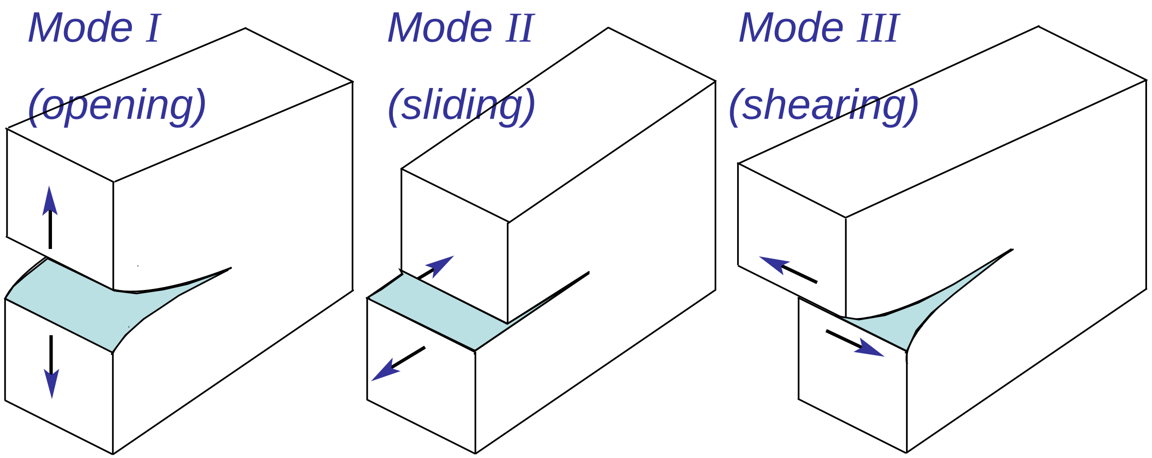
University of Liège
Aerospace & Mechanical Engineering

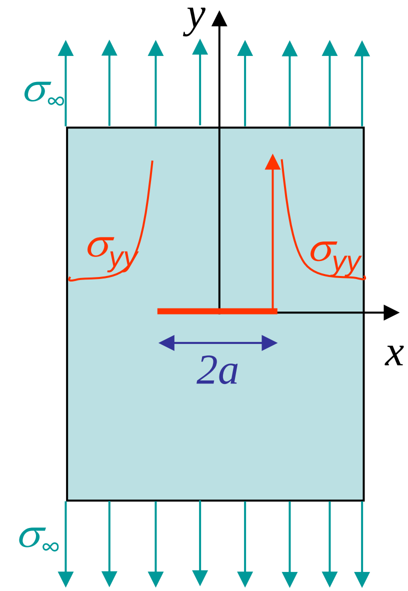
Fracture Mechanics, Damage and Fatigue
Non Linear Fracture Mechanics: J-Integral

Ludovic Noels

Computational & Multiscale Mechanics of Materials – CM3
http://www.ltas-cm3.ulg.ac.be/
Chemin des Chevreuils 1, B4000 Liège
L.Noels@ulg.ac.be

- Cracked body: summary
  - 3 failure modes

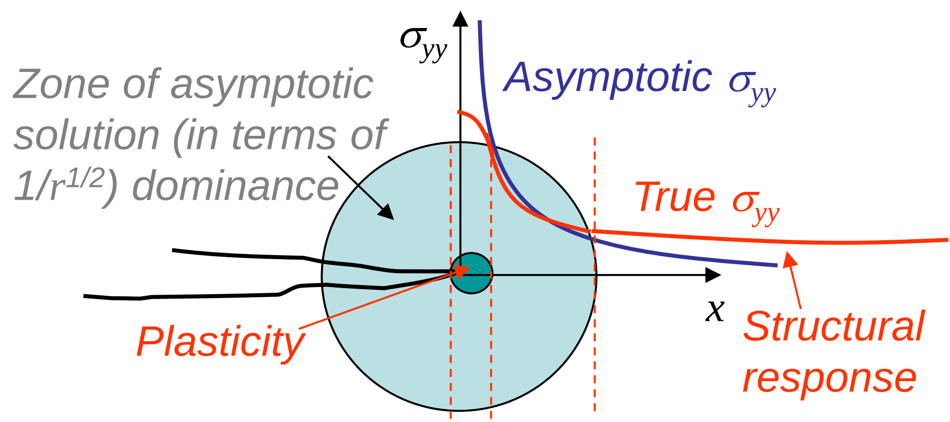




  - Asymptotic solution governed by stress intensity factors

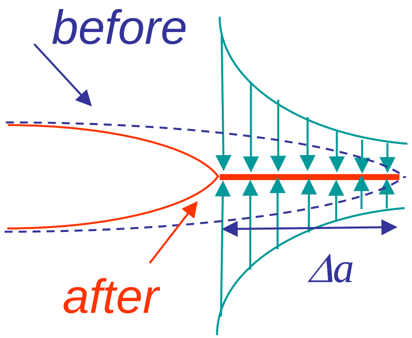
$$\boldsymbol{\sigma}^{\text{mode } i} = \frac{K_i}{\sqrt{2\pi r}} f^{\text{mode } i}(\theta) + o(r^0)$$

$$\boldsymbol{u}^{\text{mode } i} = K_i \sqrt{\frac{r}{2\pi}}\, g^{\text{mode } i}(\theta) + o(r^0)$$



- Cracked body: summary
  - Potential energy $\Pi_T = E_{\text{int}} - Qu$
  - Crack closure integral
    - Energy required to close crack tip

$$\Delta\Pi_T = \int_{\Delta A}\int_{\boldsymbol{u}}^{\boldsymbol{u}+\boldsymbol{\Delta u}} \boldsymbol{t}\cdot [\![\boldsymbol{u}']\!] d\boldsymbol{u}' \; dA$$

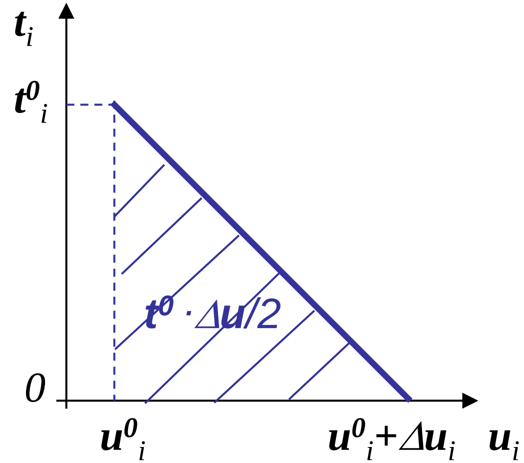


  - Energy release rate
    - Variation of potential energy in case of crack growth

$$G = -\partial_{\text{A}}\left(E_{\text{int}} - W_{\text{ext}}\right) = -\partial_A \Delta\Pi_T$$

    - In linear elasticity

$$G = -\partial_A\Delta\Pi_T = -\lim_{\Delta A\to 0}\frac{1}{\Delta A}\int_{\Delta A}\frac{1}{2}\boldsymbol{t}^{\mathbf{0}}\cdot[\![\boldsymbol{\Delta u}]\!]\,dA$$

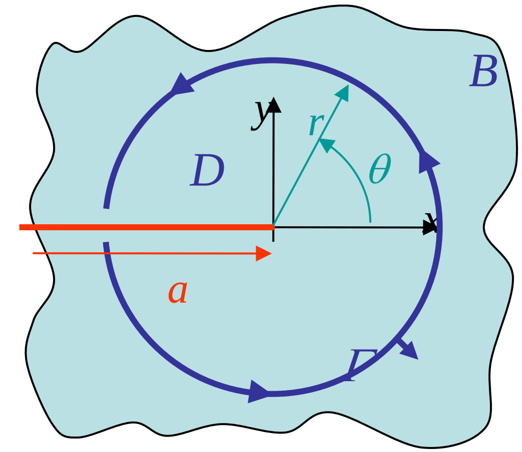


  - In linear elasticity & if crack grows straight ahead

$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \qquad E' = \frac{4E}{(1+\nu)(\kappa+1)} = \begin{cases} E & \text{P}\sigma \\ \frac{E}{1-\nu^2} & \text{P}\varepsilon \end{cases}$$

- Cracked body: summary
  - J-integral
    - Strain energy flow

$$J = \int_{\Gamma} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \right] dl$$

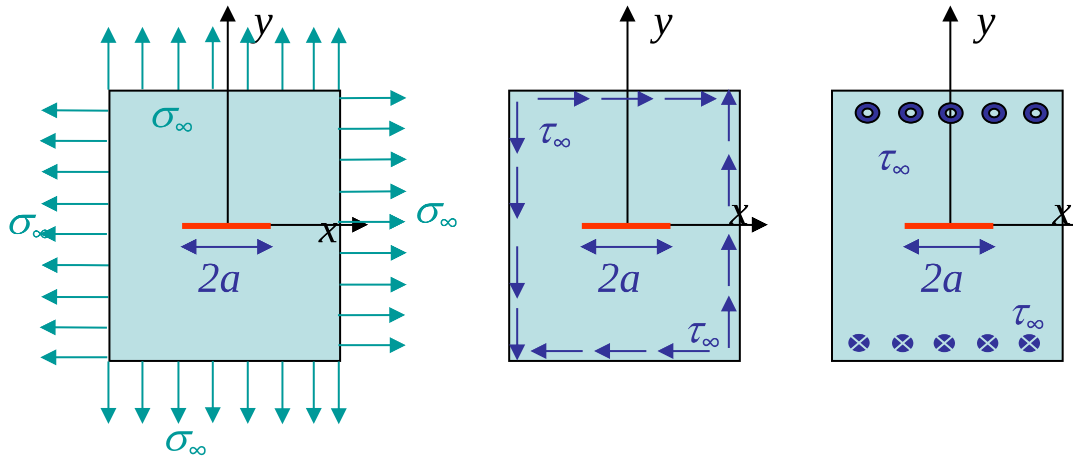


  - Exists if an internal potential exists
    - Is path independent if the contour Γ embeds a straight crack tip
    - No assumption on subsequent growth direction
    - Can be extended to plasticity if no unloading (see later)
  - If crack grows straight ahead $\Longrightarrow$ $G=J$
  - In linear elasticity (independently of crack growth direction):

$$J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \qquad E' = \frac{4E}{\left(1+\nu\right)\left(\kappa+1\right)} = \begin{cases} E & \text{P}\sigma \\ \frac{E}{1-\nu^2} & \text{P}\varepsilon \end{cases}$$

- Analytical
  - SIF from full-field solution
    - Limited cases

$$\Rightarrow \begin{cases} K_I = \sigma_\infty \sqrt{\pi a} \\ K_{II} = \tau_\infty \sqrt{\pi a} \\ K_{III} = \tau_\infty \sqrt{\pi a} \end{cases}$$



  - From energetic consideration
    - Growing straight ahead crack
    - From J-integral

$$G = -\partial_A \left(E_{\text{int}} - W_{\text{ext}}\right) \Rightarrow G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

$$J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

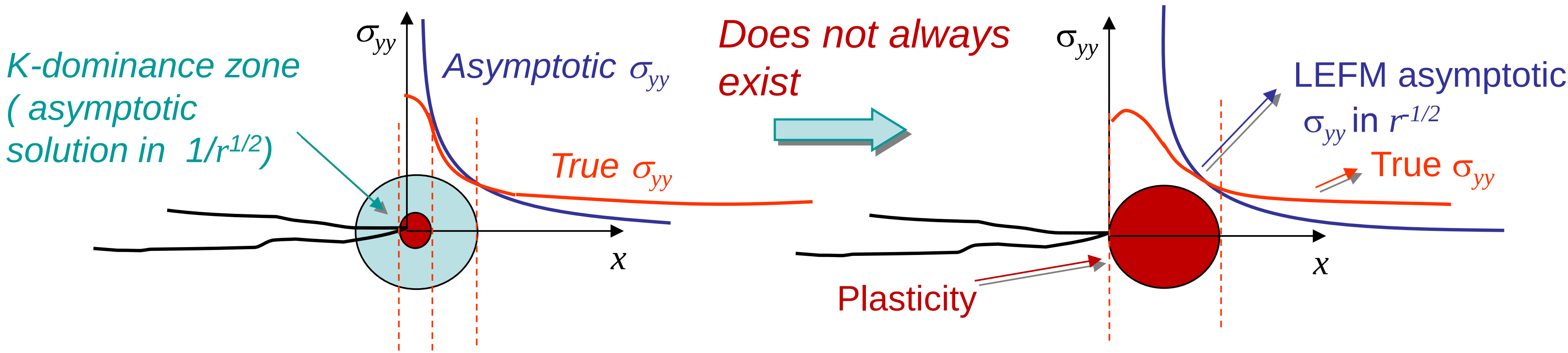
- Numerical (e.g. FEM)

$$\Rightarrow \begin{cases} K_I = \beta_I \sigma_\infty \sqrt{\pi a} \\ K_{II} = \beta_{II} \tau_\infty \sqrt{\pi a} \\ K_{III} = \beta_{III} \tau_\infty \sqrt{\pi a} \end{cases}$$

  - β_i depends on geometry & crack length
- Tabulated solutions (handbooks)
  - http://ebooks.asmedigitalcollection.asme.org/book.aspx?bookid=230

Linear Elastic Fracture Mechanics (LEFM)

- Small Scale Yielding assumption
  - LEFM: we have assumed the existence of a K-dominance zone



  - This holds of if the process zone (in which irreversible process occurs)
    - Is a small region compared to the specimen size &
    - Is localized at the crack tip
  - Validity of this approach?
    - We check the dimensions

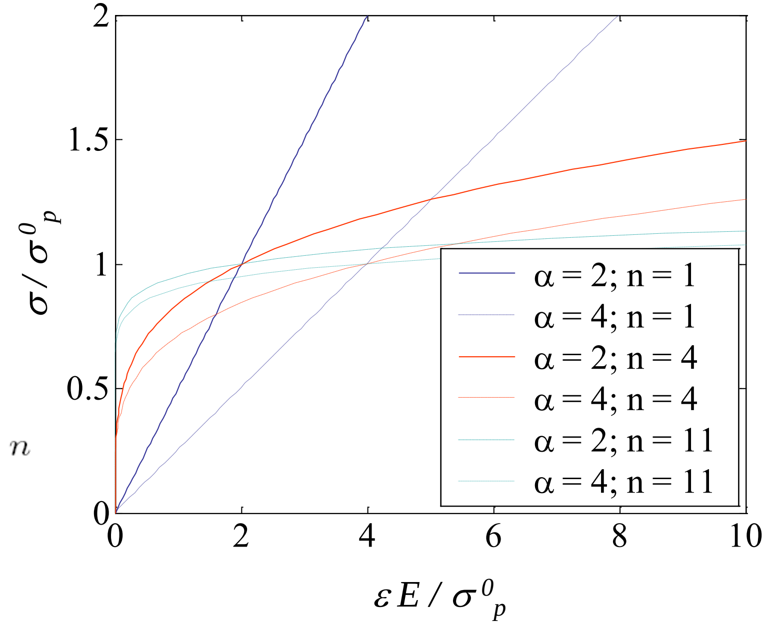
$$a, W - a > 2.5 \left(\frac{K_I}{\sigma_p^0}\right)^2$$

∝Process
zone size

- Non-linear fracture mechanics
  - Derivation of the LEFM validity criterion
  - Providing solutions when LEFM criterion is not met

- Power law
  - This law can be rewritten in terms of the total deformations
    - Yield stress is replaced by equivalent stress $\sigma_e = \sqrt{\frac{3}{2}\mathbf{s}:\mathbf{s}}$
    - Plastic strain is replaced by equivalent strain $\bar{\varepsilon} = \sqrt{\frac{2}{3}\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}}$
    - The governing law becomes

$$\sigma_e = \sigma_p^0 \left(\frac{\bar{\varepsilon}}{\alpha\sigma_p^0/E} \right)^{\frac{1}{n}} \text{ or } \bar{\varepsilon} = \frac{\alpha\sigma_p^0}{E} \left(\frac{\sigma_e}{\sigma_p^0} \right)^n$$



    - Parameter n
      - $n \to \infty$: perfect plasticity
      - $n \to 1$: “elasticity”
  - Doing so requires 2 assumptions
    - There is no unloading
    - As elastic strains are assimilated to plastic strains, the material is incompressible
  - Which are satisfied if
    - We are interested **only in crack initiation and not in crack propagation**
    - The stress components remain proportional with the loading
    - **Elastic deformations are negligible** compared to plastic ones

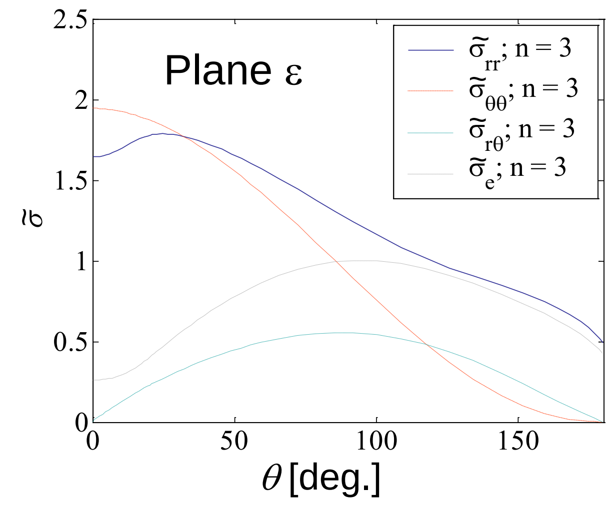
- Summary
  - Assumptions
    - J2-plasticity with power law description
    - Small deformations
    - There is no unloading and loading is proportional in all the directions (ok for **crack initiation and not for crack propagation**)
    - Elastic strains are assimilated to plastic strain (material is **incompressible**)
    - Semi-infinite crack
  - HRR results for semi-infinite **mode I crack**
    - Asymptotic stress, strain and displacement fields

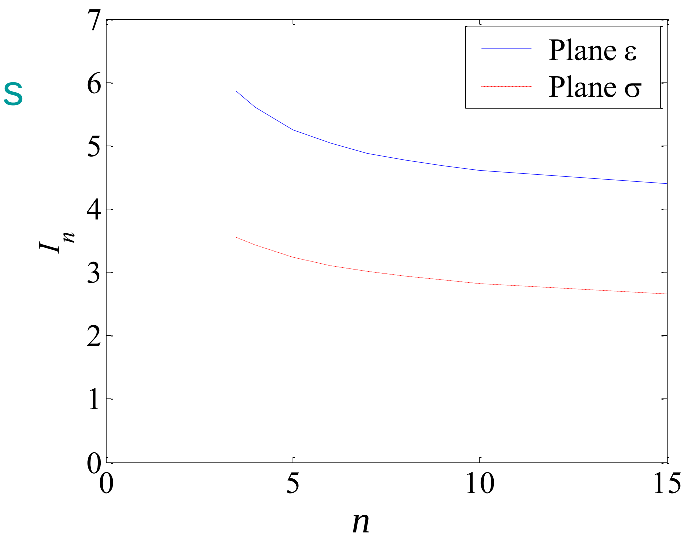
$$\boldsymbol{\sigma} = \sigma_p^0 \left(\frac{JE}{r\alpha \left(\sigma_p^0\right)^2 I_n} \right)^{\frac{1}{n+1}} \tilde{\boldsymbol{\sigma}}\left(\theta,\, n\right)$$

$$\boldsymbol{\varepsilon} = \frac{\sigma_p^0 \alpha}{E} \left(\frac{JE}{r\alpha \left(\sigma_p^0\right)^2 I_n} \right)^{\frac{n}{n+1}} \tilde{\boldsymbol{\varepsilon}}\left(\theta,\, n\right)$$

$$\boldsymbol{u}\left(r,\, \pi\right) = \frac{\alpha \sigma_p^0}{E} \left(\frac{JE}{\alpha \left(\sigma_p^0\right)^2 I_n} \right)^{\frac{n}{n+1}} r^{\frac{1}{n+1}} \tilde{\boldsymbol{u}}\left(\pi,\, n\right)$$

- J is a "plastic strain intensity factor"

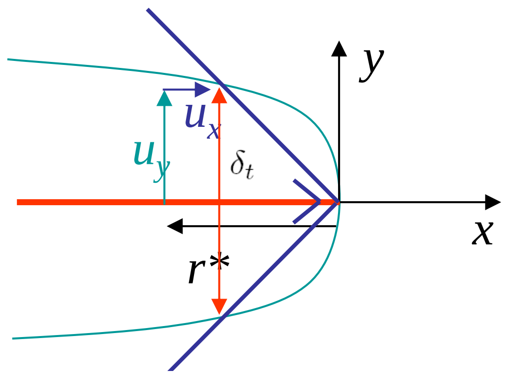




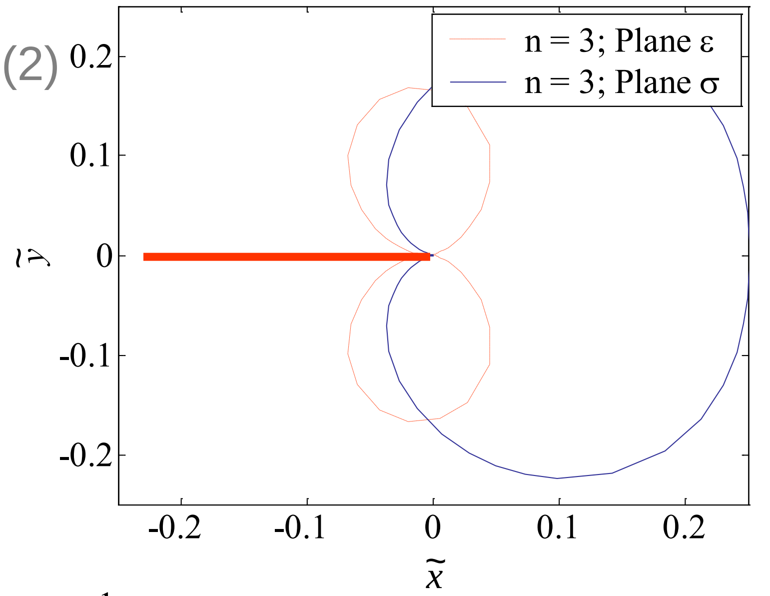
- Summary (2)
  - HRR results for semi-infinite **mode I crack** (2)
    - Process zone $\tilde{r}\left(\theta,\, n\right) = \frac{\left(\tilde{\sigma}_e\left(\theta,\, n\right)\right)^{n+1}}{I_n}$

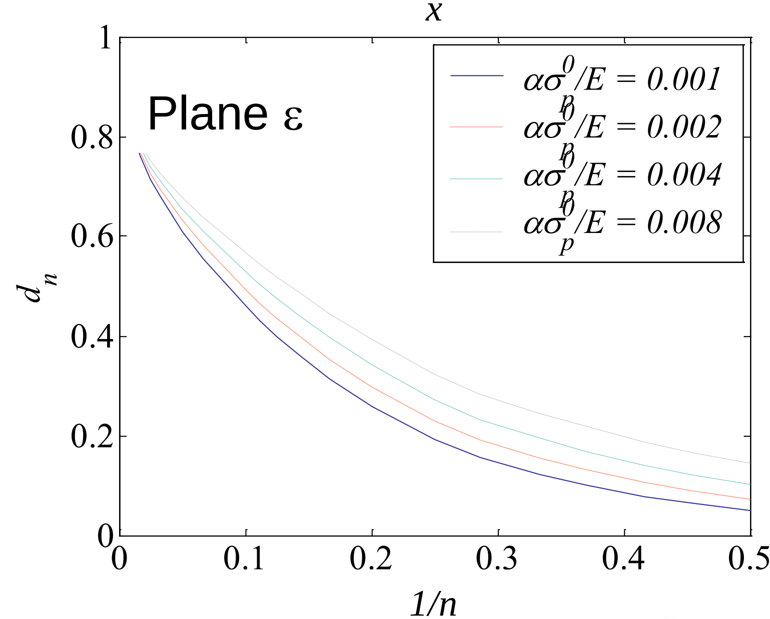
      with $\tilde{r} = \frac{r\alpha\left(\sigma_p^0\right)^2}{JE}$

      - If SSY ⟹ $r \propto \left(1-\nu^2\right)\left(\frac{K_I}{\sigma_p^0}\right)^2$
    - CTOD
      - New definition

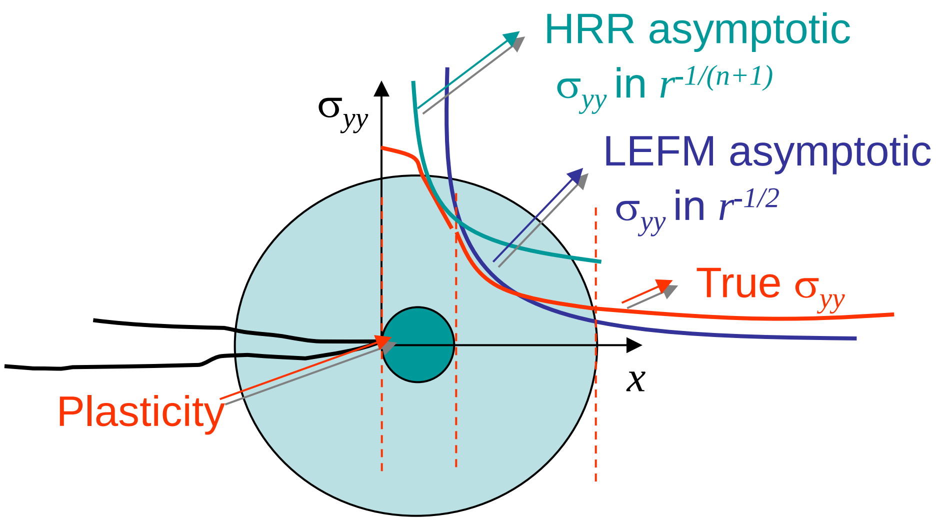


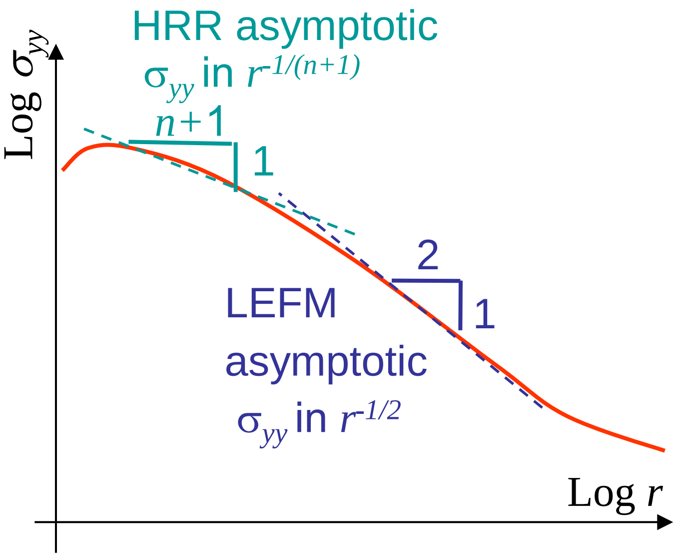
      - $\delta_t = d_n \frac{J}{\sigma_p^0}$
      - Also function of J
  - We did not assume SSY !!!





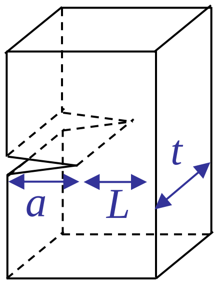
- Validity in SSY
  - We have two asymptotic solutions
    - HRR field is valid in the process zone
    - LEFM is still valid in the elastic zone close to the crack tip



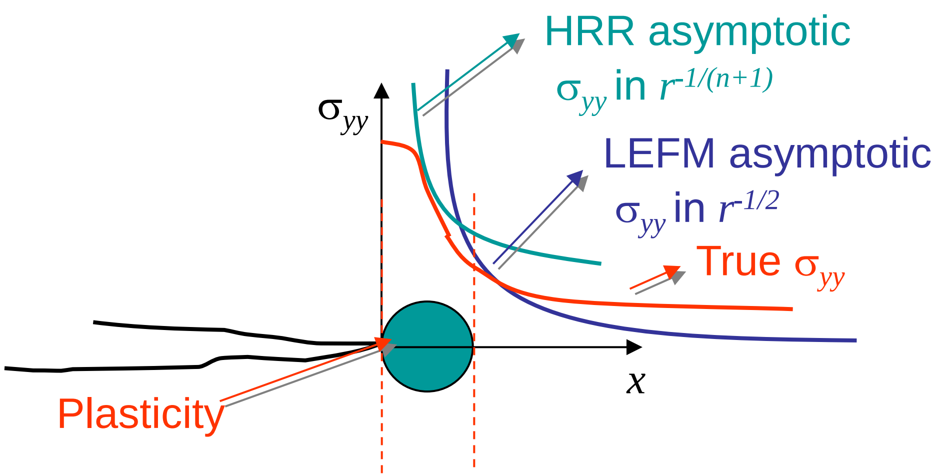


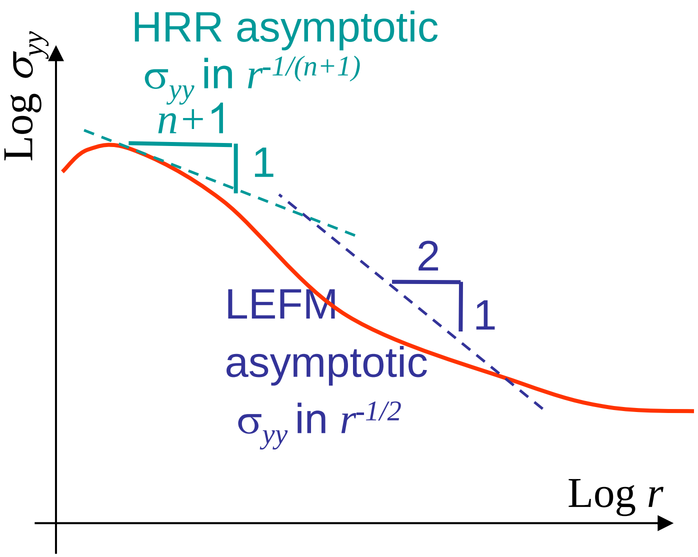
  - Conditions
    - This is the case if all sizes are 25 times larger than the plastic zone

$$a,\, t,\, L > 25 r_p \simeq \frac{25}{3\pi}\left(\frac{K_I}{\sigma_p^0}\right)^2 \simeq 2.5\left(\frac{K_I}{\sigma_p^0}\right)^2$$



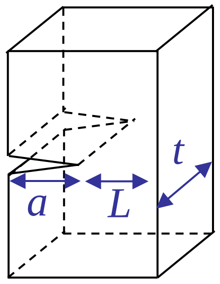
- Validity in elasto-plastic conditions
  - Deformations are small
  - We still have one asymptotic solution valid
    - HRR field is valid in the process zone
  - LEFM is **NOT** valid in the elastic zone close to the crack



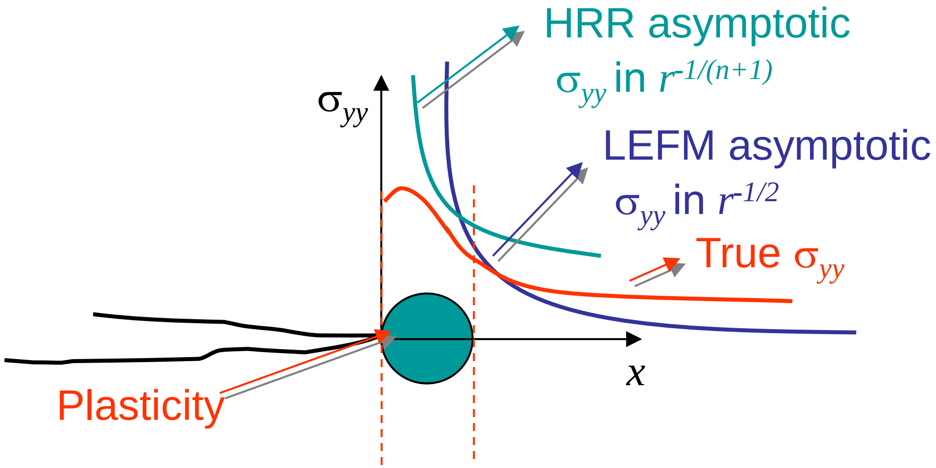


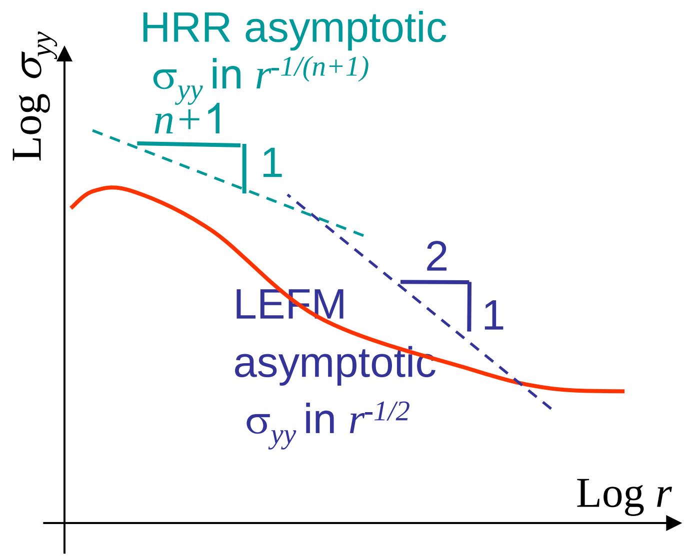
  - Conditions
    - This is the case if all sizes are 25 times larger than CTOD

$$a,\ t,\ L > 25\delta_t \simeq 25\frac{J}{\sigma_p^0}$$

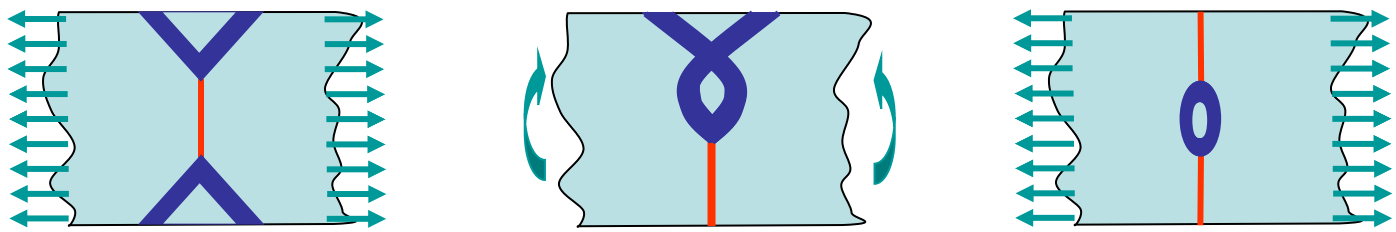


- Validity in large yielding
  - Example: ligament size is too small
  - Small deformations assumption does not hold
  - Neither HRR field nor LEFM asymptotic fields are valid





  - There is no zone of J-dominance
  - Plastic strain concentrations depend on the experiment

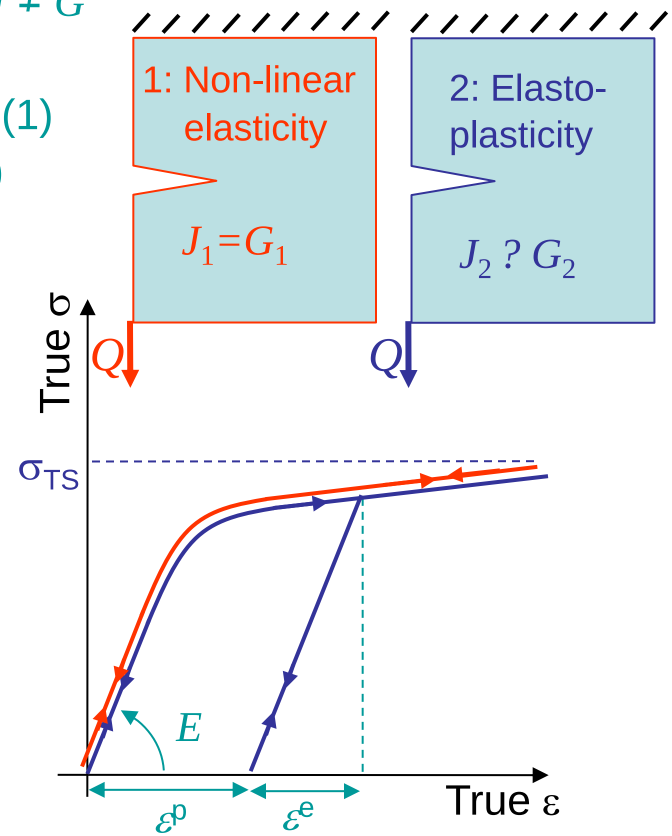


- Crack initiation criteria
  - In SSY: $a,\, t,\, L > 25 r_p \simeq \frac{25}{3\pi}\left(\frac{K_I}{\sigma_p^0}\right)^2 \simeq 2.5\left(\frac{K_I}{\sigma_p^0}\right)^2$
    - Criteria based on J or δ_t are valid: $J \geq J_C$ or $\delta_t \geq \delta_C$
      - J & δ_t depend on a, the geometry, the loading, …
    - But as the LEFM solution holds, we can still use $K(a) \geq K_C$
      - May be corrected by using the effective length a_{eff} if $\sigma_\infty < 50\%$ of σ_p^0
  - In Elasto-Plastic conditions: $a,\, t,\, L > 25\delta_t \simeq 25\frac{J}{\sigma_p^0}$
    - Criteria based on J or δ_t are valid: $J \geq J_C$ or $\delta_t \geq \delta_C$
      - J & δ_t depend on a, the geometry, the loading, …
    - The LEFM solution **DOES NOT** hold, we **CANNOT** use $K(a) \geq K_C$
  - In Large Scale Yielding
    - Plastic strain concentrations depend on the experiment
    - Zones near free boundaries or other cracks tend to be less stressed
      - Solution is no longer uniquely governed by J
      - Relation between J & δ_t is **dependent on the configuration and on the loading**
      - The critical J_C measured for an experiment might not be valid for another one
      - A 2-parameter characterization is needed

Relation J-G

- For elasto-plastic materials
  - J is a useful concept as it can be used as crack initiation criterion
  - Relation J-G
    - $J = G$ if an internal potential is defined **&** if the crack grows straight ahead
    - For an elasto-plastic material
      - Since no internal potential can be defined $J \neq G$
      - But let us consider two specimens
        - » One with a non-linear elastic material (1)
        - » One with an elasto-plastic material (2)

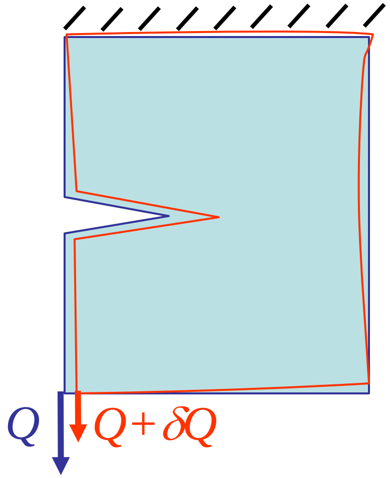
        with the same loading response
      - If the crack will grow straight ahead
        - » $J_1 = G_1$ as there is a potential defined
        - » Before crack propagation (which would involve possible unloading) the stress state is the same for the two materials $\Rightarrow J_2 = J_1 = G_1 = G_2$
    - If the crack will grow straight ahead, before crack propagation: $J = G$ for an elasto-plastic material

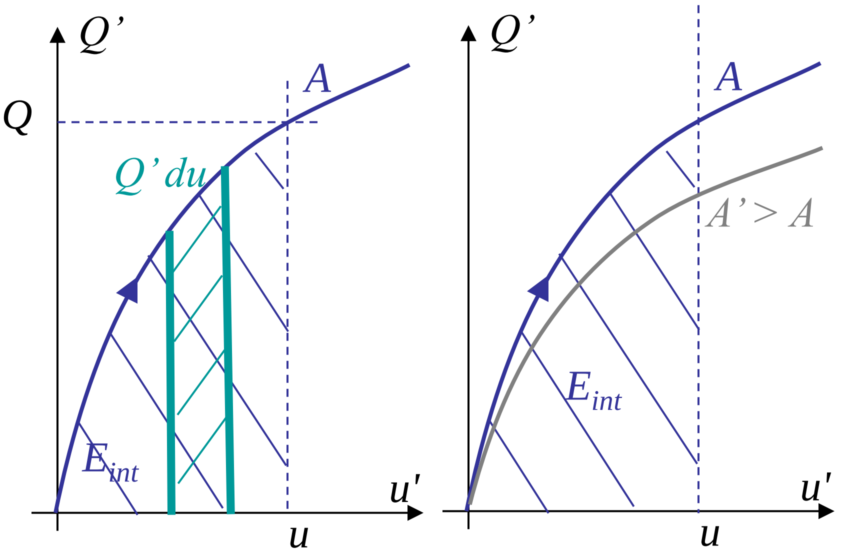


- Available methods of computing J
  - Numerical (see previous lectures)
    - Contour integral
    - Domain integral
  - Energetic approach
    - Numerical
    - Experimental
  - Experimental
    - Multiple specimen testing
    - Deeply notched SENB
    - Eta factor approach
  - Engineering
    - Use of handbooks
- If J expression is known it can be applied to
  - Fracture toughness test
  - Fracture criteria and stability prediction

- Prescribed displacement
  - Let us consider a specimen
    - With a prescribed displacement
    - A compliance depending on the crack length
  - Using compliance curves
    - Energy release rate: $G = -\partial_A E_{\text{int}}$
    - Internal energy: $\partial_u E_{\text{int}}(u) = Q$
    - Energy release rate in terms of displacements
      - We have $-\partial_u G = \partial_A Q$

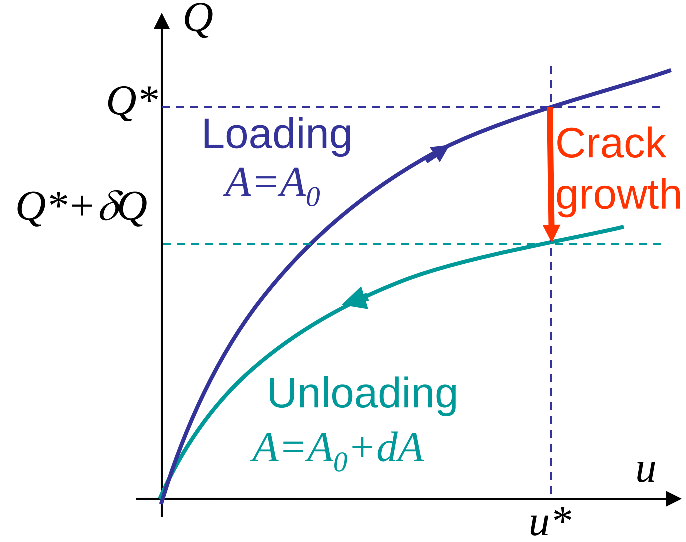
$$\Rightarrow G = -\int_0^u \partial_A Q(u',\, A)\, du'$$





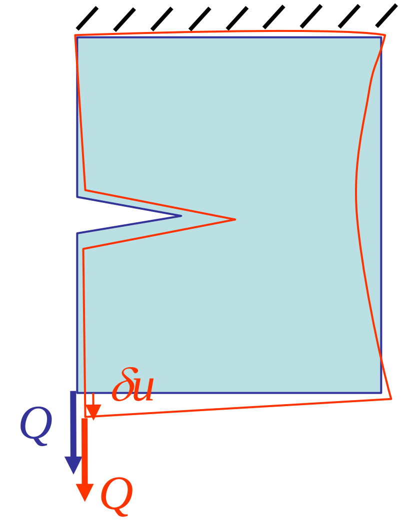
Computation of J : Energetic approach

- Prescribed displacement (2)
  - Interpretation of $G = -\int_0^u \partial_A Q\left(u',\, A\right) du'$ for an elastic material



    - Body with crack surface A_0 loaded up to Q^*
    - Crack growth dA at constant grip
      - ⇒ the specimen becomes more flexible
      - ⇒ the load decreases by $\partial_A Q dA$
    - Unload to zero
    - The area between the 2 curves is then - $G\, dA$
  - For an elasto-plastic material
    - Since $J = G$ only before crack growth, this method can be used
      - Either experimentally with 2 specimens with a different initial crack length
      - Or by computing the curve $u(a, Q)$
        - » Analytically
        - » Using FEM

Computation of J : Energetic approach



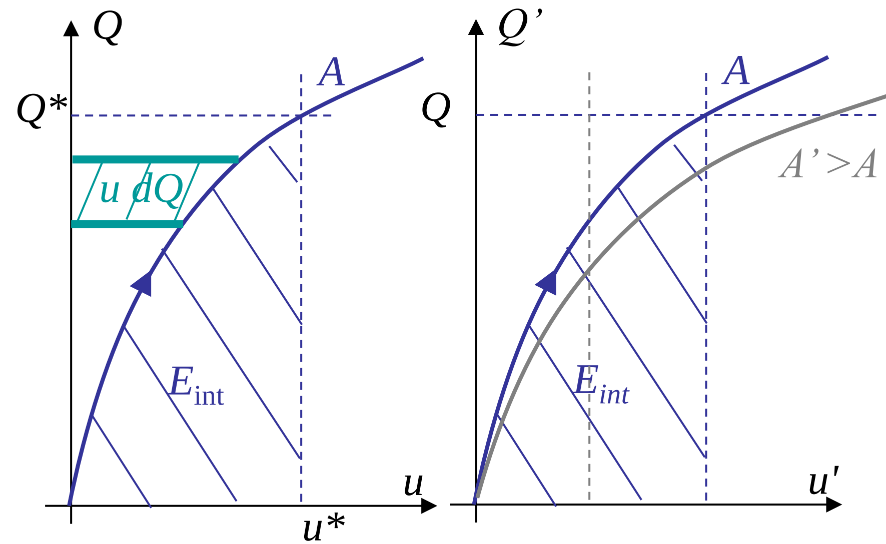
- Prescribed loading
  - Let us consider a specimen
    - With a prescribed loading (dead load)
    - A compliance depending on the crack length
  - Using compliance curves
    - Energy release rate: $G = -\partial_A \left(E_{\text{int}} - Qu\right)$
    - Complementary energy:

$$Qu - E_{\text{int}} = \int_0^Q u\left(Q',\, A\right) dQ'$$

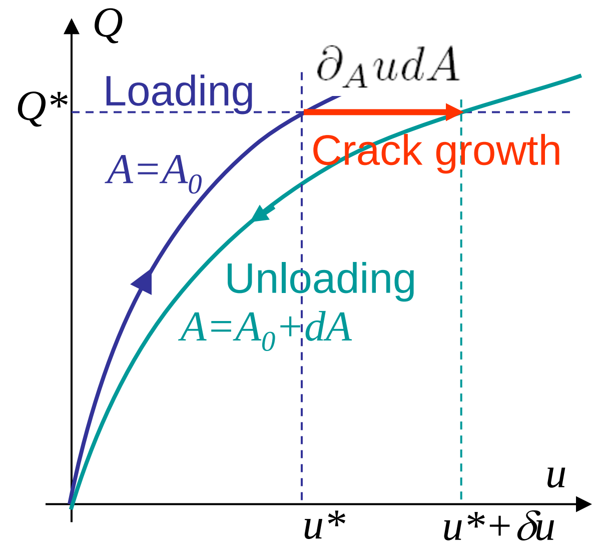
$$\Longrightarrow \quad u\left(Q,\, A\right) = -\partial_Q \left(E_{\text{int}} - Qu\right)$$

    - Energy release rate in terms of displacements
      - We have $\partial_Q G = \partial_A u$

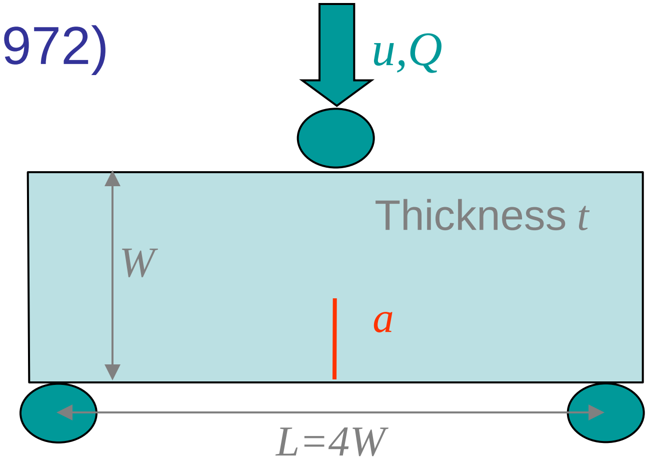
$$\Longrightarrow \quad G = \int_0^Q \partial_A u\left(Q',\, A\right) dQ'$$

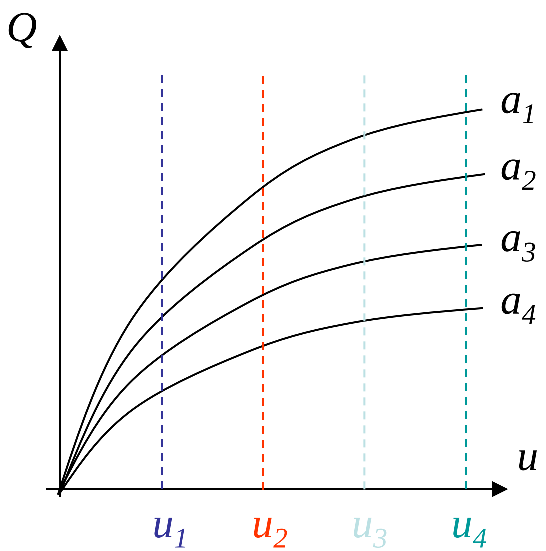


- Prescribed loading (2)
  - Interpretation of $G = \int_0^Q \partial_A u\,(Q',\, A)\, dQ'$ for an elastic material
    - Body with crack surface A_0 loaded up to Q^*
    - Crack growth dA at constant load
      - ⟹ the specimen becomes more flexible
      - ⟹ displacement increment $\partial_A u dA$
    - Unload to zero
    - The area between the 2 curves is then $G\, dA$
  - For an elasto-plastic material
    - Since $J = G$ only before crack growth, this method can be used
      - Either experimentally with 2 specimens with a different initial crack length
      - Or by computing the curve $Q(a, u)$
        - » Analytically
        - » Using FEM

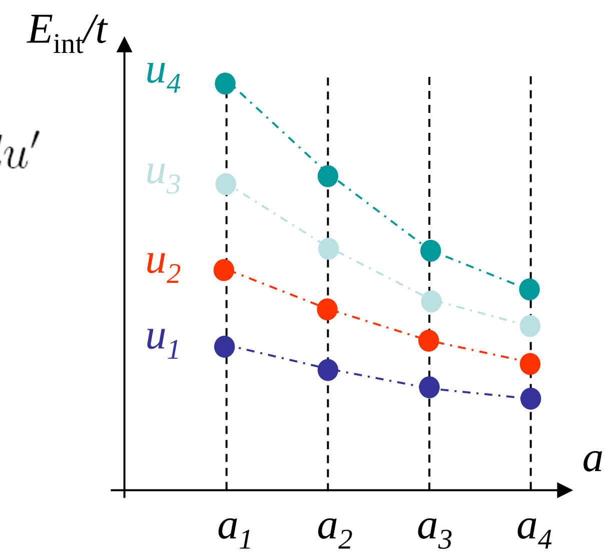


- Multiple specimen testing (Begley & Landes, 1972)
  - Consider 4 specimens with
    - 4 different crack lengths $a_1 < a_2 < a_3 < a_4$
    - Under displacement control
    - No fracture taking place
  - This leads to compliance curves
    - Integration to obtain the internal energies





$$E_{\text{int}}(a) = \int_0^u Q(u',\, a)\, du'$$

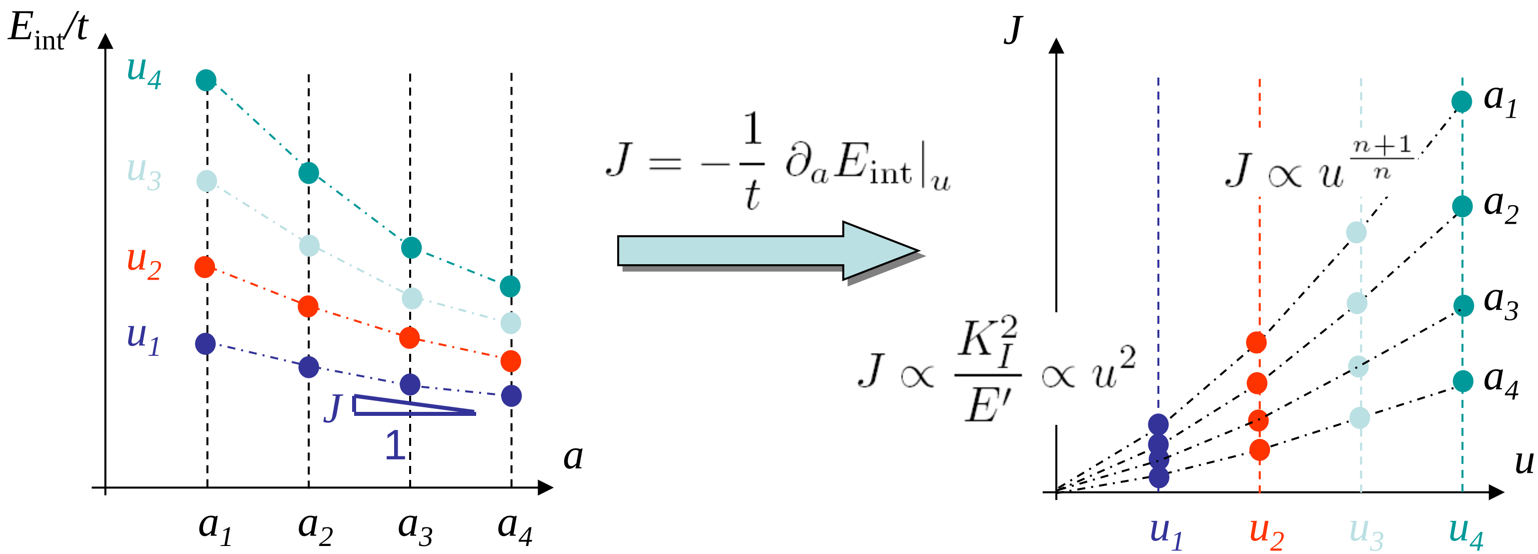


Experimental determination of J

- Multiple specimen testing (2)
  - Since $G = -\partial_A E_{\text{int}}$

    ⟹ The area between the curves $u(a, Q)$ and $u(a+da, Q)$ is equal to - $G\,t\,da$

    ⟹ The slopes of the extrapolated curves $E_{\text{int}}(u, a)$ lead to J

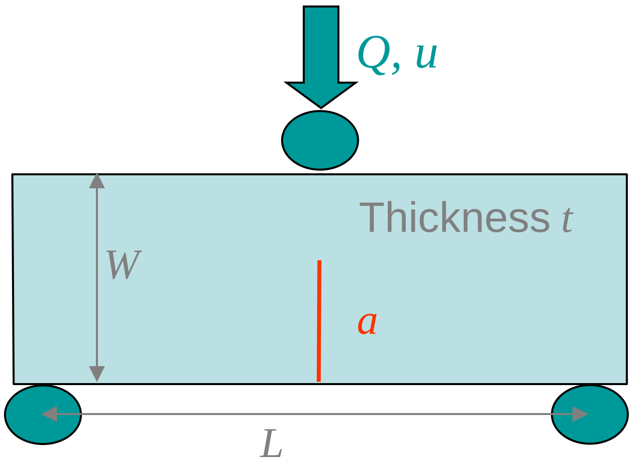


- Small displacements: LEFM holds ⟹ $J \propto \frac{K_I^2}{E'} \propto u^2$
- For large displacements

$$\boldsymbol{u}\left(r,\,\pi\right) = \frac{\alpha\sigma_p^0}{E}\left(\frac{JE}{\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{n}{n+1}} r^{\frac{1}{n+1}}\,\tilde{\boldsymbol{u}}\left(\pi,\,n\right) \Longrightarrow J \propto u^{\frac{n+1}{n}}$$

  - This technique is very time consuming but it avoids unloading at the crack

- Deeply notched specimen testing
  - Consider 1 Single Edge Notch Bend specimen
    - Prescribed loading Q
  - Non-dimensional analysis
    - 9 input variables:
      - $[\sigma_p^0] = \text{kg}\cdot\text{s}^{-2}\cdot\text{m}^{-1}$, $[W\text{-}a] = \text{m}$, $[t] = \text{m}$,
        $[E] = \text{kg}\cdot\text{s}^{-2}\cdot\text{m}^{-1}$, $[\nu] = -$, $[n] = -$,
        $[L] = \text{m}$, $[W] = \text{m}$, $[Q] = \text{kg}\cdot\text{m}\cdot\text{s}^{-2}$
    - 3 dimensions ⟹ 9-3 = 6 independent non-dimensional inputs
      - n, ν, E / σ_p^0, $QL / Et(W\text{-}a)^2$, $L/(W\text{-}a)$, $W/(W\text{-}a)$
    - 1 output variable: $[u] = \text{m}$ ⟹ 1 relation in terms of the non-dimensional inputs



$$\frac{u}{L} = f\left(\frac{QL}{Et\,(W-a)^2},\ \frac{E}{\sigma_p^0},\ \frac{L}{W-a},\ \frac{W}{W-a},\ n,\ \nu\right)$$

  - Prescribed loading
    - Before crack propagation: $J = \frac{1}{t}\int_0^Q \partial_a u\,(Q',\ a)\,dQ'$

- Deeply notched specimen testing (2)
  - J-integral $J = \frac{1}{t}\int_0^Q \partial_a u\,(Q',\, a)\, dQ'$

    with $\frac{u}{L} = f\left(\frac{QL}{Et\,(W-a)^2},\, \frac{E}{\sigma_p^0},\, \frac{L}{W-a},\, \frac{W}{W-a},\, n,\, \nu\right)$
  - We have
    - $\partial_a u = L\left(\frac{2QL}{Et\,(W-a)^3}\partial_{\frac{QL}{Et(W-a)^2}} f + \frac{L}{(W-a)^2}\partial_{\frac{L}{W-a}} f + \frac{W}{(W-a)^2}\partial_{\frac{W}{W-a}} f\right)$

      $\frac{\partial\left(\frac{QL}{Et(w-a)^2}\right)}{\partial a}$ $\frac{\partial\left(\frac{L}{w-a}\right)}{\partial a}$ $\frac{\partial\left(\frac{W}{w-a}\right)}{\partial a}$
    - $\partial_Q u = L\left(\frac{L}{Et\,(W-a)^2}\partial_{\frac{QL}{Et(W-a)^2}} f\right)$ $\Rightarrow$ $\partial_{\frac{QL}{Et(W-a)^2}} f = \frac{Et\,(W-a)^2}{L^2}\partial_Q u$

      $\frac{\partial\left(\frac{QL}{Et(w-a)^2}\right)}{\partial Q}$

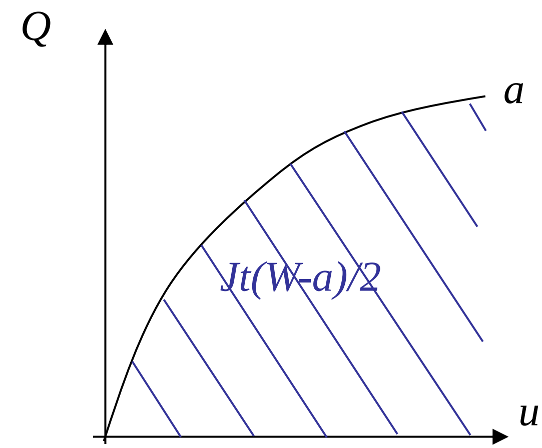
$$\partial_a u = \frac{2Q}{(W-a)}\partial_Q u + \frac{L^2}{(W-a)^2}\partial_{\frac{L}{W-a}} f + \frac{LW}{(W-a)^2}\partial_{\frac{W}{W-a}} f$$

- Deeply notched specimen testing (3)
  - J-integral $J = \frac{1}{t}\int_0^Q \partial_a u\left(Q',\, a\right) dQ'$

    with $\partial_a u = \frac{2Q}{(W-a)}\partial_Q u + \frac{L^2}{(W-a)^2}\partial_{\frac{L}{W-a}} f + \frac{LW}{(W-a)^2}\partial_{\frac{W}{W-a}} f$
  - So we have

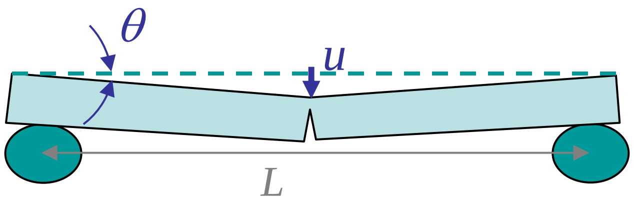
$$J = \frac{1}{t}\int_0^Q \frac{2Q'}{(W-a)}\partial_{Q'} u dQ' + \frac{1}{t}\int_0^Q \frac{L^2}{2\left(W-a\right)^2}\partial_{\frac{L}{W-a}} f dQ' + \frac{1}{t}\int_0^Q \frac{LW}{2\left(W-a\right)^2}\partial_{\frac{W}{W-a}} f dQ'$$

$$\Rightarrow \boxed{J = \frac{2}{t\left(W-a\right)}\int_0^u Q\left(u'\right) du' + I_1 + I_2}$$

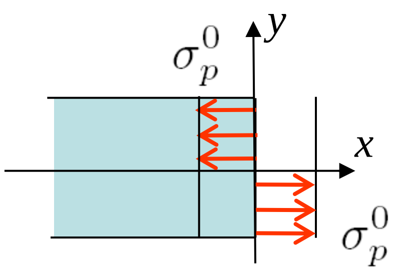


  - Can we neglect I_1 and I_2 ?
    - This would be convenient since this would involve the load-displacement curve only

- Deeply notched specimen testing (4)
  - If the Single Edge Notch Bend specimen
    - Is deeply notched
    - Has a remaining ligament W-a fully plastic



  - There is a plastic hinge
    - The curve $Q-u$ represents essentially the plastic deformations
    - Crack is deep enough so that the plastic hinge is localized between the applied load and the crack tip
    - Example: perfectly plastic material



$$M = t\int_{-\frac{W-a}{2}}^{\frac{W-a}{2}} \sigma y dy = 2t\int_{0}^{\frac{W-a}{2}} \sigma_p^0 y dy$$

$$\Longrightarrow \quad M = t\sigma_p^0 \left(\frac{W-a}{2}\right)^2$$

$$\Longrightarrow \quad M = \frac{t\sigma_p^0}{4}(W-a)^2$$

- Deeply notched specimen testing (5)
  - If the Single Edge Notch Bend specimen
    - Is deeply notched
    - Has a remaining ligament W-a fully plastic
  - There is a plastic hinge
    - Perfectly plastic material

$$M = \frac{t\sigma_p^0}{4}(W-a)^2$$

    - Static equilibrium:

$$M = \frac{QL}{4}$$

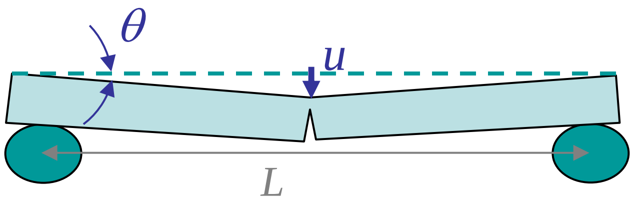
    - And we have

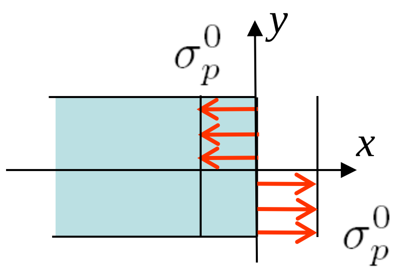
$$Q = \frac{t\sigma_p^0}{L}(W-a)^2 \Longrightarrow \frac{\partial Q}{\partial a} = -2\frac{t\sigma_p^0}{L}(W-a) = -\frac{2}{W-a}Q$$

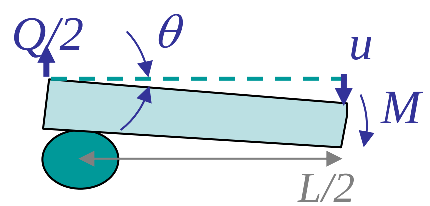
    - The J-integral becomes

$$J = -\frac{1}{t}\int_0^u \partial_a Q du' = \frac{2}{t(W-a)}\int_0^u Q(u')\,du$$

$\Longrightarrow$ $I_1 = 0$ and $I_2 = 0$

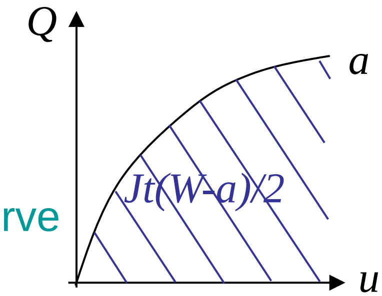




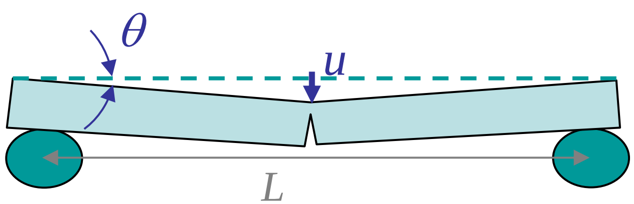


Experimental determination of J

- Eta factor approach



  – Expression $J = \frac{2}{t\,(W-a)}\int_0^u Q\,(u')\,du'$
    - Is convenient since it involves only the load-displacement curve
    - Is valid for SENB with plastic hinge



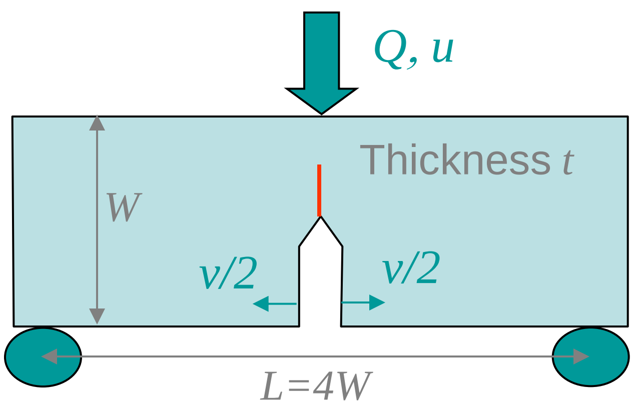
    - Elastic materials
      – SENB with L/W = 4 & a/W > 0.5: I_1 and I_2 can be neglected*
    - Elastic-plastic materials
      – SENB at low temperature, for L/W = 4 & 0.6> a/W > 0.4: I_1 and I_2 can be neglected (Comparison with multiple specimen testing**)
  – How can this expression be generalized to
    - Other geometries, loadings
    - Elastic-plastic materials

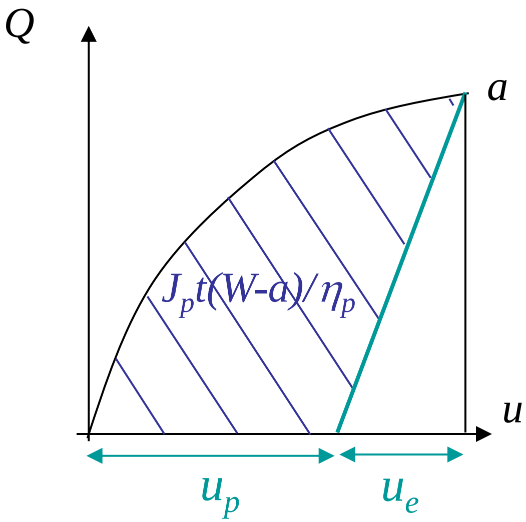
*Srawley JE (1976), On the relation of J to work done per unit uncracked area: total, or component due to crack, International Journal of Fracture 12, 470–474

**Castro P, Spurrier P, and Hancock J (1984), Comparison of J testing techniques and correlation J-COD using structural steel specimens, International Journal of Fracture 17,83–95.

- Eta factor approach (2)
  - How can $J = \frac{2}{t\left(W-a\right)}\int_0^u Q\left(u'\right)du'$ be generalized?
  - For other specimens and through-cracked structures
    - Assuming deformations are largely plastic
    - Eta factor: $J = \frac{\eta_J}{t\left(W-a\right)}\int_0^u Q\left(u'\right)du'$
    - With η_J depending on
      - Geometry & loading
      - Crack length (a/W)
      - Material hardening (n)
  - But
    - Materials are not rigidly plastic, so there is an elastic-plastic response
    - How to determine η_J?

- Eta factor approach (3)
  - How can $J = \frac{2}{t\left(W-a\right)}\int_0^u Q\left(u'\right)du'$ be generalized (2) ?
  - For elastic-plastic behavior
    - Split of elastic and plastic parts of
      - The displacement $u = u_e + u_p$
      - The J-integral $J = J_e + J_p = \frac{K_I^2\left(a_{\text{eff}}\right)}{E'} + \frac{\eta_p}{t\left(W-a\right)}\left(\int_0^u Q du' - \frac{Q u_e}{2}\right)$
      - The Crack Mouth-Opening Displacement $v = v_e + v_p$





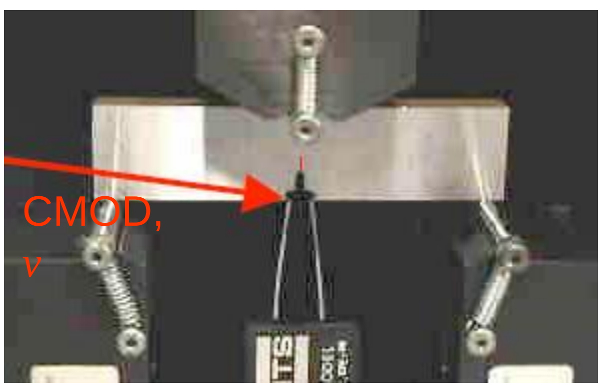
- Factor η_p still remains to be evaluated

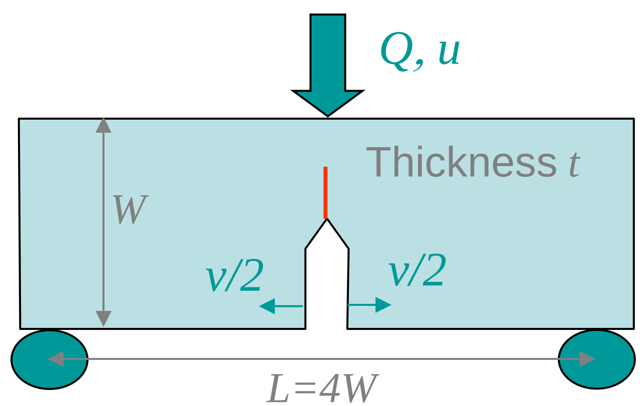
- Eta factor approach (4)
  - How can $J = \frac{2}{t\left(W-a\right)}\int_0^u Q\left(u'\right)du'$ be generalized (3) ?
  - For elastic-plastic behavior (2)
    - It is more accurate to measure the CMOD v than the displacement u

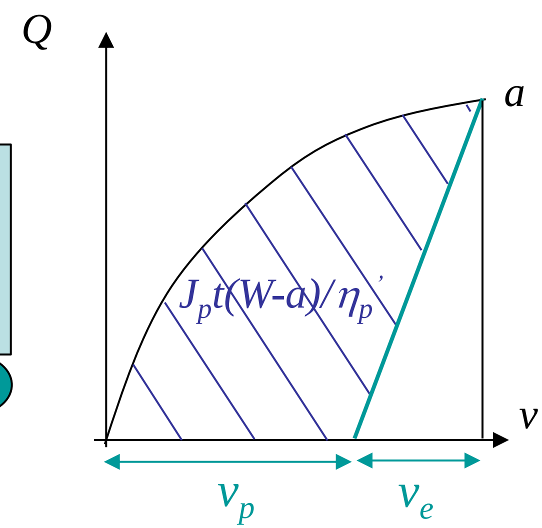
$$J = J_e + J_p = \frac{K_I^2\left(a_{\text{eff}}\right)}{E'} + \frac{\eta_p'}{t\left(W-a\right)}\left(\int_0^v Q dv' - \frac{Qv_e}{2}\right)$$

    - But v and Q are not work conjugated

⟹ the new η_p' factor has to be evaluated with FE







- Evaluate η_p' for a given geometry
  - Characterize non-linear material response
    - From uniaxial tensile test
  - Linear FE analysis
    - Extract K_I and then $J_e(Q)$

$$\Rightarrow J_e = \frac{K_I^2\left(a_{\text{eff}}\right)}{E'}$$

  - Non-linear analysis
    - Load the specimen incrementally
    - At each increment
      - Extract J (domain integral)

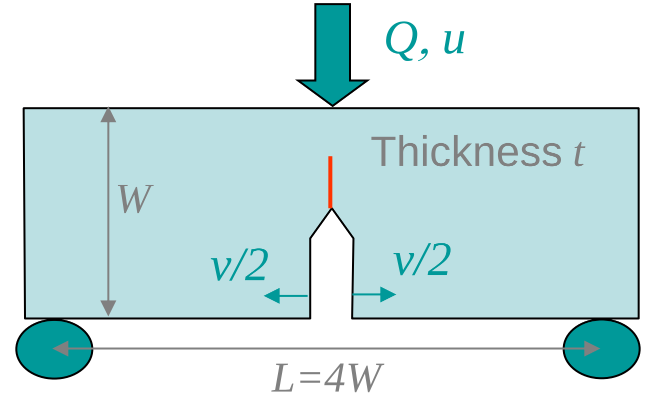
$$\Rightarrow J_p(Q) = J - J_e(Q)$$

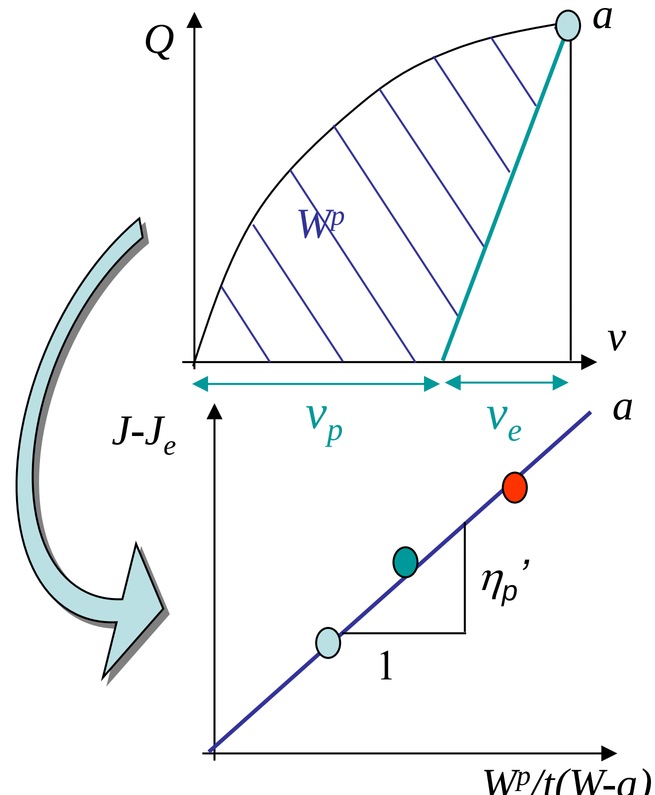
      - Compute W^p

$$W^p = \int_0^v Q dv' - \frac{Qv_e}{2}$$

  - Factor η_p' is the slope of

$$J_p = \frac{\eta_p'}{t\left(W-a\right)}\left(\int_0^v Q dv' - \frac{Qv_e}{2}\right)$$



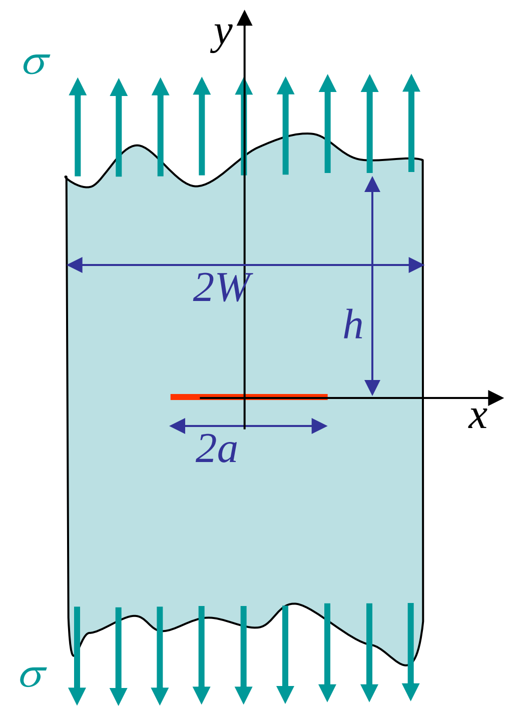


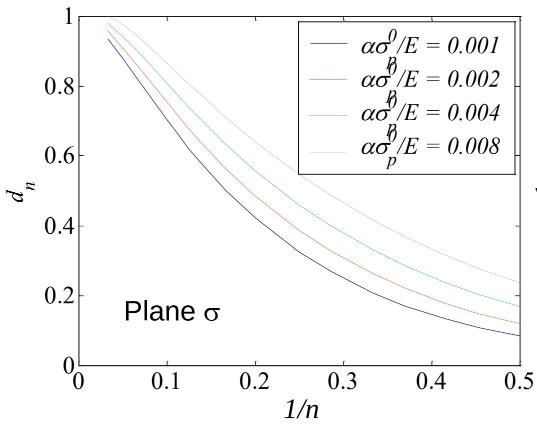
- Use of handbooks
  - Split of J: $J = J_e + J_p = \frac{K_I^2\left(a_{\text{eff}}\right)}{E'} + J_p$
  - Use of handbooks to determine J_e (see SIF lecture) and J_p:
    - HRR field $\boldsymbol{\sigma} = \sigma_p^0 \left(\frac{JE}{r\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{1}{n+1}} \tilde{\boldsymbol{\sigma}}\left(\theta,\, n\right)$
    - Since this solution holds for fully plastic solutions, we need to adapt it
      - J becomes J_p
      - r becomes W-a
      - Since $Q \propto \boldsymbol{\sigma}$
        - » σ becomes Q
        - » σ_p^0 becomes yielding load Q_0

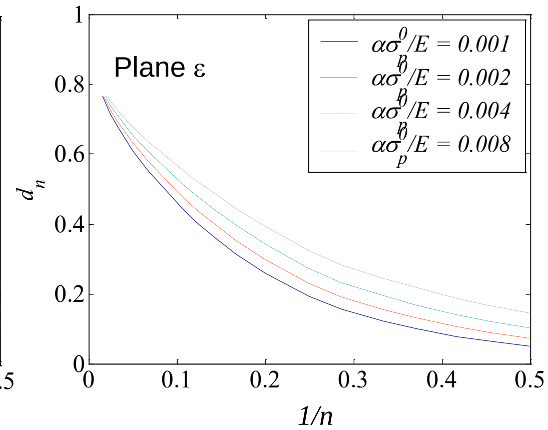
      $\Longrightarrow J_p \propto \frac{\alpha\left(\sigma_p^0\right)^2\left(W-a\right)}{E}\left(\frac{Q}{Q^0}\right)^{n+1}$
    - Tabulation of the values for different geometries
      - $J_p = \frac{\alpha\left(\sigma_p^0\right)^2\left(W-a\right)}{E} h_1\left(\frac{a}{W},\, n\right)\left(\frac{Q}{Q^0}\right)^{n+1}$

        or $J_p = \frac{\alpha\left(\sigma_p^0\right)^2\left(W-a\right)a}{EW} h_1\left(\frac{a}{W},\, n\right)\left(\frac{Q}{Q^0}\right)^{n+1}$ depending on the test
      - Similar relations for ν_p, u_p and δ_t

- Determination of J_p
  - Example of centered crack plate
    - $$\begin{cases} J_p = \dfrac{\alpha\left(\sigma_p^0\right)^2 (W-a)\, a}{EW} h_1\left(\dfrac{a}{W}, n\right)\left(\dfrac{Q}{Q^0}\right)^{n+1} \\ v_p;\, u_p = \dfrac{\alpha\left(\sigma_p^0\right)^2 a}{E} h_{2;3}\left(\dfrac{a}{W}, n\right)\left(\dfrac{Q}{Q^0}\right)^{n} \\ \delta_t = \dfrac{\alpha \quad \sigma_p^0 \quad (W-a)\, a}{EW} h_4\left(\dfrac{a}{W}, n\right)\left(\dfrac{Q}{Q^0}\right)^{n+1} \end{cases}$$
    - With $$Q^0 = \begin{cases} 4(W-a)\dfrac{\sigma_p^0}{\sqrt{3}} & \text{if plane } \varepsilon \\ 2(W-a)\,\sigma_p^0 & \text{if plane } \sigma \end{cases}$$
      $$h_4 = h_1 d_n$$
      Other h_i tabulated

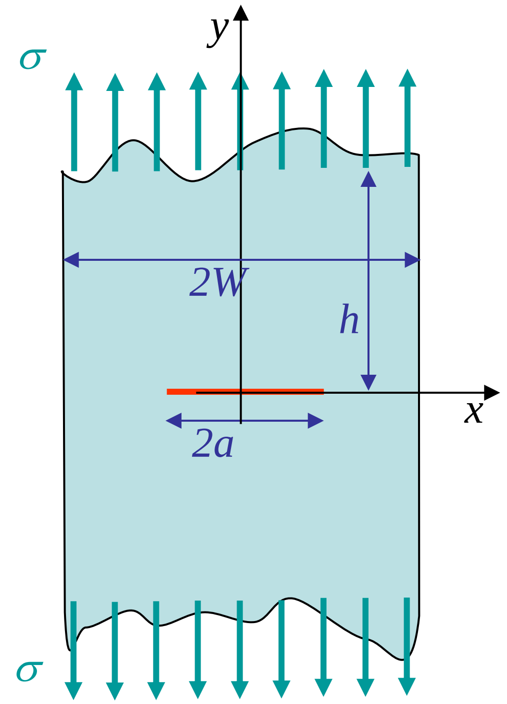






- Determination of J_p (2)
  - Example of centered crack plate (2)

Plane ε		n=1	n=2	n=3	n=5	n=7
$a/W = 1/8$	h_1	2.80	3.61	4.06	4.35	4.33
	h_2	3.05	3.62	3.91	4.06	3.93
	h_3	0.303	0.574	0.840	1.300	1.630
$a/W = 1/4$	h_1	2.54	2.62	2.65	2.51	2.28
	h_2	2.68	2.99	3.01	2.85	2.61
	h_3	0.536	0.911	1.220	1.640	1.840

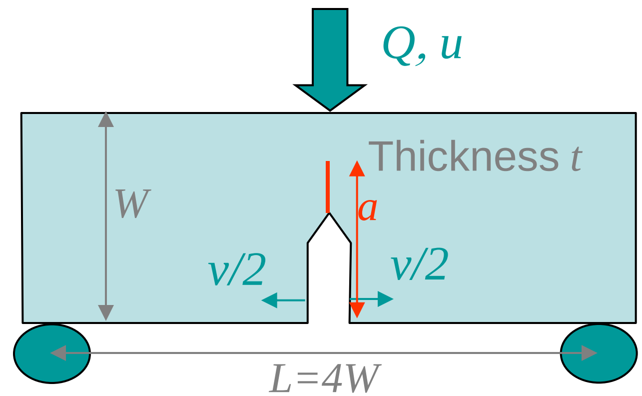


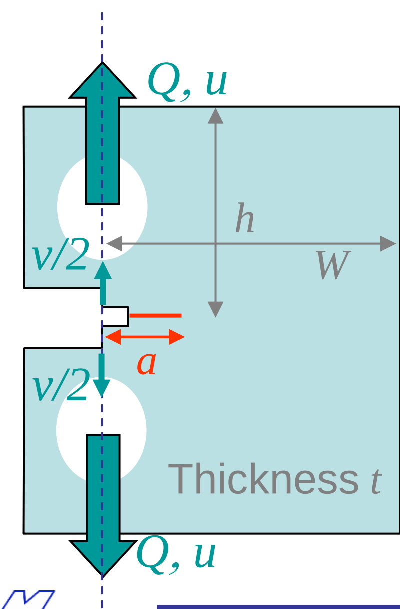
- Effective crack length
  - Evaluation of $J_e = \frac{K_I^2\left(a_{\text{eff}}\right)}{E'}$ is required in the engineering method
    - Effective crack length $a_{\text{eff}} = a + \eta\, r_p$ (η not to be mistaken for the previous one)
    - Recall in SSY, η =1/2 & $r_p = \begin{cases} \frac{n-1}{n+1}\frac{1}{\pi}\left(\frac{K_I(a_{\text{eff}})}{\sigma_p^0}\right)^2 & \text{if plane } \sigma \\ \frac{n-1}{n+1}\frac{1}{3\pi}\left(\frac{K_I(a_{\text{eff}})}{\sigma_p^0}\right)^2 & \text{if plane } \varepsilon \end{cases}$
    - For Large Scale Yielding this would lead to lengths larger than the ligament
      - Correction of η: $\eta = \frac{1}{2\left[1+\left(\frac{Q}{Q^0}\right)^2\right]}$
      - Use of the same plastic zone expression

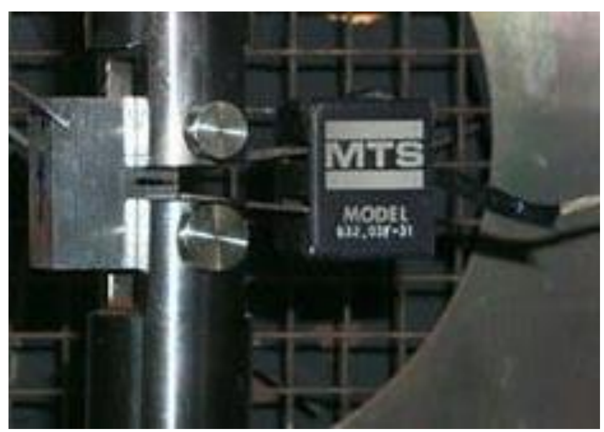
$$r_p = \begin{cases} \frac{n-1}{n+1}\frac{1}{\pi}\left(\frac{K_I(a_{\text{eff}})}{\sigma_p^0}\right)^2 & \text{if plane } \sigma \\ \frac{n-1}{n+1}\frac{1}{3\pi}\left(\frac{K_I(a_{\text{eff}})}{\sigma_p^0}\right)^2 & \text{if plane } \varepsilon \end{cases}$$

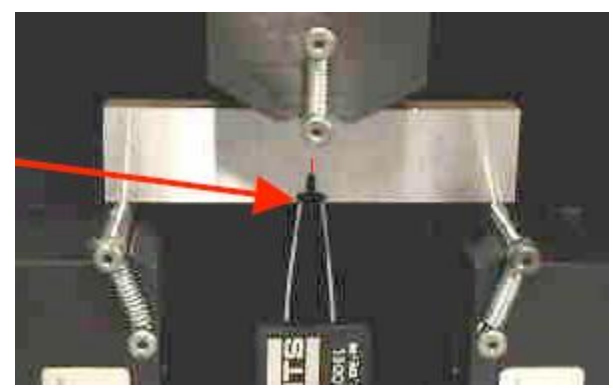
Fracture toughness testing for elastic-plastic materials

- Procedure detailed in the ASTM E1820 norm
- What is measured?
  - Plane strain value of J prior to significant crack growth: J_C
  - Plane strain value of J near the onset of stable crack growth: J_{IC}
  - J vs Δa resistance curve for stable crack growth: J_R
- Pre-cracked specimens that can be used
  - Either SENB or Compact Tension specimen

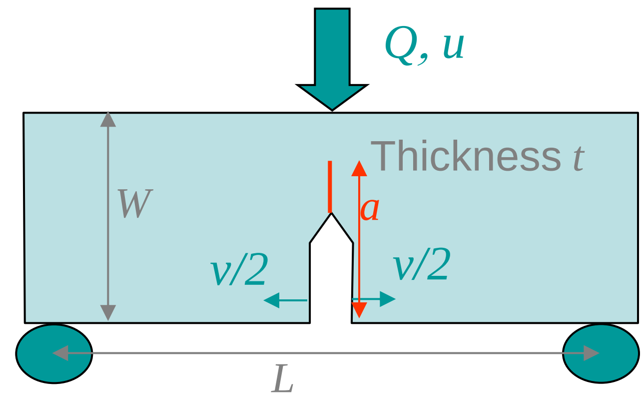


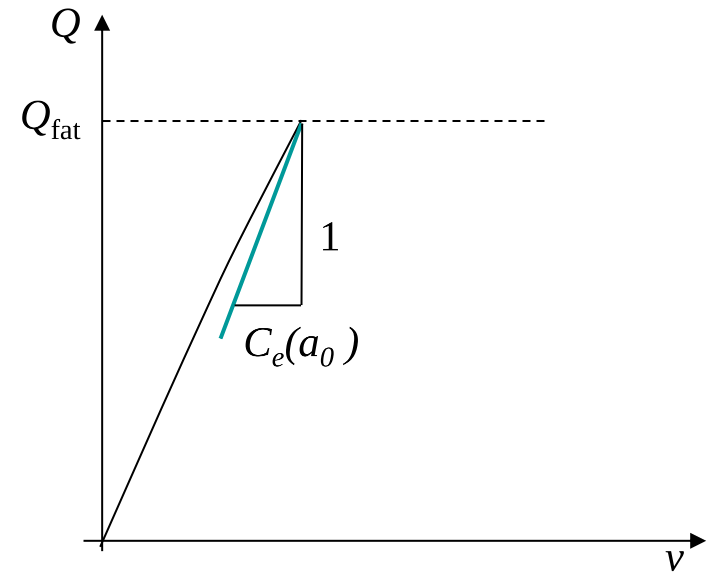




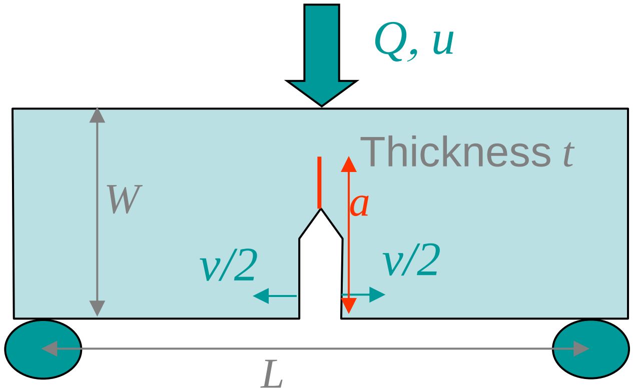


- 1^{st} step: Determine crack length a_0 after fatigue loading
  - 3 cycles of loading-unloading
    - Between Q_{fat} and $Q_{\text{fat}}/2$ with Q_{fat} the maximal loading during pre-cracking
    - Crack length is obtained using the tabulated compliance curves $C_e(a)$
    - See lecture on Energetic Approach





- 1st step: Determine crack length a_0 after fatigue loading (2)
  - Example: SENB



- If $$U = \frac{1}{1+\sqrt{\frac{4E'tvW}{LQ}}}$$
- Then

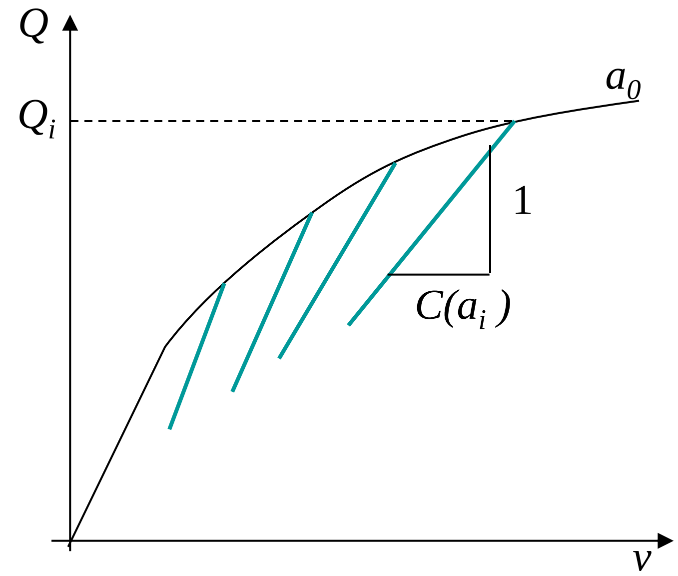
$$\frac{a}{W} = 0.9997 - 3.95U + 2.982U^2 - 3.214U^3 + 51.52U^4 - 113.0U^5$$

- In elasticity: $C(a) = C_e(a) = v/Q$
  - $$C_e = \frac{v}{Q} = 6\frac{L}{E'tW}\frac{a}{W}\left[0.76 - 2.28\frac{a}{W} + 3.87\left(\frac{a}{W}\right)^2 - 2.04\left(\frac{a}{W}\right)^3 + \frac{0.66}{\left(1. - \frac{a}{W}\right)^2}\right]$$
  - In terms of Load Line displacement:

$$C_{LLe} = \frac{u}{Q} = \frac{1}{E't}\left(\frac{L}{W-a}\right)^2$$
$$\left[1.193 - 1.98\frac{a}{W} + 4.478\left(\frac{a}{W}\right)^2 - 4.443\left(\frac{a}{W}\right)^3 + 1.739\left(\frac{a}{W}\right)^4\right]$$

Fracture toughness testing for elastic-plastic materials

- 2[nd] step: Proceed with the testing: increase loading
  - At regular intervals i: unload
    - In order to determine a_i with the compliance
    - Do not unload too much in order to avoid reverse plastic loading
    - At least 8 unloadings are required
  - After the final loading step
    - Unload to zero and
    - Mark final crack length
    - Break open (cool down if required to have brittle fracture)
    - Measure final crack length



- 3rd step: Data reduction
  - For each pair (a_i, Q_i)
    - Calculate $J_{e,i}$:
      - Using crack length $J_{e,i} = \frac{K_I^2(a_i)}{E'}$
      - E.g. SENB: $K_I = \frac{QL}{tW^{\frac{3}{2}}} \frac{3\sqrt{\frac{a}{W}}\left[1.99 - \frac{a}{W}\left(1-\frac{a}{W}\right)\left(2.15 - 3.93\frac{a}{W} + 2.7\left(\frac{a}{W}\right)^2\right)\right]}{2\left(1+2\frac{a}{W}\right)\left(1-\frac{a}{W}\right)^{\frac{3}{2}}}$
    - Calculate $J_{p,i}$:
      - Plastic displacement

$$u_{p,\,i} = u_i - Q_i C_{LLe}(a_i)$$

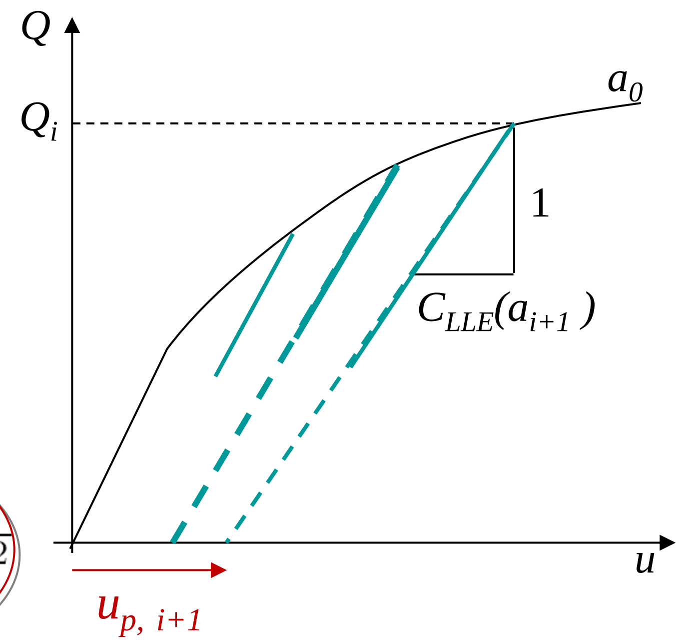
      - Increment in plastic work

$$\Delta W_i^p = \int_{u_{p,\,i-1}}^{u_{p,\,i}} Q du'_p$$

      - Since for SENB η-factor is equal to 2:

$$J_{p,\,i} = J_{p,\,i-1} + \frac{2}{t}\Delta W_i^p \frac{W - a_i}{(W - a_{i-1})^2}$$

Average ligament size



- 3[rd] step: Data reduction (2)
  - For each pair (a_i, Q_i)
    - Calculate $J_{e,i}$:

$$J_{e,i} = \frac{K_I^2\left(a_{\mathrm{eff_i}}\right)}{E'}$$

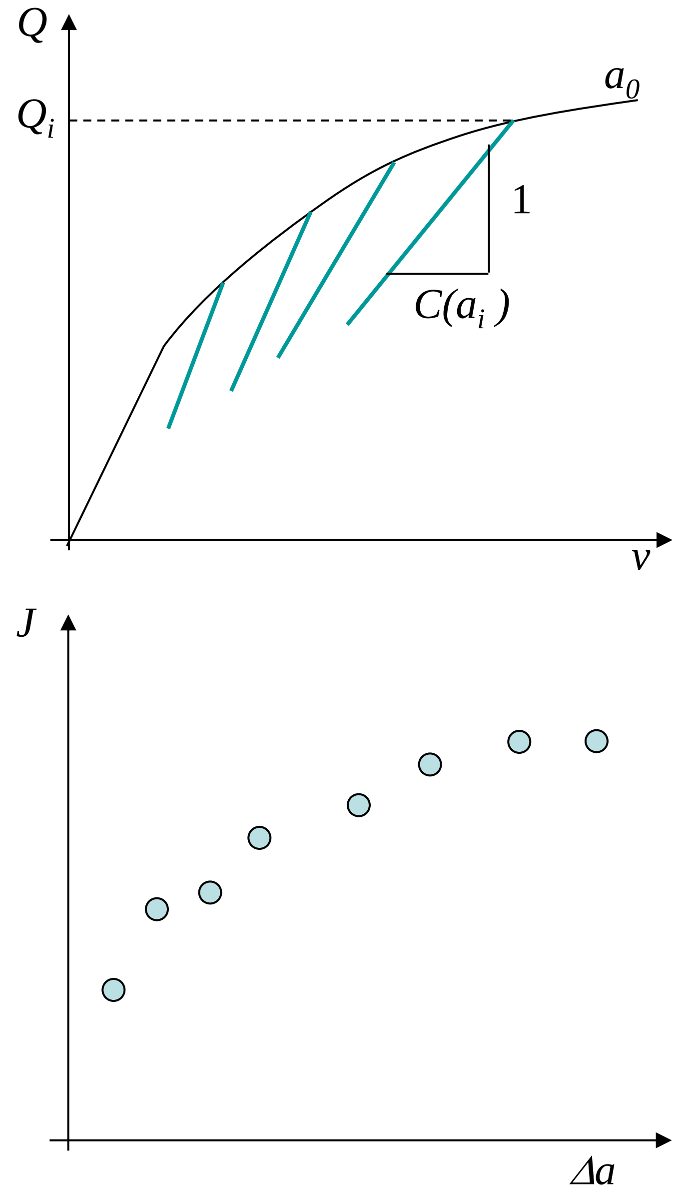
    - Calculate $J_{p,i}$ (e.g. SENB)

$$J_{p,\,i} = J_{p,\,i-1} + \frac{2}{t}\Delta W_i^p \frac{W - a_i}{\left(W - a_{i-1}\right)^2}$$

    - Total J value:

$$J_i = J_{e,\,i} + J_{p,\,i}$$

  - This allows drawing
    - The J vs. Δa=a-a_0 curve
    - This requires accurate determination of a_0

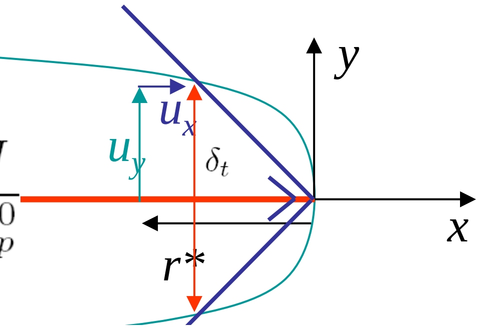


Fracture toughness testing for elastic-plastic materials

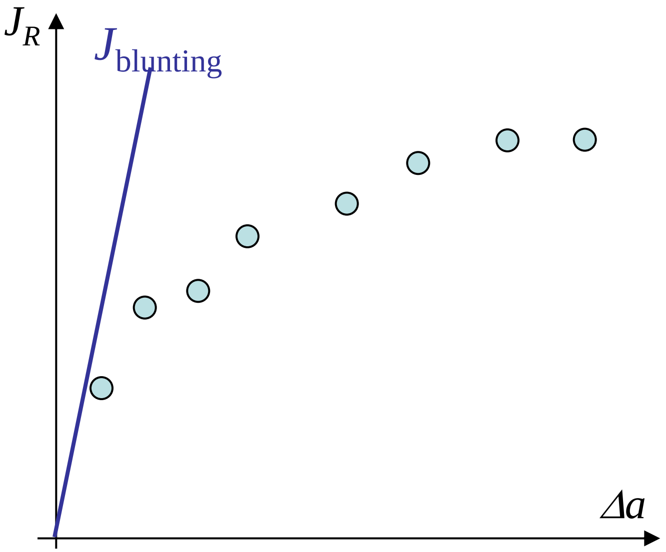
- 4th step: Analysis
  - Even before physical crack growth there is a blunting of crack tip
    - Due to plasticity
    - Which results in an apparent Δa
    - Corrected by plotting the blunting line

$$J_{\text{blunting}} = 2\sigma_p^0 \Delta a$$

$$\delta_t = d_n \frac{J}{\sigma_p^0}$$



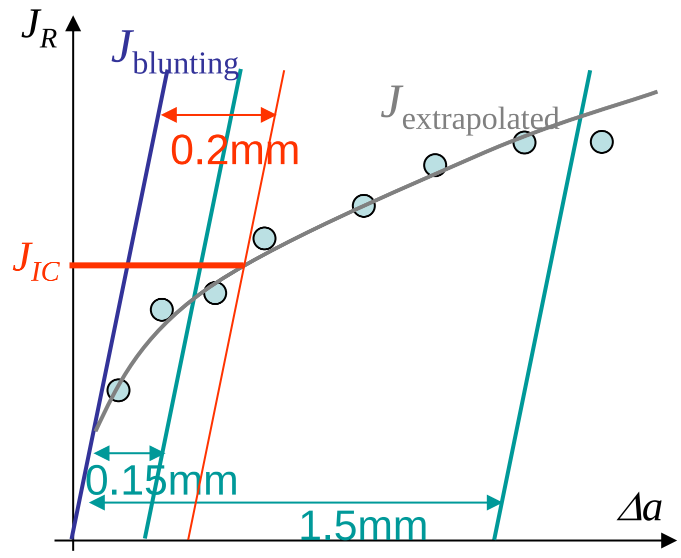
$$r^* - \boldsymbol{u}_x(r^*, \pi) = \boldsymbol{u}_y(r^*, \pi) = \frac{\delta_t}{2}$$



- 4[th] step: Analysis (2)
  - Using data points in between the 0.15 and 1.5-mm exclusion lines,
    - Extrapolate J :

$$\ln J = \ln C_1 + C_2 \ln\left(\frac{\Delta a}{1\ mm}\right)$$

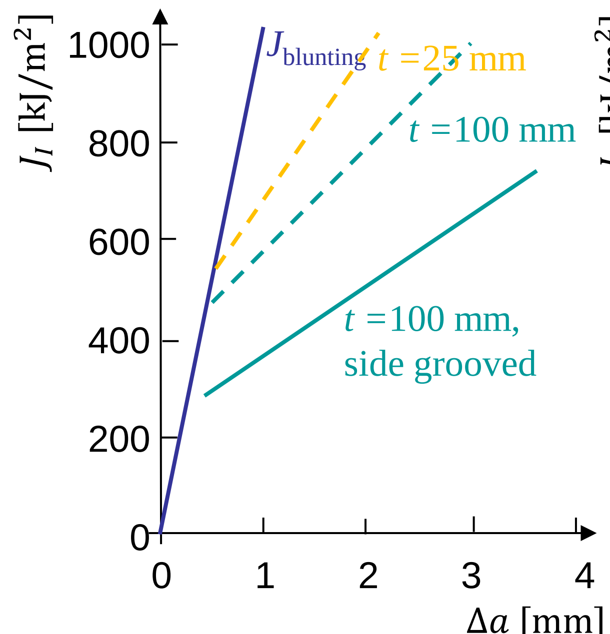
  - Plane strain value of J near the onset of stable crack growth: J_{IC}
    - Fracture toughness
    - Intersection with the 0.2mm offset line
  - Plane strain value of J prior to significant crack growth: J_C
    - Intersection with the blunting line
  - J vs Δa resistance curve for stable crack growth: J_R

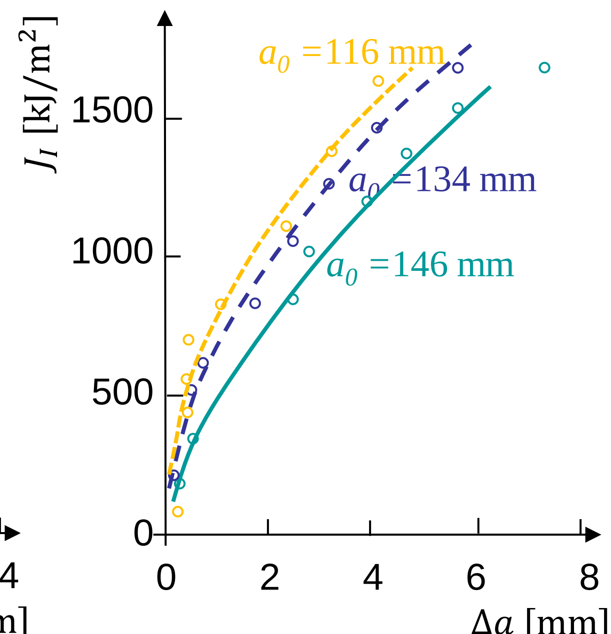


  - Are sizes correct?

$$t,\ W - a > 25\frac{J_{IC}}{\frac{\sigma_p^0 + \sigma_{TS}}{2}}$$

- Experimental curves
  - Example: steel A533-B at 93°C*
    - Thickness has an important influence
    - Slight influence of the initial crack length
    - Side grooving suppresses the plane stress state

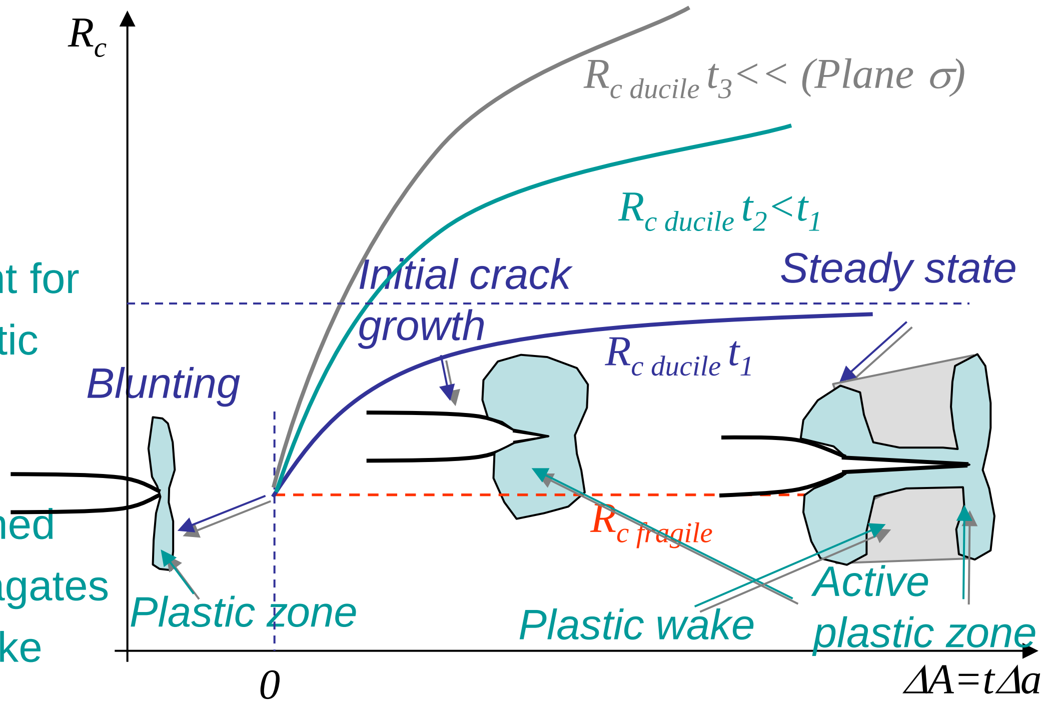




- From these curves one can extract
  - The toughness J_{IC} ⟹ crack initiation criterion $J(a, Q) = J_{IC}$
  - The resistance curve $J_R(\Delta a)$ ⟹ crack stability criterion?

*Andrews WR and Shih CF (1979), Thickness and side-groove effects on J- and δ-resistance curves for A533-B steel at 93◦C,ASTM STP 668 , 426–450.

- Non-perfectly brittle material: Resistance curve
  - For non-perfectly brittle materials G_C depends on the crack surface
    - Therefore G_C will be renamed the resistance $R_c\,(A)$
  - Elastoplastic behavior
    - Active plastic zone in front of the crack tip
    - Plastic work increases
    - ⇨ Resistance increases
    - This effect is more important for thin specimens (elasto-plastic behavior more pronounced under plane σ)
    - A steady state can be reached in which case a crack propagates in a compressive plastic wake

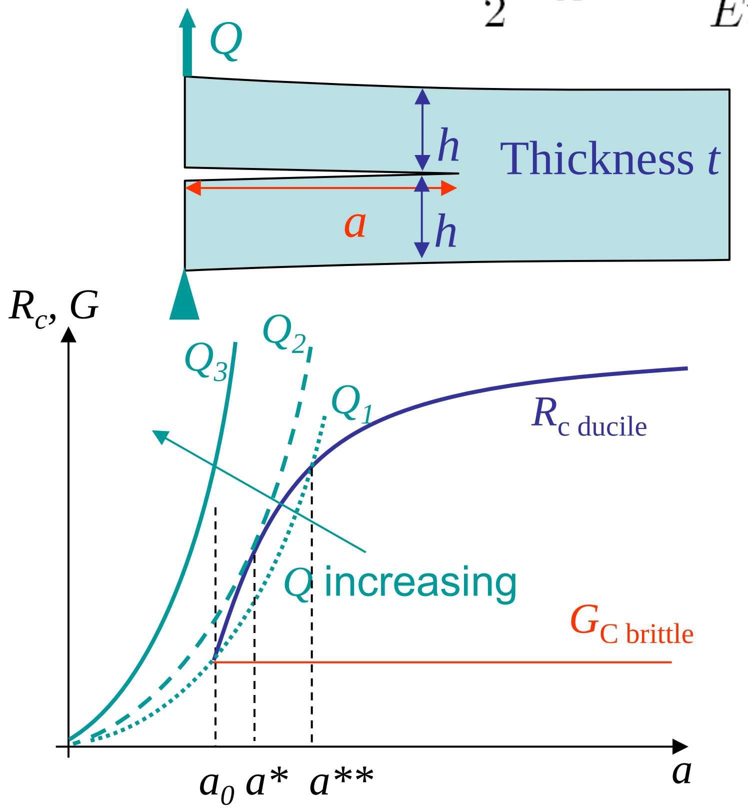


  - Stability of the crack also depends on the variation of G with crack length
    - Stable crack growth if $\partial_a G \leq \partial_a R_c$
    - Unstable crack growth if $\partial_a G > \partial_a R_c$

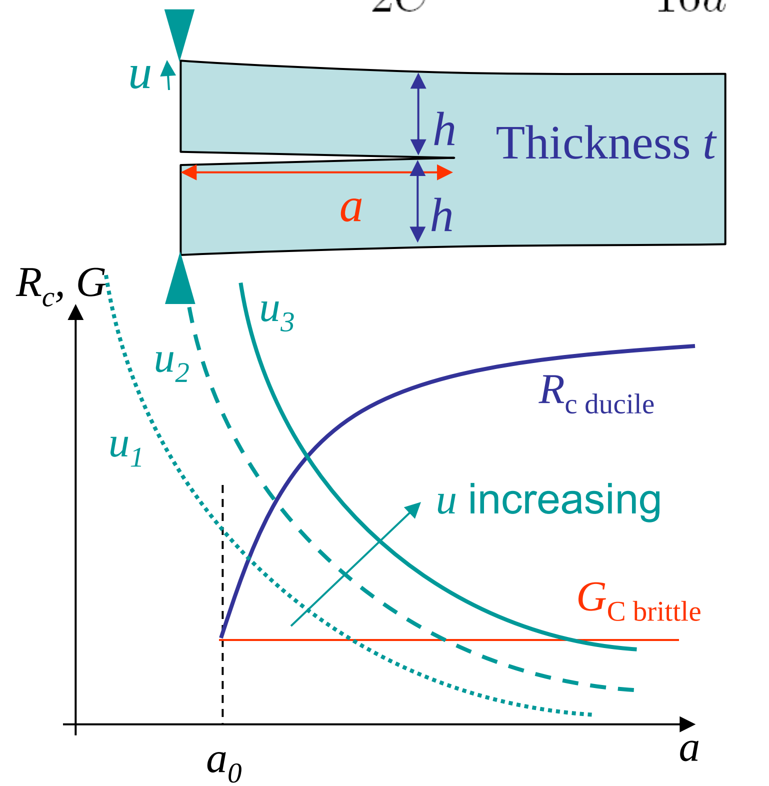
Reminder: LEFM crack grow stability

- Example: Delamination of composites with initial crack a_0

Dead load $G = \frac{Q^2}{2}\partial_A C = \frac{12Q^2a^2}{Et^2h^3}$ vs **Fixed grip** $G = \frac{u^2}{2C^2}\partial_A C = \frac{3u^2Eh^3}{16a^4}$

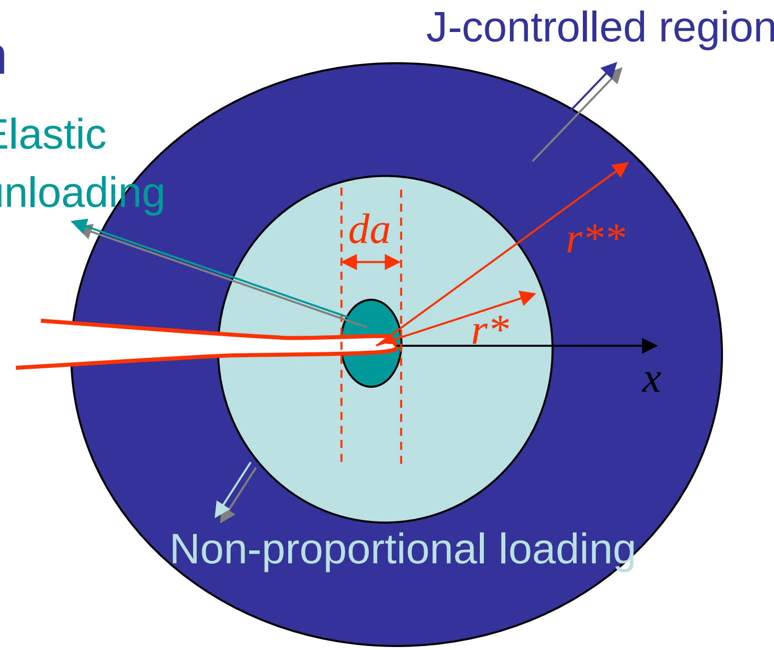


- Dead load Q_1
  - Perfectly brittle materials: unstable
  - Ductile materials: stable, but if a is larger than a^{**} it turns unstable



- Fixed grip u_1 ,u_2 or u_3
  - The crack is stable for any material

- Under which conditions is the crack growth J-controlled?
  - Recall that in order to use J
    - Unloading is prohibited
    - So the solution is valid before crack growth
  - But we wish to use the J_R curve
    - When is the crack growth still J-controlled?
  - HRR solution holds in an annular region
    - Since HRR solution is singular, there is a region not controlled by J
    - There is an annular region between r^* and r^{**} controlled by J



⇒ When can we use the HRR theory during crack growth?

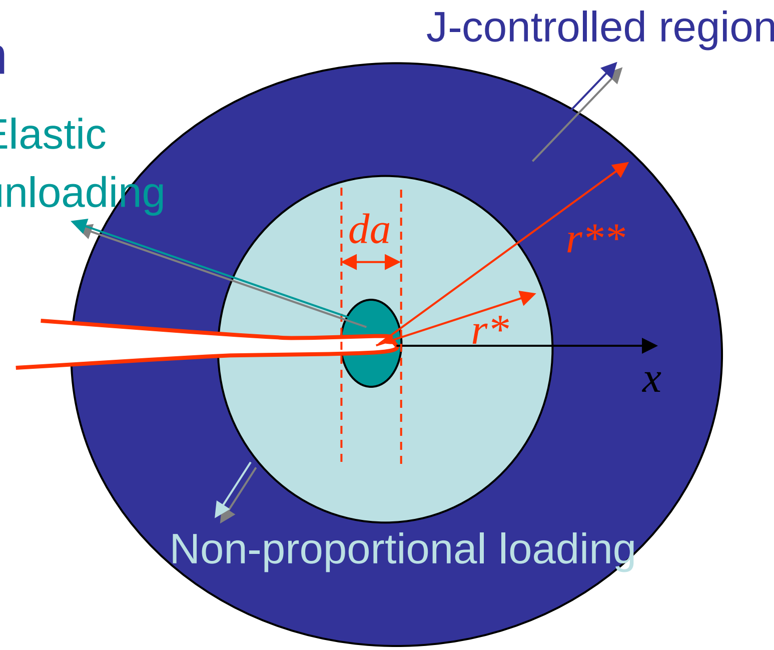
J-controlled crack growth & NLFM crack grow stability

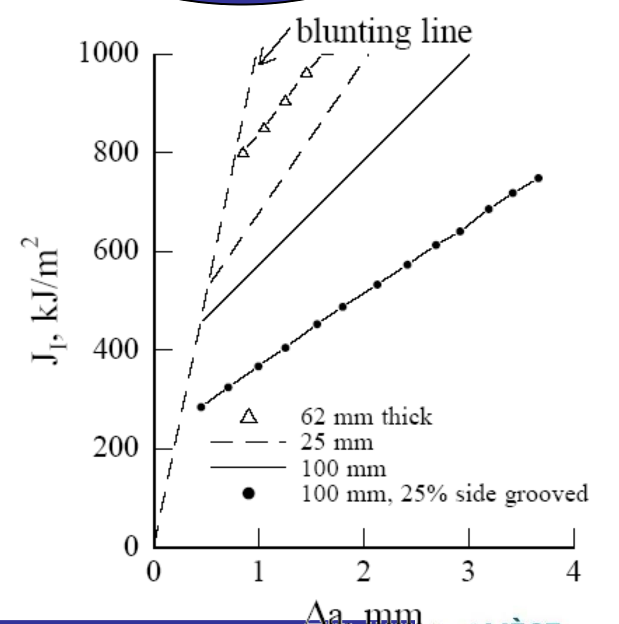
- Under which conditions is the crack growth J-controlled (2)?
  - HRR solution holds in an annular region
    - As HRR solution is singular, there is a region not controlled by J
    - There is an annular region between r^* and r^{**} controlled by J
  - If the crack grows by da
    - There is an elastic unloading in a region that scales with da
    - For a potential to be defined: no unloading
  - So for J to remains relevant if the crack grows
    - One clear condition is therefore

$$da << r^{**}$$

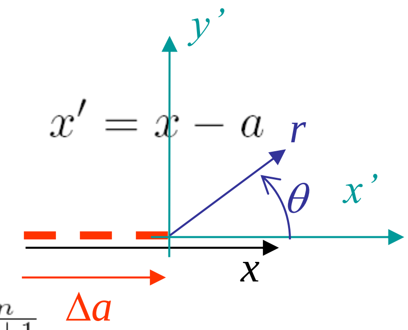
    - Another condition to keep asymptotic field proportionality (see next slides):

$$D = \frac{J_{IC}}{\frac{dJ}{d\Delta a}} \ll r^{**}$$





- Under which conditions is the crack growth *J*-controlled (3)?
  - HRR strain field (between r^* and r^{**}): $\boldsymbol{\varepsilon} = \frac{\sigma_p^0 \alpha}{E}\left(\frac{JE}{r\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{n}{n+1}} \tilde{\boldsymbol{\varepsilon}}\left(\theta,\, n\right)$
  - Since ε depends on J and a:
    - $d\boldsymbol{\varepsilon} = \partial_a \boldsymbol{\varepsilon} da + \partial_J \boldsymbol{\varepsilon} dJ$
    - The crack is moving to the right:



$$\Rightarrow \partial_a = -\partial_{x'} = -\cos\theta\partial_r + \frac{\sin\theta}{r}\partial_\theta$$

– With $$\partial_r \boldsymbol{\varepsilon} = -\frac{n}{n+1}\frac{\sigma_p^0 \alpha}{E}\left(\frac{JE}{r\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{n}{n+1}} \frac{1}{r}\tilde{\boldsymbol{\varepsilon}}\left(\theta,\, n\right)$$

& $$\partial_\theta \boldsymbol{\varepsilon} = \frac{\sigma_p^0 \alpha}{E}\left(\frac{JE}{r\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{n}{n+1}} \partial_\theta\tilde{\boldsymbol{\varepsilon}}\left(\theta,\, n\right)$$

– So one has $$\partial_a \boldsymbol{\varepsilon} = \frac{\sigma_p^0 \alpha}{E}\left(\frac{JE}{r\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{n}{n+1}} \frac{1}{r}\tilde{\tilde{\boldsymbol{\varepsilon}}}\left(\theta,\, n\right)$$

with $$\tilde{\tilde{\boldsymbol{\varepsilon}}}\left(\theta,\, n\right) = \frac{n}{n+1}\cos\theta\tilde{\boldsymbol{\varepsilon}} + \sin\theta\partial_\theta\tilde{\boldsymbol{\varepsilon}}$$

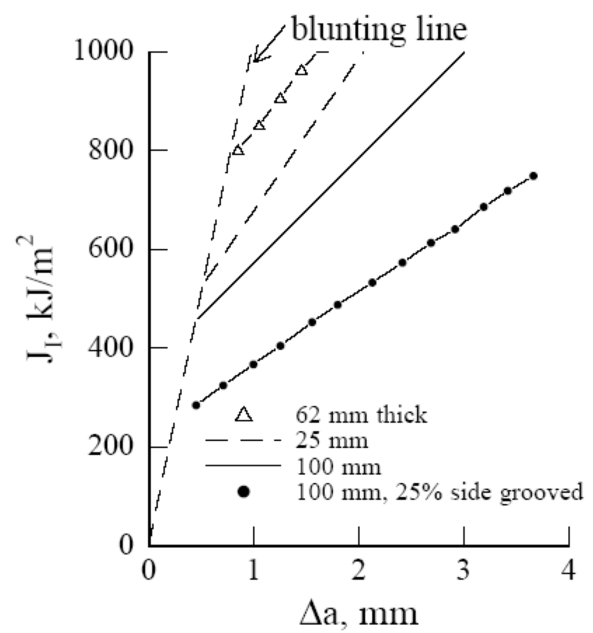
- Under which conditions is the crack growth *J*-controlled (4)?
  - HRR strain field (between r^* and r^{**}): $\boldsymbol{\varepsilon} = \frac{\sigma_p^0 \alpha}{E} \left(\frac{JE}{r\alpha \left(\sigma_p^0\right)^2 I_n} \right)^{\frac{n}{n+1}} \tilde{\boldsymbol{\varepsilon}} \left(\theta,\, n\right)$
  - Since ε depends on J and a:
    - $d\boldsymbol{\varepsilon} = \partial_a \boldsymbol{\varepsilon} da + \partial_J \boldsymbol{\varepsilon} dJ$
    - With $\partial_a \boldsymbol{\varepsilon} = \boxed{\frac{\sigma_p^0 \alpha}{E} \left(\frac{JE}{r\alpha \left(\sigma_p^0\right)^2 I_n} \right)^{\frac{n}{n+1}}} \frac{1}{r} \tilde{\tilde{\boldsymbol{\varepsilon}}} \left(\theta,\, n\right)$
    - & $\partial_J \boldsymbol{\varepsilon} = \frac{n}{n+1} \boxed{\frac{\sigma_p^0 \alpha}{E} \left(\frac{JE}{r\alpha \left(\sigma_p^0\right)^2 I_n} \right)^{\frac{n}{n+1}}} \frac{1}{J} \tilde{\boldsymbol{\varepsilon}} \left(\theta,\, n\right)$
  - At the end of the day:

$$d\boldsymbol{\varepsilon} = \boxed{\frac{\sigma_p^0 \alpha}{E} \left(\frac{JE}{r\alpha \left(\sigma_p^0\right)^2 I_n} \right)^{\frac{n}{n+1}}} \left(\frac{n}{n+1} \frac{dJ}{J} \tilde{\boldsymbol{\varepsilon}} + \frac{da}{r} \tilde{\tilde{\boldsymbol{\varepsilon}}} \right)$$

- Under which conditions is the crack growth J-controlled (5)?
  - We have $d\boldsymbol{\varepsilon} = \frac{\sigma_p^0 \alpha}{E}\left(\frac{JE}{r\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{n}{n+1}}\left(\frac{n}{n+1}\frac{dJ}{J}\tilde{\boldsymbol{\varepsilon}} + \frac{da}{r}\tilde{\tilde{\boldsymbol{\varepsilon}}}\right)$
  - Term $\frac{da}{r}\tilde{\tilde{\boldsymbol{\varepsilon}}}$ is in $1/r$ $\Longrightarrow$ it changes the proportionality of the solution
    - But $\tilde{\tilde{\boldsymbol{\varepsilon}}}$ & $\frac{n}{n+1}\tilde{\boldsymbol{\varepsilon}}$ are of the same order
    - We have J-controlled crack growth if

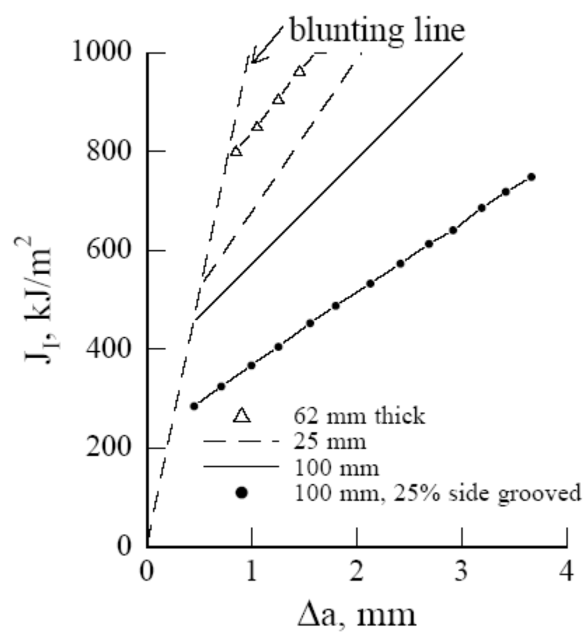
$$\frac{dJ}{J} >> \frac{da}{r} \Longrightarrow r >> \frac{J}{\frac{dJ}{da}}$$

  - From the J_R curves, the length scale can be extracted as $D = J_{IC}/\left.\frac{dJ}{d\Delta a}\right|_c$
  - The J-controlled growth criteria are
    - $da << r^{**}$ & $D << r^{**}$
    - r^{**} remains to be determined
    - Stability is not established yet



- Crack growth stability
  - The J-controlled crack growth criteria are
    - $da << r^{**}$ & $D << r^{**}$
    - r^{**} is a fraction
      - Of the remaining ligament W-a
      - Or other characteristic lengths (e.g. up to r_p / 4)
    - For SENB: da < 6% (W-a) and D < 10% (W-a) (obtained by FE analysis)
  - If these criteria are satisfied the stability of the crack can be assessed
    - J depends on the crack length and loading
    - Toughness variation with Δa is evaluated from J_R curves
    - Stable crack if $\frac{dJ\,(a,\,Q)}{da} < \left.\frac{dJ_R}{d_a}\right|_{\Delta a=0}$
    - In terms of the non-dimensional tearing:

$$T < T_R \left\{ \begin{array}{l} T = \frac{E}{\left(\sigma_p^0\right)^2}\frac{dJ(a,Q)}{da} \\ T_R = \frac{E}{\left(\sigma_p^0\right)^2}\left.\frac{dJ_R(a,Q)}{da}\right|_{a=0} \end{array} \right.$$

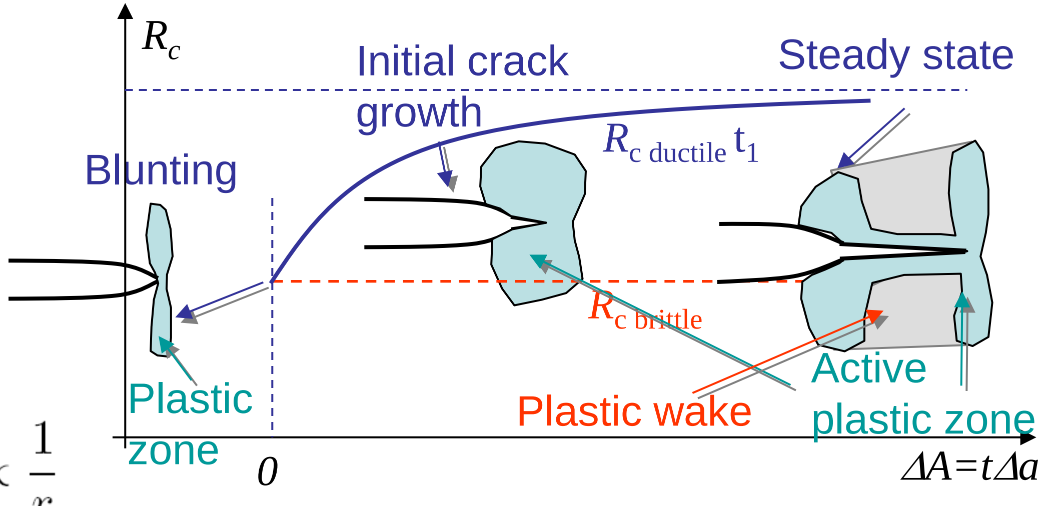


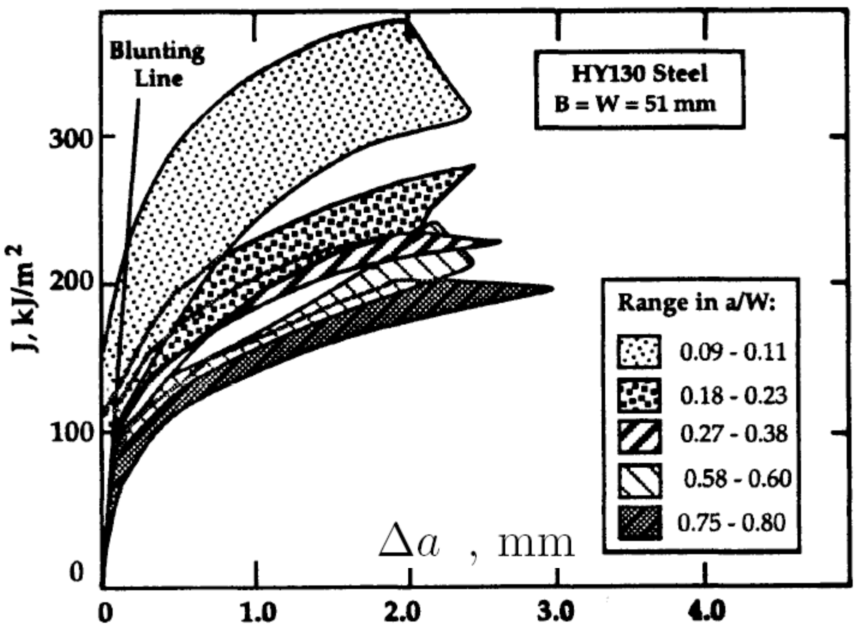
- Crack growth stability (2)
  - Tabulated values

Material	Specimen	T° [°C]	$(\sigma_p^0+\sigma_{TS})/2$ [MPa]	J_{IC} [MPa·m]	dJ/da [MPa]	T_R []	D [mm]
ASTM 470 (Cr-Mo-V)	CT	149	621	0.084	48.5	25.5	1.78
ASTM 470 (Cr-Mo-V)	CT	260	621	0.074	49	25.8	1.52
ASTM 470 (Cr-Mo-V)	CT	427	592	0.088	84	48.6	1.01
ASTM A453 Stainless steel	CT	24	931	0.124	141	32.8	
ASTM A453 Stainless steel	CT	204	820	0.107	84	25.6	1.27
ASTM A453 Stainless steel	CT	427	772	0.092	65	22.4	
6061-T651 Aluminum	CT	24	298.2	0.0178	3.45	2.79	5

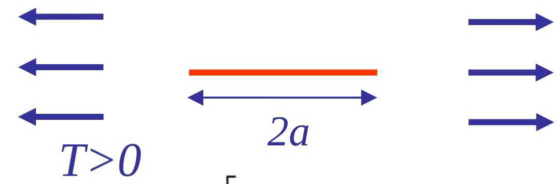
- Moving crack and elasto-plastic material
  - A moving crack sees a plastic wake
    - The nonlinear elasticity assumption (power law) is not valid anymore
    - Ex: perfectly plastic material
      - HRR $n \rightarrow \infty \Rightarrow \bar{\varepsilon}^p \propto \frac{1}{r}$
      - Steady state $\bar{\varepsilon}^p \propto \ln \frac{1}{r}$
    - Singularity is weaker in the steady state, so the apparent J limit is larger
  - J_{SS} is the steady state limit of J
    - $= J_{IC}$ for brittle materials
    - Many times J_{IC} for ductile materials
  - This is characterized by

$$T_R = \frac{E}{\left(\sigma_p^0\right)^2} \frac{dJ_R}{da}$$

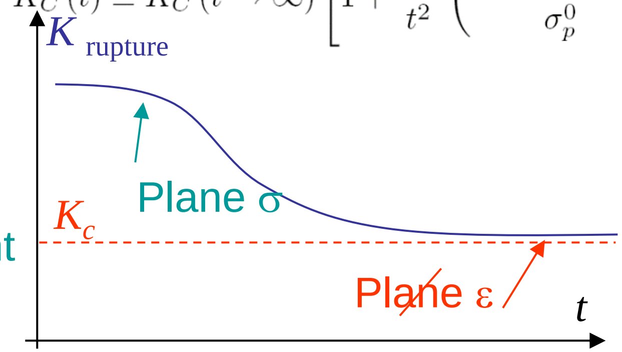


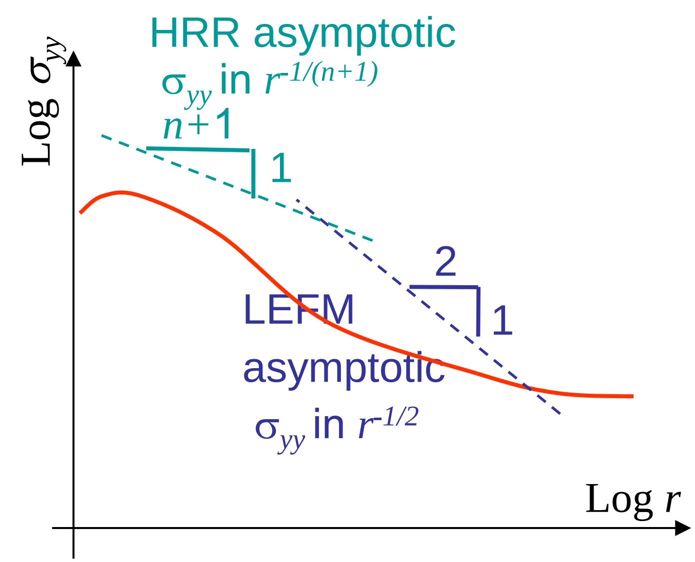


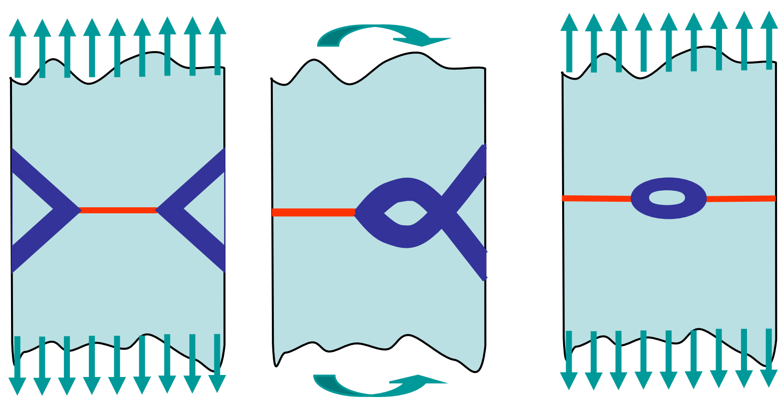
- A single parameter cannot always fully described the process
  - Example 1: Thickness effect
    - In LEFM, recall T-stress is the 0-order term, which is dominant at radius r_c
    - In general, if $T < 0$, the measured fracture K will be larger than for $T > 0$
    - If thick specimen $T > 0$
    - HRR solution is also an asymptotic behavior
    - Measure of J limit is also thickness-dependant
  - Example 2: Large Yielding
    - There is no zone of J-dominance
    - Plastic strain concentrations depend on the experiment



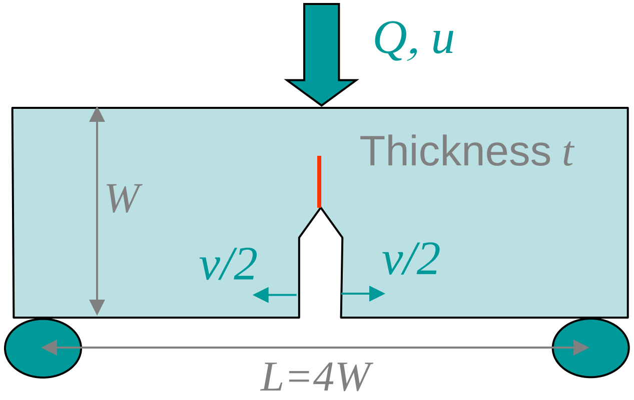
$$K_C(t) \simeq K_C(t\to\infty)\left[1+\frac{1.4}{t^2}\left(\frac{K_C(t\to\infty)}{\sigma_p^0}\right)^4\right]^{\frac{1}{2}}$$







- Consider the SENB specimen
  - Made of steel
    - E = 210 GPa, σ_p^0 = 700 MPa
    - Power law: α=0.5, n=10
  - Geometry
    - t = 25 mm, W = 75 mm, L = 300 mm
    - a = 28 mm
  - Assuming a toughness of J_{IC} = 200 kPa·m, dJ/da = 87 MPa
    - Compute the maximal loading before fracture using LEFM
    - Compute the maximal loading before fracture using NLFM
    - Evaluate the crack growth stability



- LEFM solution
  - SIF (handbook)
    - $$K_I = \frac{QL}{tW^{\frac{3}{2}}} f\left(\frac{a_{\text{eff}}}{W}\right)$$

    with $$f\left(\frac{a}{W}\right) = \frac{3\sqrt{\frac{a}{W}}\left[1.99 - \frac{a}{W}\left(1-\frac{a}{W}\right)\left(2.15 - 3.93\frac{a}{W} + 2.7\left(\frac{a}{W}\right)^2\right)\right]}{2\left(1+2\frac{a}{W}\right)\left(1-\frac{a}{W}\right)^{\frac{3}{2}}}$$
  - Effective crack length
    - $a_{\text{eff}} = a + \eta r_p$

    with $$r_p = \begin{cases} \frac{n-1}{n+1}\frac{1}{\pi}\left(\frac{K_I(a_{\text{eff}})}{\sigma_p^0}\right)^2 & \text{if plane } \sigma \\ \frac{n-1}{n+1}\frac{1}{3\pi}\left(\frac{K_I(a_{\text{eff}})}{\sigma_p^0}\right)^2 & \text{if plane } \varepsilon \end{cases}$$?

    & $$\eta = \frac{1}{2\left[1+\left(\frac{Q}{Q^0}\right)^2\right]}$$ with $$Q^0 = 0.728\sigma_p^0 \frac{2t(W-a)^2}{L}$$ (handbook)

- LEFM solution
  - Example Q=150 kN
    - Starting point

$$K_I^{(0)} = \frac{QL}{tW^{\frac{3}{2}}} f\left(\frac{a}{W}\right) = \frac{150000\,.\,0.3}{0.025\,.\,0.075^{\frac{3}{2}}} f\,(0.3733)$$
$$= \frac{150000\,.\,0.3}{0.025\,.\,0.075^{\frac{3}{2}}} 1.843 = 161.49\mathrm{MPa}\cdot\sqrt{\mathrm{m}}$$

    - Yield load

$$Q^0 = 0.728\sigma_p^0 \frac{2t(W-a)^2}{L} = 0.728 \frac{700\;10^6\;0.05 * 0.047^2}{0.3} = 187.62\mathrm{kN}$$

Exercise 1: Solution

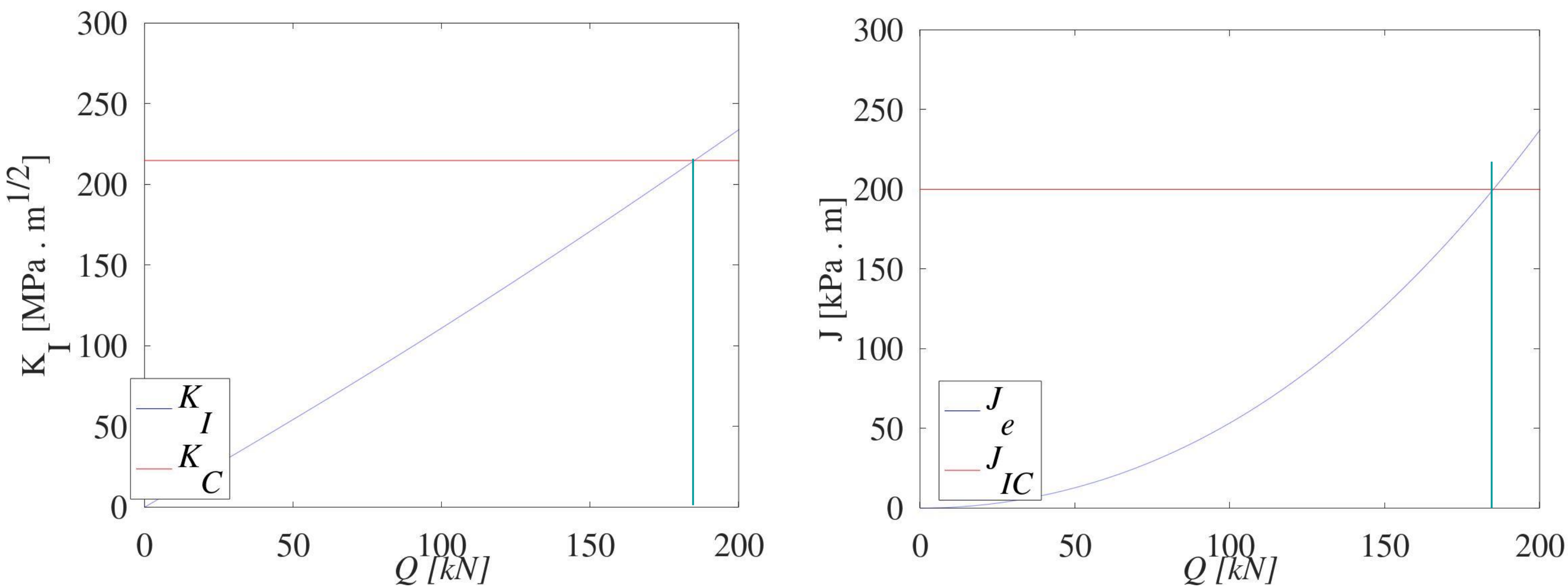
- LEFM solution (2)
  - Example Q=150 kN (2)
    - First iteration

$$
\begin{cases}
r_p^{(1)} = \dfrac{n-1}{n+1}\dfrac{1}{3\pi}\left(\dfrac{K_I^{(0)}}{\sigma_p^0}\right)^2 = \dfrac{9}{33\pi}\left(\dfrac{161.49}{700}\right)^2 = 4.62\text{mm} \\
a_{\text{eff}}^{(1)} = a + \dfrac{r_p^{(1)}}{2\left(1+150^2/187.62^2\right)} = 29.41\text{mm} \\
K_I^{(1)} = \dfrac{QL}{tW^{\frac{3}{2}}} f\left(\dfrac{a_{\text{eff}}^{(1)}}{W}\right) = \dfrac{150000\,.\,0.3}{0.025\,.\,0.075^{\frac{3}{2}}} f\left(0.39212\right) \\
\qquad = \dfrac{150000\,.\,0.3}{0.025\,.\,0.075^{\frac{3}{2}}} 1.9393 = 169.95\text{MPa}\cdot\sqrt{\text{m}}
\end{cases}
$$

    - Second iteration

$$
\begin{cases}
r_p^{(2)} = \dfrac{n-1}{n+1}\dfrac{1}{3\pi}\left(\dfrac{K_I^{(1)}}{\sigma_p^0}\right)^2 = \dfrac{9}{33\pi}\left(\dfrac{169.95}{700}\right)^2 = 5.12\text{mm} \\
a_{\text{eff}}^{(2)} = a + \dfrac{r_p^{(2)}}{2\left(1+150^2/187.62^2\right)} = 29.56\text{mm} \\
K_I^{(2)} = \dfrac{QL}{tW^{\frac{3}{2}}} f\left(\dfrac{a_{\text{eff}}^{(2)}}{W}\right) = \dfrac{150000\,.\,0.3}{0.025\,.\,0.075^{\frac{3}{2}}} f\left(0.39415\right) \\
\qquad = \dfrac{150000\,.\,0.3}{0.025\,.\,0.075^{\frac{3}{2}}} 1.95 = 170.90\text{MPa}\cdot\sqrt{\text{m}}
\end{cases}
$$

- LEFM solution (3)
  - Whole Q-range with $K_C = \sqrt{J_{IC} E'} = \sqrt{200000 \frac{210\,10^9}{1-\nu^2}} = 214.8\text{MPa} \cdot \sqrt{\text{m}}$
    - Or in terms of $J = K_I^2 / E'$



- Rupture load in LEFM (with corrected η) is 186 kN

Exercise 1: Solution

- NLFM solution
  - Handbook
    - J_p: $J_p = \frac{\alpha \left(\sigma_p^0\right)^2 (W-a)}{E} h_1 \left(\frac{a}{W}, n\right) \left(\frac{Q}{Q^0}\right)^{n+1}$
      - Yield load: $Q^0 = 0.728 \sigma_p^0 \frac{2t(W-a)^2}{L}$
      - Coefficient: $h_1 \left(\frac{a}{W} = 0.375,\ n = 10,\ \text{plane}\varepsilon\right) = 0.556$

Table 3-5

h_1, h_2 and h_3 for SECP in plane strain under three-point bending.

		n = 1	n = 2	n = 3	n = 5	n = 7	n = 10	n = 13	n = 16	n = 20
a/b = 1/8	h_1	0.936	0.869	0.805	0.687	0.580	0.437	0.329	0.245	0.165
	h_2	6.97	6.77	6.29	5.29	4.38	3.24	2.40	1.78	1.19
	h_3	3.00	22.1	20.0	15.0	11.7	8.39	6.14	4.54	3.01
a/b = 1/4	h_1	1.20	1.034	0.930	0.762	0.633	0.523	0.396	0.303	0.215
	h_2	5.80	4.67	4.01	3.08	2.45	1.93	1.45	1.09	0.758
	h_3	4.08	9.72	8.36	5.86	4.47	3.42	2.54	1.90	1.32
a/b = 3/8	h_1	1.33	1.15	1.02	0.084	0.695	$\boxed{0.556}$	0.442	0.360	0.265
	h_2	5.18	3.93	3.20	2.38	1.93	1.47	1.15	0.928	0.684
	h_3	4.51	6.01	5.03	3.74	3.02	2.30	1.80	1.45	1.07
a/b = 1/2	h_1	1.41	1.09	0.922	0.675	0.495	0.331	0.211	0.135	0.0741
	h_2	4.87	3.28	2.53	1.69	1.19	0.773	0.480	0.304	0.165
	h_3	4.69	4.33	3.49	2.35	1.66	1.08	0.669	0.424	0.230

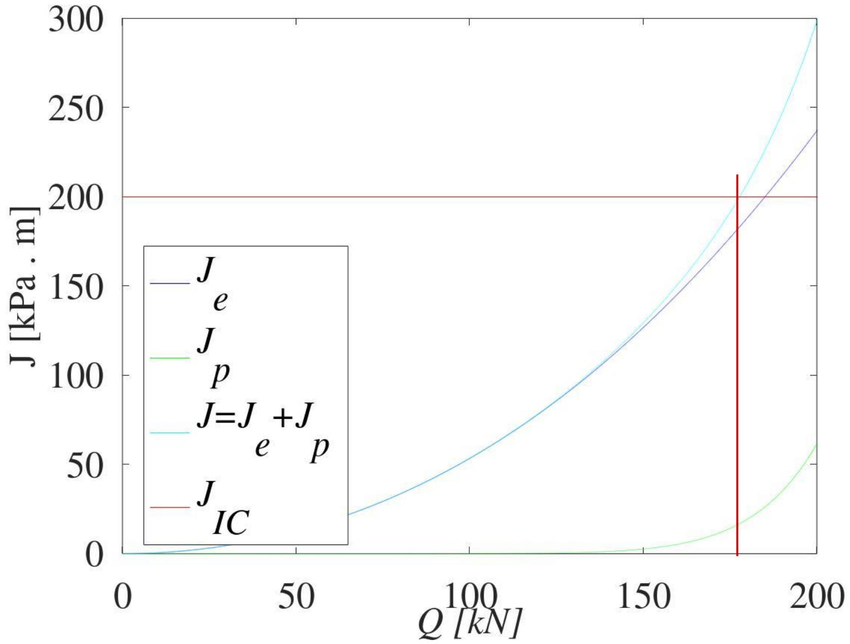
Exercise 1: Solution

- NLFM solution (2)
  - Example Q = 150 kN: $$J_p = \frac{\alpha\left(\sigma_p^0\right)^2(W-a)}{E} h_1\left(\frac{a}{W}, n\right)\left(\frac{Q}{Q^0}\right)^{n+1}$$
    $$= 0.556\,\frac{0.5\,700^2\,10^{12}\,0.047}{210\,10^9}\left(\frac{150}{187.62}\right)^{11} = 2.6\mathrm{kN{\cdot}m}$$
  - Whole range of Q:
    - Limit load 177.71 kN
  - Validity:
    - J dominance if

      a, W-a > 25 J/σ_p^0 = 4.6 mm **OK**

      Plane ε? t >> 4.6 mm
    - K dominance if

      a, W-a > 2.5 $(K/\sigma_p^0)^2$ = 0.149 m **KO**



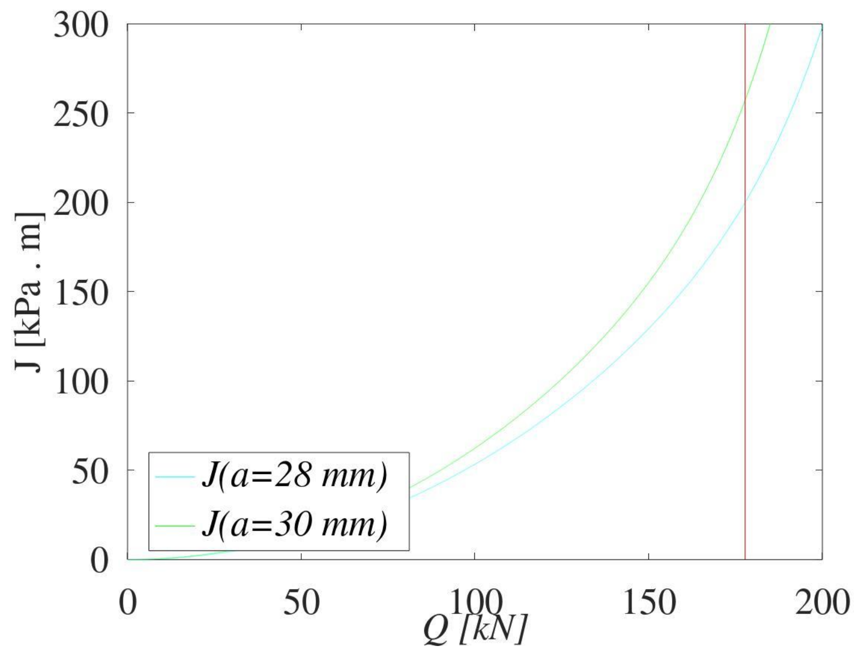
- Crack growth stability
  - Region of J-controlled crack growth
    - Material length scale: $D = J_{IC} / \frac{dJ}{d\Delta a} = 200\,10^3/87\,10^6 = 2.3\text{mm}$
    - For SENB, crack growth is J-controlled if D < 0.1 (W-a) = 4.7 mm **OK**
    - Then, stability can be studied for Δa < 0.06 (W-a) = 2.8 mm
  - 2 studies:
    - a = 28 mm: done
    - a = 30 mm so Δa=2 mm< 2.8 mm: to do

- Crack growth stability
  - 2 studies:
    - a = 28 mm: done

      Q_{rupt} = 177.75 kN, J = 200 kPa·m
    - Dead load: Q = 177.75 kN

      but with a = 30 mm

      ⟹ J = 257.22 KPa·m



  - Crack stability
    - T for this geometry and dead load

$$T = \frac{E}{\left(\sigma_p^0\right)^2} \frac{\Delta J}{\Delta a} = \frac{210\,10^9}{700^2\,10^{12}} \frac{57220}{0.002} = 12.26$$

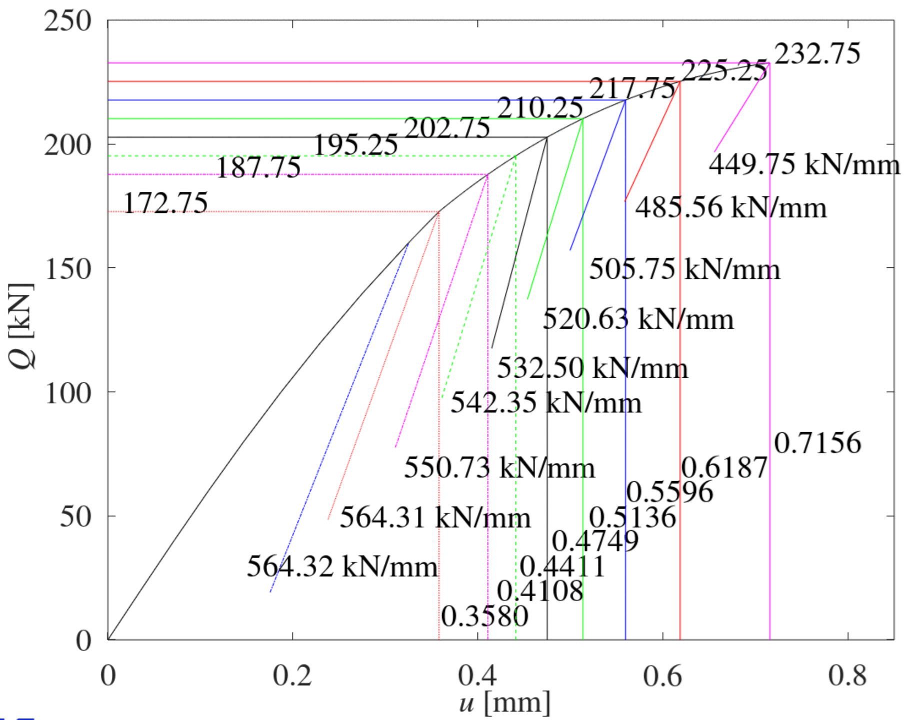
    - Tearing of the material

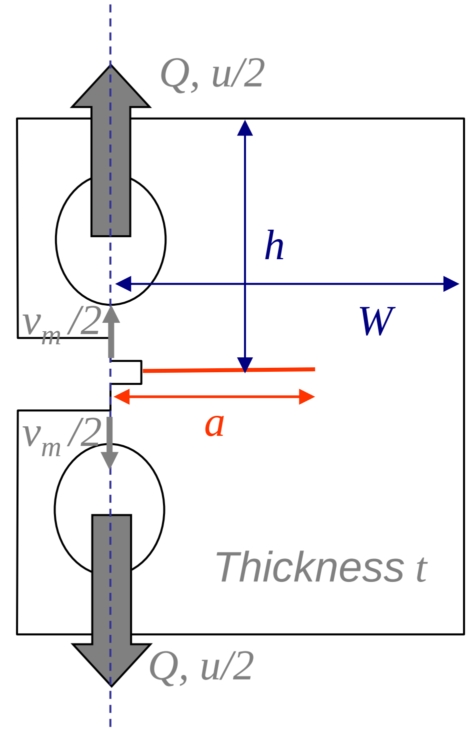
$$T_R = \frac{E}{\left(\sigma_p^0\right)^2} \frac{dJ_R}{da} = \frac{210\,10^9}{700^2\,10^{12}} 87\,10^6 = 37.3$$

⟹ stable crack (for Δa < 2.8 mm)

Exercise 2

- Toughness evaluation
  - Ductile material: steel
  - Follow the norm ASTM E1820
  - Normalized specimen
    - Compact Tension Specimen
    - Thick enough $W - a, t > 25 \frac{2J_{IC}}{\sigma_p^0 + \sigma_{TS}}$
  - Fracture test

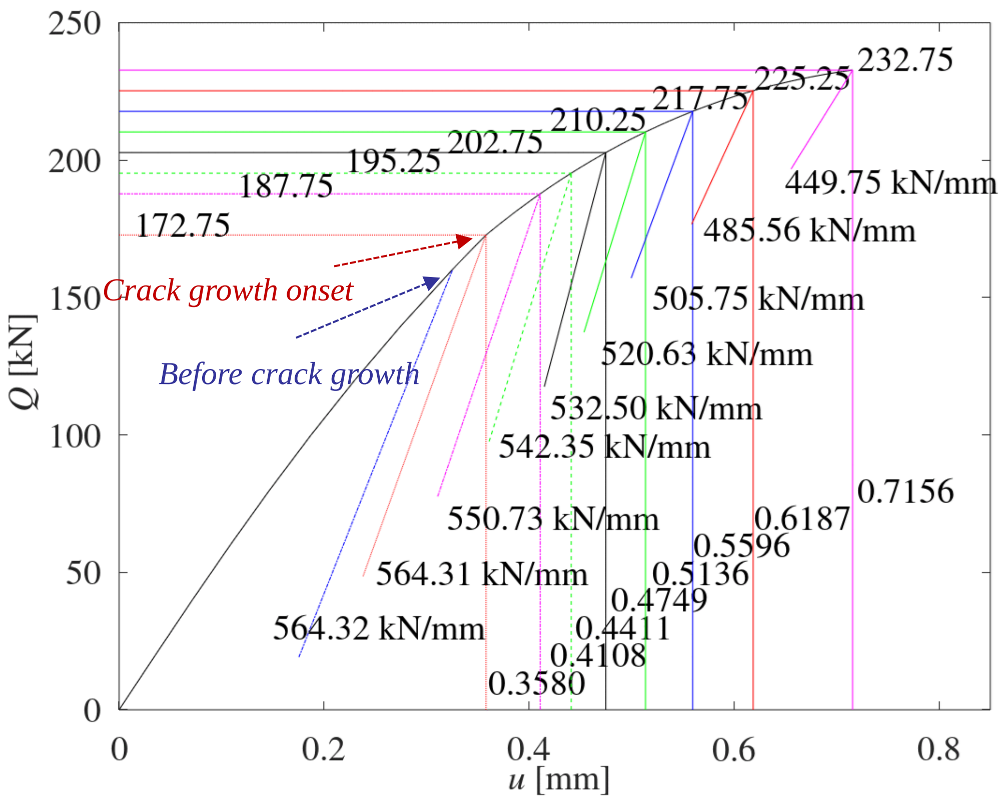


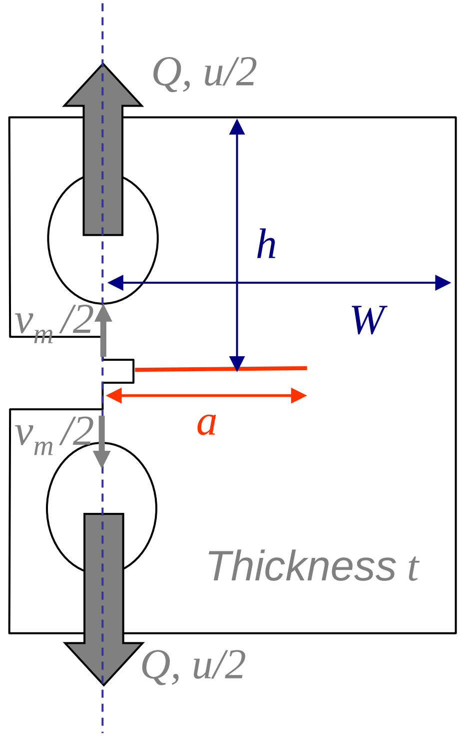


Properties	Values
Half height h	0.036 m
Width W	0.06 m
Thickness t	0.03 m
Young E	210 Gpa
Yield σ_p^0	600 Mpa
Poisson ν [-]	0.3
Power law α [-]	1
Power law n [-]	3

- 1st step: Determine crack length a_0 after fatigue loading
  - Compliance before crack growth onset
    - Check during unloading to have the elastic part

$$\frac{u_e}{Q} = \frac{1}{564.32\ 10^6} = 0.001772\ 10^{-6}\ \mathrm{m/N}$$





Exercise 2: Solution

- 1st step: Determine crack length a_0 after fatigue loading
  - Compliance
    - Calibration of the geometry following the norm (a correction for rotation should actually be introduced)

$$\begin{cases} U = \dfrac{1}{1+\sqrt{\dfrac{Etu_{,e}}{Q}}} \\ \dfrac{a}{W} = 1.000196 - 4.06391U + 11.242U^2 - 106.043U^3 + \\ \qquad 464.335U^4 - 650.677U^5 \end{cases}$$

    - For this test

$$\frac{u_e}{Q} = \frac{1}{564.32} = 0.001772 \text{ mm/kN}$$

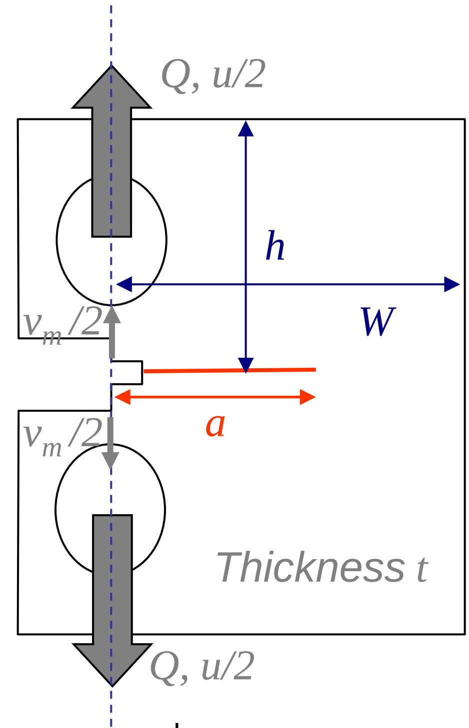
$$\Rightarrow U = 0.23035$$

$$\Rightarrow \frac{a_0}{W} = 0.2498$$

$$\Rightarrow a_0 = 0.01499 \text{ m}$$

    - Check size requirements

$a_0 > 1.3 \text{ mm},\ a_0 > 0.05\ t$ OK



Properties	Values
Half height h	0.036 m
Width W	0.06 m
Thickness t	0.03 m
Young E	210 Gpa
Yield σ_p^0	600 Mpa
Poisson ν [-]	0.3
Power law α [-]	1
Power law n [-]	3

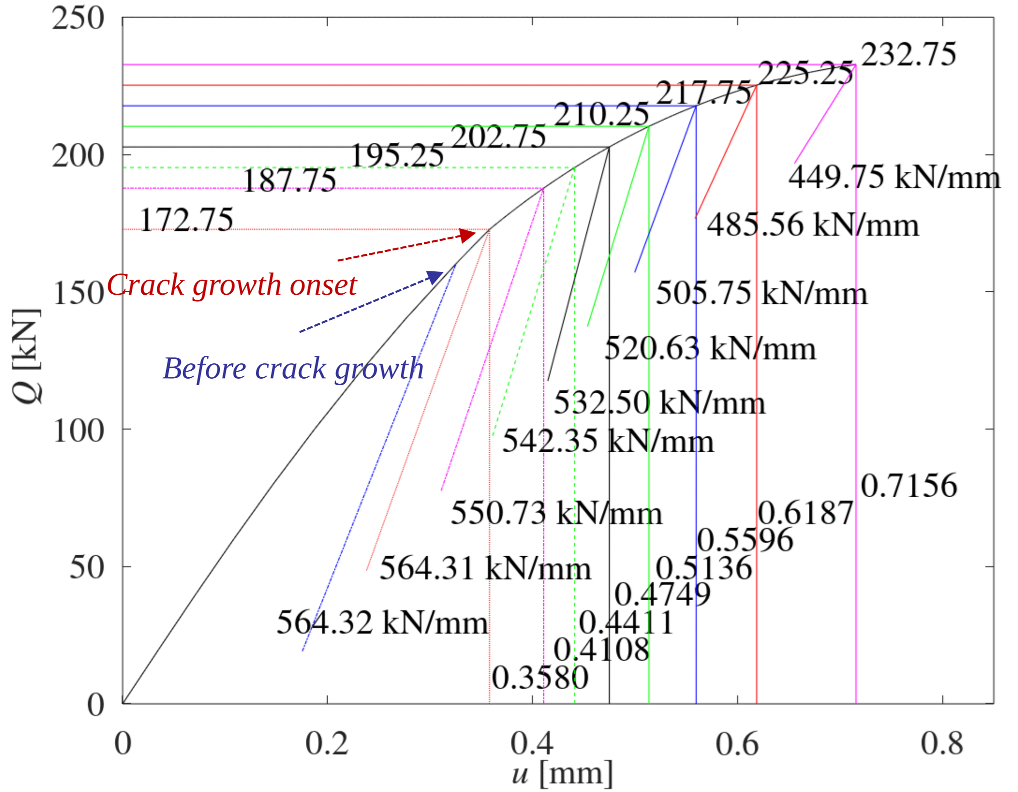
Exercise 2: Solution

- 2nd step: Evaluate the couples (a_i, Q_i)
  - Compliance data:
    - $Q_1 = 172.75$ kN

      $\frac{u_{e_1}}{Q_1} = \frac{1}{564.31}$ ⇒ $a_1 = 0.015$ m
    - $Q_2 = 187.75$ kN

      $\frac{u_{e_2}}{Q_2} = \frac{1}{550.73}$ ⇒ $a_2 = 0.015277$ m
    - …….

Q [kN]	u [mm]	$\frac{u_e}{Q}$ [kN/mm]	a [mm]
172.75	0.3580	564.31	14.988
187.75	0.4108	550.73	15.277
195.25	0.4411	542.35	15.460
202.75	0.4749	532.50	15.679
210.25	0.5136	520.63	15.950
217.75	0.5596	505.75	16.300
225.25	0.6187	485.56	16.796
232.75	0.7156	449.75	17.740

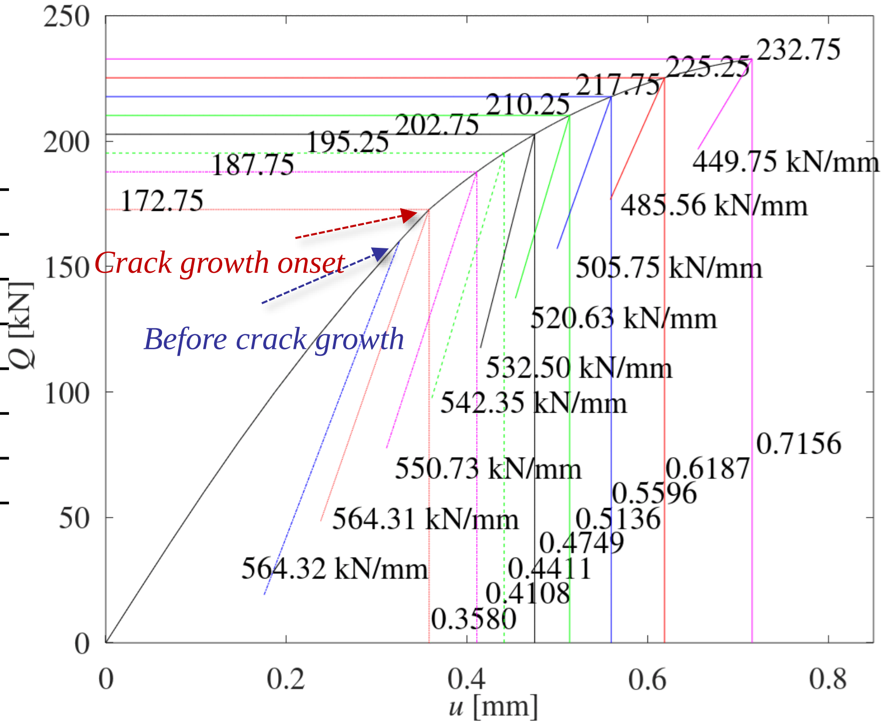


- 2nd step: Evaluate the couples (a_i, Q_i) (2)
  - Elasto-plastic responses:
    - $Q_1 = 172.75$ kN, $u = 0.3580$ mm

      $\frac{u_{e_1}}{Q_1} = \frac{1}{564.31} \Rightarrow u_{e_1} = 0.3061$ mm $\Rightarrow u_{p_1} = u_1 - u_{e_1} = 0.05189$ mm
    - $Q_2 = 187.75$ kN, $u = 0.4108$ mm

      $\frac{u_{e_2}}{Q_2} = \frac{1}{550.73} \Rightarrow u_{e_2} = 0.3409$ mm $\Rightarrow u_{p_2} = u_2 - u_{e_2} = 0.06992$ mm
    - …….

Q [kN]	u [mm]	$\frac{u_e}{Q}$ [kN/mm]	a [mm]	u_e [mm]	u_p [mm]
172.75	0.3580	564.31	14.988	0.3061	0.05189
187.75	0.4108	550.73	15.277	0.3409	0.06992
195.25	0.4411	542.35	15.460	0.3600	0.08107
202.75	0.4749	532.50	15.679	0.3808	0.09413
210.25	0.5136	520.63	15.950	0.4038	0.1098
217.75	0.5596	505.75	16.300	0.4306	0.1291
225.25	0.6187	485.56	16.796	0.4639	0.1548
232.75	0.7156	449.75	17.740	0.5175	0.1981



- 3rd step: Data reduction – Elastic part
  - Stress intensity factor:
    - $$K_I = \frac{Q}{tW^{\frac{1}{2}}} \frac{\left(2+\frac{a}{W}\right)\left(0.886+4.64\frac{a}{W}-13.32\left(\frac{a}{W}\right)^2+14.72\left(\frac{a}{W}\right)^3-5.6\left(\frac{a}{W}\right)^4\right)}{\left(1-\frac{a}{W}\right)^{\frac{3}{2}}}$$
    - $Q_1 = 172.75$ kN, $a = 14.988$ mm

      $\frac{a_1}{W} = 0.2498 \Rightarrow K_{I_1} = 115.71\ \mathrm{MPa}\sqrt{m} \Rightarrow J_{e_1} = \frac{K_{I_1}^2}{E'} = 58016\ \mathrm{J\cdot m^{-2}}$
    - $Q_2 = 187.75$ kN, $a = 15.277$ mm

      $\frac{a_2}{W} = 0.25462 \Rightarrow K_{I_2} = 127.41\ \mathrm{MPa}\sqrt{m} \Rightarrow J_{e_2} = \frac{K_{I_2}^2}{E'} = 70341\ \mathrm{J\cdot m^{-2}}$
    - ....

Q [kN]	a [mm]	u_e [mm]	K_I [Mpa m$^{1/2}$]	J_e [J/m^2]
172.75	14.988	0.3061	115.71	58016
187.75	15.277	0.3409	127.41	70341
195.25	15.460	0.3600	133.59	77332
202.75	15.679	0.3808	140.09	85039
210.25	15.950	0.4038	147.03	93679
217.75	16.300	0.4306	154.66	103649
225.25	16.796	0.4639	163.51	115856
232.75	17.740	0.5175	176.04	134291

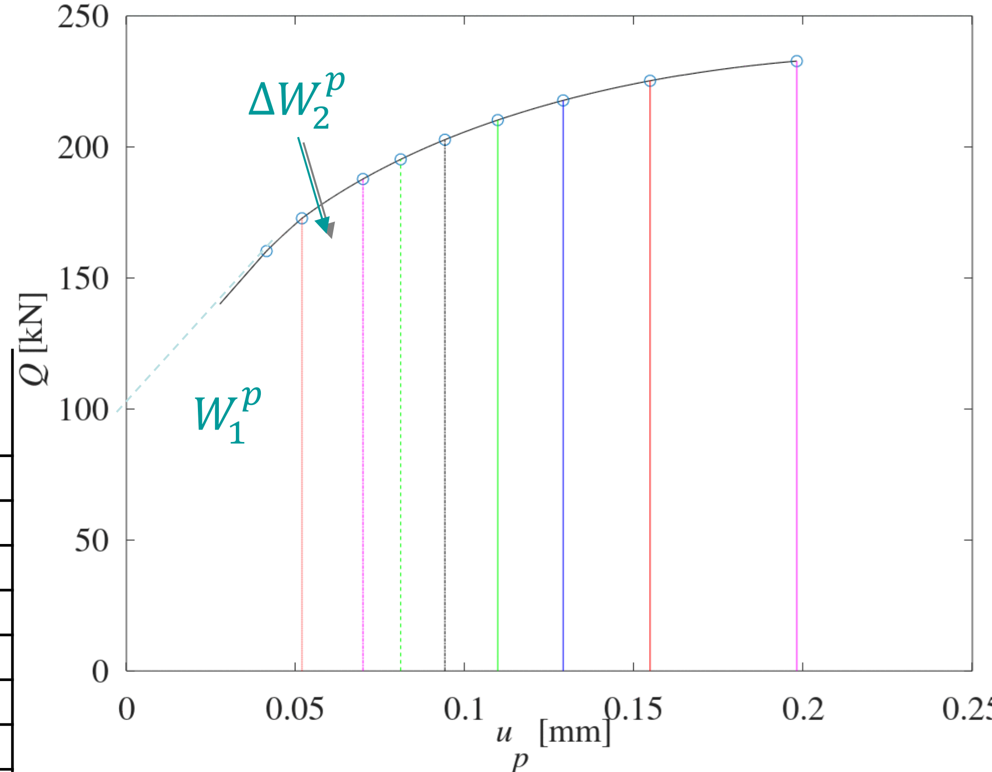
Properties	Values
Half height h	0.036 m
Width W	0.06 m
Thickness t	0.03 m
Young E	210 Gpa
Yield σ_p^0	600 Mpa
Poisson ν [-]	0.3
Power law α [-]	1
Power law n [-]	3

- 3rd step: Data reduction – Plastic part
  - Plastic energy:
    - $W_1^p = 7.3564 \text{ J}$
    - $\Delta W_2^p = \frac{Q_1+Q_2}{2}\left(u_{p\,2} - u_{p\,1}\right) = 3.2504 \text{ J}$

$$\Rightarrow \quad W_2^p = W_1^p + \Delta W_2^p = 10.6068 \text{ J}$$

    - ....

Q [kN]	a [mm]	u_p [mm]	ΔW^p [J]	W^p [J]
172.75	14.988	0.05189	-	7.3564
187.75	15.277	0.06992	3.2504	10.6068
195.25	15.460	0.08107	2.1350	12.7418
202.75	15.679	0.09413	2.5993	15.3411
210.25	15.950	0.1098	3.2250	18.5660
217.75	16.300	0.1291	4.1380	22.7040
225.25	16.796	0.1548	5.6875	28.3915
232.75	17.740	0.1981	9.9283	38.3198



- 3rd step: Data reduction – Plastic part (2)
  - J-integral by η:
    - $J_{p_1} = \left(\dfrac{2 + 0.522 \dfrac{W - a_0}{W}}{W - a_0}\right) \dfrac{W_1^p}{t} \Rightarrow J_{p_1} = 10776.06 \ \mathrm{J \cdot m^{-2}}$
    - $J_{p_i} = \left(J_{p_{i-1}} + \left(\dfrac{2 + 0.522 \dfrac{W - a_{i-1}}{W}}{W - a_{i-1}}\right) \dfrac{\Delta W_i^p}{t}\right) \left(1 - \left(1 + 0.76 \dfrac{W - a_{i-1}}{W}\right) \left(\dfrac{a_i - a_{i-1}}{W - a_{i-1}}\right)\right)$

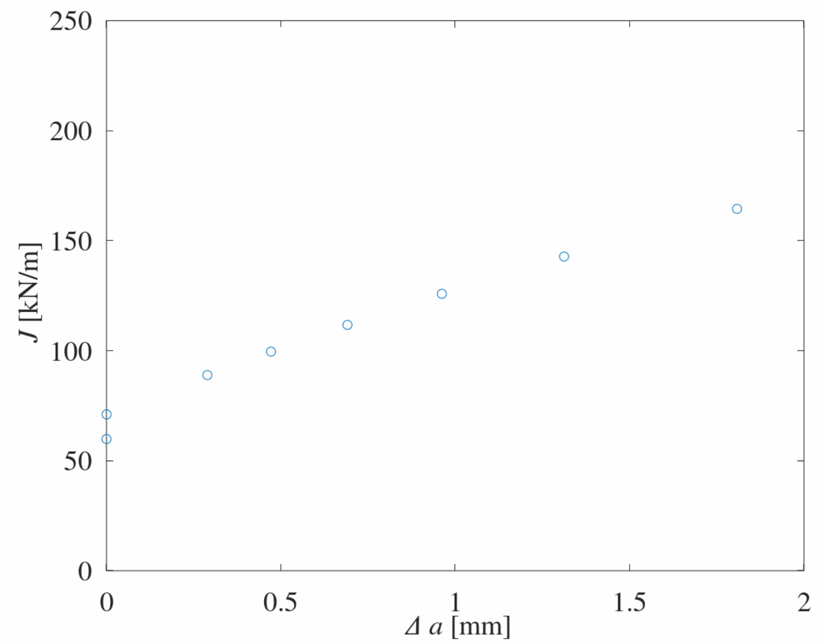
      $a_2 = 15.277 \ \mathrm{mm}, \ \Delta W_2^p = 3.2504 \ \mathrm{J} \Rightarrow J_{p_2} = 15753 \ \mathrm{J \cdot m^{-2}}$
    - …….

Q [kN]	a [mm]	u_p [mm]	ΔW^p [J]	W^p [J]	J_p [J/m^2]
172.75	14.988	0.05189	-	7.3564	10776
187.75	15.277	0.06992	3.2504	10.6068	15753
195.25	15.460	0.08107	2.1350	12.7418	18978
202.75	15.679	0.09413	2.5993	15.3411	22866
210.25	15.950	0.1098	3.2250	18.5660	27637
217.75	16.300	0.1291	4.1380	22.7040	33677
225.25	16.796	0.1548	5.6875	28.3915	41850
232.75	17.740	0.1981	9.9283	38.3198	55842

Properties	Values
Half height h	0.036 m
Width W	0.06 m
Thickness t	0.03 m
Young E	210 Gpa
Yield σ_p^0	600 Mpa
Poisson ν [-]	0.3
Power law α [-]	1
Power law n [-]	3

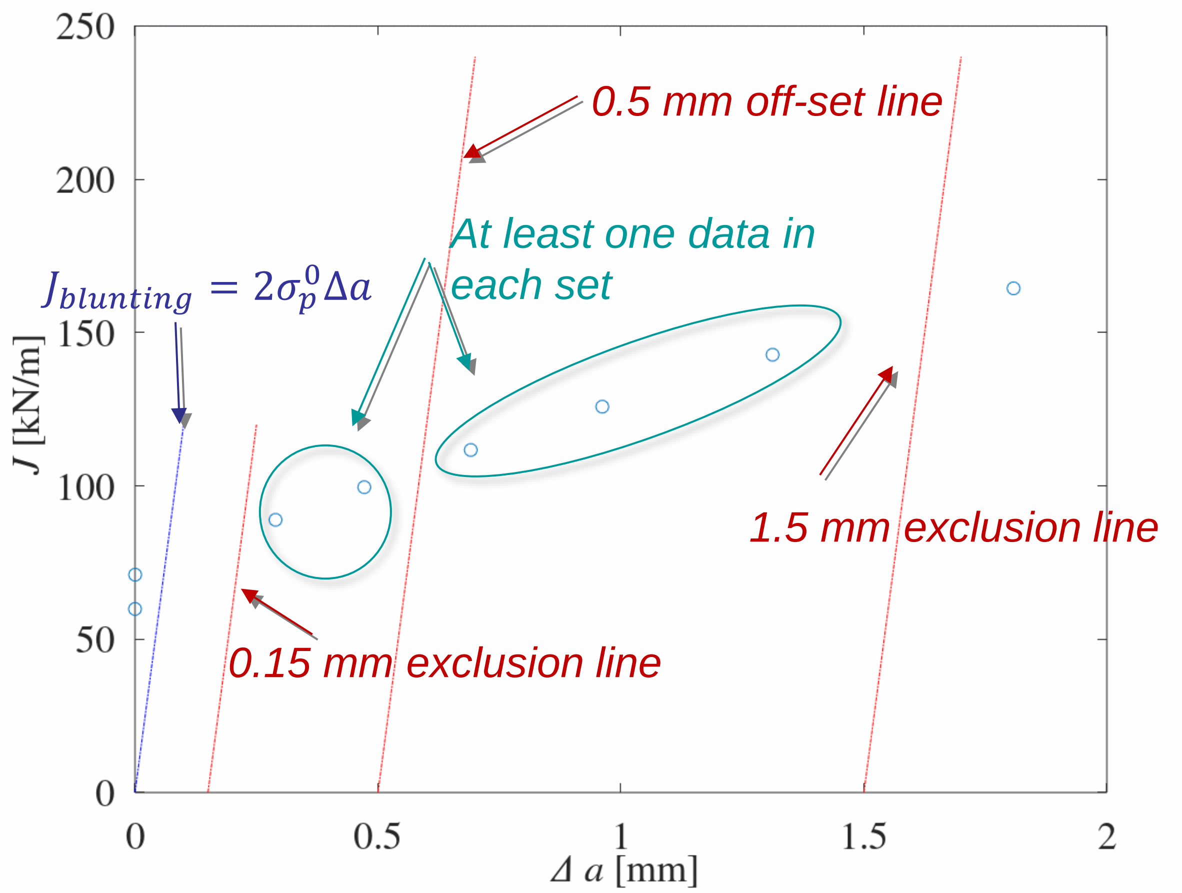
- 3rd step: Data reduction – Elasto-Plastic part
  - Total J-integral:
    - $J_1 = J_{e_1} + J_{p_1}$ ⇨ $J_1 = 68792\ \mathrm{J \cdot m^{-2}}$
    - .....

Q [kN]	a [mm]	Δa [mm]	J_e [J/m²]	J_p [J/m²]	J [J/m²]
172.75	14.988	0	58016	10776	68792
187.75	15.277	0.2892	70341	15753	86094
195.25	15.460	0.4718	77332	18978	96310
202.75	15.679	0.6910	85039	22866	107906
210.25	15.950	0.9617	93679	27637	121316
217.75	16.300	1.312	103649	33677	137326
225.25	16.796	1.808	115856	41850	157706
232.75	17.740	2.752	134291	55842	190133



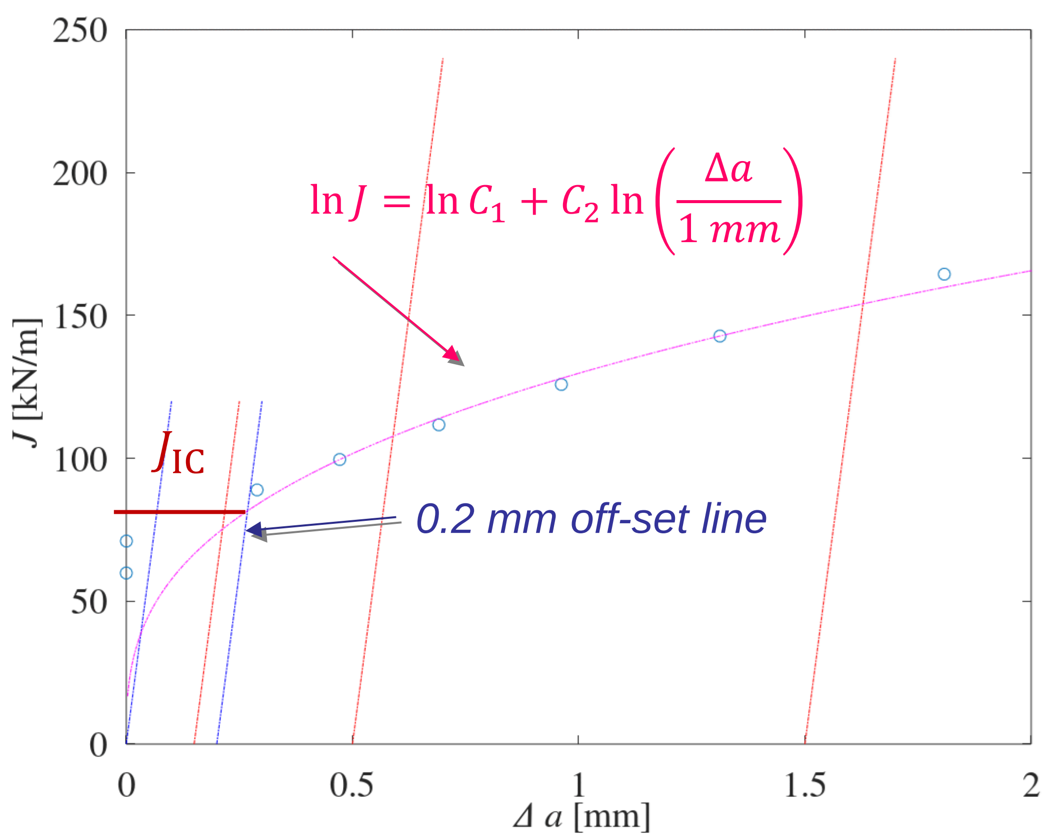
- 4^{th} step: Analysis
  - Check data:

Q [kN]	a [mm]	Δa [mm]	J [J/m^2]
172.75	14.988	0	68792
187.75	15.277	0.2892	86094
195.25	15.460	0.4718	96310
202.75	15.679	0.6910	107906
210.25	15.950	0.9617	121316
217.75	16.300	1.312	137326
225.25	16.796	1.808	157706
232.75	17.740	2.752	190133



- 4th step: Analysis
  - Extrapolate:
    - Use date between exclusions lines
  - Plane strain value of J near the onset of stable crack growth: J_{IC}
    - Fracture toughness
    - Intersection with the 0.2 mm offset line
    - $J_{\mathrm{IC}} = 80\ 193\ \ \mathrm{J} \cdot \mathrm{m}^{-2}$

$$\Rightarrow K_{\mathrm{IC}} = \sqrt{J_{\mathrm{IC}} E'} = 136\ \mathrm{MPa}\sqrt{\mathrm{m}}$$



  - Check validity

$$t, W - a > 25 \frac{J_{IC}}{\frac{\sigma_p^0}{2} + \frac{\sigma_{TS}}{2}} \simeq 3.3\ \mathrm{mm}$$

References

- Lecture notes
  - Lecture Notes on Fracture Mechanics, Alan T. Zehnder, Cornell University, Ithaca, http://hdl.handle.net/1813/3075
- Handbook
  - An engineering approach for elastic-plastic fracture analysis V. Kumar, M.D. German, C.F. Shih, NP-1931 Research project 1237-1, 1981, General electric company, https://inis.iaea.org/search/search.aspx?orig_q=RN:12640849
- Other references
  - « on-line »
    - Fracture Mechanics, Piet Schreurs, TUe, http://www.mate.tue.nl/~piet/edu/frm/sht/bmsht.html
  - Book
    - Fracture Mechanics: Fundamentals and applications, D. T. Anderson. CRC press, 1991.
    - An Engineering Approach for Elastic-Plastic Fracture Analysis, V. Kumar, M.D. German, C.F. and Shih CF (1981), Tech. Rep. NP-1931, Electric Power Research Institute.
    - Fatigue of Materials, S. Suresh, Cambridge press, 1998.