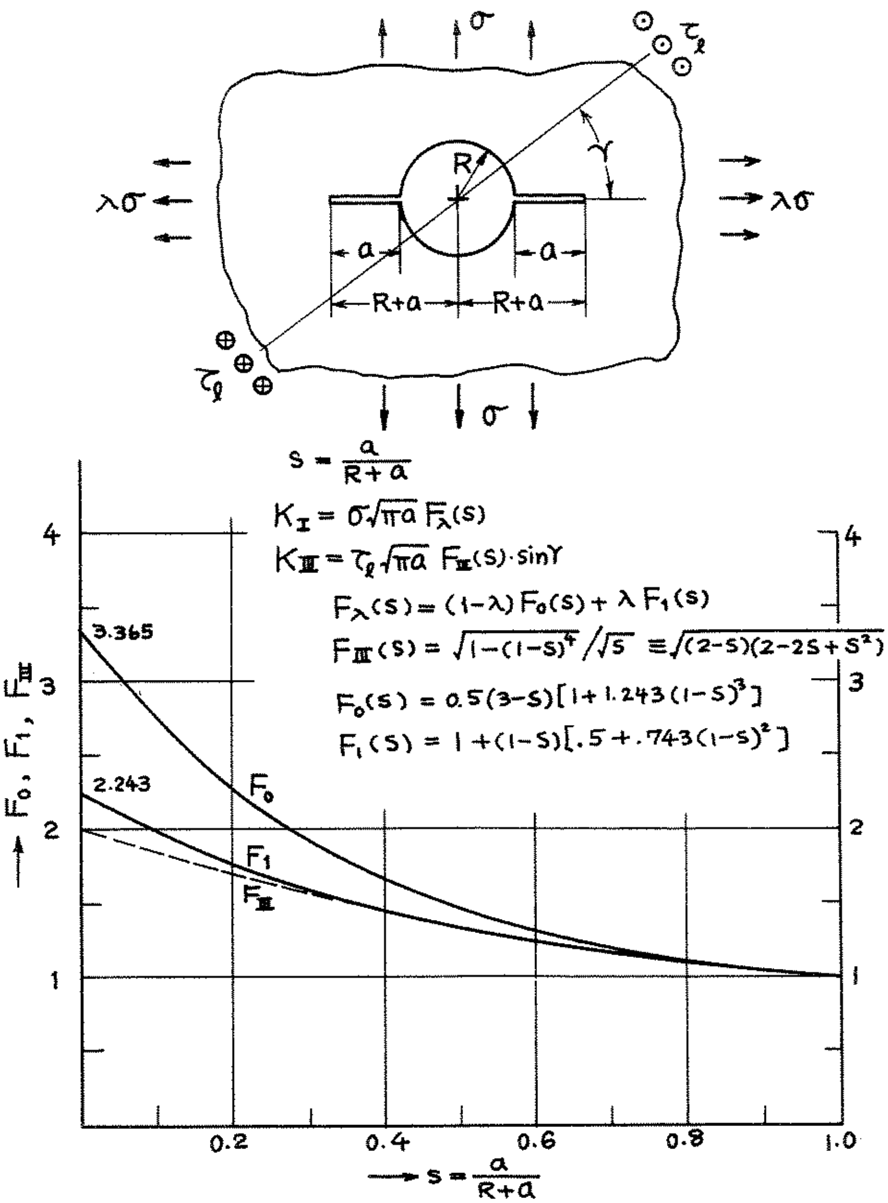
Empirical Formulas

a. Accuracy
b. Method, reference

For $s/b = 4$,

$$F(a/b) = \frac{1}{\sqrt{\pi}} \cdot \frac{1.99 - a/b(1 - a/b)\left(2.15 - 3.93\,a/b + 2.7\,(a/b)^2\right)}{(1 + 2\,a/b)(1 - a/b)^{3/2}}$$

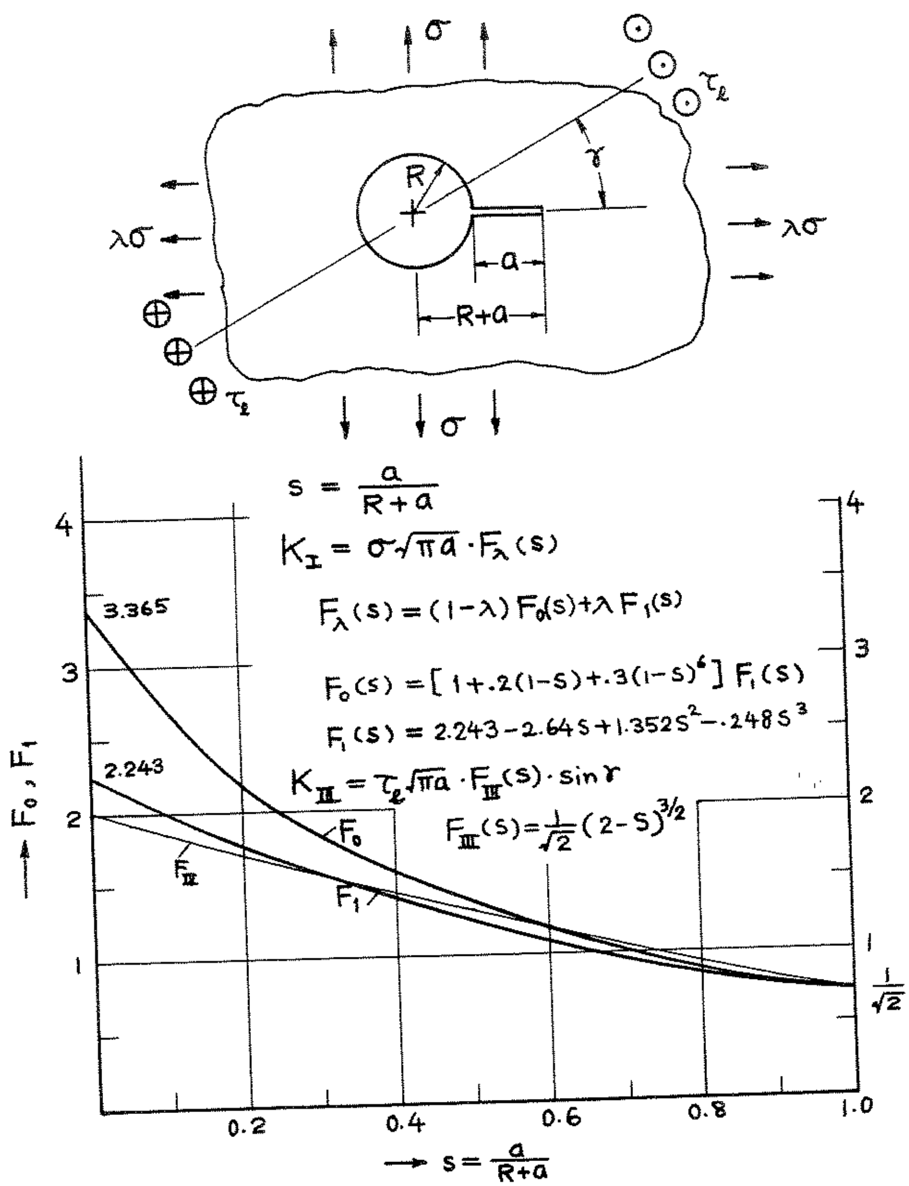
a. 0.5% for any a/b
b. **Srawley 1976**



Methods: Mapping Function Methods (Bowie — Mode I; Sih — Mode III), Boundary Collocation Method (Newman)

Accuracy: F_0 and F_1 curves are based on numerical values with expected accuracy of 0.1%. Formulas F_0 and F_1 1%; F_{III} Exact

References: **Bowie 1956; Sih 1965a; Newman 1971; Tada 1985**

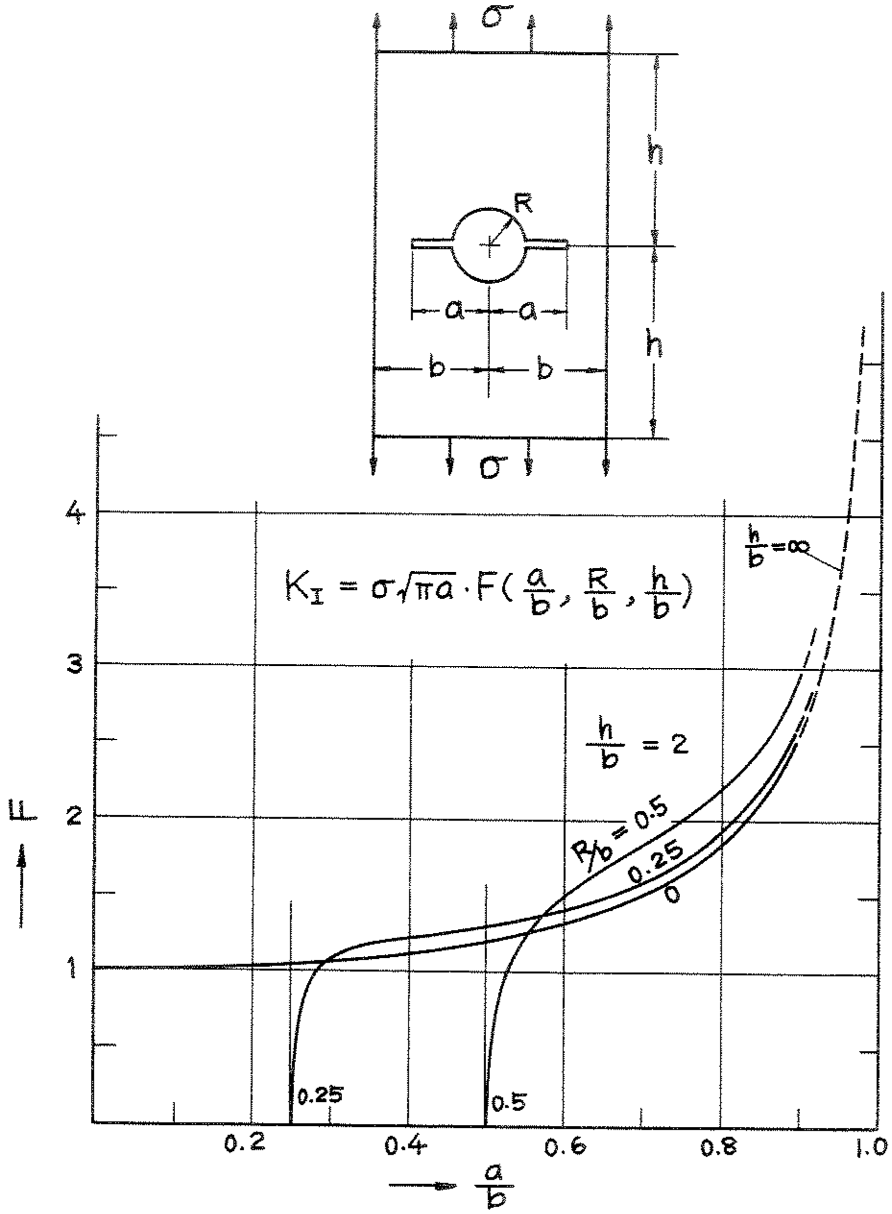


Method: Mapping Function Method

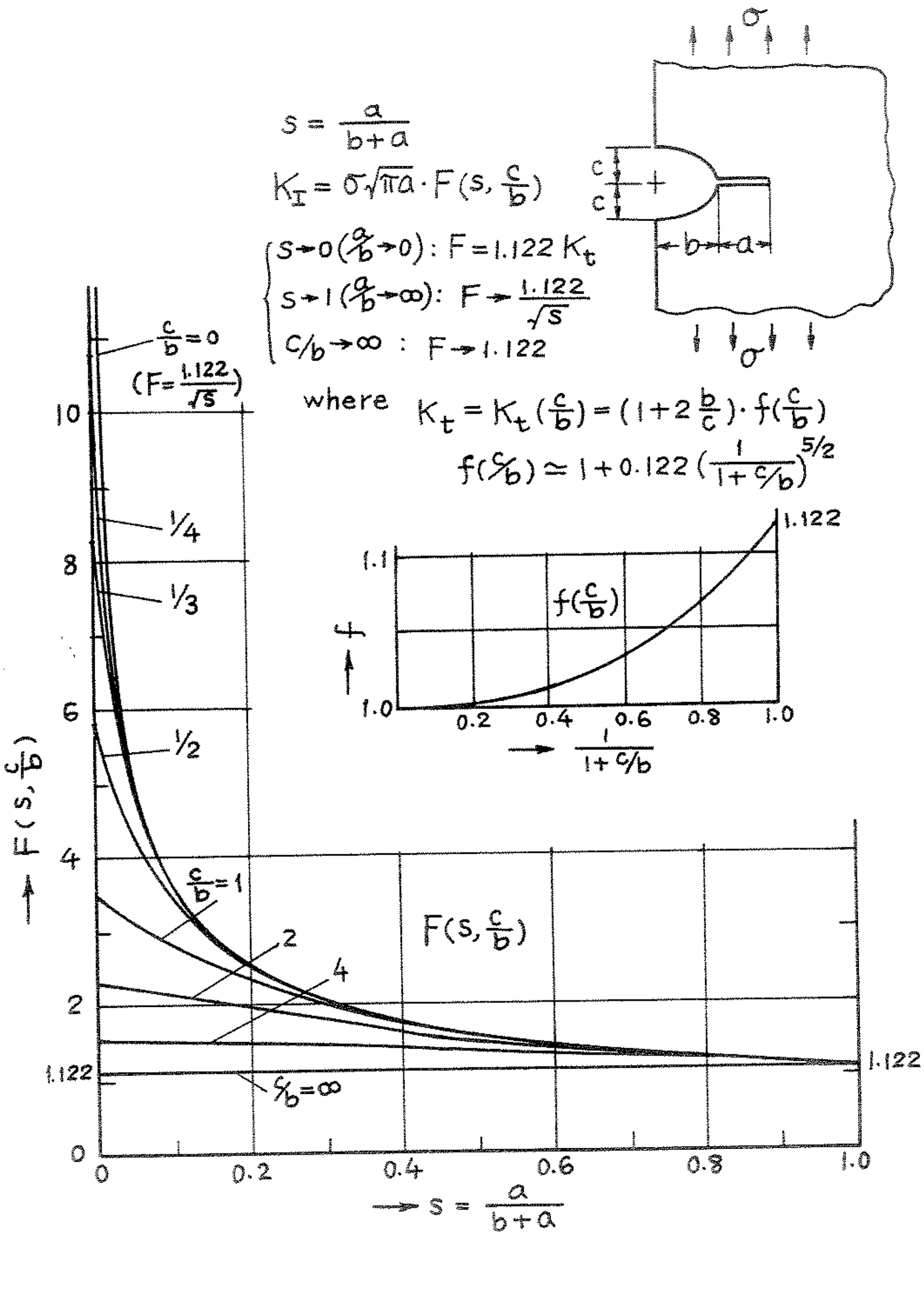
Accuracy: F_0 and F_I Better than 1%

F_{III} Exact

References: **Bowie 1956; Yokobori 1972 (or Kamei 1974); Tada 1985**



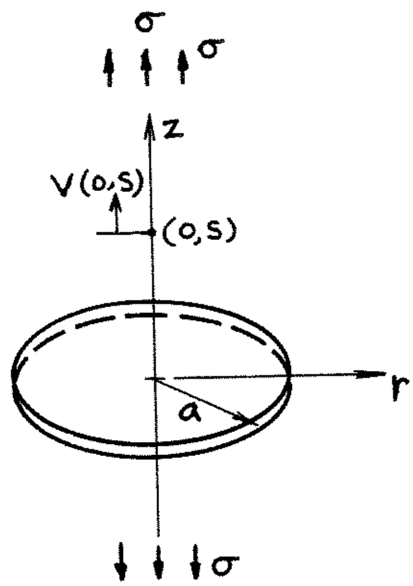
Method: Boundary Collocation Method.
Accuracy: Curves were drawn based on the results having better than 0.1% accuracy.
Reference: **Newman 1971**
See also **page 19.11**.



Methods: Stress Relaxation (Superposition) (Nishitani; $c/b = 1/2$, 1, 2 and $0.2 \le a/b \le 1$), Estimated by Interpolation (Tada)

Accuracy: Better than 2%

References: **Nishitani 1973; Tada 1973**



$$K_I = \frac{2}{\pi}\sigma\sqrt{\pi a} \qquad (K_{II} = K_{III} = 0)$$

Volume of Crack:

$$V = \frac{16\left(1-\nu^2\right)}{3E}\sigma a^3$$

Crack Opening Shape:

$$2v\underset{r\le a}{(r,0)} = \frac{8\left(1-\nu^2\right)}{\pi E}\sigma\sqrt{a^2-r^2}$$

Opening at Center:

$$\delta_0 = 2v(0,0) = \frac{8\left(1-\nu^2\right)}{\pi E}\sigma a$$

Additional Displacement at $(0,s)$ due to Crack:

$$v(0,s) = \frac{4\left(1-\nu^2\right)}{\pi E}\sigma a\left\{1-(1-\alpha)\frac{s}{a}\tan^{-1}\frac{a}{s} - \alpha\frac{s^2}{s^2+a^2}\right\}$$

where

$$\alpha = \frac{1}{2(1-\nu)}$$

Methods: Integral Transform, Integration of **page 24.5 or 24.11**, Paris' Equation (see **Appendix B**), Reciprocity (see **page 24.7**)

Accuracy: Exact

Reference: **Tada 1985**

NOTE: $V(0,s)$ Is the displacement at $(0,s)$ when uniform pressure σ is applied on crack surfaces.

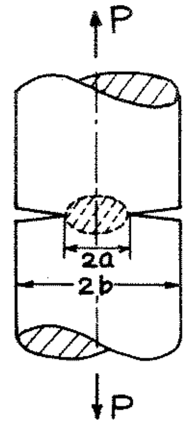
$$\sigma_{net} = \frac{P}{\pi a^2}$$

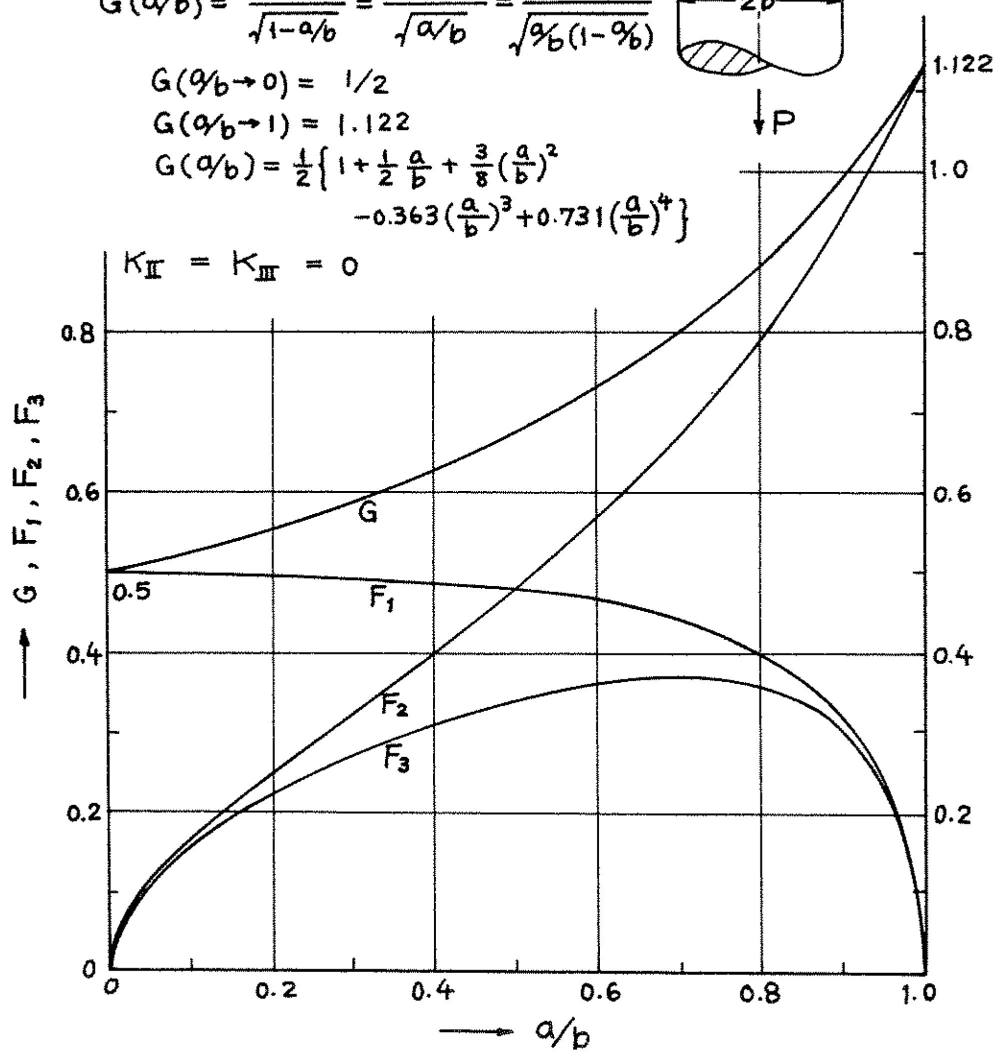
$$K_I = \sigma_{net}\sqrt{\pi a}\, F_1(a/b)$$
$$= \sigma_{net}\sqrt{\pi(b-a)}\, F_2(a/b)$$
$$= \sigma_{net}\sqrt{\pi b}\, F_3(a/b)$$

$$G(a/b) = \frac{F_1(a/b)}{\sqrt{1-a/b}} = \frac{F_2(a/b)}{\sqrt{a/b}} = \frac{F_3(a/b)}{\sqrt{a/b\,(1-a/b)}}$$

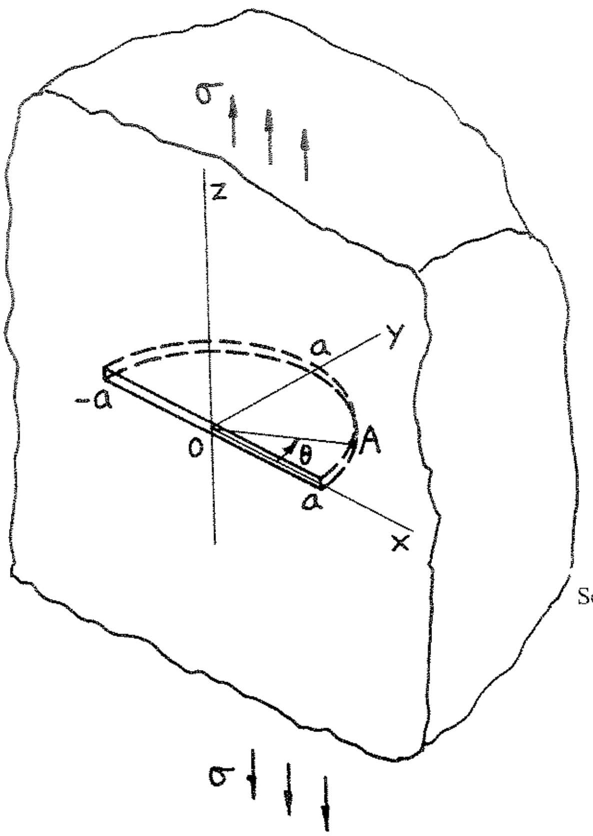
$$G(a/b \to 0) = 1/2$$
$$G(a/b \to 1) = 1.122$$
$$G(a/b) = \frac{1}{2}\left\{1 + \frac{1}{2}\frac{a}{b} + \frac{3}{8}\left(\frac{a}{b}\right)^2 - 0.363\left(\frac{a}{b}\right)^3 + 0.731\left(\frac{a}{b}\right)^4\right\}$$

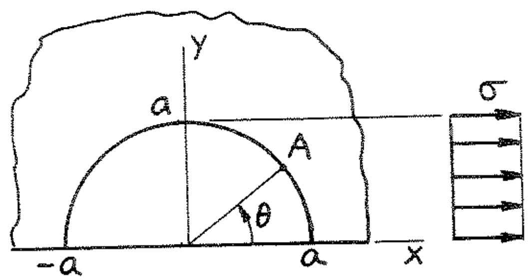
$$K_{II} = K_{III} = 0$$





Method: Singular Integral Equation (Bueckner), Asymptotic Approximation (Benthem)
Accuracy: Better than 1%
Referencees: **Bueckner 1965, 1972; Benthem 1972**
Other References: **Lubahn 1959; Wundt 1959; Irwin 1961; Paris 1965; Zahn 1965; Harris 1967**





Semi-infinite body ($y \geq 0$) ($y = 0$: Free surface)

$$K_{IA} = \frac{2}{\pi}\sigma\sqrt{\pi a}\ F(\theta)$$

$$F(\theta) = 1.211 - .186\sqrt{\sin\theta} \quad \left(10^\circ < \theta < 170^\circ\right)$$

Methods: Alternating Method (Smith, Hartranft), Finite Element Method (Tracey, Raju); $F(\theta)$ is based on Smith's result (Merkle)

Accuracy: 2%

References: **Smith 1967; Hartranft 1973; Tracey 1973; Merkle 1973; Raju 1979**

A.7 Large plates with elliptical and semi-elliptical cracks

Consider a plate specimen containing an elliptical crack. Let the major and minor axes of the ellipse be $2c$ and $2a$, respectively, as shown in Figure A.3. The stress intensity factor for the embedded elliptical flaw varies along the crackfront as a function of the angle ϕ (see Fig. A.3*b*). When the dimensions of the cracked body are much larger than a and c,

$$K_I = \frac{\sigma\sqrt{\pi a}}{\Psi}\left(\sin^2\phi + \frac{a^2}{c^2}\cos^2\phi\right)^{1/4}, \tag{A.12a}$$

where Ψ is the elliptical integral of the second kind, which is given by

$$\Psi = \int_0^{\pi/2}\left\{1 - \left(1 - \frac{a^2}{c^2}\right)\sin^2\phi\right\}^{1/2} d\phi. \tag{A.12b}$$

K_I is maximum when $\phi = 90°$. Using a series expansion for Ψ, it can be shown that

$$\Psi \approx \frac{3\pi}{8} + \frac{\pi}{8}\left(\frac{a^2}{c^2}\right). \tag{A.12c}$$

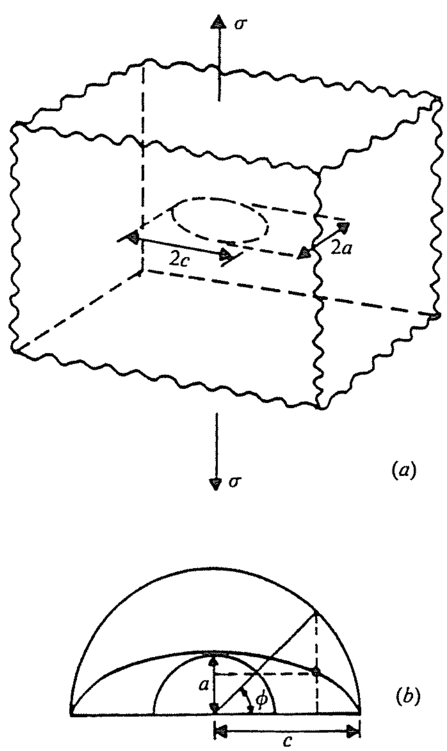


Fig. A.3. (*a*) A large plate containing an embedded elliptical crack. (*b*) Details of the crackfront.

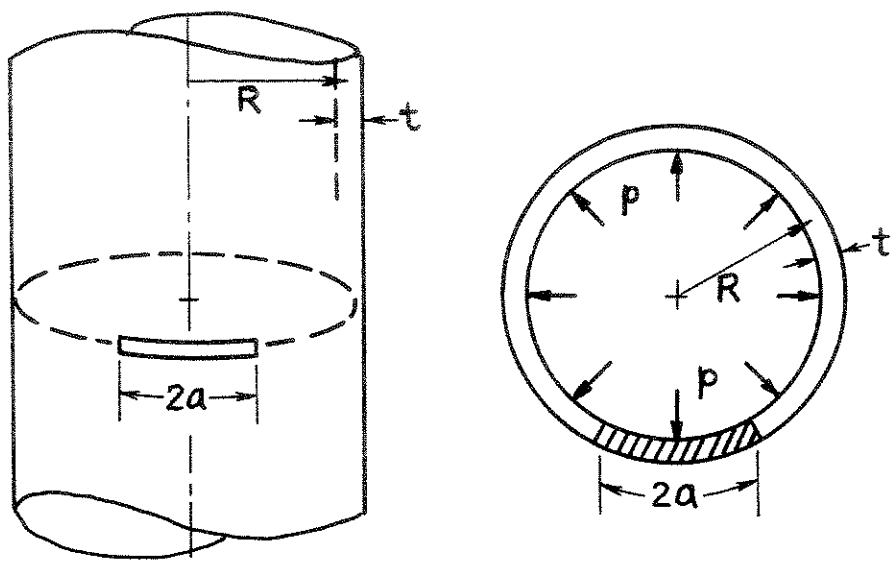
When $a = c$, we obtain the solution for a circular (penny-shaped) crack. In this case, Eqs. A.12 reduce to

$$K_{\mathrm{I}} = \frac{2}{\pi}\sigma\sqrt{\pi a}. \tag{A.13}$$

The solutions given by Eqs. A.12 can also be applied to the case of semi-elliptical surface cracks. For the semi-elliptical surface flaw (thumb-nail crack) in a finite size plate, the stress intensity factor at the mid-point (i.e. end of the minor axis, $\phi = \pi/2$) is

$$K_{\mathrm{I}} = \frac{1.12\sigma\sqrt{\pi a}}{\sqrt{Q}}, \tag{A.14}$$

where the pre-multiplier 1.12 is the free-surface correction factor, Q is the flaw shape parameter extracted from Ψ in Eq. A.12*b*, $2c$ is the surface length of the crack, and a is the maximum depth (at $\phi = \pi/2$) of the crack into the material. $Q = \Psi^2$ in the elastic limit, $\sigma/\sigma_{\mathrm{y}} \to 0$, where σ is the applied stress and σ_{y} is the yield strength of the material. Additional corrections may have to be made to Eq. A.14 to account for the proximity of the free surface to the crackfront (depending on the relative magnitudes of a and the specimen thickness B) and for crack-tip plasticity. The modified value of Q incorporating the plasticity correction is usually taken to be $Q \approx \Psi^2 - 0.212(\sigma^2/\sigma_{\mathrm{y}}^2)$.



$$\sigma = \frac{1}{2} p \frac{R}{t}$$

$$\lambda = a \Big/ \sqrt{Rt}$$

$$K_I = \sigma\sqrt{\pi a} \cdot F(\lambda)$$

$$F(\lambda) = \left(1 + .3225\lambda^2\right)^{1/2} \qquad 0 < \lambda \leq 1$$

$$= 0.9 + 0.25\lambda \qquad 1 \leq \lambda \leq 5$$

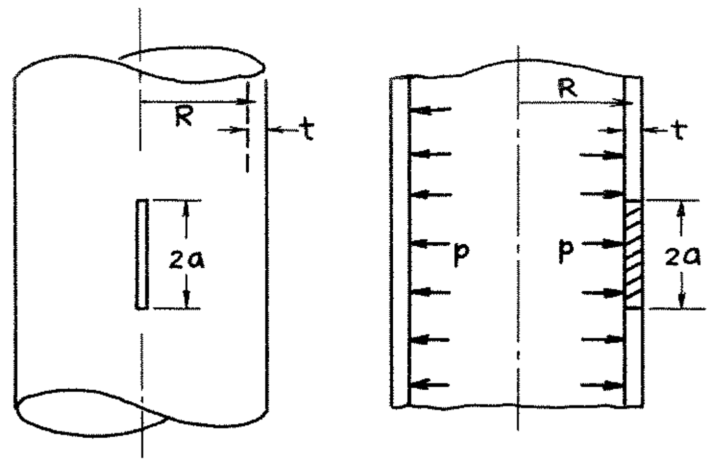
Crack Opening Area:

$$A = \frac{\sigma}{E'} (2\pi Rt) \cdot G(\lambda)$$

$$G(\lambda) = \lambda^2 + 0.16\lambda^4 \qquad 0 < \lambda \leq 1$$

$$= .02 + .81\lambda^2 + .30\lambda^3 + .03\lambda^4 \qquad 1 \leq \lambda \leq 5$$

Methods: K_I Integral Equation; A Paris' Equation (see **Appendix B**)
Accuracy: K_I 1%; A 2%
References: **Folias 1967; Fama 1972; Tada 1983a**



$$\sigma = p\frac{R}{t}$$

$$\lambda = a\Big/\sqrt{Rt}$$

$$K_I = \sigma\sqrt{\pi a}\cdot F(\lambda)$$

$$F(\lambda) = (1 + 1.25\lambda^2)^{1/2} \qquad 0 < \lambda \leq 1$$

$$= 0.6 + 0.9\,\lambda \qquad 1 \leq \lambda \leq 5$$

Crack Opening Area:

$$A = \frac{\sigma}{E'}(2\pi Rt)\cdot G(\lambda)$$

$$G(\lambda) = \lambda^2 + .625\lambda^4 \qquad 0 < \lambda \leq 1$$

$$= .14 + .36\lambda^2 + .72\lambda^3 + .405\lambda^4 \qquad 1 \leq \lambda \leq 5$$

Methods: K Integral Transform; A Paris' Equation (see **Appendix B**)
Accuracy: K_I 1%; A 2%
References: **Folias 1965; Erdogan 1969; Tada 1983a**

NOTE: As $a \to \infty$, $K_I \to \sigma\sqrt{R}\left(\frac{\sqrt{27\pi}}{2}\cdot\frac{R}{t}\right)$ **(Harris 1997)**.