University of Liège
Aerospace & Mechanical Engineering

Fracture Mechanics, Damage and Fatigue
Non Linear Fracture Mechanics – HRR Theory

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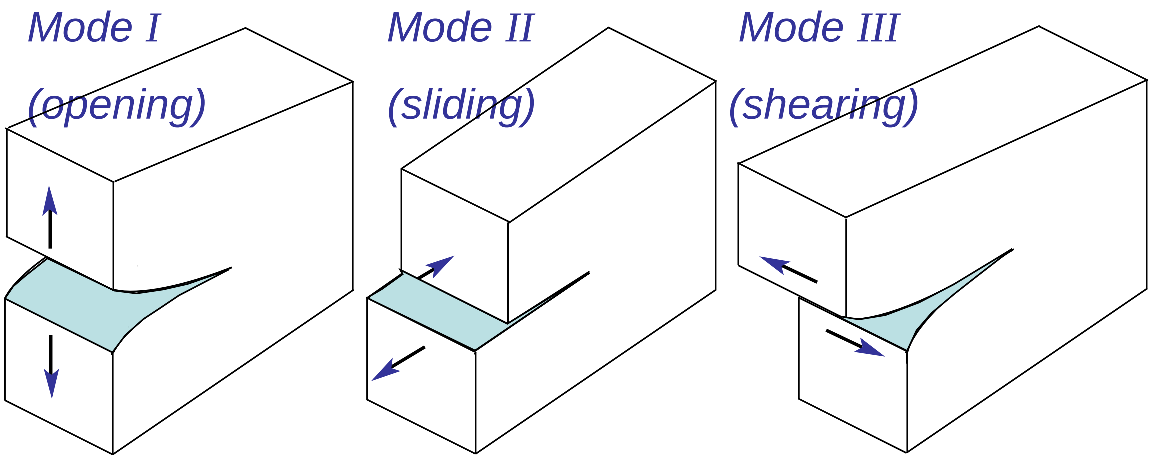
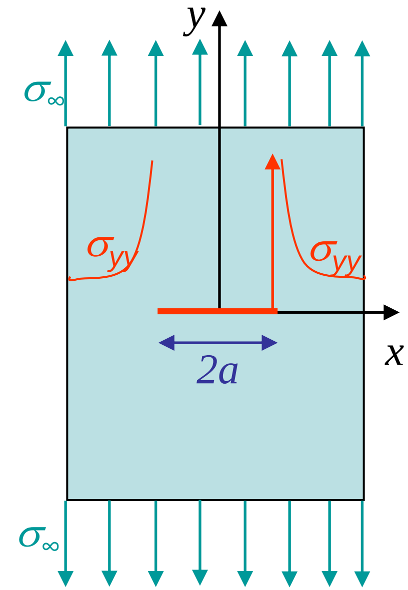
- Cracked body: summary
  - 3 failure modes

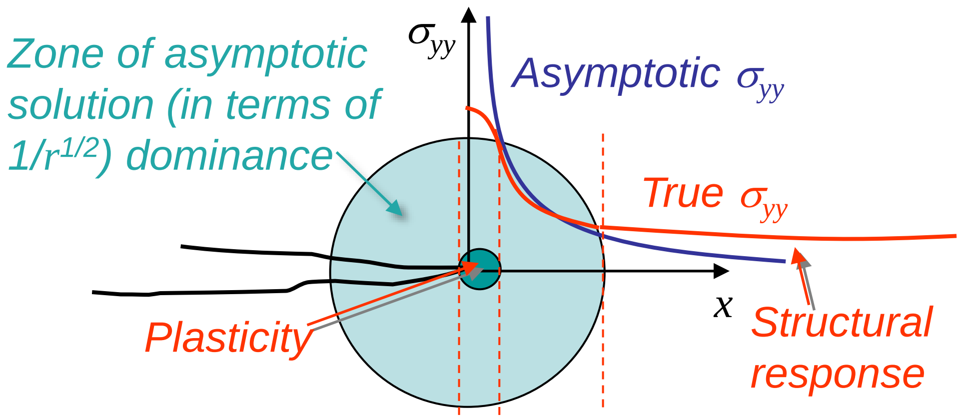




  - Asymptotic solution governed by stress intensity factors

$$\boldsymbol{\sigma}^{\text{mode } i} = \frac{K_i}{\sqrt{2\pi r}} f^{\text{mode } i}(\theta) + o(r^0)$$

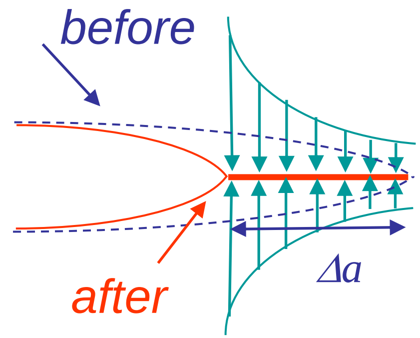
$$\boldsymbol{u}^{\text{mode } i} = K_i \sqrt{\frac{r}{2\pi}}\, g^{\text{mode } i}(\theta) + o(r^0)$$



Linear Elastic Fracture Mechanics (LEFM)

- Cracked body: summary
  - Potential energy $\Pi_T = E_{\text{int}} - Qu$
  - Crack closure integral
    - Energy required to close crack tip

$$\Delta\Pi_T = \int_{\Delta A}\int_{\boldsymbol{u}}^{\boldsymbol{u}+\boldsymbol{\Delta u}} \boldsymbol{t}\cdot[\![\boldsymbol{u}']\!]d\boldsymbol{u}'\ dA$$

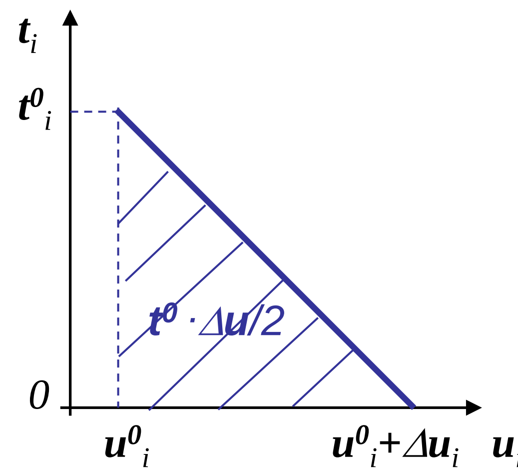


  - Energy release rate
    - Variation of potential energy in case of crack growth

$$G = -\partial_{\text{A}}\left(E_{\text{int}} - W_{\text{ext}}\right) = -\partial_A\Delta\Pi_T$$

    - In linear elasticity

$$G = -\partial_A\Delta\Pi_T = -\lim_{\Delta A\to 0}\frac{1}{\Delta A}\int_{\Delta A}\frac{1}{2}\boldsymbol{t}^{\mathbf{0}}\cdot[\![\boldsymbol{\Delta u}]\!]\,dA$$

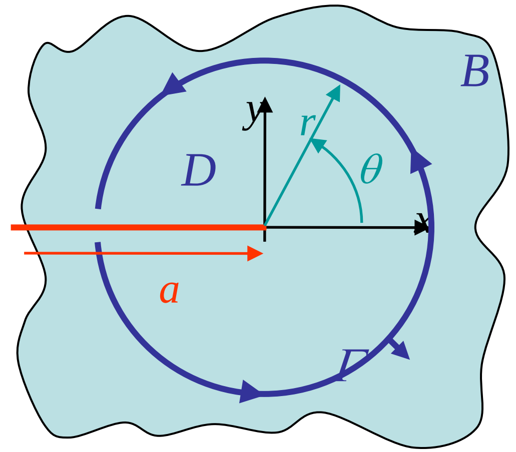


  - In linear elasticity & if crack grows straight ahead

$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \qquad E' = \frac{4E}{(1+\nu)(\kappa+1)} = \begin{cases} E & \text{P}\sigma \\ \frac{E}{1-\nu^2} & \text{P}\varepsilon \end{cases}$$

- Cracked body: summary
  - J-integral
    - Strain energy flow

$$J = \int_{\Gamma} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \right] dl$$

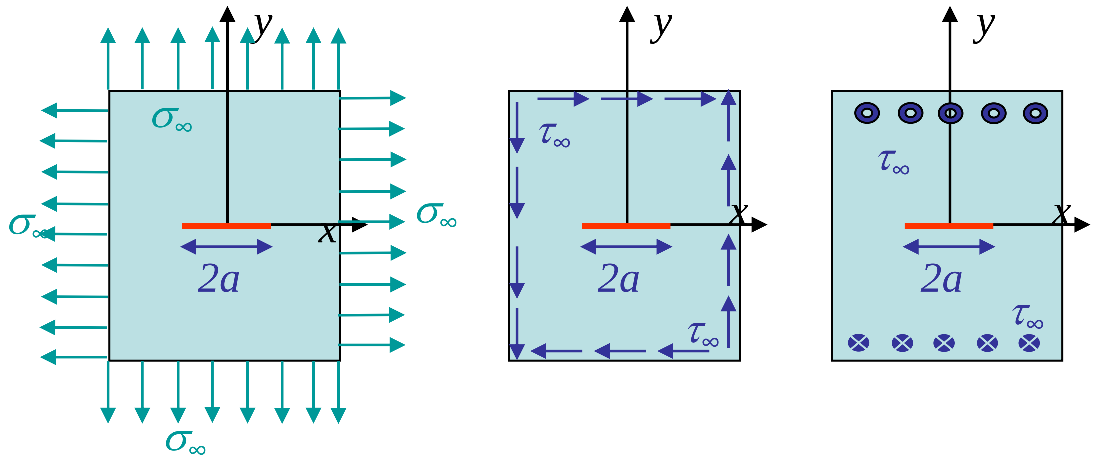


  - Exists if an internal potential exists
    - Is path independent if the contour Γ embeds a straight crack tip
    - No assumption on subsequent growth direction
    - Can be extended to plasticity if no unloading (see lecture on cohesive zone)
  - If crack grows straight ahead $\Longrightarrow$ $G=J$
  - In linear elasticity (independently of crack growth direction):

$$J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \qquad E' = \frac{4E}{(1+\nu)(\kappa+1)} = \begin{cases} E & \mathrm{P}\sigma \\ \frac{E}{1-\nu^2} & \mathrm{P}\varepsilon \end{cases}$$

- Analytical
  - SIF from full-field solution
    - Limited cases

$$\Rightarrow \begin{cases} K_I = \sigma_\infty \sqrt{\pi a} \\ K_{II} = \tau_\infty \sqrt{\pi a} \\ K_{III} = \tau_\infty \sqrt{\pi a} \end{cases}$$



  - From energetic consideration
    - Growing straight ahead crack
    - From J-integral

$$G = -\partial_A \left(E_{\text{int}} - W_{\text{ext}}\right) \Rightarrow G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

$$J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

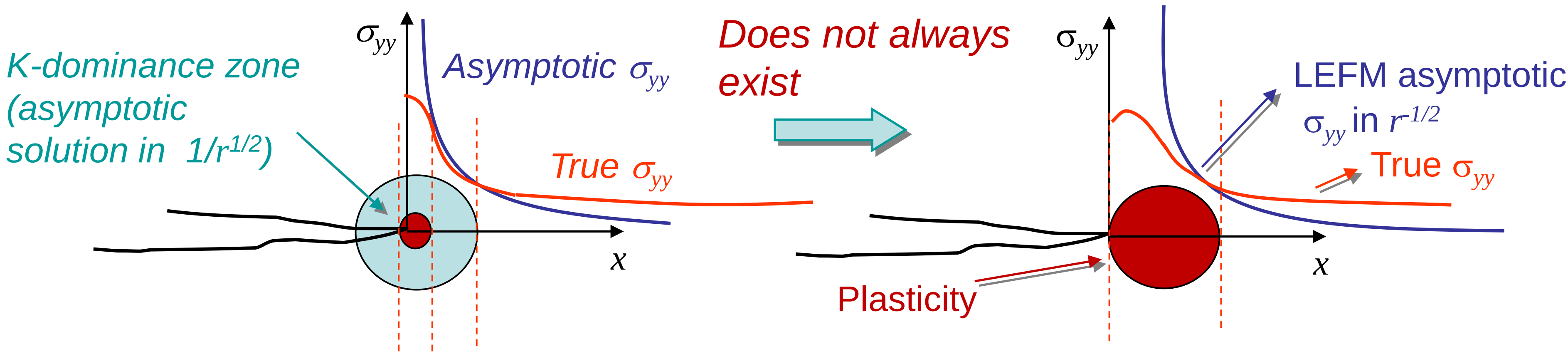
- Numerical (e.g. FEM)

$$\Rightarrow \begin{cases} K_I = \beta_I \sigma_\infty \sqrt{\pi a} \\ K_{II} = \beta_{II} \tau_\infty \sqrt{\pi a} \\ K_{III} = \beta_{III} \tau_\infty \sqrt{\pi a} \end{cases}$$

  - β_i depends on geometry & crack length
- Tabulated solutions (handbooks)
  - http://ebooks.asmedigitalcollection.asme.org/book.aspx?bookid=230

Linear Elastic Fracture Mechanics (LEFM)

- Small Scale Yielding assumption
  - LEFM: we have assumed the existence of a K-dominance zone



    - This holds if the process zone (in which irreversible process occurs)
      - Is a small region compared to the specimen size &
      - Is localized at the crack tip
  - Validity of this approach?
    - We check the dimensions

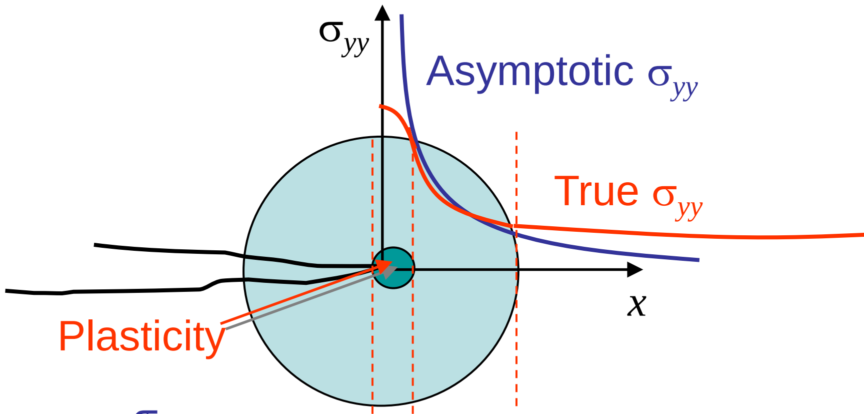
$$a, W - a > 2.5 \left(\frac{K_I}{\sigma_p^0}\right)^2$$

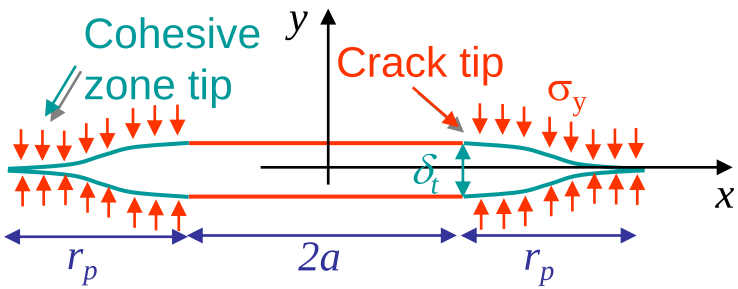
∝Process zone size

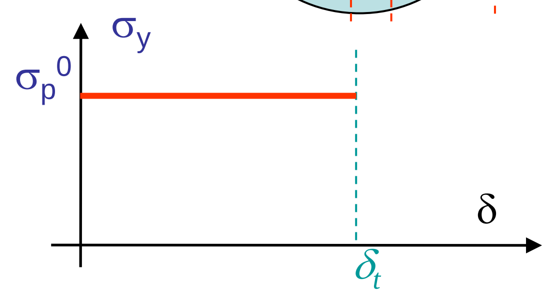
- Non-linear fracture mechanics
  - Derivation of the LEFM validity criterion
  - Providing solutions when LEFM criterion is not met

Dugdale (1960) & Barenblatt (1962)'s cohesive model

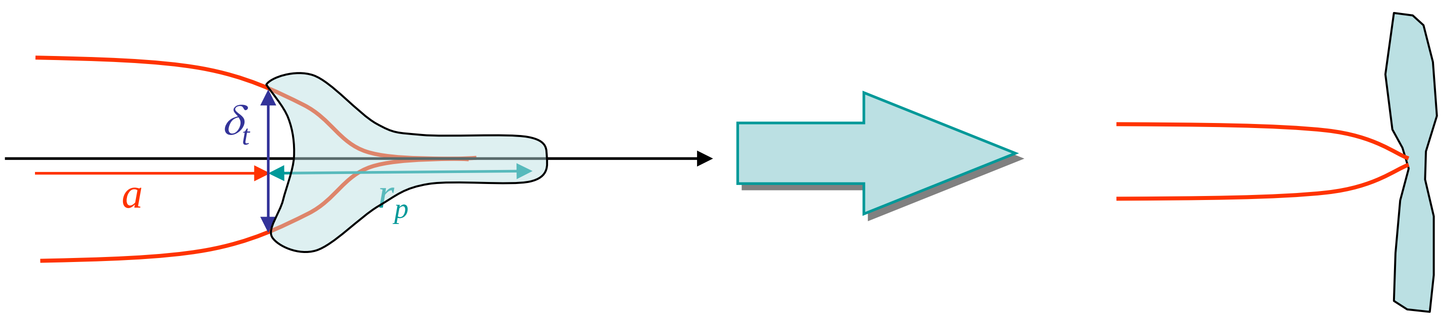
- Dugdale cohesive zone/Yielding strip
  - Remove the singularity
  - Introduce plasticity at crack tip
  - Nonlinearities localized at crack tip
  - Elastic perfectly plastic material ($h^p=0$)



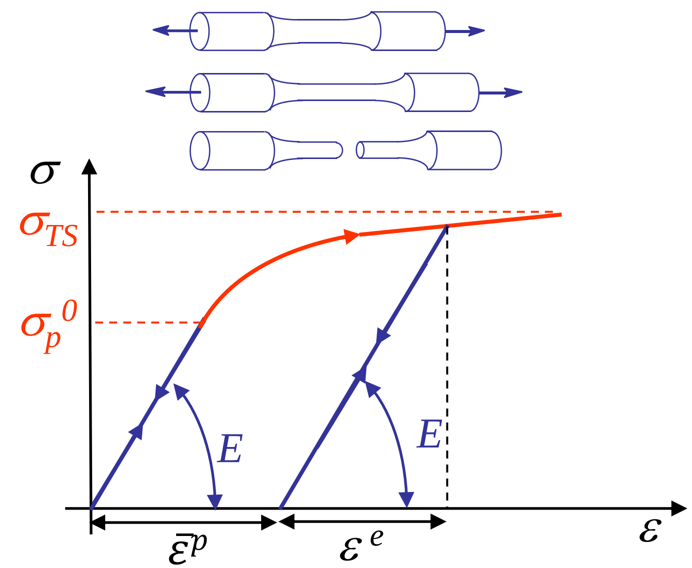




- The plastic zone has actually a different shape in most cases
  - Depends on the stress state, hardening law etc.



- Elasto-plasticity (small deformations)
  - Beyond a threshold the material experiences irreversible deformations
  - Typical behavior at low/room temperature
    - Curves σ-ε independent of time
    - At higher temperature creep …

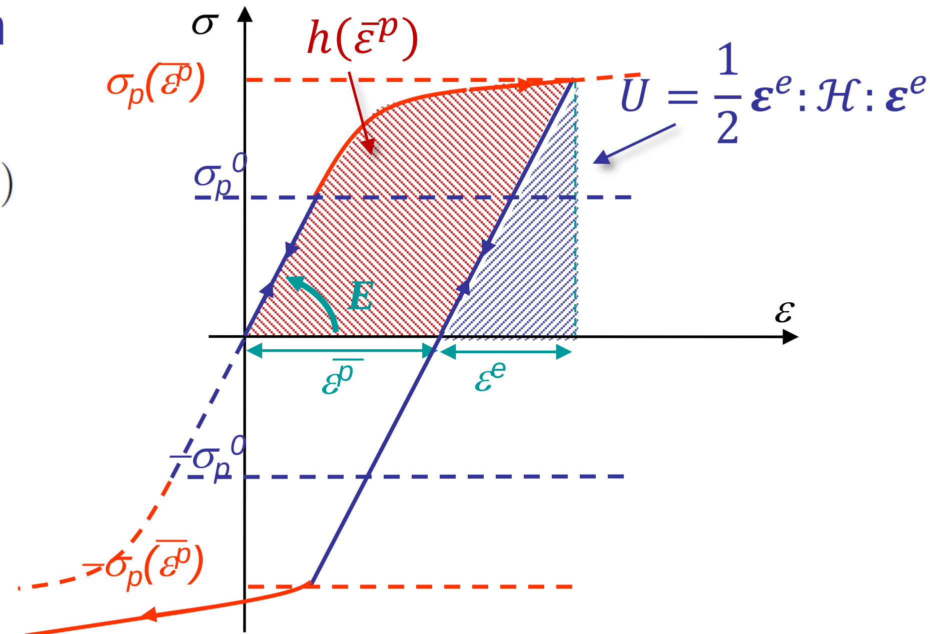


  - Yield surface

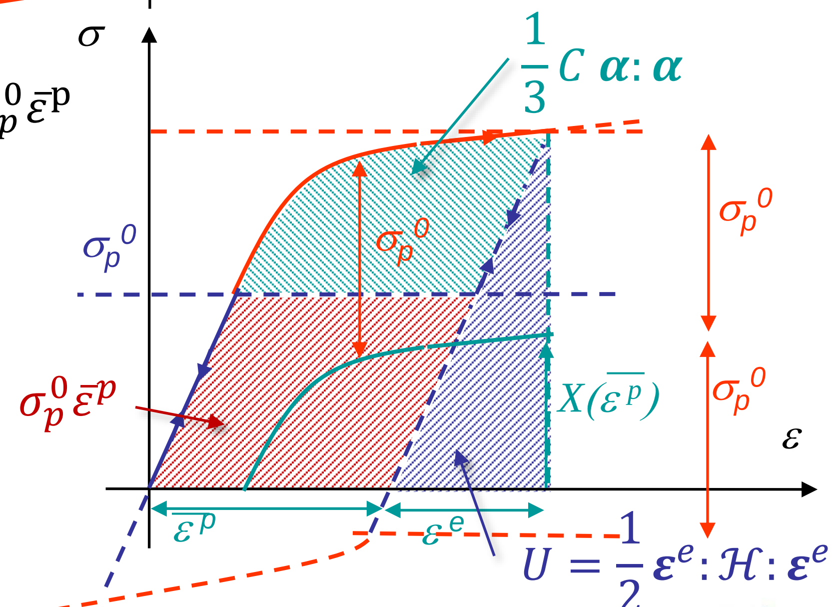
$$f(\boldsymbol{\sigma}) \leq 0 \quad \begin{cases} f < 0\text{: elastic region} \\ f = 0\text{: plasticity} \end{cases}$$

  - Plastic flow
    - Assumption: deformations can be added $d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^{\mathrm{e}} + d\boldsymbol{\varepsilon}^{\mathrm{p}} \Longrightarrow d\boldsymbol{\sigma} = \mathcal{H} : d\boldsymbol{\varepsilon}^{\mathrm{e}}$
    - Normal plastic flow $d\boldsymbol{\varepsilon}^{\mathrm{p}} = d\lambda \dfrac{\partial f}{\partial \boldsymbol{\sigma}}$
  - Path dependency (incremental equations in d)

- Existence of a free energy function
  - Isotropic hardening
    - $\Psi\left(\boldsymbol{\varepsilon}^e,\, \bar{\varepsilon}^p\right) = \frac{1}{2}\boldsymbol{\varepsilon}^e : \mathcal{H} : \boldsymbol{\varepsilon}^e + h\left(\bar{\varepsilon}^p\right)$
    - $\begin{cases} \boldsymbol{\sigma} = \dfrac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^e} \\ \sigma_p = \dfrac{\partial \Psi}{\partial \bar{\varepsilon}^p} = h' \end{cases}$



  - Kinematic hardening
    - $\psi(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}) = \frac{1}{2}\boldsymbol{\varepsilon}^e : \mathcal{H} : \boldsymbol{\varepsilon}^e + \frac{1}{3} C \boldsymbol{\alpha} : \boldsymbol{\alpha} + \sigma_p^0 \bar{\varepsilon}^{\mathrm{p}}$
    - $\begin{cases} \boldsymbol{\sigma} = \dfrac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^e} \\ \sigma_p = \dfrac{\partial \Psi}{\partial \bar{\varepsilon}^p} = \sigma_p^0 \\ \boldsymbol{X} = \dfrac{\partial \Psi}{\partial \boldsymbol{\alpha}} = \dfrac{2}{3} C \boldsymbol{\alpha} \end{cases}$



- Existence of energy release rate?
  - Free energy
    - $\Psi\left(\boldsymbol{\varepsilon}^{e},\, \bar{\varepsilon}^{p}\right) = \frac{1}{2}\boldsymbol{\varepsilon}^{e} : \mathcal{H} : \boldsymbol{\varepsilon}^{e} + h\left(\bar{\varepsilon}^{p}\right)$
    - $\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^{e}}$ & $\sigma_p = \frac{\partial \Psi}{\partial \bar{\varepsilon}^{p}} = h'$
  - But $\boldsymbol{\sigma} = \frac{\partial U}{\partial \boldsymbol{\varepsilon}^{e}} \neq \frac{\partial U}{\partial \boldsymbol{\varepsilon}}$
    - No unique value of $\boldsymbol{\sigma}$ for a given $\boldsymbol{\varepsilon}$
    - Since $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}^{e})$ depends on the history

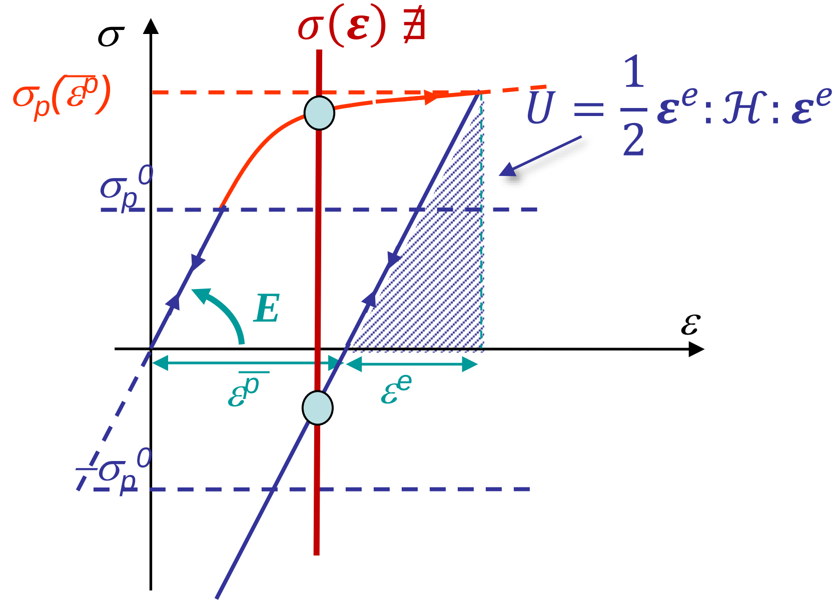
    ⟹ neither G nor J exists
  - We assume no possible unloading

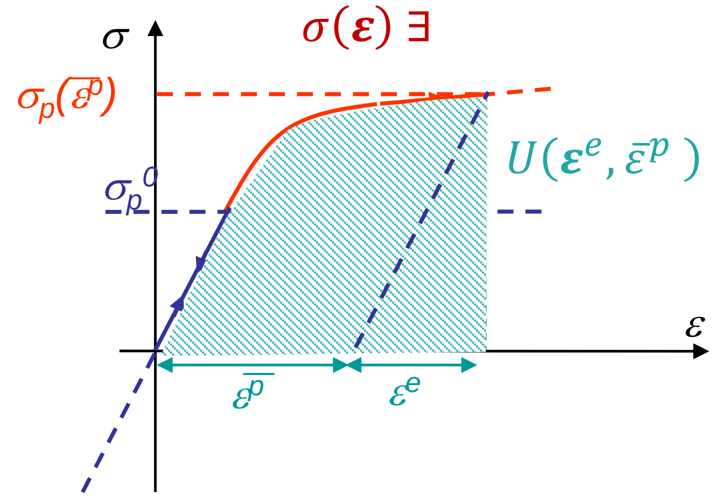
⟹ We recover an energy $U(\boldsymbol{\varepsilon})$ with

$$U(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon}^{e}(\boldsymbol{\varepsilon}) : \mathcal{H} : \boldsymbol{\varepsilon}^{e}(\boldsymbol{\varepsilon}) + h\left(\bar{\varepsilon}^{p}(\boldsymbol{\varepsilon})\right)$$

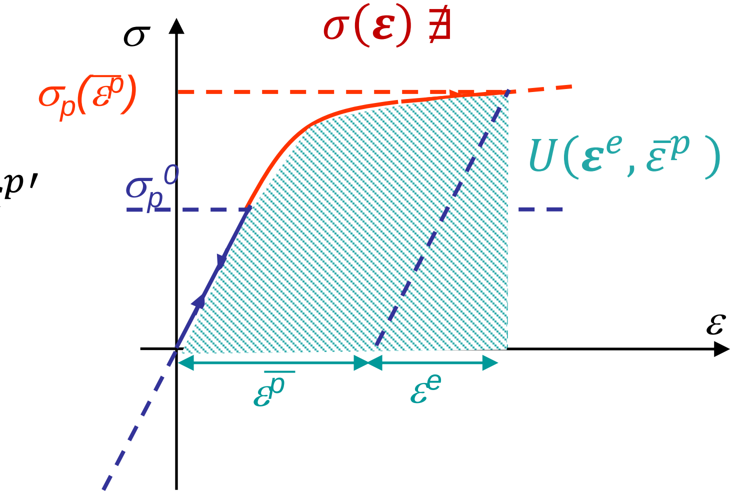
and with (see next slide)

$$\boldsymbol{\sigma} = \frac{\partial U}{\partial \boldsymbol{\varepsilon}}$$





- Assuming no unloading
  - Internal Potential



$$U(\boldsymbol{\varepsilon}) = \int_0^{\boldsymbol{\varepsilon}} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}' = \int_0^{\boldsymbol{\varepsilon}^e} \boldsymbol{\varepsilon}^{e\prime} : \mathcal{H} : d\boldsymbol{\varepsilon}^{e\prime} + \int_0^{\boldsymbol{\varepsilon}^p} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^{p\prime}$$

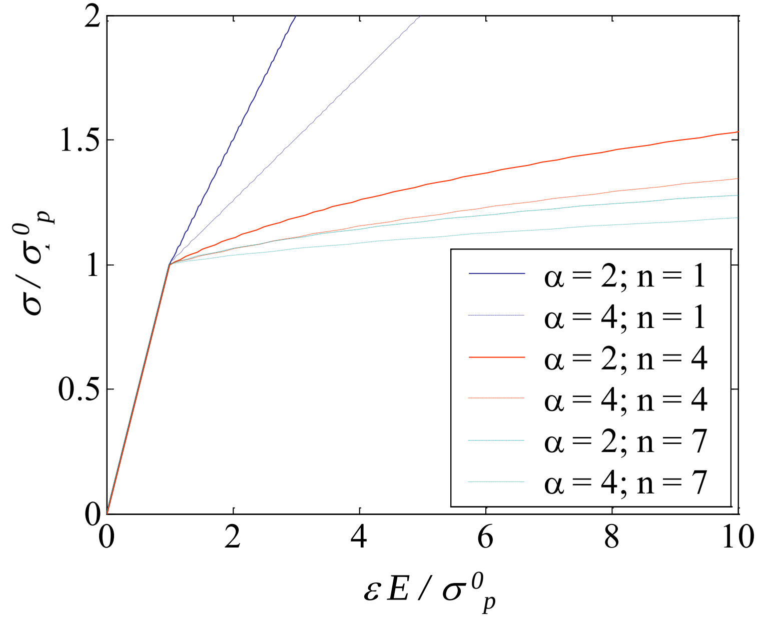
- One has

$$\begin{cases} d\boldsymbol{\varepsilon}^{\mathrm{p}} = d\lambda \dfrac{\partial f}{\partial \boldsymbol{\sigma}} \\ d\bar{\varepsilon}^{\mathrm{p}} = \sqrt{\dfrac{2}{3} d\boldsymbol{\varepsilon}^{\mathrm{p}} : d\boldsymbol{\varepsilon}^{\mathrm{p}}} \\ \dfrac{\partial f}{\partial \boldsymbol{\sigma}} = \sqrt{\dfrac{3}{2}} \dfrac{\mathbf{s}}{\sqrt{\mathbf{s} : \mathbf{s}}} \\ \mathbf{s} = \boldsymbol{\sigma} - \dfrac{\mathrm{tr}(\boldsymbol{\sigma})}{3} \mathbf{I} \end{cases} \Longrightarrow \begin{cases} d\lambda = d\bar{\varepsilon}^p \\ d\boldsymbol{\varepsilon}^{\mathrm{p}} = d\bar{\varepsilon}^{\mathrm{p}} \sqrt{\dfrac{3}{2}} \dfrac{\mathbf{s}}{\sqrt{\mathbf{s} : \mathbf{s}}} \end{cases}$$

- So $\displaystyle \int_0^{\boldsymbol{\varepsilon}^p} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^{p\prime} = \int_0^{\boldsymbol{\varepsilon}^p} \mathbf{s} : d\boldsymbol{\varepsilon}^{p\prime} = \int_0^{\boldsymbol{\varepsilon}^p} \frac{\sqrt{\frac{3}{2}}\, \mathbf{s} : \mathbf{s}}{\sqrt{\mathbf{s} : \mathbf{s}}} d\bar{\varepsilon}^p = \int_0^{\bar{\varepsilon}^p} \sigma_p d\bar{\varepsilon}^p = h(\bar{\varepsilon}^p)$

$$\Longrightarrow U(\boldsymbol{\varepsilon}) = \frac{1}{2} \boldsymbol{\varepsilon}^e(\boldsymbol{\varepsilon}) : \mathcal{H} : \boldsymbol{\varepsilon}^e(\boldsymbol{\varepsilon}) + h\left(\bar{\varepsilon}^p(\boldsymbol{\varepsilon})\right) \quad \& \quad \boldsymbol{\sigma} = \frac{\partial U}{\partial \boldsymbol{\varepsilon}}$$

- J2-plasticity without unloading
  - Internal energy
    - $U(\boldsymbol{\varepsilon}) = \dfrac{1}{2}\boldsymbol{\varepsilon}^e(\boldsymbol{\varepsilon})\!:\!\mathcal{H}\!:\!\boldsymbol{\varepsilon}^e(\boldsymbol{\varepsilon}) + h\big(\bar{\varepsilon}^p(\boldsymbol{\varepsilon})\big)$
    - With $h\big(\bar{\varepsilon}^p(\boldsymbol{\varepsilon})\big) = \int_{\bar{\varepsilon}^p} \sigma_p d\bar{\varepsilon}^p$
  - Example: Power law hardening
    - $\sigma_p\left(\bar{\varepsilon}^{\mathrm{p}}\right) = \sigma_p^0\left(\dfrac{\bar{\varepsilon}^p}{\alpha\sigma_p^0/E}+1\right)^{\frac{1}{n}}$
    - Parameters α & n
    - Term σ_p^0/E represents the elastic deformation before yield



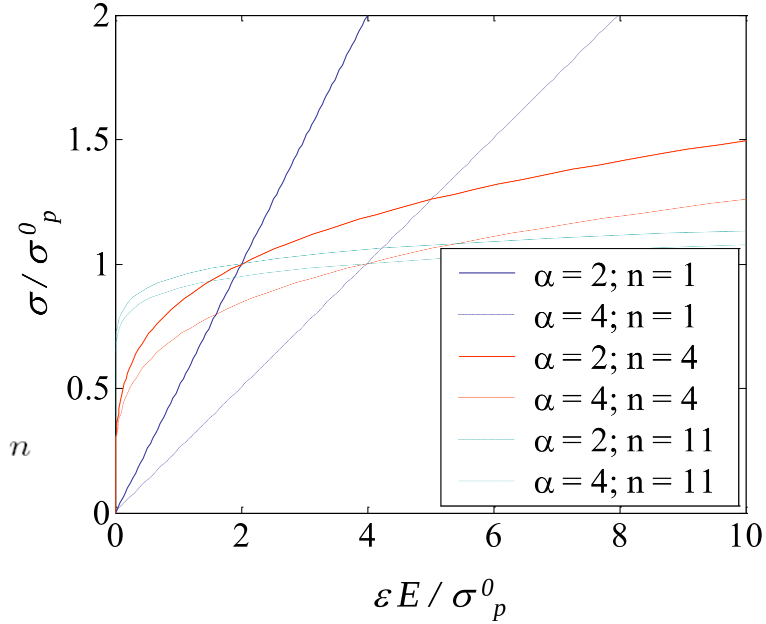
$$\Longrightarrow U = \frac{\boldsymbol{\varepsilon}^{\mathrm{e}}:\mathcal{H}:\boldsymbol{\varepsilon}^{\mathrm{e}}}{2} + \frac{n\alpha\left(\sigma_p^0\right)^2}{E\,(n+1)}\left[\left(\frac{\bar{\varepsilon}^p}{\alpha\sigma_p^0/E}+1\right)^{\frac{n+1}{n}}-1\right]$$

- Power law
  - This law can be rewritten in terms of the total deformations
    - Yield stress is replaced by

      equivalent stress $\sigma_e = \sqrt{\frac{3}{2}\mathbf{s}:\mathbf{s}}$
    - Plastic strain is replaced by

      equivalent strain $\bar{\varepsilon} = \sqrt{\frac{2}{3}\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}}$
    - The governing law becomes

$$\sigma_e = \sigma_p^0 \left(\frac{\bar{\varepsilon}}{\alpha\sigma_p^0/E}\right)^{\frac{1}{n}} \text{ or } \bar{\varepsilon} = \frac{\alpha\sigma_p^0}{E}\left(\frac{\sigma_e}{\sigma_p^0}\right)^n$$

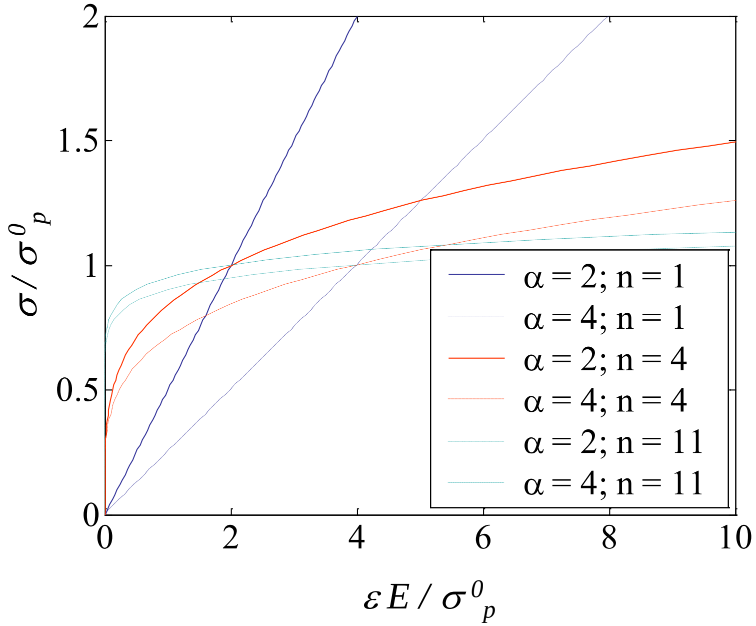


    - Parameter n
      - $n \to \infty$: perfect plasticity
      - $n \to 1$: “elasticity”
  - Doing so requires 2 assumptions
    - There is no unloading
    - As elastic strains are assimilated to plastic strains, the material is incompressible
  - Which are satisfied if
    - We are interested **only in crack initiation and not in crack propagation**
    - The stress components remain proportional with the loading
    - **Elastic deformations are negligible** compared to plastic ones

- Power law (2)
  - Internal potential for J2-elasto plasticity
    - $U(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon}^e(\boldsymbol{\varepsilon}):\mathcal{H}:\boldsymbol{\varepsilon}^e(\boldsymbol{\varepsilon}) + h\left(\bar{\varepsilon}^p(\boldsymbol{\varepsilon})\right)$

    with $h\left(\bar{\varepsilon}^p(\boldsymbol{\varepsilon})\right) = \int_{\bar{\varepsilon}^p} \sigma_p d\bar{\varepsilon}^p$

  - Here we have a non-linear incompressible response
    - $\sigma_e = \sigma_p^0 \left(\frac{\bar{\varepsilon}}{\alpha\sigma_p^0/E}\right)^{\frac{1}{n}}$
    - Obeys plastic flow equations

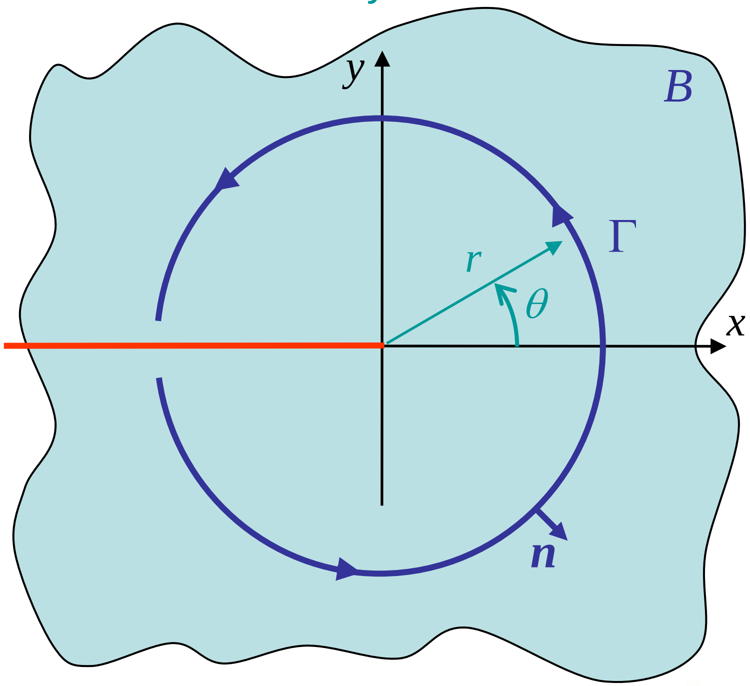


$\Longrightarrow U(\bar{\varepsilon}) = \int_{\bar{\varepsilon}} \sigma_e d\bar{\varepsilon}$

$$\Longrightarrow U(\bar{\varepsilon}) = \frac{n\alpha\left(\sigma_p^0\right)^2}{E(n+1)}\left(\frac{\bar{\varepsilon}}{\alpha\frac{\sigma_p^0}{E}}\right)^{\frac{n+1}{n}} = \frac{n\alpha\left(\sigma_p^0\right)^2}{E(n+1)}\left(\frac{\sigma_e}{\sigma_p^0}\right)^{n+1}$$

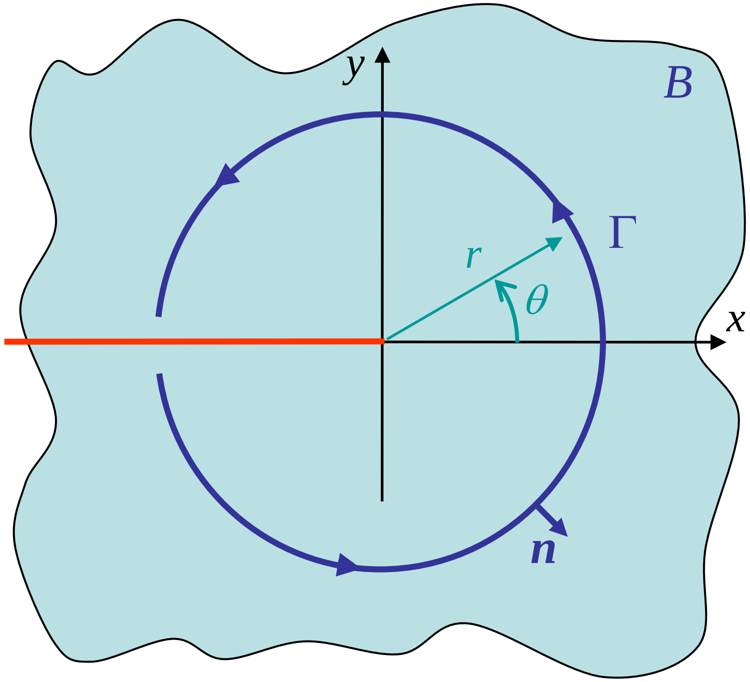
- The HRR theory
  - From Hutchinson, Rice and Rosengren, 1967-1968
  - Assumptions
    - Semi-infinite crack
    - Loading increased monotonically $\Rightarrow$ only for crack initiation
    - Power law $\Rightarrow$ only if plastic deformations are large compared to elastic ones
  - J-integral
    - Elasto-plastic model & these assumptions = non-linear elasticity
    - The J-integral can be used
    - It is path independent, so one may choose

$$J = \int_{-\pi}^{\pi} \left(U \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \right) r d\theta$$



- Near fields
  - Evaluation of J-integral
    - Choice of a circle to evaluate J

$$\Rightarrow J = \int_{-\pi}^{\pi} \left(U \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \right) r d\theta$$



    - Since J is path independent, it is the same whatever the value of r

$$\Rightarrow J = \lim_{r \longrightarrow 0} \int_{-\pi}^{\pi} \left(U \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \right) r d\theta$$

      - So the integrant should be independent of r (at the limit)

$$\Rightarrow \lim_{r \longrightarrow 0} \left(U \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \right) = \frac{h(\theta)}{r}$$

      - The terms involve stress times strain

$$\Rightarrow \lim_{r \longrightarrow 0} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \frac{l(\theta)}{r}$$

- Near fields
  - Because of the path independence of J
    - $\lim_{r \to 0} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \frac{l(\theta)}{r}$
  - Generally speaking, the stress tensor can be expanded in a power series
    - $\boldsymbol{\sigma}(r, \theta) = \sum_s r^s \hat{\boldsymbol{\sigma}}(\theta, s)$

      where $\hat{\boldsymbol{\sigma}}(\theta, s)$ depends only on θ and s
  - Let s' be the dominant exponent near the crack tip

    $\Longrightarrow \boldsymbol{\sigma}(r, \theta) \simeq r^{s'} \hat{\boldsymbol{\sigma}}(\theta, s')$

    where $\hat{\boldsymbol{\sigma}}(\theta, s')$ depends only on θ and s'
    - Strain field:

      $\bar{\varepsilon} = \frac{\alpha \sigma_p^0}{E} \left(\frac{\sigma_e}{\sigma_p^0} \right)^n \Longrightarrow \boldsymbol{\varepsilon} \simeq \left(r^{s'} \right)^n \hat{\boldsymbol{\varepsilon}}(\theta, s')$
  - Therefore
    - $\lim_{r \to 0} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \frac{l(\theta)}{r} \Longrightarrow ns' + s' = -1 \Longrightarrow \boxed{s' = \frac{-1}{n+1}}$

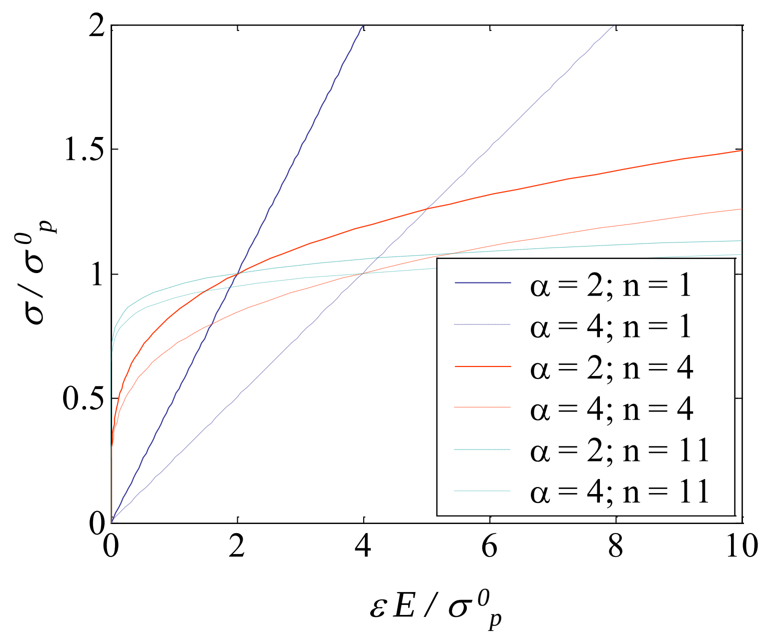
- Near fields (2)
  - Near fields are obtained using $s' = -\frac{1}{n+1}$
    - Rewrite stress field

$$\boldsymbol{\sigma}(r,\,\theta) \simeq r^{s'}\hat{\boldsymbol{\sigma}}(\theta,\,s') \Longrightarrow \boxed{\boldsymbol{\sigma}(r,\,\theta) \rightarrow \sigma_p^0 k_n \frac{\tilde{\boldsymbol{\sigma}}(\theta,\,n)}{r^{\frac{1}{n+1}}}}$$

    - k_n is a plastic stress intensity factor which
      - Allows defining $\tilde{\boldsymbol{\sigma}}(\theta,\,n)$ independently of the loading amplitude
      - Depends on n
    - Limit cases
      - "Elasticity" ($n \rightarrow 1$):

      $\Longrightarrow$ LEFM solution in $1/r^{1/2}$ recovered
      - Perfect plasticity ($n \rightarrow \infty$):

      $\Longrightarrow$ stress remains finite



- Near fields (3)
  - Near fields are obtained using $s' = -\frac{1}{n+1}$
    - Stress field

$$\boldsymbol{\sigma}(r,\,\theta) \to \sigma_p^0 k_n \frac{\tilde{\boldsymbol{\sigma}}(\theta,\,n)}{r^{\frac{1}{n+1}}}$$

    - Strain field

$$\bar{\varepsilon} = \frac{\alpha\sigma_p^0}{E}\left(\frac{\sigma_e}{\sigma_p^0}\right)^n \Longrightarrow \boldsymbol{\varepsilon} \to \frac{\alpha\sigma_p^0}{E}\frac{k_n^n}{r^{\frac{n}{n+1}}}\tilde{\boldsymbol{\varepsilon}}(\theta,\,n)$$

    - Internal energy

$$U(\bar{\varepsilon}) = \frac{n\alpha\left(\sigma_p^0\right)^2}{E(n+1)}\left(\frac{E\bar{\varepsilon}}{\alpha\sigma_p^0}\right)^{\frac{n+1}{n}} \Longrightarrow U \to \frac{n\alpha\left(\sigma_p^0\right)^2}{E(n+1)}\frac{k_n^{n+1}}{r}\tilde{U}(\theta,n)$$

    - J-integral

$$J = \lim_{r\to 0}\int_{-\pi}^{\pi}\left(U\boldsymbol{n}_x - \boldsymbol{u}_{,x}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) r d\theta = \frac{\alpha\left(\sigma_p^0\right)^2}{E}k_n^{n+1}\int_{-\pi}^{\pi}\tilde{h}(\theta,\,n)\,d\theta$$

$$\Longrightarrow \boxed{k_n = \left(\frac{JE}{\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{1}{n+1}}} \text{ with } I_n = \int_{-\pi}^{\pi}\tilde{h}(\theta,\,n)\,d\theta$$

- Solution in terms of J-integral
  - Near fields are obtained using $s' = -\frac{1}{n+1}$
    - Plastic stress intensity factor

$$\boxed{k_n = \left(\frac{JE}{\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{1}{n+1}}}$$

    - Stress field

$$\boldsymbol{\sigma}(r,\theta) \rightarrow \sigma_p^0 k_n \frac{\tilde{\boldsymbol{\sigma}}(\theta, n)}{r^{\frac{1}{n+1}}} \Longrightarrow \boxed{\boldsymbol{\sigma} = \sigma_p^0 \left(\frac{JE}{r\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{1}{n+1}} \tilde{\boldsymbol{\sigma}}(\theta, n)}$$

    - Strain field

$$\boldsymbol{\varepsilon} \rightarrow \frac{\alpha\sigma_p^0}{E}\frac{k_n^n}{r^{\frac{n}{n+1}}}\tilde{\boldsymbol{\varepsilon}}(\theta, n) \Longrightarrow \boxed{\boldsymbol{\varepsilon} = \frac{\sigma_p^0\alpha}{E}\left(\frac{JE}{r\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{n}{n+1}} \tilde{\boldsymbol{\varepsilon}}(\theta, n)}$$

  - J is governing the stress field near the crack tip
    - Limit case of $n \rightarrow 1$

$$J = \frac{K_I^2}{E'} \Longrightarrow \boldsymbol{\sigma} \propto \frac{K_I}{\sqrt{r}}$$

- Resolution for a semi-infinite crack in plane ε state
  - Airy functions
    - Linear momentum $\boldsymbol{\sigma}_{\alpha\beta,\beta} + \boldsymbol{b}_\alpha = 0$
      - If $\boldsymbol{b} = 0$, there exists an Airy function Φ: $\boldsymbol{\sigma}_{\alpha\beta} = -\Phi_{,\alpha\beta} + \delta_{\alpha\beta}\Phi_{,\gamma\gamma}$
      - We are not in linear elasticity ⟹ we cannot say $\nabla^2 \cancel{\nabla^2} \Phi = 0$
    - In polar coordinates $\begin{cases} \boldsymbol{\sigma}_{rr} = \frac{1}{r}\Phi_{,r} + \frac{1}{r^2}\Phi_{,\theta\theta} \\ \boldsymbol{\sigma}_{\theta\theta} = \Phi_{,rr} \\ \boldsymbol{\sigma}_{r\theta} = -\frac{1}{r}\Phi_{,r\theta} + \frac{1}{r^2}\Phi_{,\theta} \end{cases}$
    - We are seeking a solution in $r^{-1/(n+1)}$ ⟹ we choose $\Phi = \frac{(n+1)^2}{n(2n+1)} r^{\frac{2n+1}{n+1}} f(\theta)$

$$\frac{-1}{n+1} + 2$$

$$\left(\frac{2n+1}{n+1}\right)^{-1} \cdot \left(\frac{2n+1}{n+1} - 1\right)^{-1}$$

    - So the stress expressions satisfying the balance equation

$$\begin{cases} \boldsymbol{\sigma}_{rr} = r^{\frac{-1}{n+1}} \left(\frac{n+1}{n} f + \frac{(n+1)^2}{n(2n+1)} f'' \right) \\ \boldsymbol{\sigma}_{\theta\theta} = r^{\frac{-1}{n+1}} f \\ \boldsymbol{\sigma}_{r\theta} = r^{\frac{-1}{n+1}} \left(-\frac{n+1}{n} + \frac{(n+1)^2}{n(2n+1)} \right) f' = r^{\frac{-1}{n+1}} \frac{-(n+1)}{(2n+1)} f' \end{cases}$$

- Resolution for a semi-infinite crack in plane ε state (2)
  - Stream function
    - Since we are incompressible $\boldsymbol{\varepsilon}_{\alpha\beta} = \frac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2}$ & $\boldsymbol{\varepsilon}_{\alpha\alpha} = \boldsymbol{u}_{\alpha,\alpha} = 0$

      with **plane strain** assumption (if not: $\boldsymbol{\varepsilon}_{zz} \neq 0$)
    - So displacements derive from a stream function Ψ with $\begin{cases} \boldsymbol{u}_x = \Psi_{,y} \\ \boldsymbol{u}_y = -\Psi_{,x} \end{cases}$
    - In polar coordinates
      - $\boldsymbol{u}_x = \Psi_{,y} = \sin\theta\Psi_{,r} + \frac{\cos\theta}{r}\Psi_{,\theta}$ & $\boldsymbol{u}_y = -\Psi_{,x} = -\cos\theta\Psi_{,r} + \frac{\sin\theta}{r}\Psi_{,\theta}$

    ⟹ the displacements become $\begin{cases} \boldsymbol{u}_r = \cos\theta\boldsymbol{u}_x + \sin\theta\boldsymbol{u}_y = \frac{\Psi_{,\theta}}{r} \\ \boldsymbol{u}_\theta = -\sin\theta\boldsymbol{u}_x + \cos\theta\boldsymbol{u}_y = -\Psi_{,r} \end{cases}$

      - Strains are obtained from

$$\begin{cases} \boldsymbol{\varepsilon}_{rr} = \boldsymbol{u}_{r,r} = \frac{\Psi_{,r\theta}}{r} - \frac{\Psi_{,\theta}}{r^2} \\ \boldsymbol{\varepsilon}_{\theta\theta} = \frac{\boldsymbol{u}_r}{r} + \frac{1}{r}\boldsymbol{u}_{\theta,\theta} = -\boldsymbol{\varepsilon}_{rr} \\ \boldsymbol{\varepsilon}_{r\theta} = \frac{1}{2}\left(\frac{\boldsymbol{u}_{r,\theta}}{r} + \boldsymbol{u}_{\theta,r} - \frac{\boldsymbol{u}_\theta}{r}\right) = \frac{1}{2}\left(\frac{\Psi_{,\theta\theta}}{r^2} - \Psi_{,rr} + \frac{\Psi_{,r}}{r}\right) \end{cases}$$

    - We are seeking a solution in $r^{-n/(n+1)}$ ⟹ we choose $\Psi = -r^{\frac{n+2}{n+1}} g(\theta)$

      $\frac{-n}{n+1} + 2$

- Resolution for a semi-infinite crack in plane ε state (3)
  - Strain tensor
    - Stream function $\Psi = -r^{\frac{n+2}{n+1}} g\left(\theta\right)$
    - So the strain components are

$$
\left\{
\begin{aligned}
\varepsilon_{rr} &= \frac{\Psi_{,r\theta}}{r} - \frac{\Psi_{,\theta}}{r^2} = r^{\frac{-n}{n+1}} \left(-\frac{n+2}{n+1} g' + g'\right) = -\frac{1}{n+1} r^{\frac{-n}{n+1}} g' \\
\varepsilon_{\theta\theta} &= \frac{1}{n+1} r^{\frac{-n}{n+1}} g' \\
\varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{\Psi_{,\theta\theta}}{r^2} - \Psi_{,rr} + \frac{\Psi_{,r}}{r}\right) = \frac{r^{\frac{-n}{n+1}}}{2}\left(-g'' + \frac{n+2}{\left(n+1\right)^2} g - \frac{n+2}{n+1} g\right) \\
&= -\frac{r^{\frac{-n}{n+1}}}{2}\left(g'' + n\frac{n+2}{\left(n+1\right)^2} g\right)
\end{aligned}
\right.
$$

- Resolution for a semi-infinite crack in plane ε state (4)
  - Stress & Strain tensors

$$\begin{cases} \boldsymbol{\sigma}_{rr} = r^{\frac{-1}{n+1}}\left(\frac{n+1}{n}f + \frac{(n+1)^2}{n(2n+1)}f''\right) \quad \& \quad \boldsymbol{\sigma}_{\theta\theta} = r^{\frac{-1}{n+1}}f \\ \boldsymbol{\sigma}_{r\theta} = r^{\frac{-1}{n+1}}\left(-\frac{n+1}{n} + \frac{(n+1)^2}{n(2n+1)}\right)f' = r^{\frac{-1}{n+1}}\frac{-(n+1)}{(2n+1)}f' \end{cases}$$

$$\begin{cases} \boldsymbol{\varepsilon}_{rr} = -\frac{1}{n+1}r^{\frac{-n}{n+1}}g' \quad \& \quad \boldsymbol{\varepsilon}_{\theta\theta} = \frac{1}{n+1}r^{\frac{-n}{n+1}}g' \\ \boldsymbol{\varepsilon}_{r\theta} = -\frac{r^{\frac{-n}{n+1}}}{2}\left(g'' + n\frac{n+2}{(n+1)^2}g\right) \end{cases}$$

  - 2 unknown functions *f* and *g* $\Longrightarrow$ need for 2 equations
    - The power law $\bar{\varepsilon} = \frac{\alpha\sigma_p^0}{E}\left(\frac{\sigma_e}{\sigma_p^0}\right)^n$
    - The normality equation $\boldsymbol{\varepsilon} = \bar{\varepsilon}\sqrt{\frac{3}{2}}\frac{\mathbf{s}}{\sqrt{\mathbf{s}:\mathbf{s}}} = \frac{3\bar{\varepsilon}}{2\sigma_e}\mathbf{s}$

- Resolution for a semi-infinite crack in plane ε state (5)
  - Power law $\bar{\varepsilon} = \frac{\alpha\sigma_p^0}{E}\left(\frac{\sigma_e}{\sigma_p^0}\right)^n$ with normality $\boldsymbol{\varepsilon} = \bar{\varepsilon}\sqrt{\frac{3}{2}}\frac{\mathbf{s}}{\sqrt{\mathbf{s}:\mathbf{s}}} = \frac{3\bar{\varepsilon}}{2\sigma_e}\mathbf{s}$
    - These two equations lead to (see annex 1 for details)
      - $\left[\frac{4}{3\left(1+n\right)^2}g'^2 + \frac{1}{3}\left(g'' + n\frac{n+2}{\left(n+1\right)^2}g\right)^2\right] =$

$$\left(\frac{\alpha\sigma_p^0}{E\left(\sigma_p^0\right)^n}\right)^2\left[\frac{3}{4}\left(\frac{1}{n}f + \frac{\left(n+1\right)^2}{n\left(2n+1\right)}f''\right)^2 + 3\left(\frac{n+1}{2n+1}\right)^2 f'^2\right]^n$$

      - $\frac{2}{2n+1}g'f' = -\frac{1}{2}\left[\frac{1}{n}f + \frac{\left(n+1\right)^2}{n\left(2n+1\right)}f''\right]\left[g'' + n\frac{n+2}{\left(n+1\right)^2}g\right]$
  - Back to HRR field
    - Near the crack tip, the fields were normalized by introducing an intensity factor

$$\boldsymbol{\sigma}\left(r,\,\theta\right) \rightarrow \sigma_p^0 k_n \frac{\tilde{\boldsymbol{\sigma}}\left(\theta,\,n\right)}{r^{\frac{1}{n+1}}} \quad \& \quad \boldsymbol{\varepsilon} \rightarrow \frac{\alpha\sigma_p^0}{E}\frac{k_n^n}{r^{\frac{n}{n+1}}}\tilde{\boldsymbol{\varepsilon}}\left(\theta,\,n\right)$$

    - Therefore, the functions f and g are rewritten

$$f\left(\theta\right) = \sigma_p^0 k_n \tilde{f}\left(\theta\right) \quad \& \quad g\left(\theta\right) = \frac{\alpha\sigma_p^0}{E}k_n^n \tilde{g}\left(\theta\right)$$

- Resolution for a semi-infinite crack in plane ε state (6)
  - Using the new non-dimensional functions leads to

$$\begin{cases} \left[\tilde{f}'' + \frac{2n+1}{(n+1)^2}\tilde{f}\right]\left[\tilde{g}'' + n\frac{n+2}{(n+1)^2}\tilde{g}\right] = -\frac{4n}{(n+1)^2}\tilde{g}'\tilde{f}' \\ \left[\frac{4}{(1+n)^2}\tilde{g}'^2 + \left(\tilde{g}'' + n\frac{n+2}{(n+1)^2}\tilde{g}\right)^2\right]^{\frac{1}{n}} = \\ \frac{3^{\frac{n+1}{n}}}{4}\frac{(n+1)^4}{n^2(2n+1)^2}\left[\left(\frac{2n+1}{(n+1)^2}\tilde{f} + \tilde{f}''\right)^2 + \frac{4n^2}{(n+1)^2}\tilde{f}'^2\right] \end{cases}$$

  - These equations can be reduced to a differential equation of the 4th order in $\tilde{f}$
    - See appendix I
    - $\tilde{f}'''' + \frac{2n+1}{(1+n)^2}\tilde{f}'' + \frac{1}{n\tilde{l}^2 + 4\tilde{f}'^2}\left\{\frac{n^2}{(1+n)^2}\left[\frac{4}{n}\tilde{f}'' + \frac{n+2}{1+n}\tilde{l}\right]\left[\tilde{l}^2 + 4\tilde{f}'^2\right] + \right.$

$$\frac{4n(n-1)}{(n+1)^2}\tilde{f}'\left[\tilde{l}\tilde{l}' + 4\tilde{f}'\tilde{f}''\right] + \frac{n(n-1)}{n+1}\left[3\tilde{l}\tilde{l}'^2 + 8\tilde{l}'\tilde{f}'\tilde{f}'' + 4\tilde{l}\left(\tilde{f}''^2 + \tilde{f}'\tilde{f}'''\right)\right]$$

$$\left. + \frac{n(n-1)(n-3)}{n+1}\frac{\tilde{l}}{\tilde{l}^2 + 4\tilde{f}'^2}\left[\tilde{l}\tilde{l}' + 4\tilde{f}'\tilde{f}''\right]^2\right\} \ \& \ \tilde{l} = \frac{1}{n}\left[(n+1)\tilde{f}'' + \frac{2n+1}{n+1}\tilde{f}\right]$$

- Resolution for a semi-infinite crack in plane ε state (7)
  - New differential equation of the 4th order in $\tilde{f}$
    - Stress field from $\tilde{f}$

$$\begin{cases} \boldsymbol{\sigma}_{rr} = r^{\frac{-1}{n+1}} \left(\dfrac{n+1}{n} f + \dfrac{(n+1)^2}{n(2n+1)} f'' \right) \quad \& \quad \boldsymbol{\sigma}_{\theta\theta} = r^{\frac{-1}{n+1}} f \\ \boldsymbol{\sigma}_{r\theta} = r^{\frac{-1}{n+1}} \left(-\dfrac{n+1}{n} + \dfrac{(n+1)^2}{n(2n+1)} \right) f' = r^{\frac{-1}{n+1}} \dfrac{-(n+1)}{(2n+1)} f' \end{cases}$$

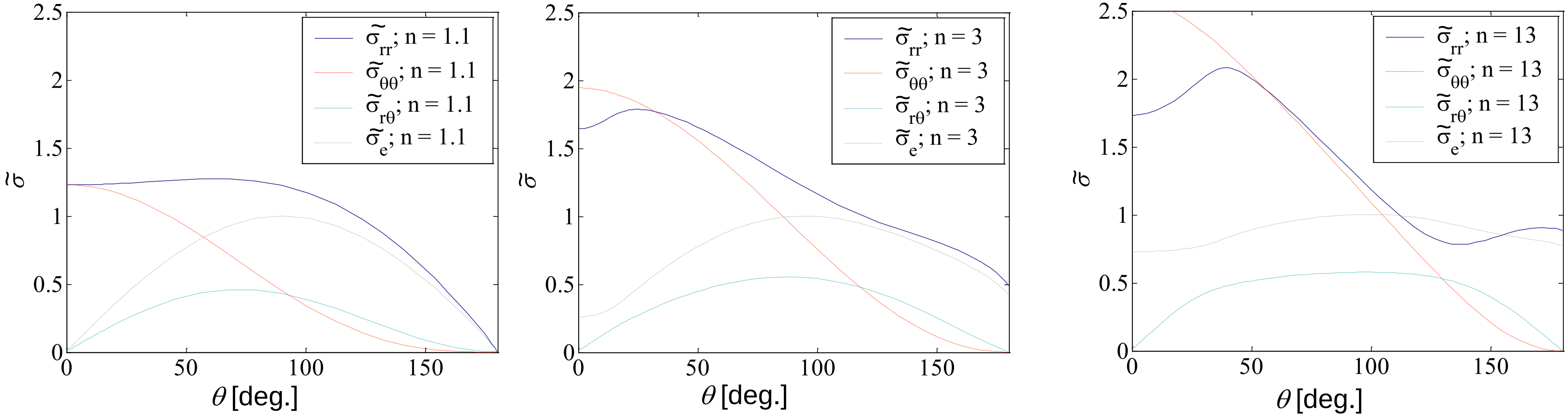
  - Range-Kutta resolution of differential equation of the 4th order in $\tilde{f}$
    - Initial boundary conditions
      - Mode I symmetry: $\boldsymbol{\sigma}_{r\theta}(\theta = 0) = \boldsymbol{\sigma}_{rr,\theta}(\theta = 0) = \boldsymbol{\sigma}_{\theta\theta,\theta}(\theta = 0) = 0$

        $\Longrightarrow$ $\tilde{f}'(0) = \tilde{f}'''(0) = 0$
      - Stress free lips: $\boldsymbol{\sigma}_{r\theta}(\theta = \pi) = \boldsymbol{\sigma}_{\theta\theta}(\theta = \pi) = 0 \Longrightarrow \tilde{f}(\pi) = \tilde{f}'(\pi) = 0$

        $\Longrightarrow$ iterations on $\tilde{f}''(0)$ until these 2 conditions are satisfied
      - It remains to choose $\tilde{f}(0)$
    - The differential equation is valid for any multiple of
      - Initial condition $\tilde{f}(0)$ is chosen so that $\max\limits_{\theta} \tilde{\sigma}_e(\theta,\, n) = 1$

Semi-infinite mode I crack in plane ε state

- Resolution for a semi-infinite crack in plane ε state (8)
  - Solution of the plane ε problem



  - The problem fields can now be determined
    - $\boldsymbol{\sigma} = \sigma_p^0 \left(\frac{JE}{r\alpha \left(\sigma_p^0\right)^2 I_n} \right)^{\frac{1}{n+1}} \tilde{\boldsymbol{\sigma}}\left(\theta,\, n\right)$ & $\boldsymbol{\varepsilon} = \frac{\sigma_p^0 \alpha}{E} \left(\frac{JE}{r\alpha \left(\sigma_p^0\right)^2 I_n} \right)^{\frac{n}{n+1}} \tilde{\boldsymbol{\varepsilon}}\left(\theta,\, n\right)$
    - J characterizes the loading intensity on the crack
      - Problem dependent
    - I_n is a normalizing factor allowing to express the dependency with respect to J
      - Still needs to be evaluated

- Determination of stress and strain fields
  - We sought the approximations

$$\begin{cases} \boldsymbol{\sigma}(r,\,\theta) \to \sigma_p^0 k_n \dfrac{\tilde{\boldsymbol{\sigma}}(\theta,\,n)}{r^{\frac{1}{n+1}}} \\ \boldsymbol{\varepsilon} \to \dfrac{\alpha\sigma_p^0}{E}\dfrac{k_n^n}{r^{\frac{n}{n+1}}}\tilde{\boldsymbol{\varepsilon}}(\theta,\,n) \end{cases} \quad \text{with} \quad \begin{cases} k_n = \left(\dfrac{JE}{\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{1}{n+1}} \\ I_n = \displaystyle\int_{-\pi}^{\pi} \tilde{h}(\theta,\,n)\,d\theta \end{cases}$$

- Similarly

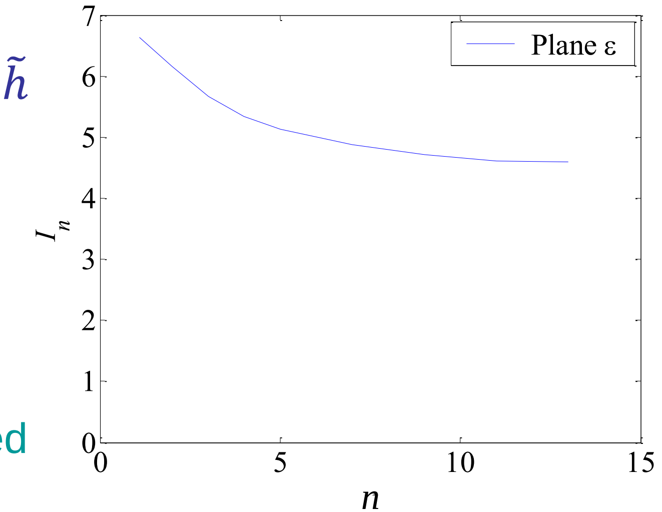
$$\begin{cases} \boldsymbol{u}_{,x} \propto \boldsymbol{\varepsilon} \\ U = \dfrac{n\alpha\left(\sigma_p^0\right)^2}{E(n+1)}\left(\dfrac{\sigma_e}{\sigma_p^0}\right)^{n+1} \end{cases} \quad \begin{aligned} &\Rightarrow \boldsymbol{u}_{,x} \to \frac{\alpha\sigma_p^0}{E}\frac{k_n^n}{r^{\frac{n}{n+1}}}\tilde{\boldsymbol{u}}_{,x} \\ &\Rightarrow U\boldsymbol{n}_x \to n\frac{\alpha\left(\sigma_p^0\right)^2}{E(n+1)}k_n^{n+1}r^{-1}\tilde{\sigma}_e^{n+1}\cos\theta \end{aligned}$$

- Since J is path independent

$$J = \lim_{r\to 0}\int_{-\pi}^{\pi}\left(U\boldsymbol{n}_x - \boldsymbol{u}_{,x}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\right)r\,d\theta \qquad \left(U\boldsymbol{n}_x - \boldsymbol{u}_{,x}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) = \frac{\alpha\left(\sigma_p^0\right)^2}{E}\frac{k_n^{n+1}}{r}\tilde{h}$$

$$= \frac{\alpha\left(\sigma_p^0\right)^2}{E}k_n^{n+1}\int_{-\pi}^{\pi}\tilde{h}(\theta,\,n)\,d\theta$$

$\Rightarrow$ $I_n = \displaystyle\int_{-\pi}^{\pi}\tilde{h}(\theta,\,n)\,d\theta$ is numerically evaluated



Semi-infinite mode I crack in plane ε state

- Process zone shape
  - The elastic-plastic boundary corresponds to $\sigma_e = \sigma_p^0$
    - Since $\boldsymbol{\sigma} = \sigma_p^0 \left(\frac{JE}{r\alpha \left(\sigma_p^0\right)^2 I_n} \right)^{\frac{1}{n+1}} \tilde{\boldsymbol{\sigma}} \left(\theta,\ n\right)$

$$\Longrightarrow \quad 1 = \frac{\sigma_e}{\sigma_p^0} = \left(\frac{JE}{\alpha \left(\sigma_p^0\right)^2} \right)^{\frac{1}{n+1}} r^{-\frac{1}{n+1}} \tilde{\sigma}_e I_n^{-\frac{1}{n+1}}$$

- Definition of the non-dimentional radius $\tilde{r} = \frac{r\alpha \left(\sigma_p^0\right)^2}{JE}$,

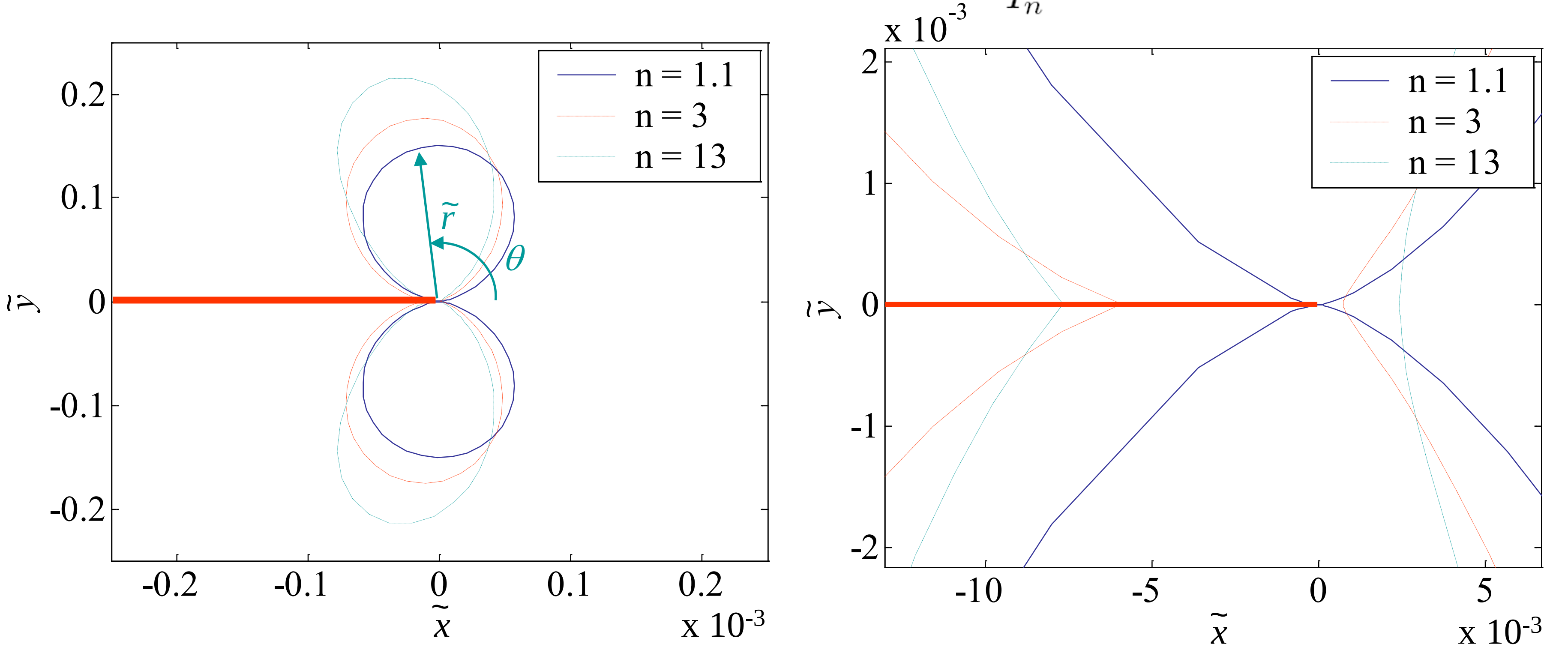
$$\Longrightarrow \quad 1 = \frac{\sigma_e}{\sigma_p^0} = \left(\frac{JE}{\alpha \left(\sigma_p^0\right)^2} \right)^{\frac{1}{n+1}} r^{-\frac{1}{n+1}} \tilde{\sigma}_e I_n^{-\frac{1}{n+1}}$$

$\tilde{r}^{-\frac{1}{n+1}}$

$\Longrightarrow$ the boundary is obtained for $\tilde{r}\left(\theta,\ n\right) = \frac{\left(\tilde{\sigma}_e \left(\theta,\ n\right)\right)^{n+1}}{I_n}$

- Process zone shape (2)
  - The elastic-plastic boundary corresponds to $\sigma_e = \sigma_p^0$
    - In terms of $\tilde{r} = \dfrac{r\alpha\left(\sigma_p^0\right)^2}{JE}$,

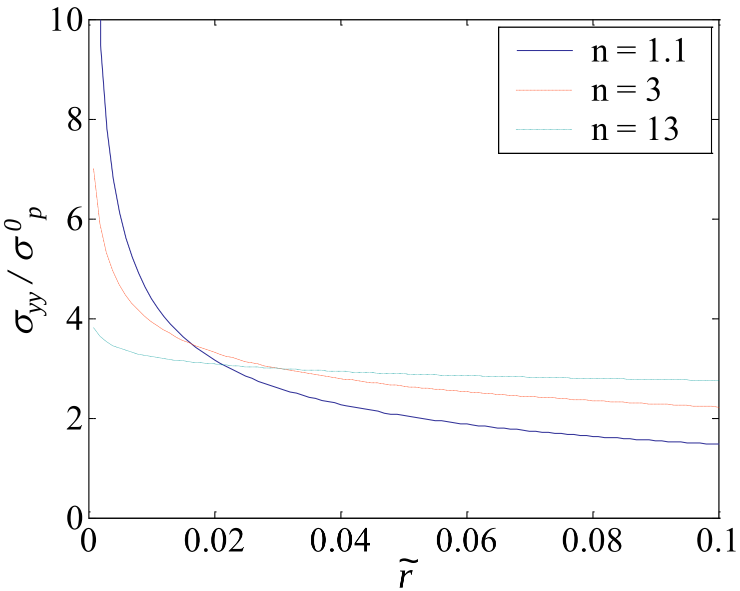
      the boundary is obtained for $\tilde{r}\left(\theta,\, n\right) = \dfrac{\left(\tilde{\sigma}_e\left(\theta,\, n\right)\right)^{n+1}}{I_n}$

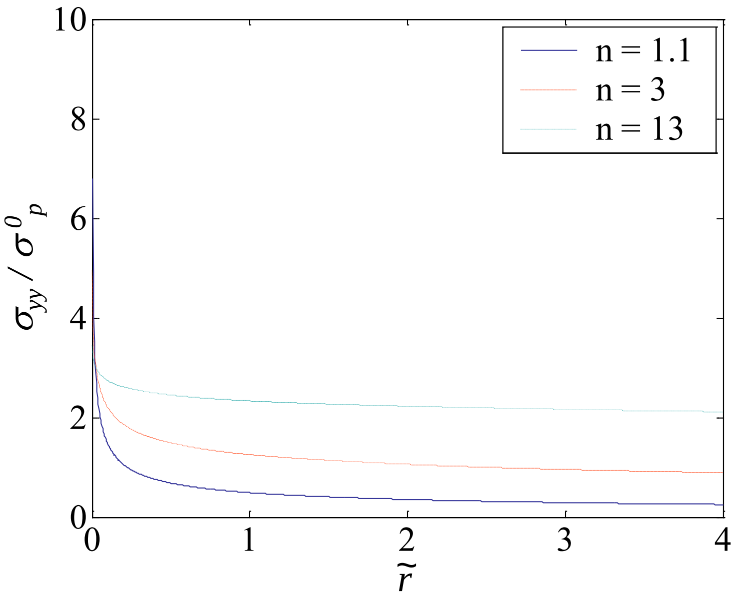


- Limit case of $n \rightarrow 1$: $J = \dfrac{K_I^2}{E}\left(1-\nu^2\right) \Longrightarrow r \propto \left(1-\nu^2\right)\left(\dfrac{K_I}{\sigma_p^0}\right)^2$

- Stress field at crack tip
  - Since $\boldsymbol{\sigma} = \sigma_p^0 \left(\frac{JE}{r\alpha \left(\sigma_p^0\right)^2 I_n} \right)^{\frac{1}{n+1}} \tilde{\boldsymbol{\sigma}}(\theta,\, n)$ & $\tilde{r} = \frac{r\alpha \left(\sigma_p^0\right)^2}{JE}$

    the stress field at crack tip is $\boldsymbol{\sigma}_{yy}(\theta = 0) = \boldsymbol{\sigma}_p^0 \left(\frac{1}{\tilde{r} I_n} \right)^{\frac{1}{n+1}} \tilde{\boldsymbol{\sigma}}_{\theta\theta}(\theta = 0,\, n)$



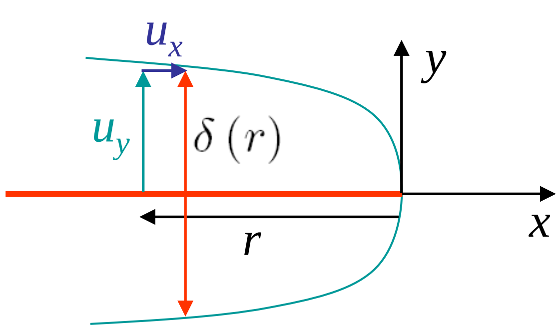


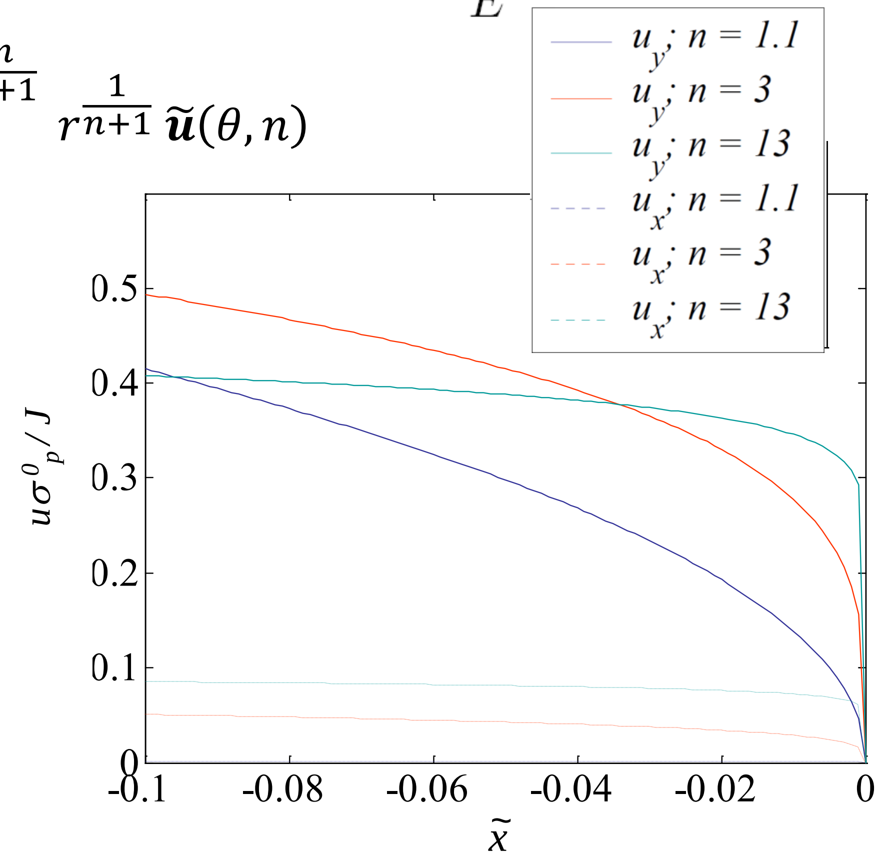
- The method does not remove the stress singularity (except for $n \to \infty$)
- Since the abscissa is $\tilde{r} = \frac{r\alpha \left(\sigma_p^0\right)^2}{JE}$, it appears that J is a measure of the intensity of the singular crack tip field (except for $n \to \infty$)

- Crack opening displacement (COD)
  - Strain field $\boldsymbol{\varepsilon} \rightarrow \frac{\alpha\sigma_p^0}{E}\frac{k_n^n}{r^{\frac{n}{n+1}}}\tilde{\boldsymbol{\varepsilon}}(\theta,\,n)$ with $k_n = \left(\frac{JE}{\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{1}{n+1}}$
  - Displacement field
    - $\left\|\begin{array}{l}\boldsymbol{u}_x = \Psi_{,y}\\ \boldsymbol{u}_y = -\Psi_{,x}\end{array}\right\| \quad \psi \propto r^{\frac{n+2}{n+1}} k_n^n \frac{\alpha\sigma_p^0}{E}\,\tilde{g}(\theta) \Longrightarrow \boldsymbol{u}(r,\theta) \rightarrow \frac{\alpha\sigma_p^0}{E} k_n^n r^{\frac{1}{n+1}}\tilde{\boldsymbol{u}}$ $\quad \frac{n+2}{n+1} - 1$

$$\Longrightarrow \boldsymbol{u}(r,\theta) = \frac{\alpha\sigma_p^0}{E}\left(\frac{JE}{\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{n}{n+1}} r^{\frac{1}{n+1}}\,\tilde{\boldsymbol{u}}(\theta,n)$$

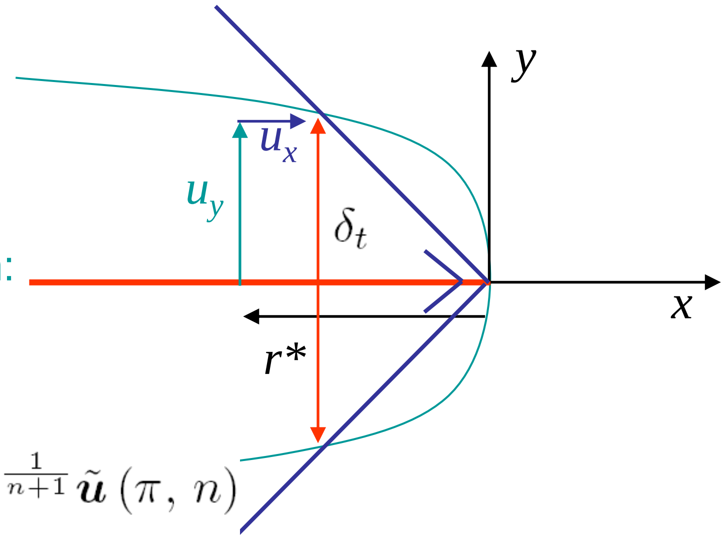
  - COD:
    - $\delta(r) = 2\boldsymbol{u}_y(r,\,\pi)$





  - For Dugdale's model, a CTOD has been defined. Quid for HRR model?

- Crack tip opening displacement (CTOD)
  - The CTOD is defined as $\delta(r)$ such that a 90°-angle is intercepted
    - Let r^* be the value of r satisfying this criterion:



$$r^* - \boldsymbol{u}_x(r^*, \pi) = \boldsymbol{u}_y(r^*, \pi) = \frac{\delta_t}{2}$$

- Since $\boldsymbol{u}(r, \pi) = \frac{\alpha\sigma_p^0}{E}\left(\frac{JE}{\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{n}{n+1}} r^{\frac{1}{n+1}} \tilde{\boldsymbol{u}}(\pi, n)$

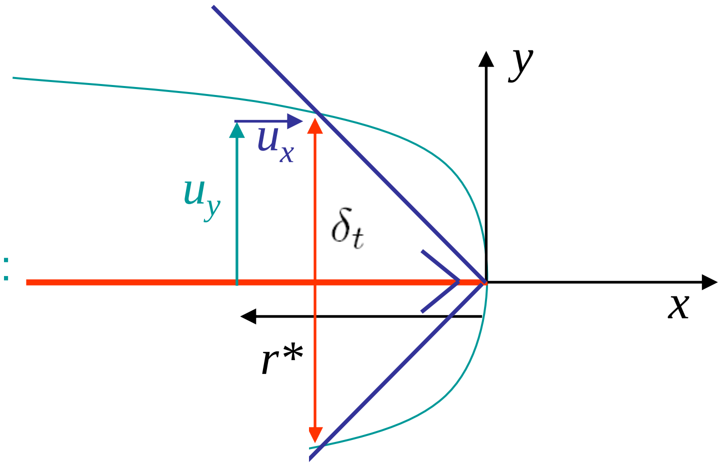
$$\Rightarrow\ r^* = \boldsymbol{u}_x(r^*,\pi) + \boldsymbol{u}_y(r^*,\pi) = \frac{\alpha\sigma_p^0}{E}\left(\frac{JE}{\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{n}{n+1}} r^{*\frac{1}{n+1}}\left[\tilde{\boldsymbol{u}}_x(\pi,n) + \tilde{\boldsymbol{u}}_y(\pi,n)\right]$$

$1 - \frac{1}{n+1}$

$$\Rightarrow\ r^{*\frac{n}{n+1}} = \frac{\alpha\sigma_p^0}{E}\left(\frac{JE}{\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{n}{n+1}}\left[\tilde{\boldsymbol{u}}_x(\pi,n) + \tilde{\boldsymbol{u}}_y(\pi,n)\right]$$

$$\Rightarrow\ r^* = \left(\frac{\alpha\sigma_p^0}{E}\right)^{\frac{1}{n}}\left(\frac{J}{\sigma_p^0 I_n}\right)\left[\tilde{\boldsymbol{u}}_x(\pi, n) + \tilde{\boldsymbol{u}}_y(\pi, n)\right]^{\frac{n+1}{n}}$$

- Crack tip opening displacement (CTOD) (2)
  - The CTOD is defined as $\delta(r)$ such that a 90°-angle is intercepted (2)
    - Let r^* be the value of r satisfying this criterion:



$$r^* - \boldsymbol{u}_x(r^*, \pi) = \boldsymbol{u}_y(r^*, \pi) = \frac{\delta_t}{2}$$

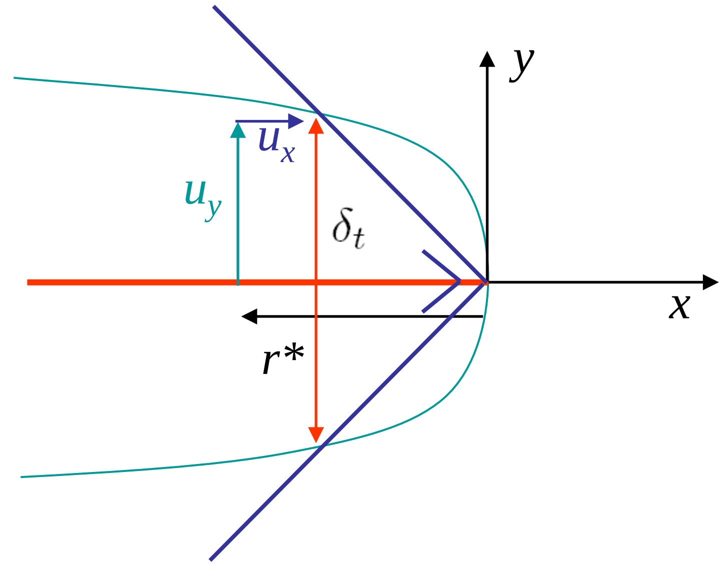
- Since $r^* = \left(\frac{\alpha\sigma_p^0}{E}\right)^{\frac{1}{n}} \left(\frac{J}{\sigma_p^0 I_n}\right) \left[\tilde{\boldsymbol{u}}_x(\pi, n) + \tilde{\boldsymbol{u}}_y(\pi, n)\right]^{\frac{n+1}{n}}$

$$\Rightarrow \boxed{r^{*\frac{1}{n+1}} = \left(\frac{\alpha\sigma_p^0}{E}\right)^{\frac{1}{n(n+1)}} \left(\frac{J}{\sigma_p^0 I_n}\right)^{\frac{1}{n+1}} \left[\tilde{\boldsymbol{u}}_x(\pi, n) + \tilde{\boldsymbol{u}}_y(\pi, n)\right]^{\frac{1}{n}}}$$

- Finally $\delta_t = 2\boldsymbol{u}_y(r^*, \pi) = 2\frac{\alpha\sigma_p^0}{E}\left(\frac{JE}{\alpha\left(\sigma_p^0\right)^2 I_n}\right)^{\frac{n}{n+1}} \boxed{r^{*\frac{1}{n+1}}} \tilde{\boldsymbol{u}}_y(\pi, n)$

$$\Rightarrow \delta_t = 2\boldsymbol{u}_y(r^*, \pi) = 2\left(\frac{\alpha\sigma_p^0}{E}\right)^{1-\frac{n}{n+1}+\frac{1}{n(n+1)} = \frac{1}{n}} \left(\frac{J}{\sigma_p^0 I_n}\right)^{\frac{1}{n+1}+\frac{n}{n+1} = 1} \left[\tilde{\boldsymbol{u}}_x(\pi, n) + \tilde{\boldsymbol{u}}_y(\pi, n)\right]^{\frac{1}{n}} \tilde{\boldsymbol{u}}_y(\pi, n)$$

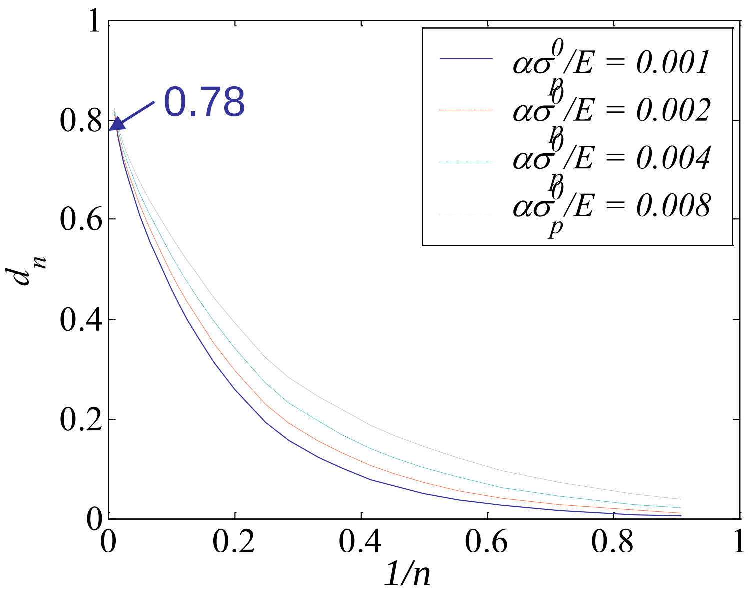
- Crack tip opening displacement (CTOD) (3)
  - The CTOD is defined as $\delta(r)$ such that a 90°-angle is intercepted (3)



- $$\delta_t = 2\boldsymbol{u}_y(r^*, \pi) = 2\left(\frac{\alpha\sigma_p^0}{E}\right)^{\frac{1}{n}}\left(\frac{J}{\sigma_p^0 I_n}\right)$$
$$[\tilde{\boldsymbol{u}}_x(\pi, n) + \tilde{\boldsymbol{u}}_y(\pi, n)]^{\frac{1}{n}}\,\tilde{\boldsymbol{u}}_y(\pi, n)$$

$$\Longrightarrow \delta_t = d_n \frac{J}{\sigma_p^0}$$

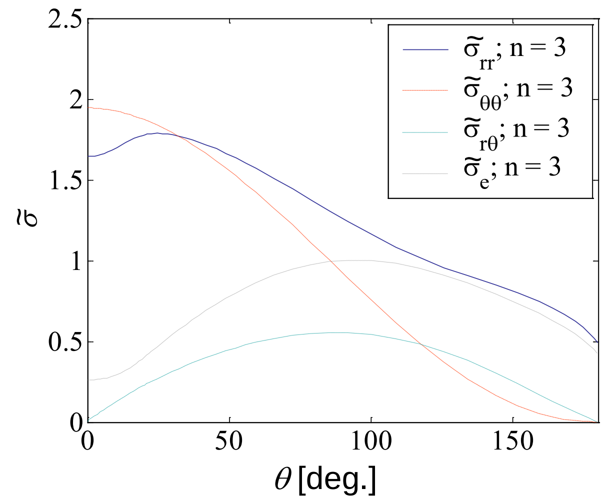
- d_n depends on
  - n
  - but also on $\frac{\alpha\sigma_p^0}{E}$
- Since for a given material J is uniquely related to the CTOD, this last one can be used as a crack initiation criterion

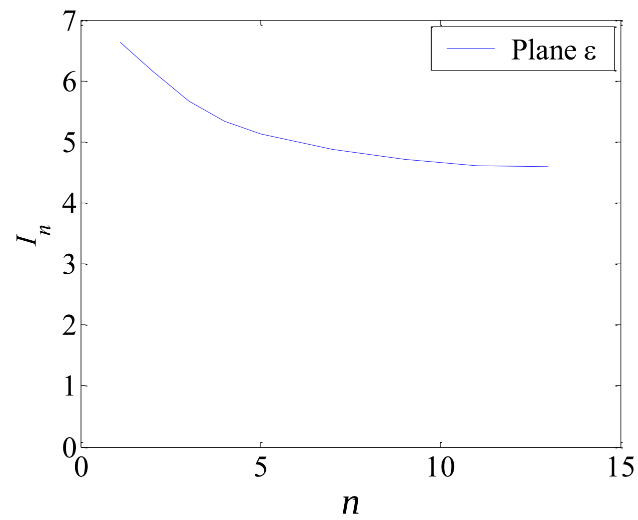


- Summary
  - Assumptions
    - J2-plasticity with power law description
    - Small deformations
    - There is no unloading and loading is proportional in all the directions (ok for **crack initiation and not for crack propagation**)
    - Elastic strains are assimilated to plastic strain (material is **incompressible**)
    - Semi-infinite crack
  - HRR results for semi-infinite **mode I crack in plane ε state**
    - Asymptotic stress, strain and displacement fields

$$\boldsymbol{\sigma} = \sigma_p^0 \left(\frac{JE}{r\alpha \left(\sigma_p^0\right)^2 I_n} \right)^{\frac{1}{n+1}} \tilde{\boldsymbol{\sigma}}\left(\theta,\ n\right)$$

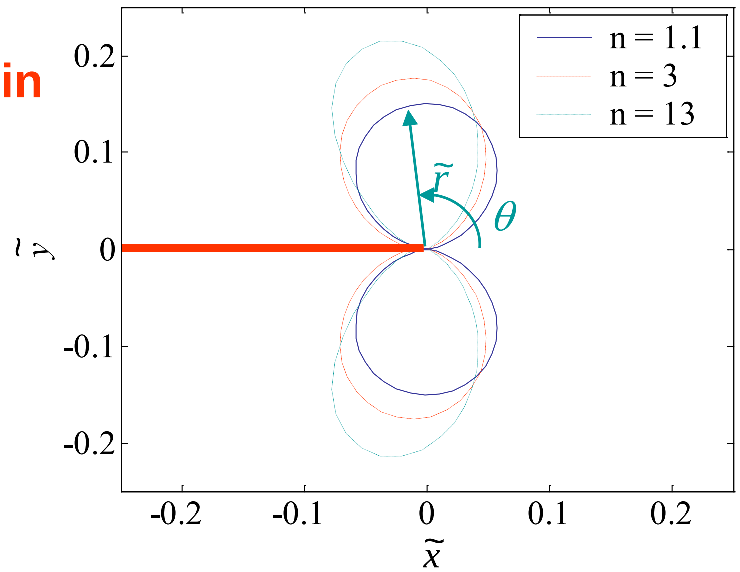
      - The J-integral plays the role of an equivalent "plastic strain intensity factor"
      - Stress field evolves in a proportional way, so this is applicable to incremental plasticity as long as J increases



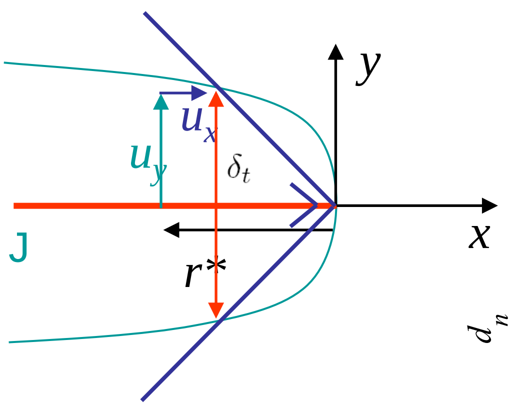


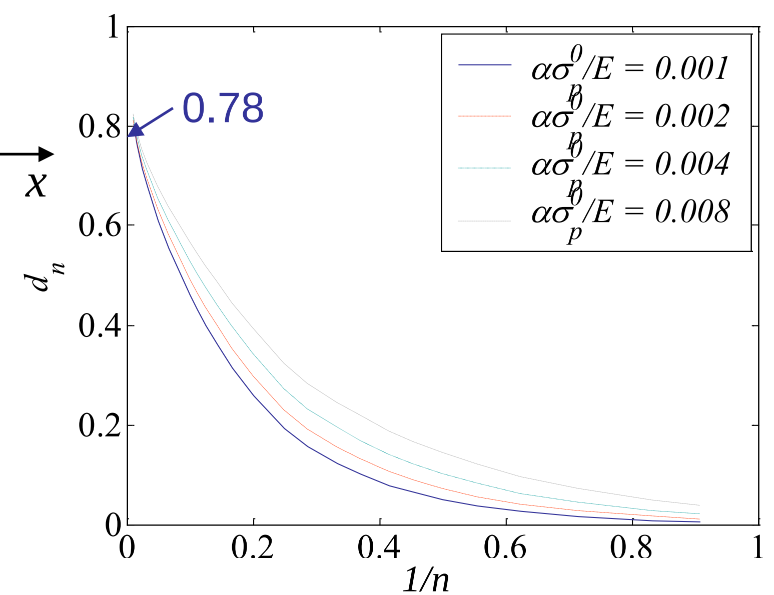
- Summary (2)
  - HRR results for semi-infinite **mode I crack in plane ε state** (2)
    - Process zone $\tilde{r}(\theta,\, n) = \frac{(\tilde{\sigma}_e(\theta,\, n))^{n+1}}{I_n}$

      with $\tilde{r} = \frac{r\alpha\left(\sigma_p^0\right)^2}{JE}$
      - If SSY ⟹ $r \propto (1-\nu^2)\left(\frac{K_I}{\sigma_p^0}\right)^2$



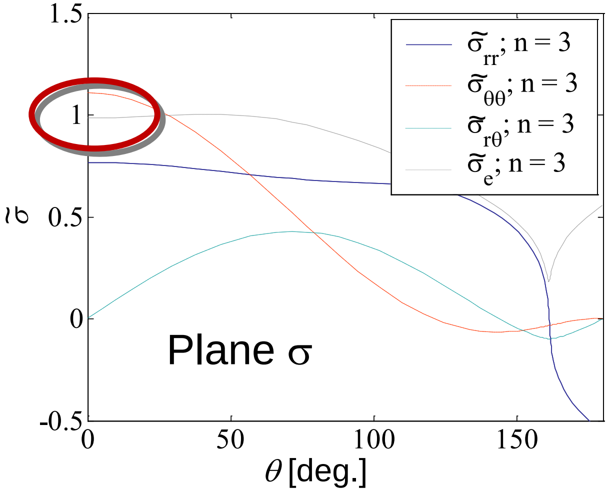
    - CTOD
      - New definition
      - $\delta_t = d_n \frac{J}{\sigma_p^0}$
      - Also function of J

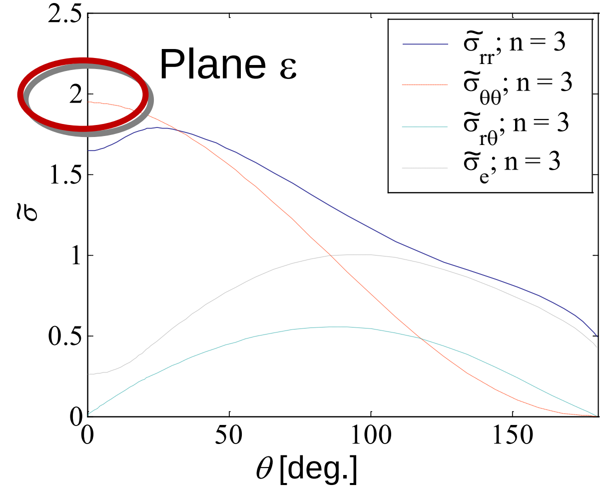


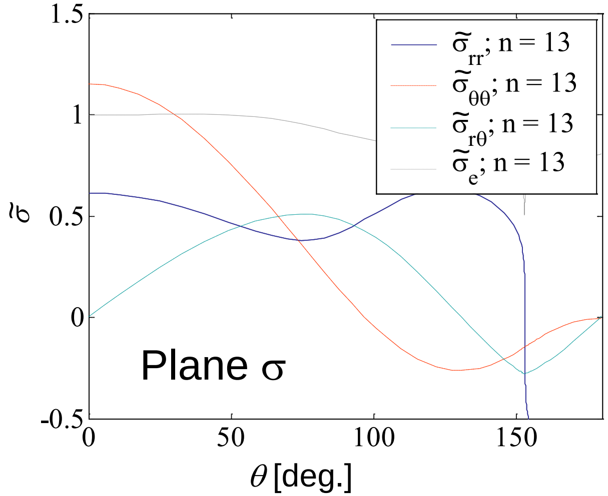


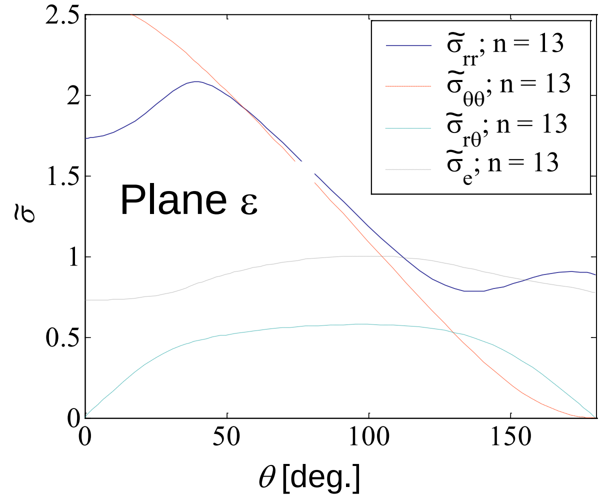
    - We did not assume SSY !!!
  - Questions
    - What is happening for other configurations?
    - What is the range of validity?

- Mode I crack in plane σ state
  - Analysis is the same with other $\tilde{\boldsymbol{\sigma}}(\theta, n)$, $\tilde{\boldsymbol{\varepsilon}}(\theta, n)$ & $\tilde{\boldsymbol{u}}(\theta, n)$ fields









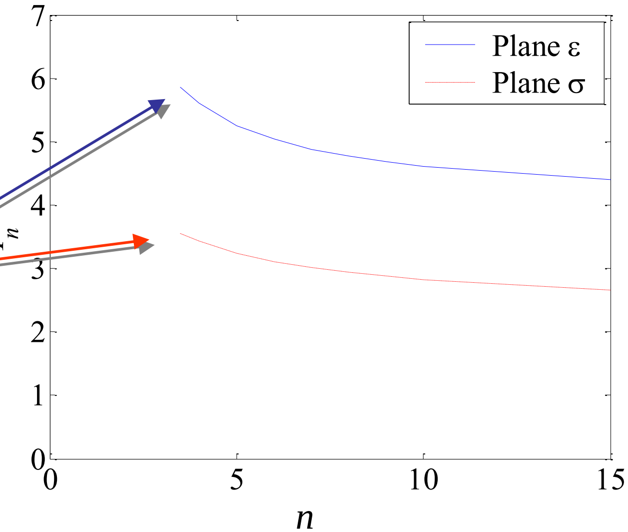
- Mode I crack in plane σ state (2)
  - Stress

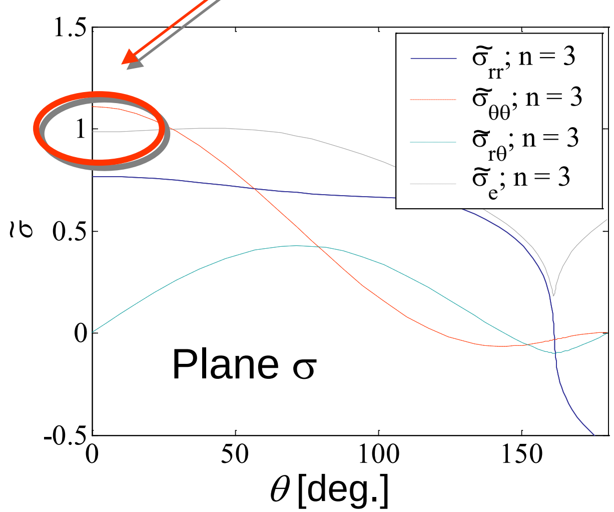
$$\boldsymbol{\sigma} = \sigma_p^0 \left(\frac{JE}{r\alpha \left(\sigma_p^0\right)^2 I_n} \right)^{\frac{1}{n+1}} \tilde{\boldsymbol{\sigma}}\left(\theta,\, n\right)$$

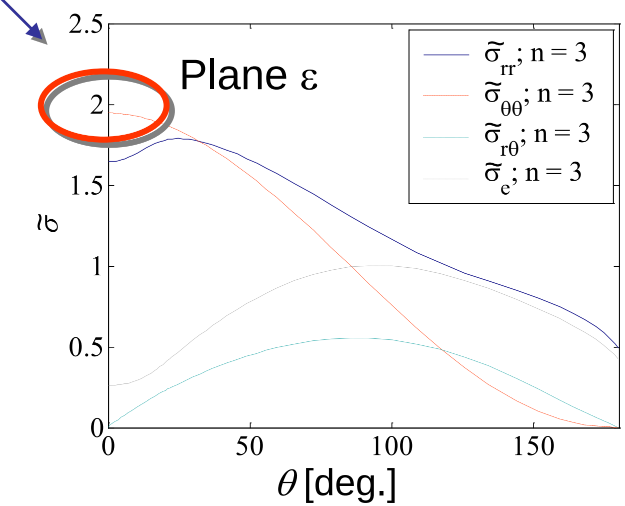
  - Asymptotic σ_{yy} stress

$$\frac{\boldsymbol{\sigma}_{\theta\theta}\left(0,\, r\right)|_{n=3,\text{ plane } \varepsilon}}{\boldsymbol{\sigma}_{\theta\theta}\left(0,\, r\right)|_{n=3,\text{ plane } \sigma}} \simeq \frac{1.9}{1.1}\left(\frac{3.86}{5.81}\right)^{\frac{1}{4}} = 1.56$$

$$\frac{\boldsymbol{\sigma}_{\theta\theta}\left(0,\, r\right)|_{n=13,\text{ plane } \varepsilon}}{\boldsymbol{\sigma}_{\theta\theta}\left(0,\, r\right)|_{n=13,\text{ plane } \sigma}} \simeq \frac{2.6}{1.2}\left(\frac{2.87}{3.4}\right)^{\frac{1}{14}} = 2.13$$





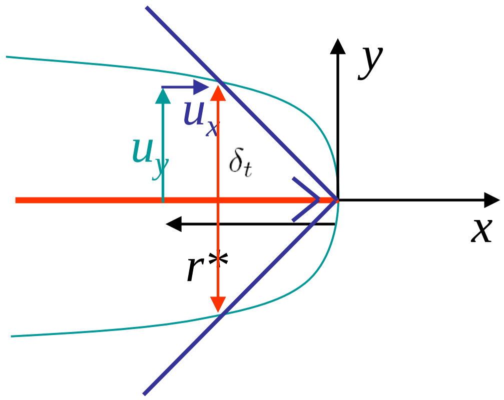


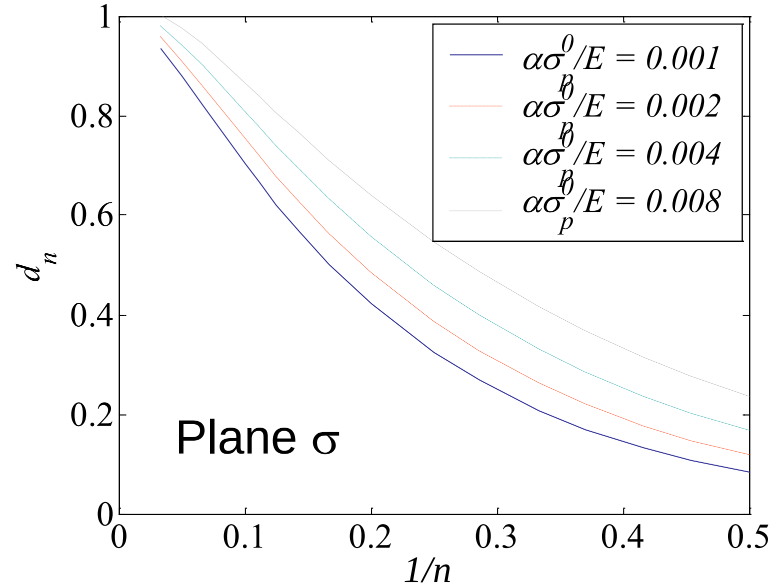
- So for the same J, a thin specimen is less stressed at crack tip

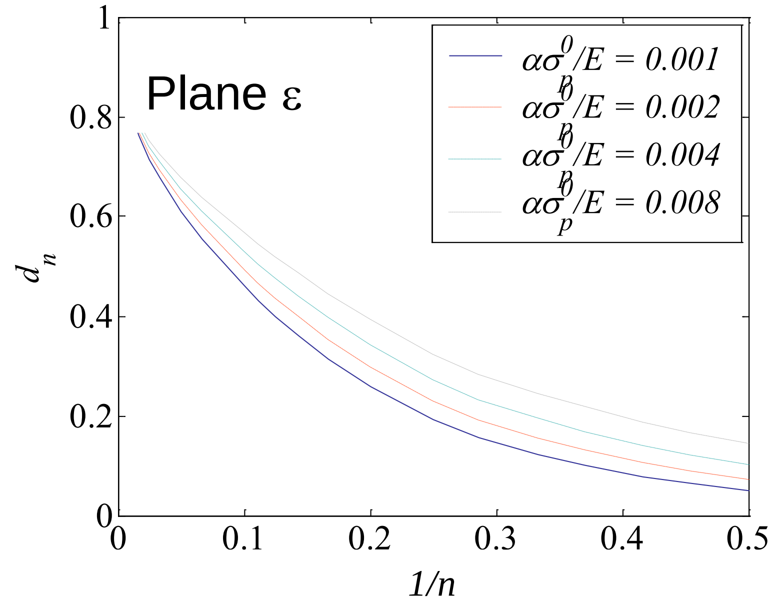
Other HRR solutions

- Mode I crack in plane σ state (3)
  – Crack Tip Opening Displacement

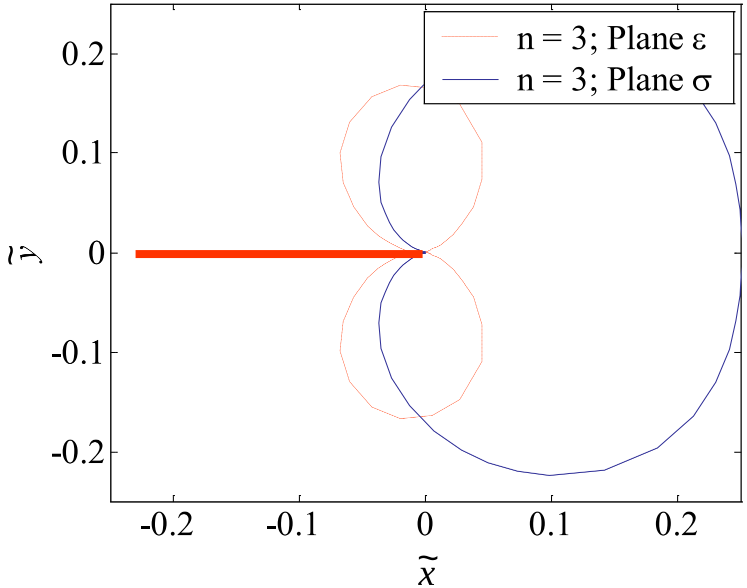
  - $\delta_t = d_n \dfrac{J}{\sigma_p^0}$



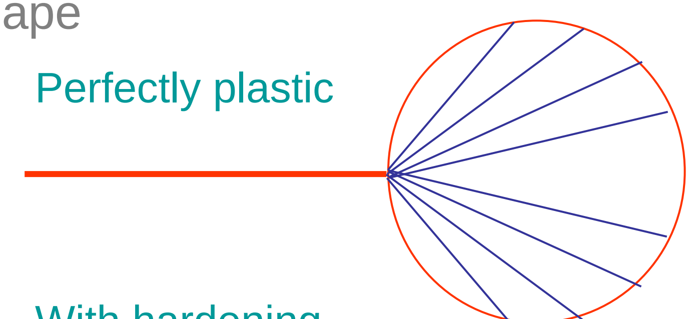




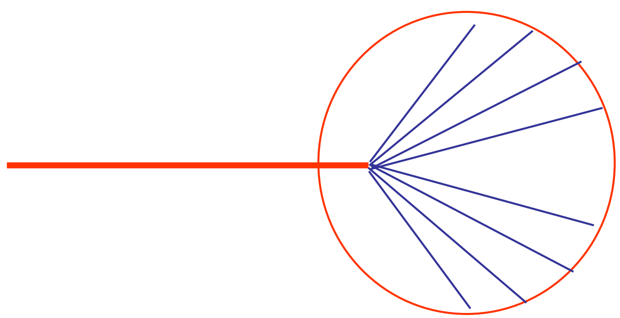
- Mode I crack in plane σ state (4)
  - Process zone shapes
    - The process zones
      - Have a different shape
      - Are more diffuse in plane σ



- Mode III
  - In SSY, the process zone has a circular shape
    - Perfectly plastic



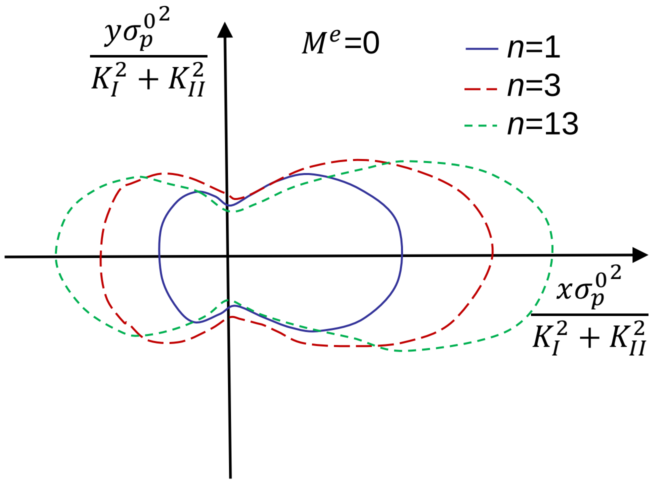
    - With hardening

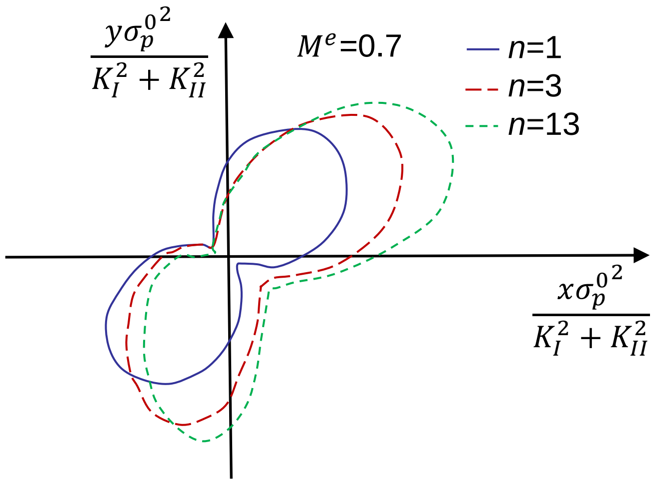


- Mode II and mixed mode crack
  - In SSY, the solution depends on the elastic mixity parameter

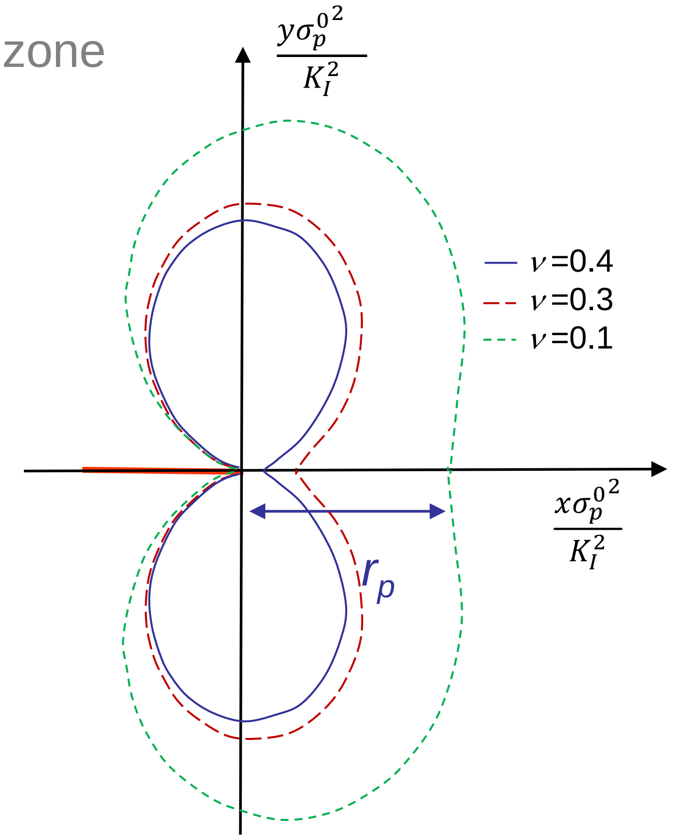
$$M^e = \frac{2}{\pi}\arctan\frac{K_I}{K_{II}}$$

  - Examples
    - Process zones in plane ε state
    - SSY

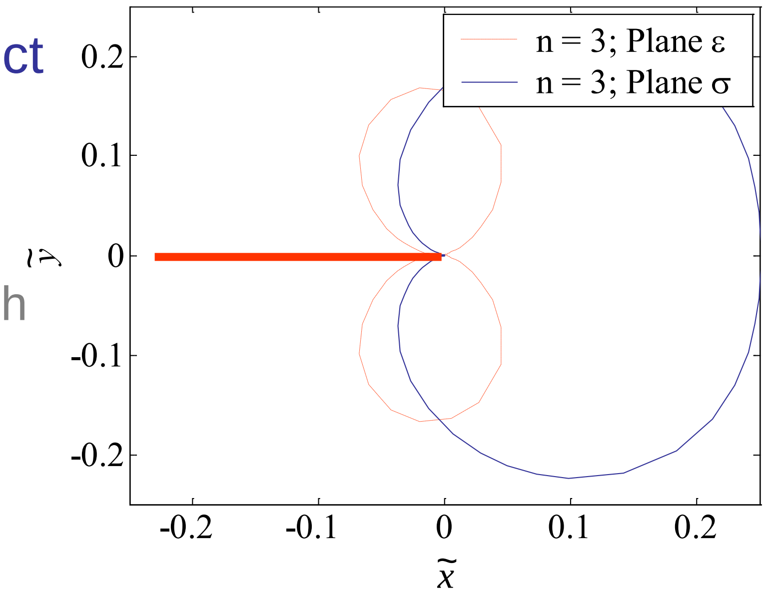


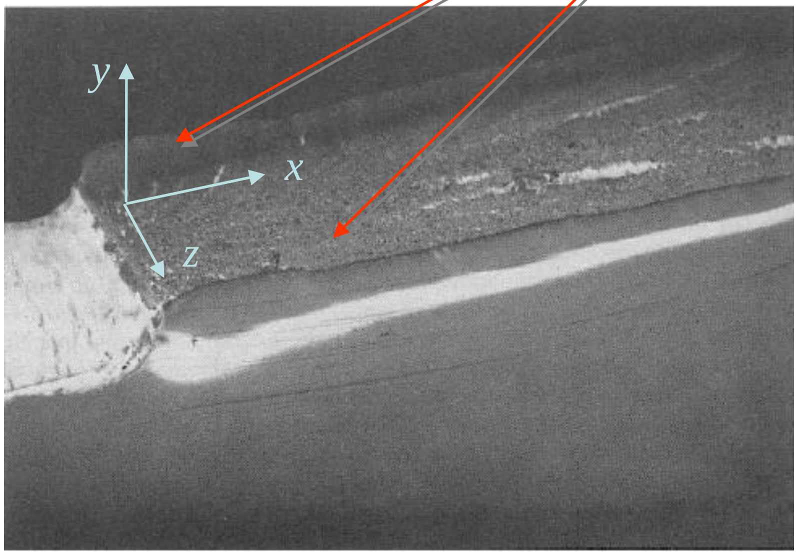


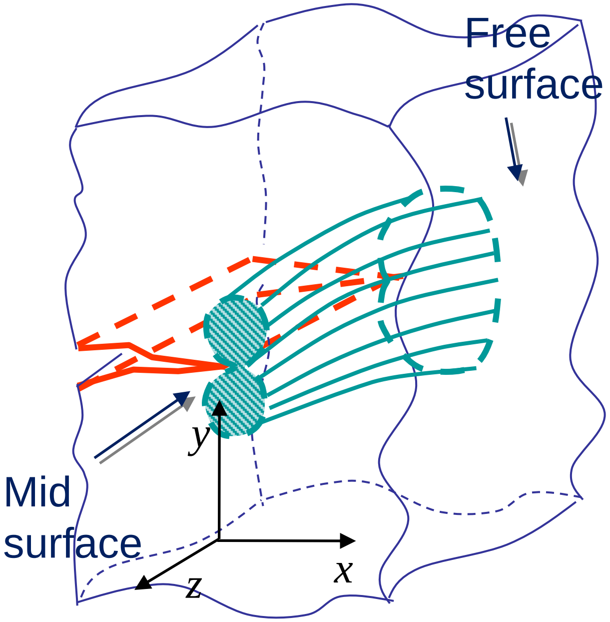
- Compressibility
  - Elastic part of the deformations in not incompressible
  - Considering this effect will diffuse the plastic zone
    - Example: Mode I, plane ε state & SSY
    - The plastic zone size r_p is defined as the length of the plastic zone ahead of the crack



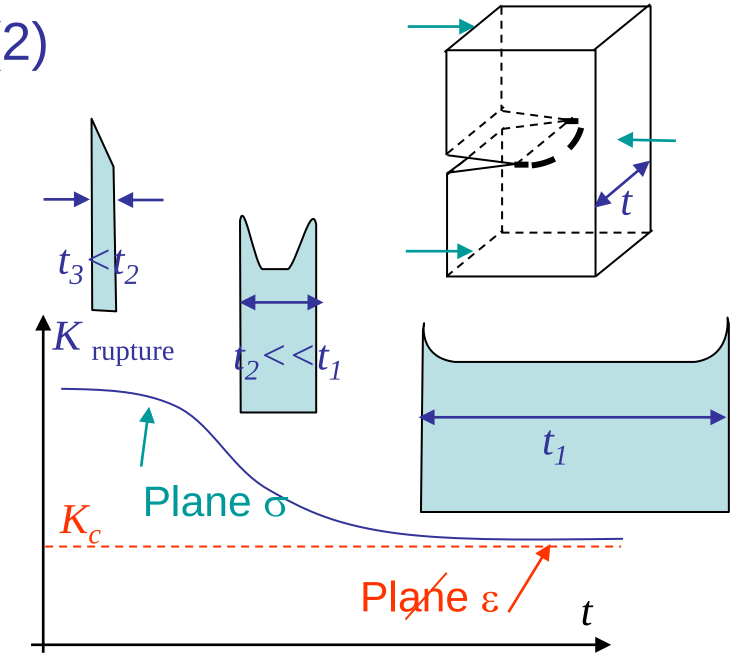
- HRR solution in SSY can explain 3D effect
  - Let us assume that the solution is
    - Plane-σ-like near free surfaces
    - Plane-ε-like near mid-plane of specimen
  - Although K and σ are not constant through the thickness, it remains that
    - There is a transition in the plastic zone shape and size
    - This transition is responsible for shear lips
    - Shear lips form a 45-degree-angle since σ_{zz}=0 (so max shear at 45° in plane Oyz)

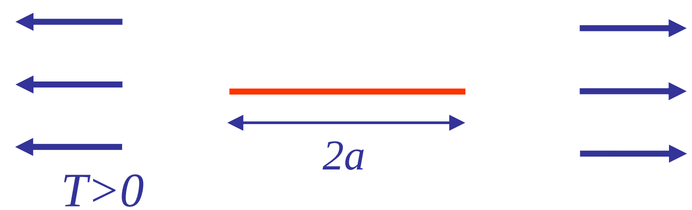






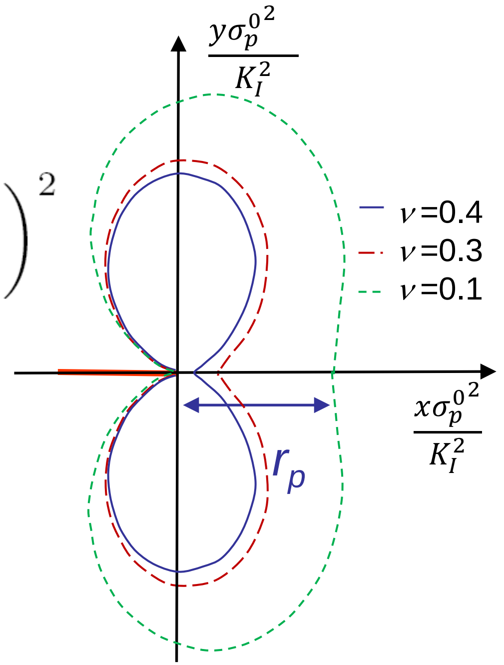
- HRR solution in SSY can explain 3D effect (2)
  - During a toughness test, the K measured is an average one
    - It is more important for thin specimen
    - The process is never really plane ε
  - There are actually complex 3D effects
    - In SSY, even for a thin specimen, near the mid-plane a plane-ε-state is developed
    - If the load increases, the plastic zone can be plane-σ–like near the mid-plane
  - Effect of T-stress should also be considered
    - Recall T-stress is the 0-order term obtained with the asymptotic solution, which is dominant at radius r_c
    - In general, if the test is such that $T < 0$, the measured fracture K will be larger than for $T > 0$, independently of the thickness *
    - For ASTM toughness tests, the thickness is large $t > 2.5\left(\frac{K_C}{\sigma_p^0}\right)^2$ so that $T > 0$





*D.J Smith, M.R Ayatollahi and M.J Pavier, *Proc. R. Soc. A, 2006, vol.* 462, pp 415-2437

- Effective crack length for SSY
  - If SSY assumption holds
    - J can be expressed in terms of K
    - Then the plastic size can be written $r \propto \frac{JE}{\left(\sigma_p^0\right)^2} \propto \left(\frac{K_I}{\sigma_p^0}\right)^2$
    - However, there are dependencies on
      - Parameters n & ν
      - Whether it is plane ε or plane σ

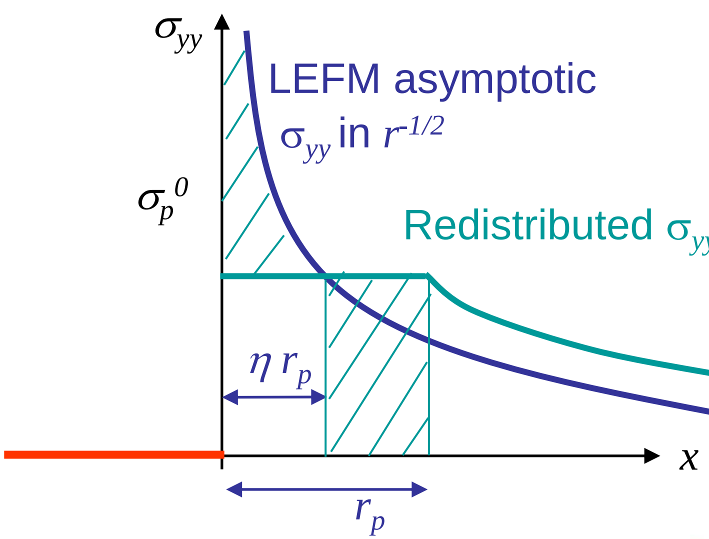


  - Rice's model for perfectly plastic material
    - Intersection of the linear elastic solution with the yield stress leads at $\eta\, r_p$

$$\frac{K_I}{\sqrt{2\pi\eta r_p}} = \sigma_p^0 \Longrightarrow \sqrt{\eta r_p} = \frac{1}{\sqrt{2\pi}}\frac{K_I}{\sigma_p^0}$$

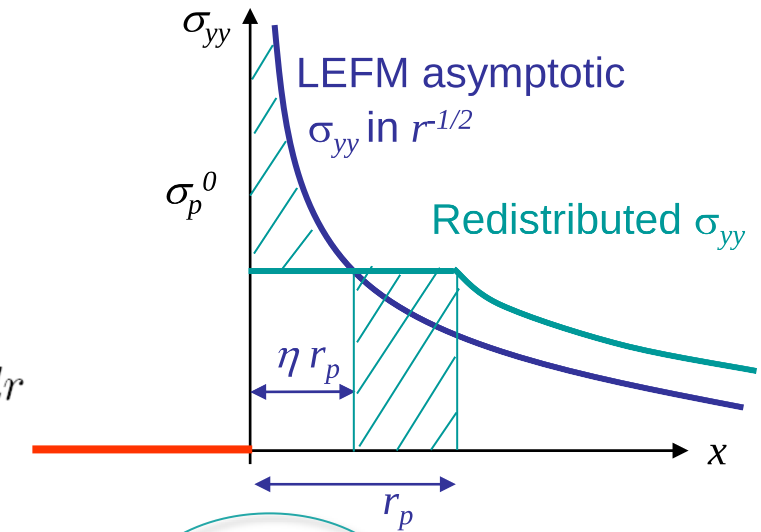
    - But there is a redistribution of the stresses so that the traction remains the same
      - First order approximation: stress distribution is shifted

$$\int_0^{\eta r_p} \left(\boldsymbol{\sigma}_{yy} - \sigma_p^0\right) dr = \sigma_p^0 \left(1-\eta\right) r_p$$



- Effective crack length for SSY (2)
  - Rice's model (2)
    - 2 equations with 2 unknowns

$$\begin{cases} \sqrt{\eta r_p} = \dfrac{1}{\sqrt{2\pi}} \dfrac{K_I}{\sigma_p^0} \\ \sigma_p^0 (1-\eta)\, r_p = \displaystyle\int_0^{\eta r_p} \left(\boldsymbol{\sigma}_{yy} - \sigma_p^0\right) dr \end{cases}$$



$$\sigma_p^0 (1-\cancel{\eta})\, r_p = \int_0^{\eta r_p} \left(\overset{\frac{K_I}{\sqrt{2\pi r}}}{\boldsymbol{\sigma}_{yy}} - \cancel{\sigma_p^0}\right) dr \Longrightarrow \sigma_p^0 r_p = \int_0^{\eta r_p} \frac{K_I}{\sqrt{2\pi r}} dr \quad \frac{2K_I\sqrt{\eta r_p}}{\sqrt{2\pi}}$$

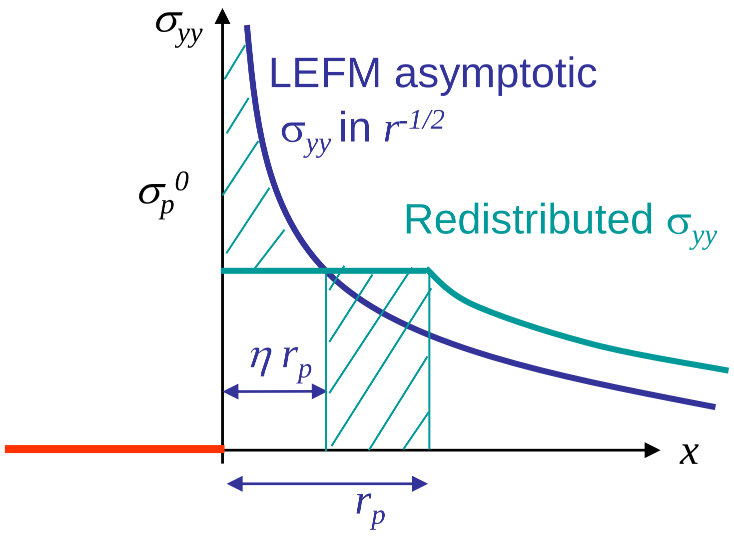
- System:

$$\begin{cases} \sqrt{\eta r_p} = \dfrac{1}{\sqrt{2\pi}} \dfrac{K_I}{\sigma_p^0} \\ \sigma_p^0 r_p = \dfrac{2K_I\sqrt{\eta r_p}}{\sqrt{2\pi}} \end{cases}$$

- So the plastic zone size is $r_p = \dfrac{2K_I}{\sqrt{2\pi}\sigma_p^0}\sqrt{\eta r_p} = \dfrac{K_I^2}{\pi}\dfrac{1}{\left(\sigma_p^0\right)^2}$ & $\eta = \dfrac{1}{2}$
- So everything is as if the crack had an effective length $a + \eta\, r_p = a + r_p/2$

- Effective crack length for SSY (3)
  - Rice's model (2)
    - So everything is as if the crack had an effective length $a + \eta\, r_p = a + r_p/2$

$$\begin{cases} r_p = \dfrac{1}{\pi}\left(\dfrac{K_I}{\sigma_p^0}\right)^2 \\ \eta = \dfrac{1}{2} \end{cases}$$



  - From HRR models, numerical simulations, etc …
    - Considering blunting, compressibility, hardening, …, an **estimation** is

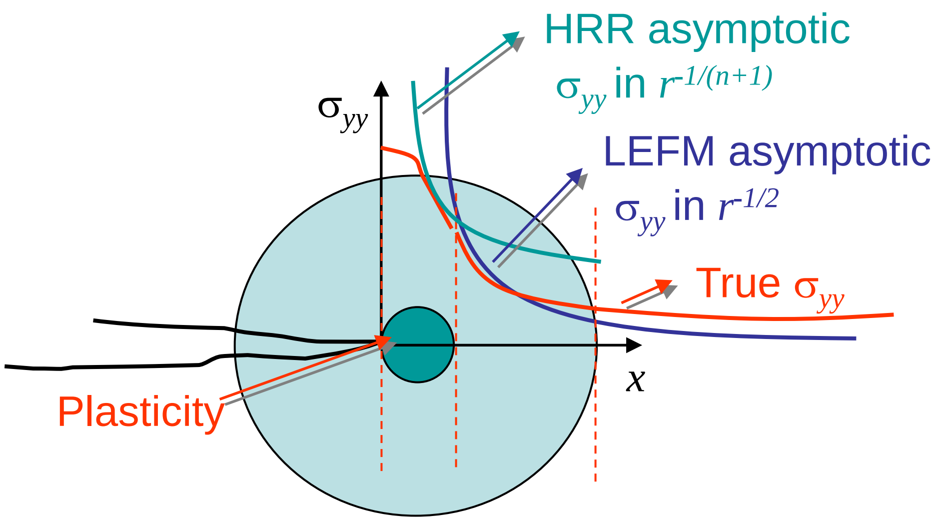
$$\eta r_p = \frac{r_p}{2} = \begin{cases} \dfrac{n-1}{n+1}\dfrac{1}{2\pi}\left(\dfrac{K_I(a_{\mathrm{eff}})}{\sigma_p^0}\right)^2 & \text{if plane } \sigma \\ \dfrac{n-1}{n+1}\dfrac{1}{6\pi}\left(\dfrac{K_I(a_{\mathrm{eff}})}{\sigma_p^0}\right)^2 & \text{if plane } \varepsilon \end{cases}$$

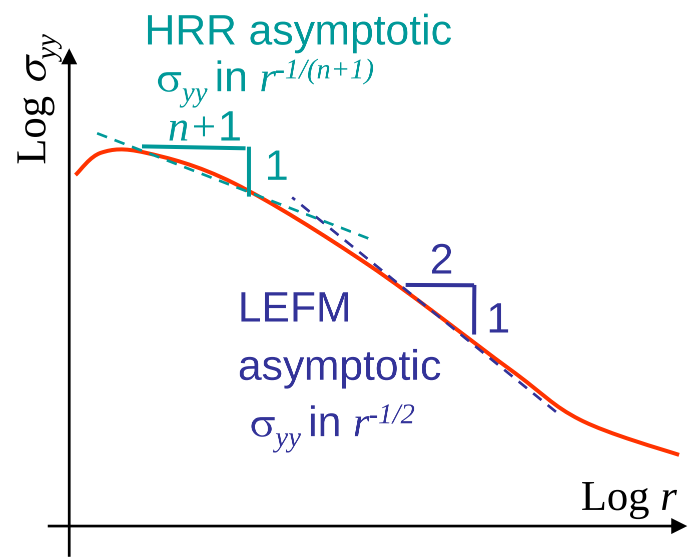
- Effective crack length for SSY (4)
  - If σ_∞< 50% of σ_p^0 then a second order SSY assumption holds
    - The cohesive zone remains small compared to crack size
    - The effective crack size can be stated as $a + \eta\, r_p = a + r_p/2$

with $\eta r_p = \dfrac{r_p}{2} = \begin{cases} \dfrac{n-1}{n+1}\dfrac{1}{2\pi}\left(\dfrac{K_I(a_{\text{eff}})}{\sigma_p^0}\right)^2 & \text{if plane } \sigma \\ \dfrac{n-1}{n+1}\dfrac{1}{6\pi}\left(\dfrac{K_I(a_{\text{eff}})}{\sigma_p^0}\right)^2 & \text{if plane } \varepsilon \end{cases}$ for cracks in finite plates

    - So there is an iterative procedure to follow:
      - a) compute K from a
      - b) compute effective crack size
      - c) compute new K from a_{eff} and back to b) if needed
  - This method is a correction for linear fracture mechanics, but does not allow considering problems with large yielding

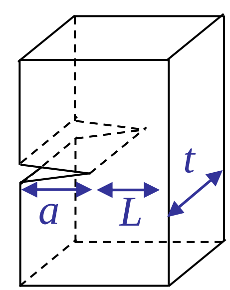
- Validity in SSY
  - We have two asymptotic solutions
    - HRR field is valid in the process zone
    - LEFM is still valid in the elastic zone close to the crack tip





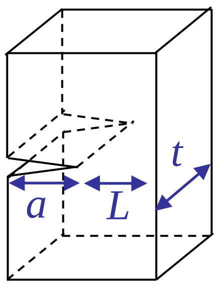
  - Conditions
    - This is the case if all sizes are 25 times larger than the plastic zone

$$a,\ t,\ L > 25 r_p \simeq \frac{25}{3\pi}\left(\frac{K_I}{\sigma_p^0}\right)^2 \simeq 2.5\left(\frac{K_I}{\sigma_p^0}\right)^2$$



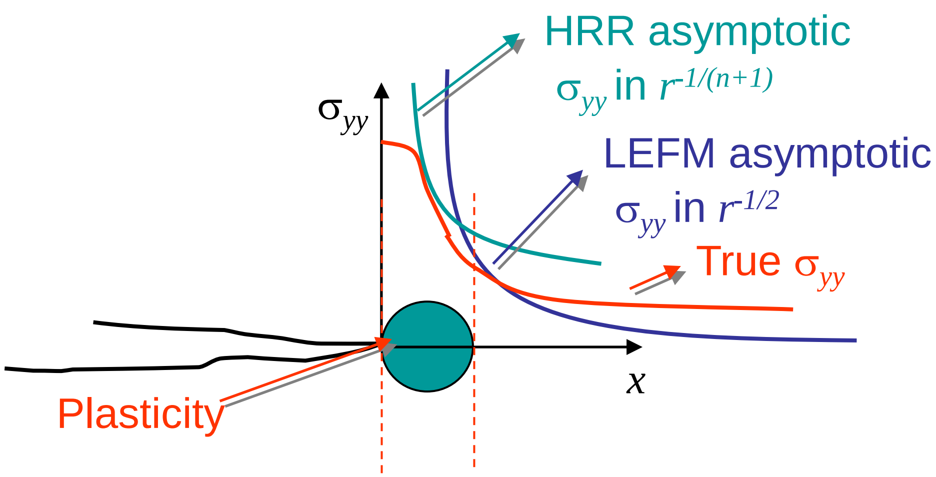
- Validity in SSY (2)
  - Crack initiation criteria
    - Criteria based on J or δ_t are valid: $J \geq J_C$ or $\delta_t \geq \delta_C$
      - J & δ_t depend on a, the geometry, the loading, …
    - But as the LEFM solution holds, we can still use $K(a) \geq K_C$
      - Might be corrected by using the effective length a_{eff} if $\sigma_\infty < 50\%$ of σ_p^0
  - Sizes for K–based toughness test

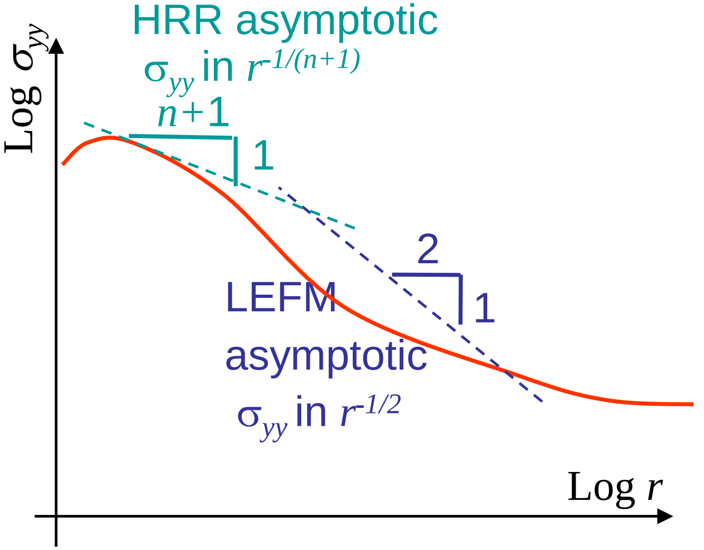
$$a,\ t,\ L > 2.5 \left(\frac{K_C}{\sigma_p^0} \right)^2$$



- Examples:
  - Titanium alloy 6%Al-4%V
    - » Yield: 830 MPa
    - » Toughness: 55 $\text{MPa} \cdot \text{m}^{1/2}$
    - » $a, t, L > 1.1$ cm
  - Strength steel
    - » Yield: 350 MPa
    - » Toughness: 250 $\text{MPa} \cdot \text{m}^{1/2}$
    - » $a, t, L > 1.27$ m !!!

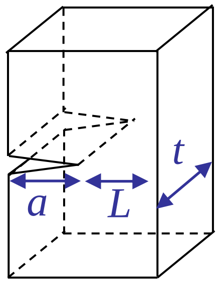
- Validity in elasto-plastic conditions
  - Deformations are small
  - We still have one asymptotic solution valid
    - HRR field is valid in the process zone
  - LEFM is **NOT** valid in the elastic zone close to the crack





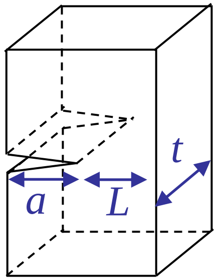
  - Conditions
    - This is the case if all sizes are 25 times larger than CTOD

$$a,\ t,\ L > 25\delta_t \simeq 25\frac{J}{\sigma_p^0}$$



- Validity in elasto-plastic conditions (2)
  - Crack initiation criteria
    - Criteria based on J or δ_t are valid: $J \geq J_C$ or $\delta_t \geq \delta_C$
      - J & δ_t depend on a, the geometry, the loading, …
    - The LEFM solution **DOES NOT** hold, we **CANNOT** use $K(a) \geq K_C$
  - Sizes for J–based toughness test

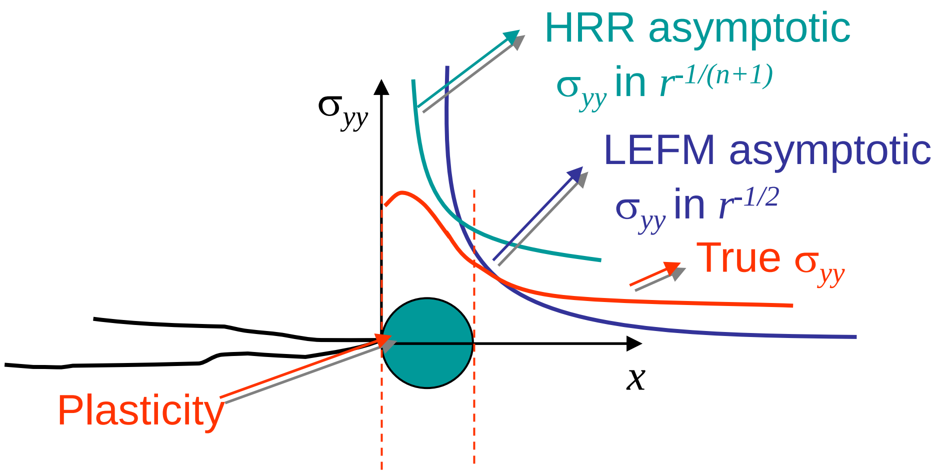
$$a,\ t,\ L > 25\frac{J_C}{\sigma_p^0}$$

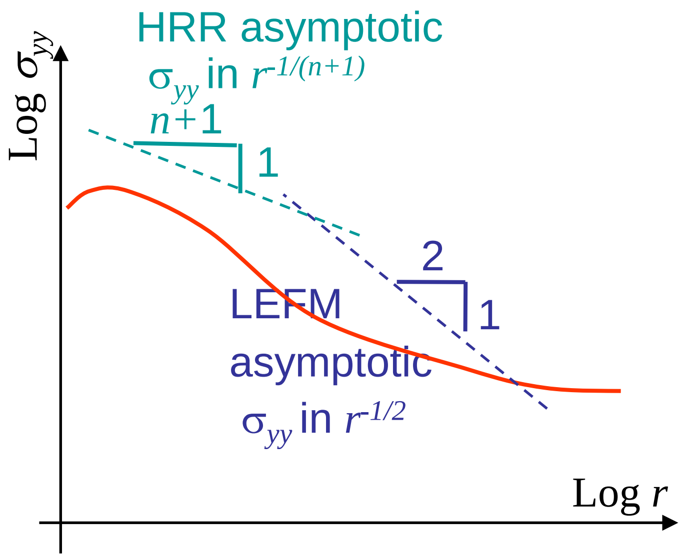


- If K_C is computed from J_C: $a,\ t,\ L > 25\frac{K_C^2\left(1-\nu^2\right)}{E\sigma_p^0}$
- Examples:
  - Titanium alloy 6%Al-4%V
    - » Yield: 830 MPa
    - » Toughness: 55 $\text{MPa} \cdot \text{m}^{1/2}$
    - » Young: 110 GPa
    - » $a, t, L > 0.75$ mm
  - Strength steel
    - » Yield: 350 MPa
    - » Toughness: 250 $\text{MPa} \cdot \text{m}^{1/2}$
    - » Young: 210 GPa
    - » $a, t, L > 1.93$ cm

Validity of HRR field

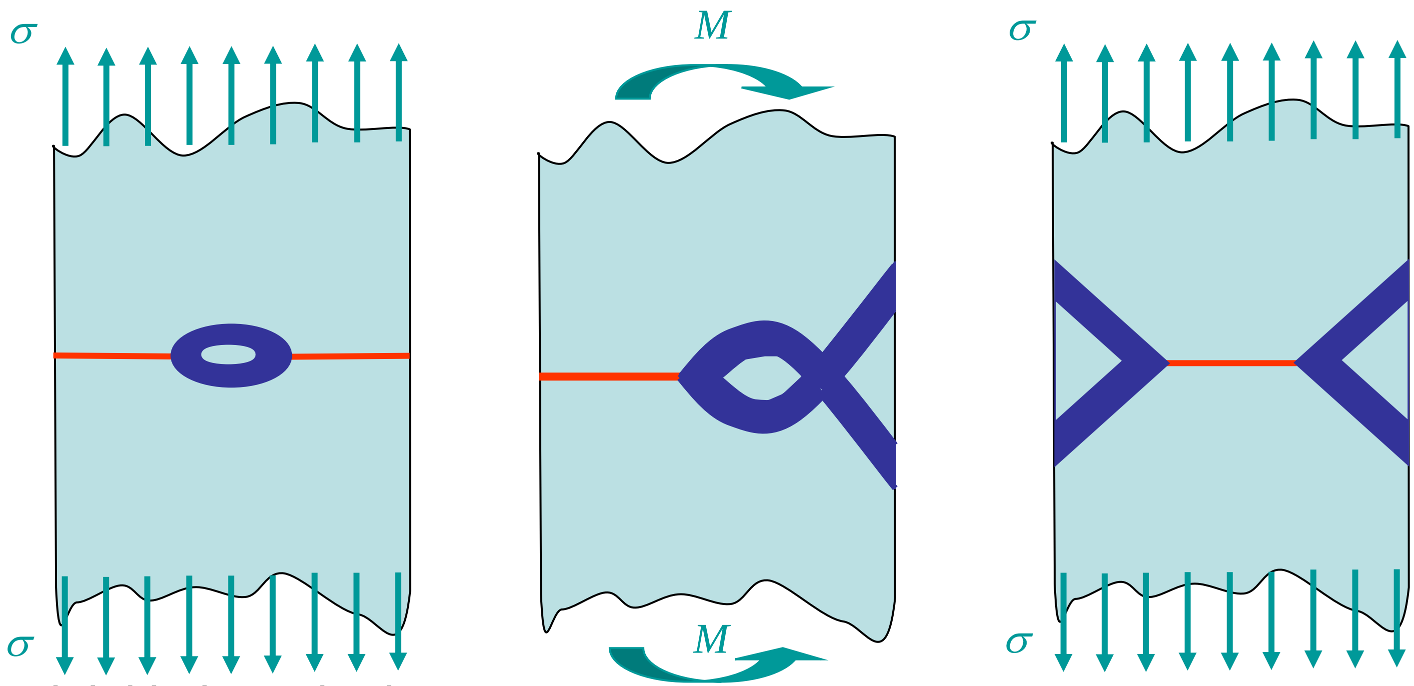
- Validity in large yielding
  - Example: ligament size is too small
  - Small deformations assumption does not hold
  - Neither HRR field nor LEFM asymptotic fields are valid



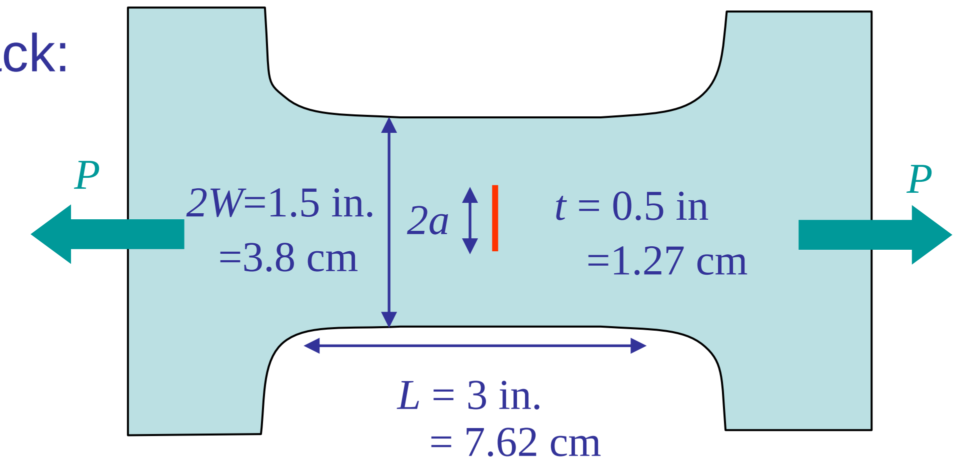


  - Crack initiation criterion?
    - As there is no zone of J-dominance can J still be used?

- Validity in large yielding (2)
  - Plastic strain concentrations depend on the experiment
  - Zones near free boundaries or other cracks tend to be less stressed



  - Crack initiation criterion?
    - Solution is no longer uniquely governed by J
    - Relation between J & δ_t is **dependent on the configuration and on the loading**
    - The critical J_C measured for an experiment might not be valid for another one
    - A 2-parameter characterization is needed



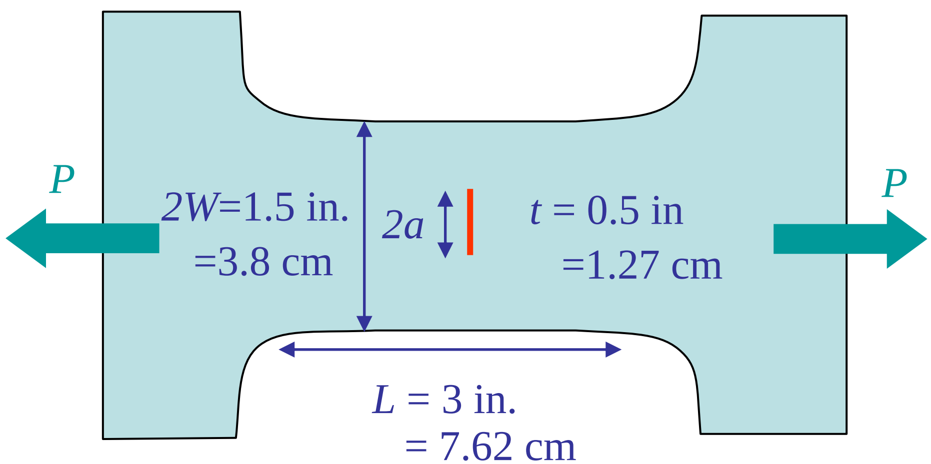
- Thick specimen with centered crack:
  - Steel
    - Yield: 350 MPa
    - Toughness: 250 $\text{MPa} \cdot \text{m}^{1/2}$
    - Young: 210 GPa
    - Hardening exponent >>
  - Loading
    - P = 25, 50 & 150 kN
  - Compute the stress intensity factor in terms of crack size a
    - Compare the solutions obtained by
      - LEFM
      - LEFM with effective crack length
    - Check validity

Exercise 1: Solution

- P= 50 kN
  - Far stress field
    - $$\sigma_\infty = \frac{P}{2Wt} = \frac{50000}{0.038\,0.0127} = 103.6\ \text{MPa}$$
  - Solution for a = 7 mm
    - SIF: $K_I = \sigma_\infty \sqrt{\pi a} f\left(\frac{a}{W}\right)$

      with $f\left(\frac{a}{W}\right) = \left[1 - 0.025\left(\frac{a}{W}\right)^2 + 0.06\left(\frac{a}{W}\right)^4\right]\sqrt{\frac{1}{\cos\frac{\pi a}{2W}}}$

      $\Longrightarrow$ $K_I = 103.6\,10^6\sqrt{\pi 0.007} f(0.3684) = 16.75\ \text{MPa}\sqrt{\text{m}}$



- P= 50 kN (2)
  - Solution for a = 7 mm (2)
    - Effective crack length
      - First iteration

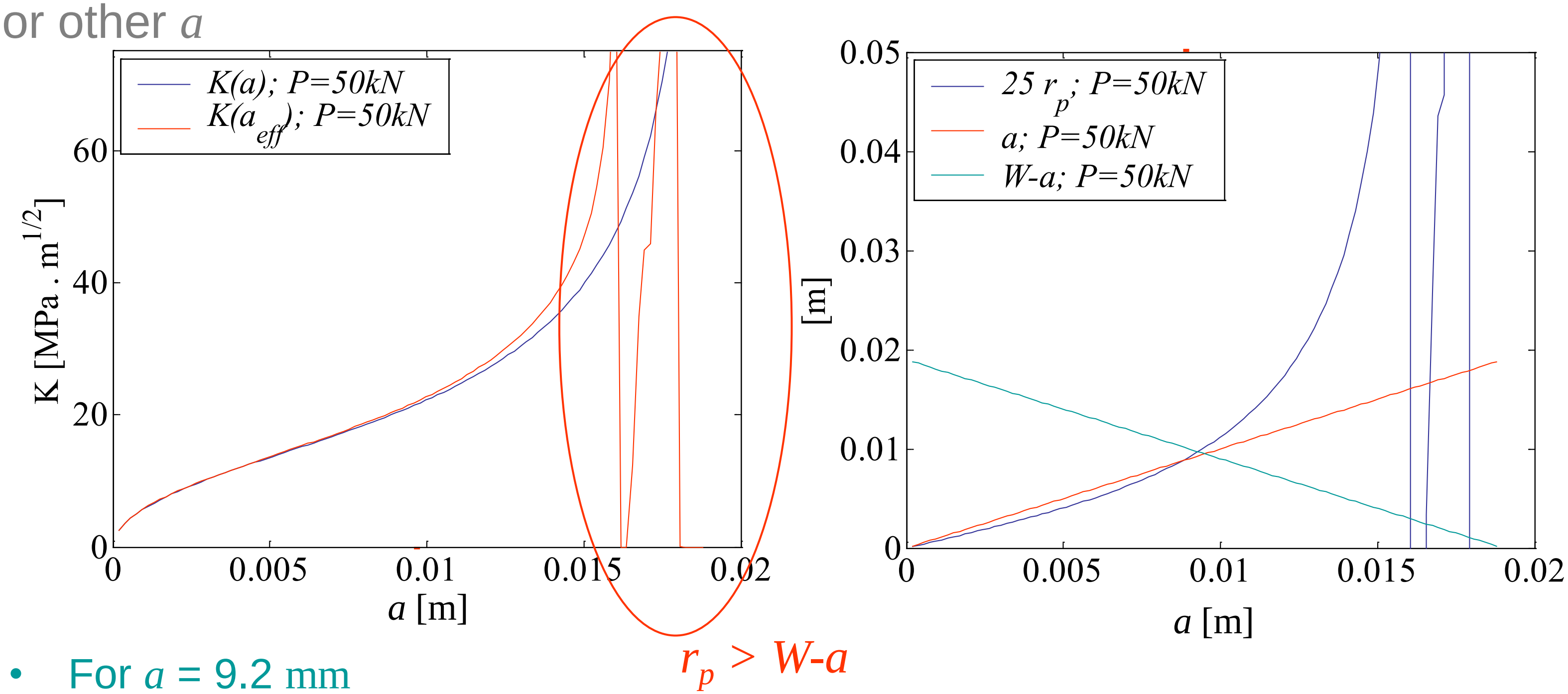
$$\begin{cases} r_p^{(1)} = \frac{1}{3\pi}\left(\frac{K_I}{\sigma_p^0}\right)^2 = \frac{1}{3\pi}\left(\frac{16.75}{350}\right)^2 = 0.24 \text{ mm} \\ a_{\text{eff}}^{(1)} = a + \frac{r_p}{2} = 7.12 \text{ mm} \\ K_I^{(1)} = 103.6\, 10^6 \sqrt{\pi 0.00712} f(0.3747) = 16.95 \text{ MPa}\sqrt{\text{m}} \end{cases}$$

      - Second iteration

$$\begin{cases} r_p^{(2)} = \frac{1}{3\pi}\left(\frac{K_I^{(1)}}{\sigma_p^0}\right)^2 = \frac{1}{3\pi}\left(\frac{16.95}{350}\right)^2 = 0.249 \text{ mm} \\ a_{\text{eff}}^{(2)} = a + \frac{r_p}{2} = 7.1245 \text{ mm} \\ K_I^{(2)} = 103.6\, 10^6 \sqrt{\pi 0.0071245} f(0.3750) = 16.96 \text{ MPa}\sqrt{\text{m}} \end{cases}$$

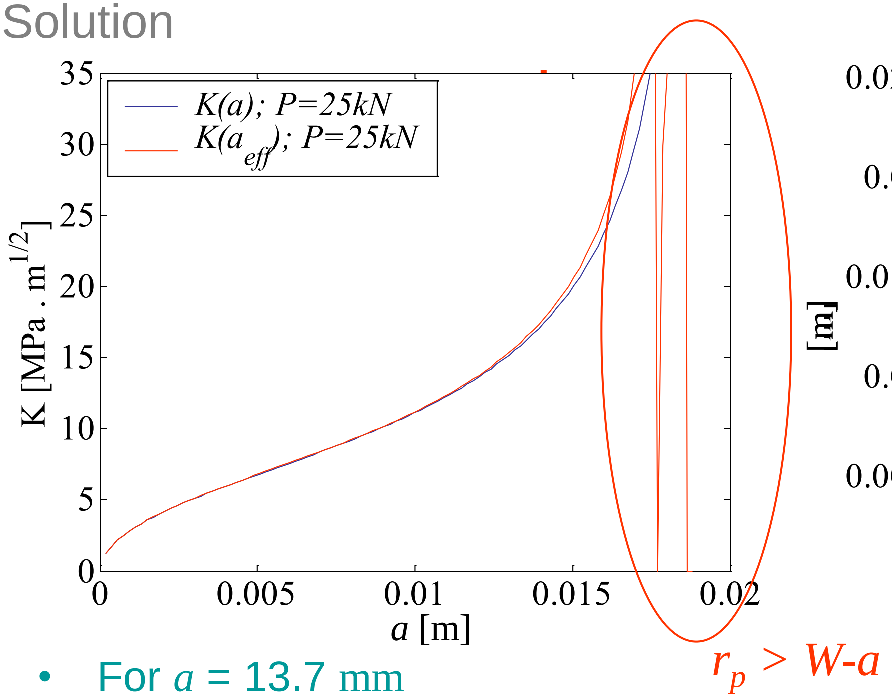
      - It has converged
    - The correction is of about 1%
    - Validity:
      - $\sigma_\infty < 0.5\, \sigma_p^0$: OK
      - a = 7 mm, t = 12.7 mm, & W-a = 12 mm > 25 r_p = 6.2 mm: OK

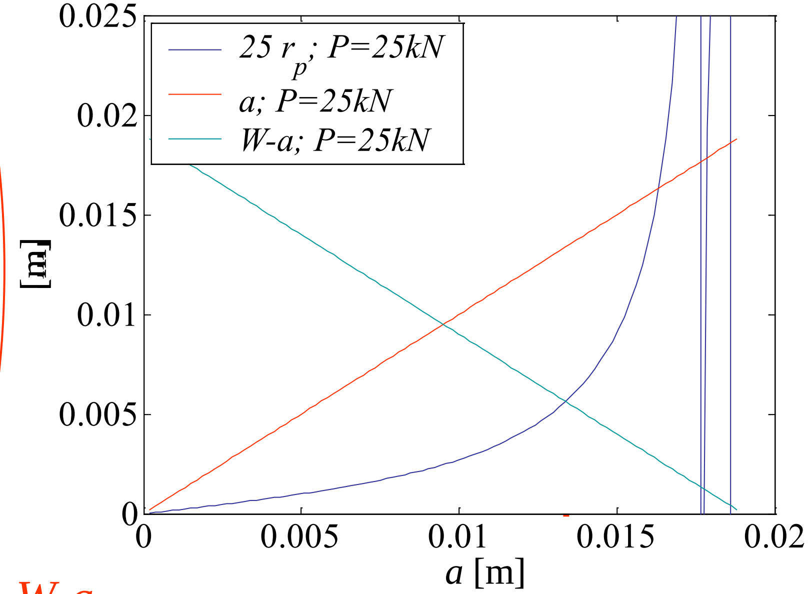
- P= 50 kN (3)
  - For other a



- For a = 9.2 mm
  - r_p = 0.383 mm $\Rightarrow$ 25 r_p = 9.6 mm > a : solution not valid anymore

- P= 25 kN
  - Solution

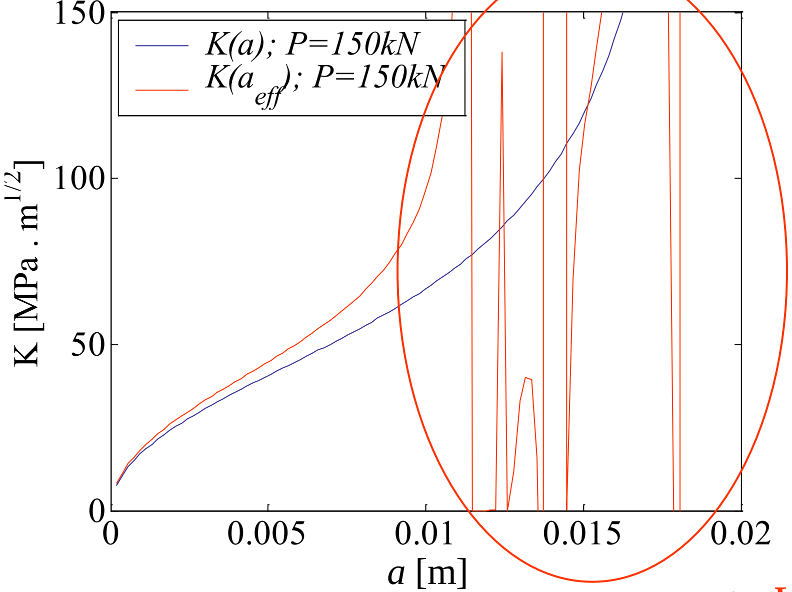


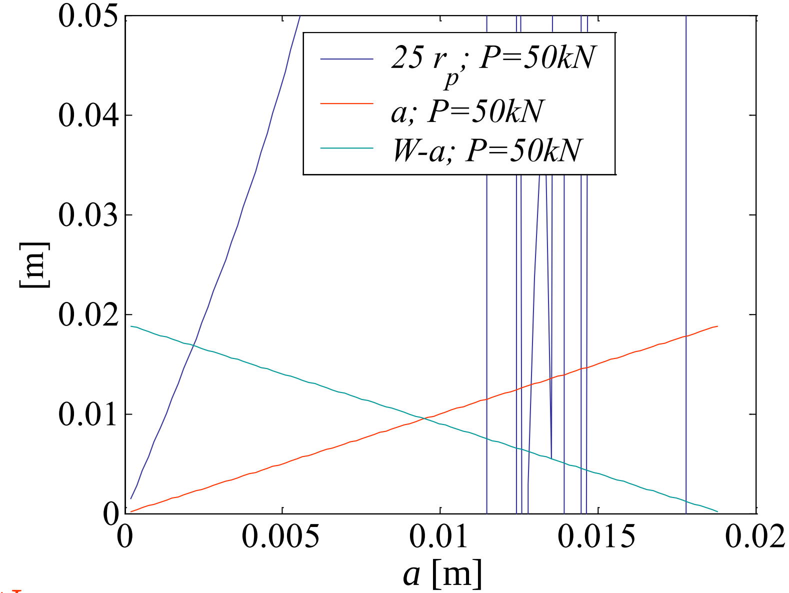


- For a = 13.7 mm
  - r_p = 0.235 mm ⟹ 25 r_p = 5.9 mm > W- a = 5.3 mm: solution not valid anymore

Exercise 1: Solution

- P= 150 kN
  - Solution





$r_p > W\text{-}a$

- For this loading
  - 25 r_p > a always
  - As σ_∞= 0.88 σ_p^0 SSY theory does not hold, the size of the plastic zone cannot be approximated by $r_p = \frac{1}{3\pi}\left(\frac{K_I}{\sigma_p^0}\right)^2$
- We will use the HRR theory, but we need to evaluate the J-integral

References

- Lecture notes
  - Lecture Notes on Fracture Mechanics, Alan T. Zehnder, Cornell University, Ithaca, http://hdl.handle.net/1813/3075
- Other references
  - « on-line »
    - Fracture Mechanics, Piet Schreurs, TUe, http://www.mate.tue.nl/~piet/edu/frm/sht/bmsht.html
  - Book
    - Fracture Mechanics: Fundamentals and applications, D. T. Anderson. CRC press, 1991.
    - Fatigue of Materials, S. Suresh, Cambridge press, 1998.

- Resolution for a semi-infinite crack in plane ε state (A1)
  - Power law $\bar{\varepsilon} = \frac{\alpha \sigma_p^0}{E} \left(\frac{\sigma_e}{\sigma_p^0}\right)^n$ with $\sigma_e = \sqrt{\frac{3}{2}\mathbf{s}:\mathbf{s}}$
    - Deviatoric tensor, plane ε & incompressible material:
      - As $\boldsymbol{\varepsilon} = \bar{\varepsilon}\sqrt{\frac{3}{2}}\frac{\mathbf{s}}{\sqrt{\mathbf{s}:\mathbf{s}}} = \frac{3\bar{\varepsilon}}{2\sigma_e}\mathbf{s} \Longrightarrow \mathbf{s}_{zz} = 0$ or $\boldsymbol{\sigma}_{zz} - \frac{\boldsymbol{\sigma}_{rr} + \boldsymbol{\sigma}_{\theta\theta} + \boldsymbol{\sigma}_{zz}}{3} = 0$
      - Out of plane stress $\boldsymbol{\sigma}_{zz} = \frac{\boldsymbol{\sigma}_{rr} + \boldsymbol{\sigma}_{\theta\theta}}{2}$ & $\mathrm{tr}\boldsymbol{\sigma} = \frac{3}{2}\left(\boldsymbol{\sigma}_{rr} + \boldsymbol{\sigma}_{\theta\theta}\right)$
      - Non zero components of deviatoric tensor: $\mathbf{s} = \begin{pmatrix} \frac{\boldsymbol{\sigma}_{rr} - \boldsymbol{\sigma}_{\theta\theta}}{2} & \boldsymbol{\sigma}_{r\theta} \\ \boldsymbol{\sigma}_{r\theta} & \frac{\boldsymbol{\sigma}_{\theta\theta} - \boldsymbol{\sigma}_{rr}}{2} \end{pmatrix}$
    - Equivalent von Mises stress $\sigma_e = \sqrt{\frac{3}{2}}\sqrt{\mathbf{s}:\mathbf{s}} = \sqrt{\frac{3}{4}\left(\boldsymbol{\sigma}_{rr} - \boldsymbol{\sigma}_{\theta\theta}\right)^2 + 3\boldsymbol{\sigma}_{r\theta}^2}$
      - In terms of r and f

$$\Longrightarrow \sigma_e^2 = r^{\frac{-2}{n+1}} \left[\frac{3}{4}\left(\frac{1}{n}f + \frac{(n+1)^2}{n(2n+1)}f''\right)^2 + 3\left(\frac{n+1}{2n+1}\right)^2 f'^2\right]$$

- Resolution for a semi-infinite crack in plane ε state (A2)
  - Power law (2)
    - Equivalent deformation in terms of r and g

$$\bar{\varepsilon}^2 = \frac{2}{3}\left(\boldsymbol{\varepsilon}_{rr}^2 + \boldsymbol{\varepsilon}_{\theta\theta}^2 + 2\boldsymbol{\varepsilon}_{r\theta}^2\right) = \frac{2}{3} r^{\frac{-2n}{n+1}} \left[\frac{2}{(1+n)^2} g'^2 + \frac{1}{2}\left(g'' + n\frac{n+2}{(n+1)^2} g \right)^2 \right]$$

- So the power law $\bar{\varepsilon}^2 = \left(\frac{\alpha\sigma_p^0}{E}\right)^2 \left(\frac{\sigma_e}{\sigma_p^0}\right)^{2n}$ becomes

$$\left[\frac{4}{3(1+n)^2} g'^2 + \frac{1}{3}\left(g'' + n\frac{n+2}{(n+1)^2} g \right)^2 \right] =$$

$$\left(\frac{\alpha\sigma_p^0}{E\left(\sigma_p^0\right)^n}\right)^2 \left[\frac{3}{4}\left(\frac{1}{n} f + \frac{(n+1)^2}{n(2n+1)} f'' \right)^2 + 3\left(\frac{n+1}{2n+1}\right)^2 f'^2 \right]^n$$

- We have a differential equation in terms of functions depending on θ & n
- But we are still missing a relation between the functions f & g

$\Longrightarrow$ We will now study the normality relation $\boldsymbol{\varepsilon} = \bar{\varepsilon}\sqrt{\frac{3}{2}}\frac{\mathbf{s}}{\sqrt{\mathbf{s}:\mathbf{s}}} = \frac{3\bar{\varepsilon}}{2\sigma_e}\mathbf{s}$

Annex 1: Mode I crack and HRR theory

- Resolution for a semi-infinite crack in plane ε state (A3)
  - Normality $\boldsymbol{\varepsilon} = \bar{\varepsilon}\sqrt{\frac{3}{2}}\frac{\mathbf{s}}{\sqrt{\mathbf{s}:\mathbf{s}}} = \frac{3\bar{\varepsilon}}{2\sigma_e}\mathbf{s} = \frac{3\bar{\varepsilon}}{2\sigma_e}\begin{pmatrix} \frac{\boldsymbol{\sigma}_{rr}-\boldsymbol{\sigma}_{\theta\theta}}{2} & \boldsymbol{\sigma}_{r\theta} \\ \boldsymbol{\sigma}_{r\theta} & \frac{\boldsymbol{\sigma}_{\theta\theta}-\boldsymbol{\sigma}_{rr}}{2} \end{pmatrix}$
    - Strain-stress relations $\boldsymbol{\varepsilon}_{r\theta} = \frac{3\bar{\varepsilon}}{2\sigma_e}\boldsymbol{\sigma}_{r\theta}$ & $\boldsymbol{\varepsilon}_{rr} - \boldsymbol{\varepsilon}_{\theta\theta} = \frac{3\bar{\varepsilon}}{2\sigma_e}\left(\boldsymbol{\sigma}_{rr} - \boldsymbol{\sigma}_{\theta\theta}\right)$

      imply $\left(\boldsymbol{\varepsilon}_{rr} - \boldsymbol{\varepsilon}_{\theta\theta}\right)\boldsymbol{\sigma}_{r\theta} = \left(\boldsymbol{\sigma}_{rr} - \boldsymbol{\sigma}_{\theta\theta}\right)\boldsymbol{\varepsilon}_{r\theta}$
    - In terms of f & g: $\frac{2}{2n+1}g'f' = -\frac{1}{2}\left[\frac{1}{n}f + \frac{(n+1)^2}{n(2n+1)}f''\right]\left[g'' + n\frac{n+2}{(n+1)^2}g\right]$
  - Back to HRR field
    - Near the crack tip, the fields were normalized by introducing an intensity factor

      $$\boldsymbol{\sigma}(r,\,\theta) \rightarrow \sigma_p^0 k_n \frac{\tilde{\boldsymbol{\sigma}}(\theta,\,n)}{r^{\frac{1}{n+1}}} \quad \& \quad \boldsymbol{\varepsilon} \rightarrow \frac{\alpha\sigma_p^0}{E}\frac{k_n^n}{r^{\frac{n}{n+1}}}\tilde{\boldsymbol{\varepsilon}}(\theta,\,n)$$
    - Therefore, the functions f and g are rewritten

      $$f(\theta) = \sigma_p^0 k_n \tilde{f}(\theta) \quad \& \quad g(\theta) = \frac{\alpha\sigma_p^0}{E}k_n^n\tilde{g}(\theta)$$
    - Which allows writing the differential equations in terms of $\tilde{f}$ & $\tilde{g}$

- The two differential equations are
  - $$\left[\tilde{f}'' + \frac{2n+1}{(n+1)^2}\tilde{f}\right]\left[\tilde{g}'' + n\frac{n+2}{(n+1)^2}\tilde{g}\right] = -\frac{4n}{(n+1)^2}\tilde{g}'\tilde{f}'$$ &
  - $$\left[\frac{4}{(1+n)^2}\tilde{g}'^2 + \left(\tilde{g}'' + n\frac{n+2}{(n+1)^2}\tilde{g}\right)^2\right]^{\frac{1}{n}} =$$
    $$\frac{3^{\frac{n+1}{n}}}{4}\frac{(n+1)^4}{n^2(2n+1)^2}\left[\left(\frac{2n+1}{(n+1)^2}\tilde{f} + \tilde{f}''\right)^2 + \frac{4n^2}{(n+1)^2}\tilde{f}'^2\right]$$
  - Let $\tilde{l} = \frac{1}{n}\left[(n+1)\tilde{f}'' + \frac{2n+1}{n+1}\tilde{f}\right]$ then, these equations become
    - $$\left[\frac{4}{(1+n)^2}\tilde{g}'^2 + \left(\tilde{g}'' + n\frac{n+2}{(n+1)^2}\tilde{g}\right)^2\right]^{\frac{1}{n}} = \frac{3^{\frac{n+1}{n}}}{4}\frac{(n+1)^2}{(2n+1)^2}\left[\tilde{l}^2 + 4\tilde{f}'^2\right]$$
    - $$\tilde{l}\left[\tilde{g}'' + n\frac{n+2}{(n+1)^2}\tilde{g}\right] = -\frac{4}{(n+1)}\tilde{g}'\tilde{f}'$$

- Theses equations can be reduced to a single ODE as

$$-\left[\frac{4}{(1+n)^2}+\frac{16}{(n+1)^2\,\tilde{l}^2}\tilde{f}'^2\right]^{\frac{1}{n}}\tilde{g}'^{\frac{2}{n}}=\frac{3^{\frac{n+1}{n}}}{4}\frac{(n+1)^2}{(2n+1)^2}\left[\tilde{l}^2+4\tilde{f}'^2\right]$$

$$\Rightarrow \tilde{g}'=\sqrt{\left[\frac{\frac{3^{n+1}}{4^n}\frac{(n+1)^{2n}}{(2n+1)^{2n}}\left[\tilde{l}^2+4\tilde{f}'^2\right]^n}{\left[\frac{4}{(1+n)^2}+\frac{16}{(n+1)^2\tilde{l}^2}\tilde{f}'^2\right]}\right]}$$

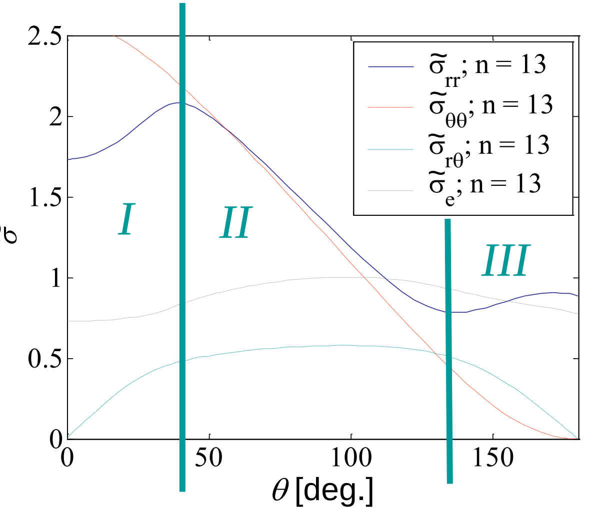
– After differentiating $\tilde{l}\left[\tilde{g}''+n\frac{n+2}{(n+1)^2}\tilde{g}\right]=-\frac{4}{(n+1)}\tilde{g}'\tilde{f}'$ and substituting $\tilde{g}'$ and its derivatives, it yields the fourth order ODE

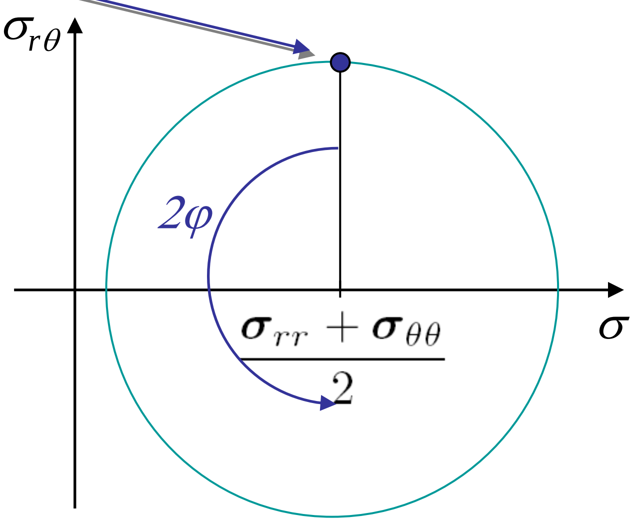
$$\tilde{f}''''+\frac{2n+1}{(1+n)^2}\tilde{f}''+\frac{1}{n\tilde{l}^2+4\tilde{f}'^2}\left\{\frac{n^2}{(1+n)^2}\left[\frac{4}{n}\tilde{f}''+\frac{n+2}{1+n}\tilde{l}\right]\left[\tilde{l}^2+4\tilde{f}'^2\right]+\right.$$

$$\frac{4n(n-1)}{(n+1)^2}\tilde{f}'\left[\tilde{l}\tilde{l}'+4\tilde{f}'\tilde{f}''\right]+\frac{n(n-1)}{n+1}\left[3\tilde{l}\tilde{l}'^2+8\tilde{l}'\tilde{f}'\tilde{f}''+4\tilde{l}\left(\tilde{f}''^2+\tilde{f}'\tilde{f}'''\right)\right]$$

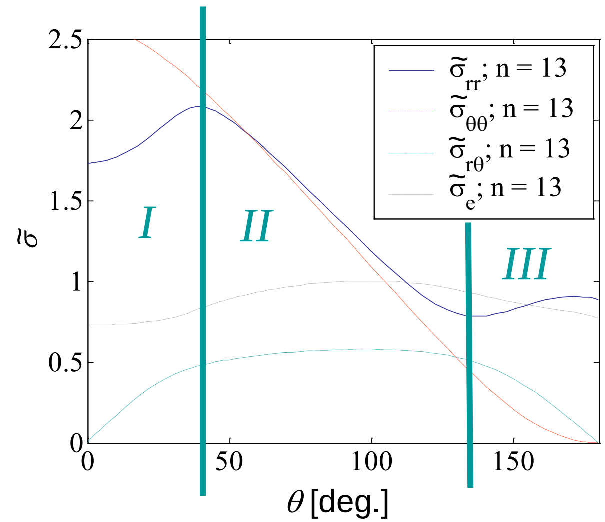
$$\left.+\frac{n(n-1)(n-3)}{n+1}\frac{\tilde{l}}{\tilde{l}^2+4\tilde{f}'^2}\left[\tilde{l}\tilde{l}'+4\tilde{f}'\tilde{f}''\right]^2\right\}$$

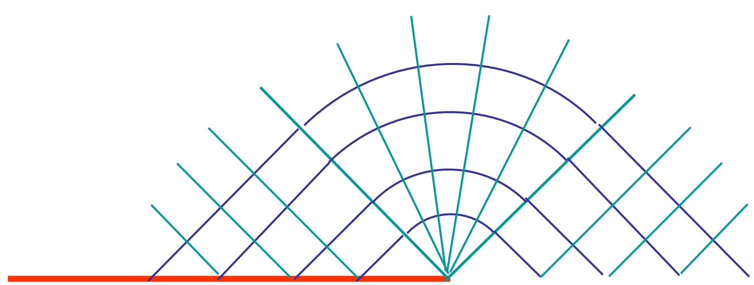
- Semi-infinite mode I crack in plane ε state
  - Von Mises stress $\sigma_e = \sqrt{\frac{3}{2}}\sqrt{\mathbf{s}:\mathbf{s}} = \sqrt{\frac{3}{4}\left(\boldsymbol{\sigma}_{rr} - \boldsymbol{\sigma}_{\theta\theta}\right)^2 + 3\boldsymbol{\sigma}_{r\theta}^2}$ (Plane ε)
  - Perfectly plastic material
    - Von Mises stress: $\left(\frac{\sigma_p^0}{\sqrt{3}}\right)^3 = \left(\frac{\boldsymbol{\sigma}_{rr} - \boldsymbol{\sigma}_{\theta\theta}}{2}\right)^2 + \boldsymbol{\sigma}_{r\theta}^2$
    - The Mohr's circle is centered on $\frac{\boldsymbol{\sigma}_{rr} + \boldsymbol{\sigma}_{\theta\theta}}{2}$ with radius $\frac{\sigma_p^0}{\sqrt{3}}$
  - In Region II $\boldsymbol{\sigma}_{rr} = \boldsymbol{\sigma}_{\theta\theta} \Longrightarrow$ in frame
    - e_r, e_θ, we have pure shearing in the directions 0 and φ=π/2
    - e_x, e_y, we have pure shearing in the directions θ and φ=θ+π/2

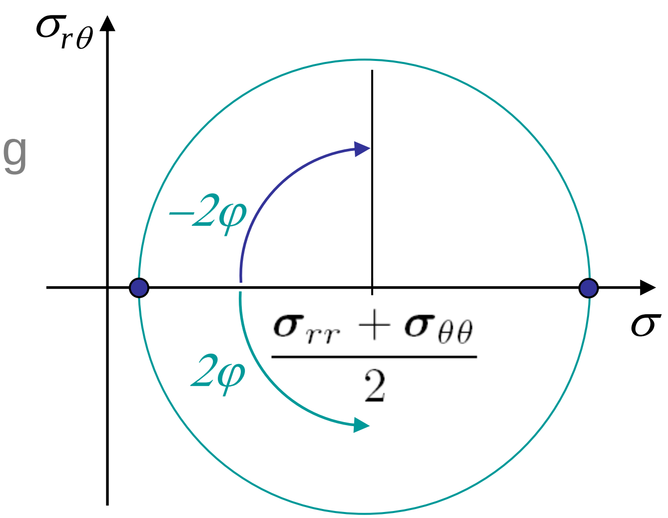




- Semi-infinite mode I crack in plane ε state (2)
  - In Region I for $\theta \to 0$: $\boldsymbol{\sigma}_{r\theta} \to 0$ ⟹ in frame
    - e_r, e_θ, we have pure shearing in the directions $\varphi=\pi/4$ and $\varphi=-\pi/4$
    - e_x, e_y, we have pure shearing in the directions $\varphi=\pi/4$ and $\varphi=-\pi/4$
  - In Region I for $\theta \to \pi$: $\boldsymbol{\sigma}_{r\theta} \to 0$ ⟹ in frame
    - e_r, e_θ, we have pure shearing in the directions $\varphi=\pi/4$ and $\varphi=-\pi/4$
    - e_x, e_y, we have pure shearing in the directions $\varphi=-3\pi/4$ and $\varphi=3\pi/4$
  - Slip directions are the ones of maximal shearing







- Mode I crack in plane σ state
  - Slip lines (perfectly plastic material)