University of Liège
Aerospace & Mechanical Engineering

Fracture Mechanics, Damage and Fatigue
Non Linear Fracture Mechanics – Cohesive Zone Model

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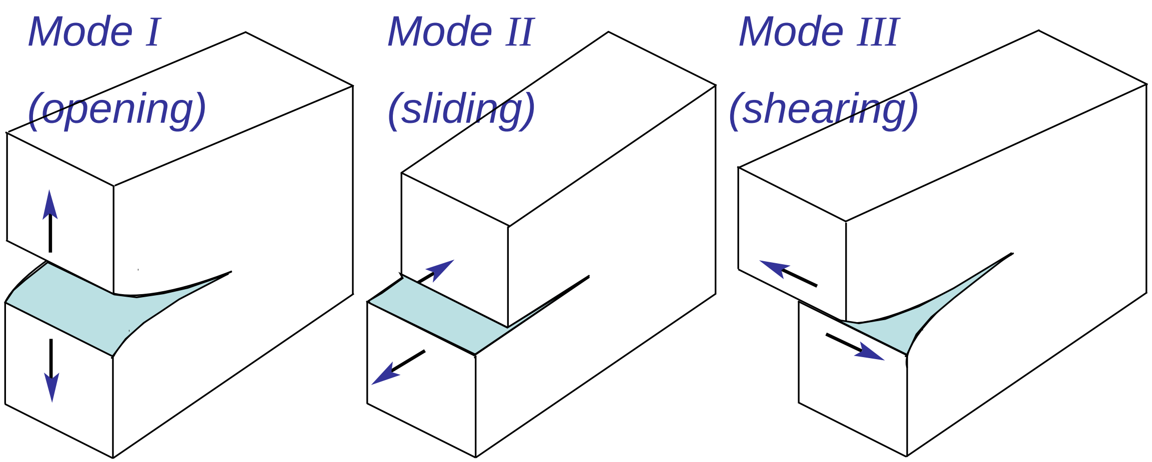
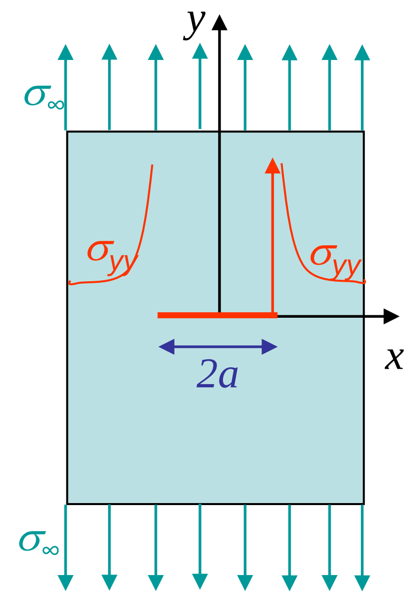
- Cracked body: summary
  - 3 failure modes

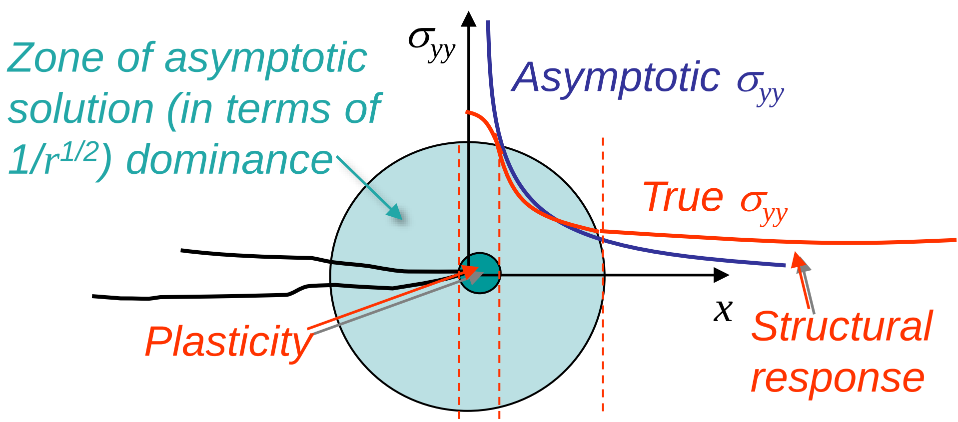




  - Asymptotic solution governed by stress intensity factors

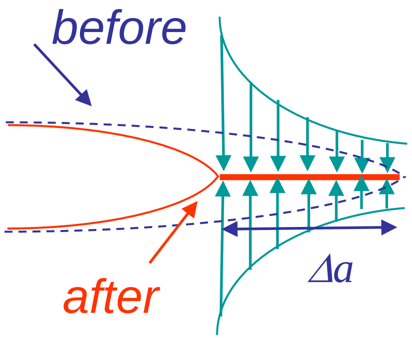
$$\boxed{\boldsymbol{\sigma}^{\text{mode } i} = \frac{K_i}{\sqrt{2\pi r}} f^{\text{mode } i}(\theta) + o(r^0)}$$

$$\boxed{\boldsymbol{u}^{\text{mode } i} = K_i \sqrt{\frac{r}{2\pi}}\, g^{\text{mode } i}(\theta) + o(r^0)}$$



- Cracked body: summary
  - Potential energy $\Pi_T = E_{\text{int}} - Qu$
  - Crack closure integral
    - Energy required to close crack tip

$$\Delta\Pi_T = \int_{\Delta A}\int_{\boldsymbol{u}}^{\boldsymbol{u}+\boldsymbol{\Delta u}} \boldsymbol{t}\cdot[\![\boldsymbol{u}']\!]d\boldsymbol{u}'\ dA$$

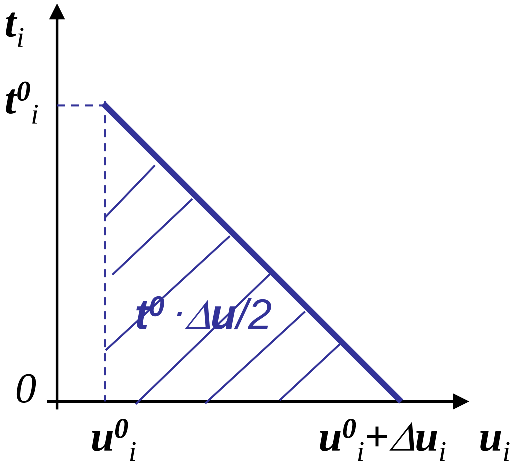


  - Energy release rate
    - Variation of potential energy in case of crack growth

$$G = -\partial_{\text{A}}\left(E_{\text{int}} - W_{\text{ext}}\right) = -\partial_A \Delta\Pi_T$$

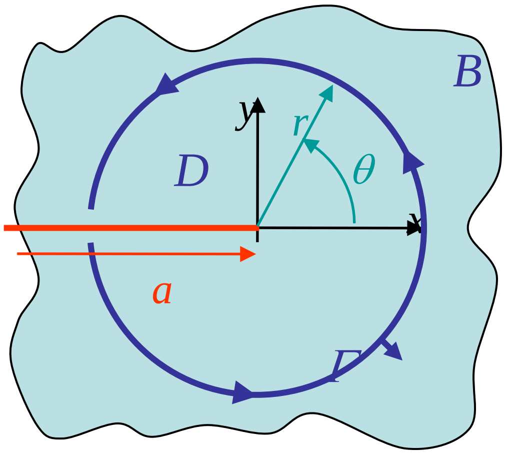
    - In linear elasticity

$$G = -\partial_A \Delta\Pi_T = -\lim_{\Delta A\to 0}\frac{1}{\Delta A}\int_{\Delta A}\frac{1}{2}\boldsymbol{t}^{\mathbf{0}}\cdot[\![\boldsymbol{\Delta u}]\!]\ dA$$



  - In linear elasticity & if crack grows straight ahead

$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \qquad E' = \frac{4E}{(1+\nu)(\kappa+1)} = \begin{cases} E & \text{P}\sigma \\ \frac{E}{1-\nu^2} & \text{P}\varepsilon \end{cases}$$



- Cracked body: summary
  - J-integral
    - Strain energy flow

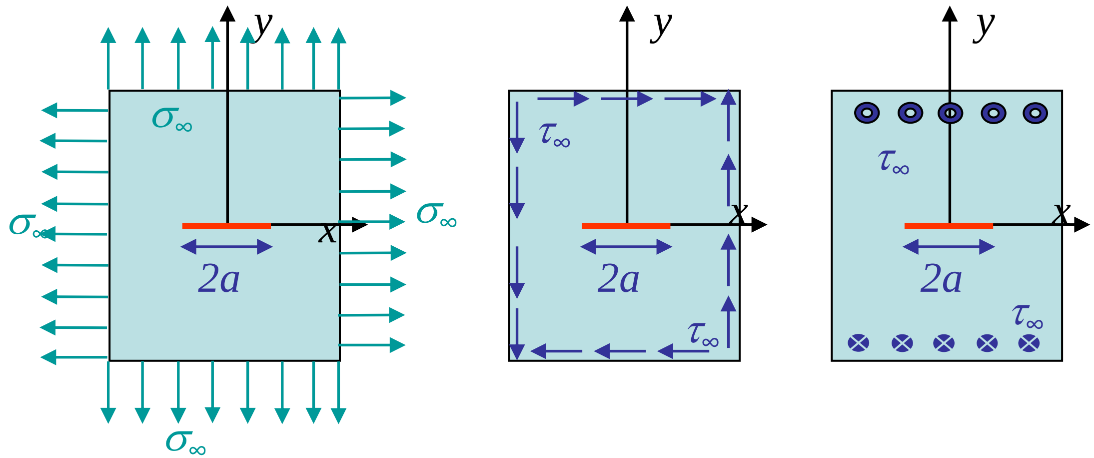
$$J = \int_{\Gamma} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \right] dl$$

  - Exists if an internal potential exists
    - Is path independent if the contour Γ embeds a straight crack tip
    - No assumption on subsequent growth direction
    - Can be extended to plasticity if no unloading (see later)
  - If crack grows straight ahead $\Longrightarrow$ $G=J$
  - In linear elasticity (independently of crack growth direction):

$$J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \qquad E' = \frac{4E}{(1+\nu)(\kappa+1)} = \begin{cases} E & \mathrm{P}\sigma \\ \frac{E}{1-\nu^2} & \mathrm{P}\varepsilon \end{cases}$$

- Analytical
  - SIF from full-field solution
    - Limited cases

$$\Rightarrow \begin{cases} K_I = \sigma_\infty \sqrt{\pi a} \\ K_{II} = \tau_\infty \sqrt{\pi a} \\ K_{III} = \tau_\infty \sqrt{\pi a} \end{cases}$$



  - From energetic consideration
    - Growing straight ahead crack
    - From J-integral

$$G = -\partial_{\mathrm{A}} \left(E_{\mathrm{int}} - W_{\mathrm{ext}}\right) \Rightarrow G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

$$J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

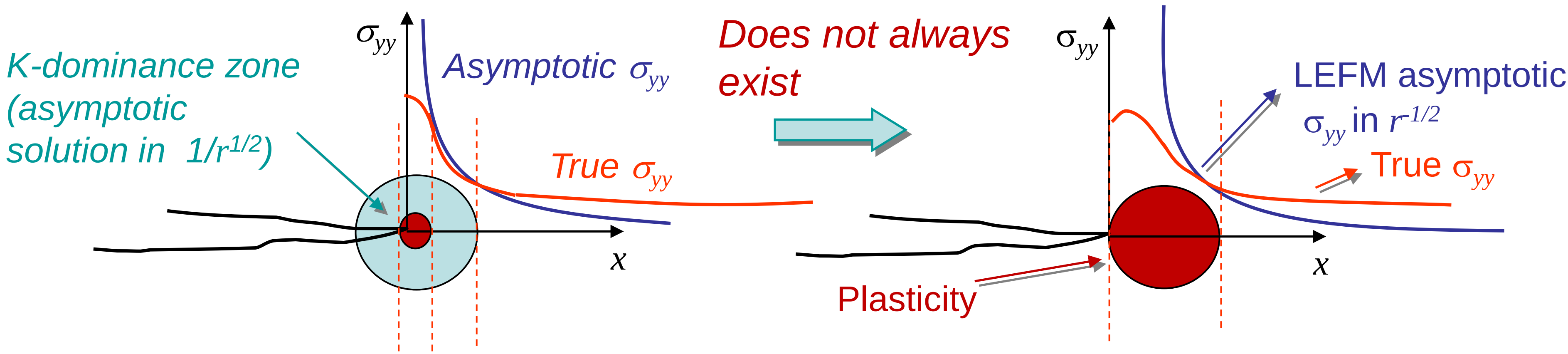
- Numerical (e.g. FEM)

$$\Rightarrow \begin{cases} K_I = \beta_I \sigma_\infty \sqrt{\pi a} \\ K_{II} = \beta_{II} \tau_\infty \sqrt{\pi a} \\ K_{III} = \beta_{III} \tau_\infty \sqrt{\pi a} \end{cases}$$

  - β_i depends on geometry & crack length
- Tabulated solutions (handbooks)
  - http://ebooks.asmedigitalcollection.asme.org/book.aspx?bookid=230

Linear Elastic Fracture Mechanics (LEFM)

- Small Scale Yielding assumption
  - LEFM: we have assumed the existence of a K-dominance zone



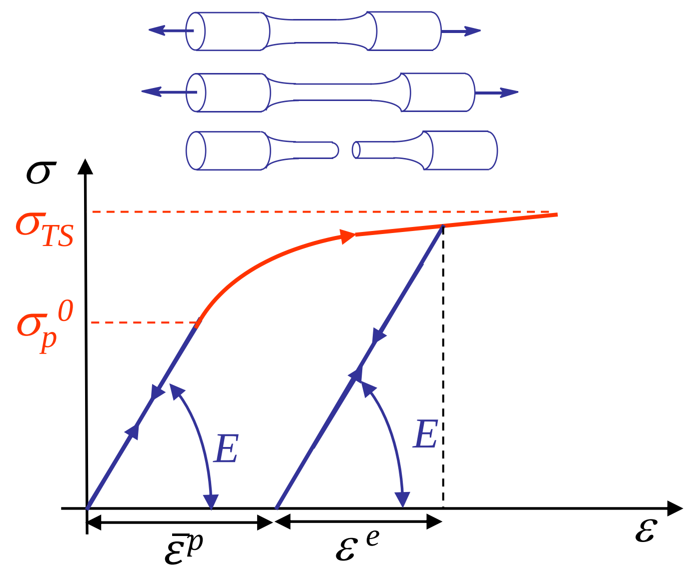
  - This holds if the process zone (in which irreversible process occurs)
    - Is a small region compared to the specimen size &
    - Is localized at the crack tip
  - Validity of this approach?
    - We check the dimensions

$$a, W - a > 2.5 \left(\frac{K_I}{\sigma_p^0}\right)^2$$

∝Process zone size

- Non-linear fracture mechanics
  - Derivation of the LEFM validity criterion
  - Providing solutions when LEFM criterion is not met

- Elasto-plasticity (small deformations)
  - Beyond a threshold the material experiences irreversible deformations
  - Typical behavior at low/room temperature
    - Curves σ-ε independent of time
    - At higher temperature creep …

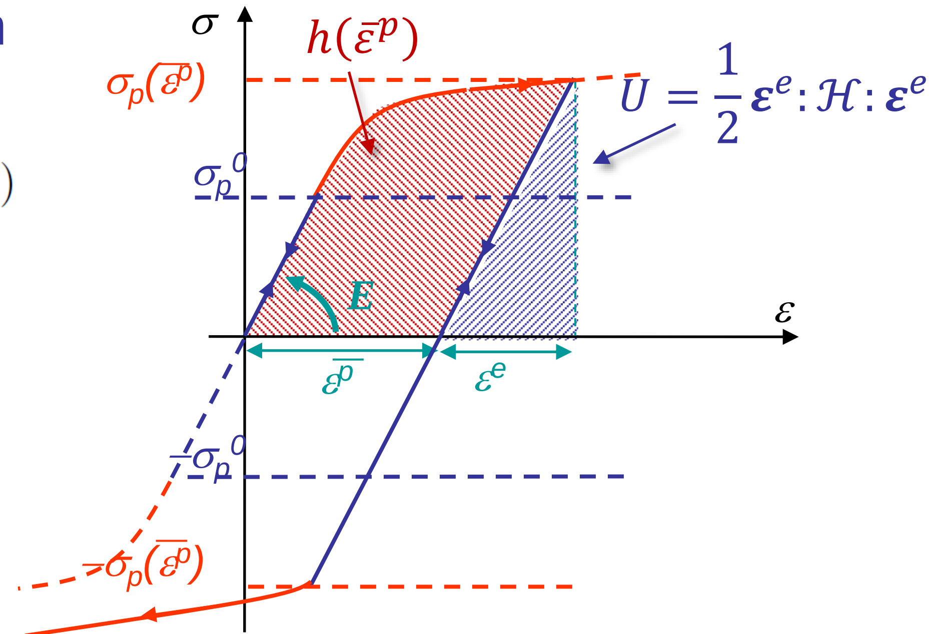


  - Yield surface

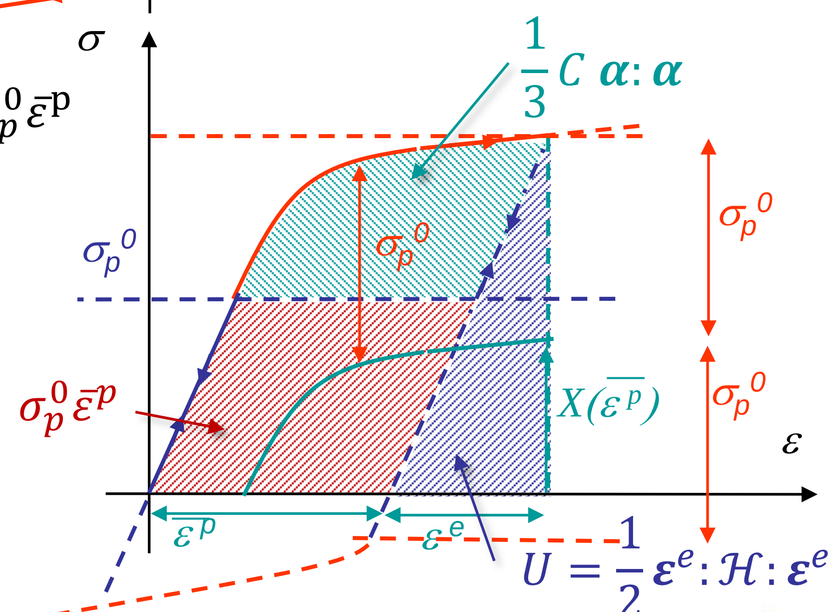
$$f(\boldsymbol{\sigma}) \leq 0 \quad \begin{cases} f < 0\text{: elastic region} \\ f = 0\text{: plasticity} \end{cases}$$

  - Plastic flow
    - Assumption: deformations can be added $d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^{\mathrm{e}} + d\boldsymbol{\varepsilon}^{\mathrm{p}} \Longrightarrow d\boldsymbol{\sigma} = \mathcal{H} : d\boldsymbol{\varepsilon}^{\mathrm{e}}$
    - Normal plastic flow $d\boldsymbol{\varepsilon}^{\mathrm{p}} = d\lambda \dfrac{\partial f}{\partial \boldsymbol{\sigma}}$
  - Path dependency (incremental equations in d)

- Existence of a free energy function
  - Isotropic hardening
    - $\Psi\left(\boldsymbol{\varepsilon}^{e},\, \bar{\varepsilon}^{p}\right) = \frac{1}{2}\boldsymbol{\varepsilon}^{e} : \mathcal{H} : \boldsymbol{\varepsilon}^{e} + h\left(\bar{\varepsilon}^{p}\right)$
    - $\left\{ \begin{array}{l} \boldsymbol{\sigma} = \dfrac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^{e}} \\ \sigma_{p} = \dfrac{\partial \Psi}{\partial \bar{\varepsilon}^{p}} = h' \end{array} \right.$



  - Kinematic hardening
    - $\psi(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}) = \frac{1}{2}\boldsymbol{\varepsilon}^e : \mathcal{H} : \boldsymbol{\varepsilon}^e + \frac{1}{3}C\boldsymbol{\alpha} : \boldsymbol{\alpha} + \sigma_p^0 \bar{\varepsilon}^{\mathrm{p}}$
    - $\left\{ \begin{array}{l} \boldsymbol{\sigma} = \dfrac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^{e}} \\ \sigma_{p} = \dfrac{\partial \Psi}{\partial \bar{\varepsilon}^{p}} = \sigma_p^0 \\ \boldsymbol{X} = \dfrac{\partial \Psi}{\partial \boldsymbol{\alpha}} = \dfrac{2}{3}C\boldsymbol{\alpha} \end{array} \right.$



- Existence of energy release rate?
  - Free energy
    - $\Psi\left(\boldsymbol{\varepsilon}^{e},\, \bar{\varepsilon}^{p}\right) = \frac{1}{2}\boldsymbol{\varepsilon}^{e} : \mathcal{H} : \boldsymbol{\varepsilon}^{e} + h\left(\bar{\varepsilon}^{p}\right)$
    - $\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^{e}}$ & $\sigma_p = \frac{\partial \Psi}{\partial \bar{\varepsilon}^{p}} = h'$
  - But $\boldsymbol{\sigma} = \frac{\partial U}{\partial \boldsymbol{\varepsilon}^{e}} \neq \frac{\partial U}{\partial \boldsymbol{\varepsilon}}$
    - No unique value of $\boldsymbol{\sigma}$ for a given $\boldsymbol{\varepsilon}$
    - Since $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}^{e})$ depends on the history

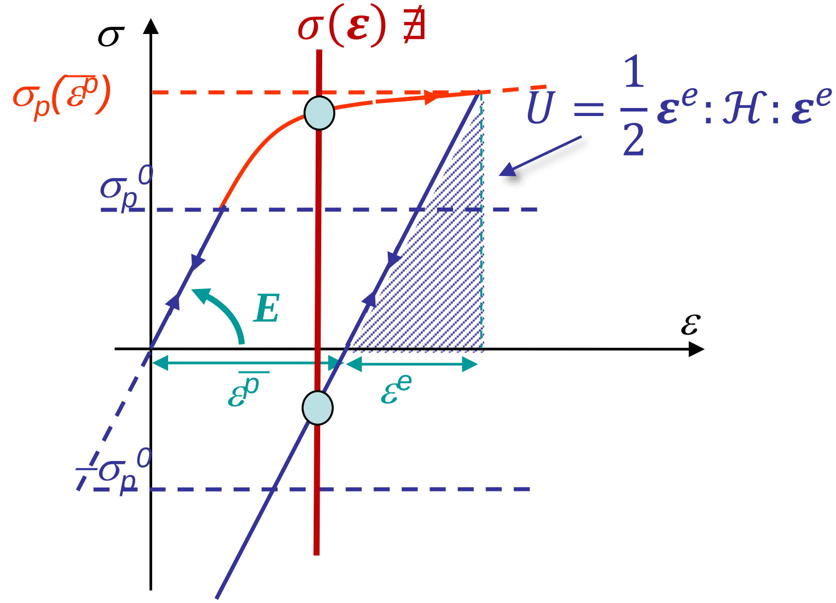
    ⇨ neither G nor J exists
  - We assume no possible unloading

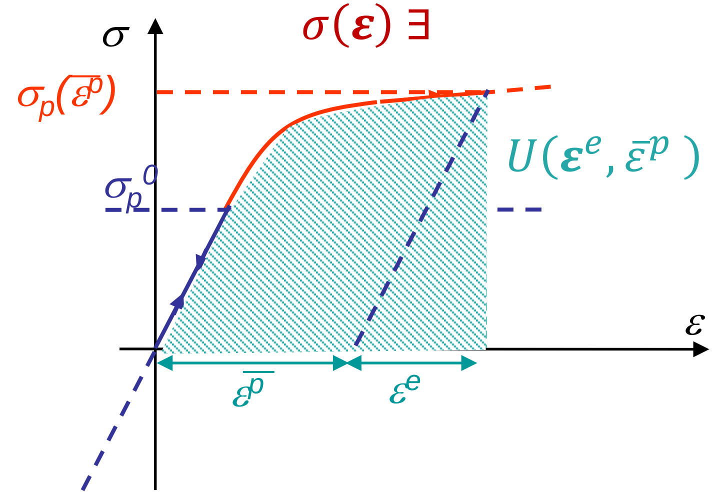
⇨ We recover an energy $U(\boldsymbol{\varepsilon})$ with

$$U(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon}^{e}(\boldsymbol{\varepsilon}) : \mathcal{H} : \boldsymbol{\varepsilon}^{e}(\boldsymbol{\varepsilon}) + h\left(\bar{\varepsilon}^{p}(\boldsymbol{\varepsilon})\right)$$

and with (see next slide)

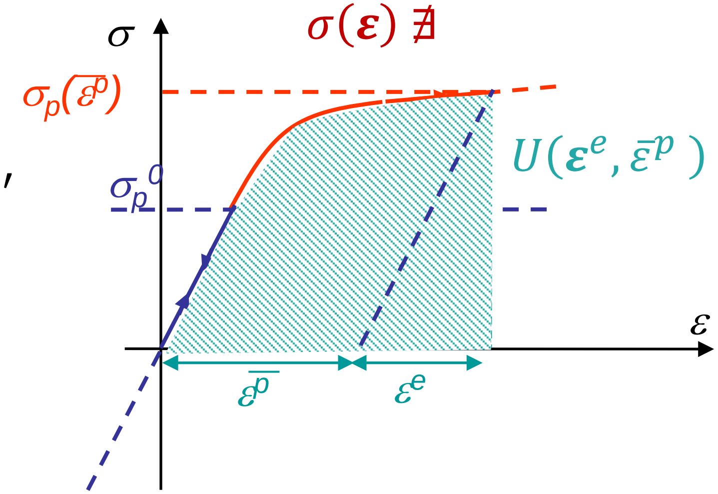
$$\boldsymbol{\sigma} = \frac{\partial U}{\partial \boldsymbol{\varepsilon}}$$





- Assuming no unloading
  - Internal Potential

$$U(\boldsymbol{\varepsilon}) = \int_0^{\boldsymbol{\varepsilon}} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}' = \int_0^{\boldsymbol{\varepsilon}^e} \boldsymbol{\varepsilon}^{e\prime} : \mathcal{H} : d\boldsymbol{\varepsilon}^{e\prime} + \int_0^{\boldsymbol{\varepsilon}^p} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^{p\prime}$$



- One has

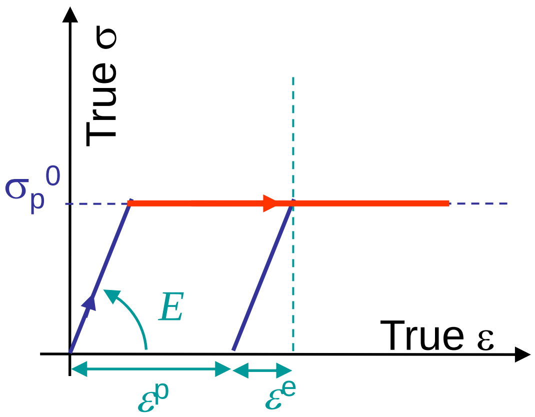
$$\begin{cases} d\boldsymbol{\varepsilon}^{\mathrm{p}} = d\lambda \dfrac{\partial f}{\partial \boldsymbol{\sigma}} \\ d\bar{\varepsilon}^{\mathrm{p}} = \sqrt{\dfrac{2}{3} d\boldsymbol{\varepsilon}^{\mathrm{p}} : d\boldsymbol{\varepsilon}^{\mathrm{p}}} \\ \dfrac{\partial f}{\partial \boldsymbol{\sigma}} = \sqrt{\dfrac{3}{2}} \dfrac{\mathbf{s}}{\sqrt{\mathbf{s} : \mathbf{s}}} \\ \mathbf{s} = \boldsymbol{\sigma} - \dfrac{\mathrm{tr}(\boldsymbol{\sigma})}{3} \mathbf{I} \end{cases} \Longrightarrow \begin{cases} d\lambda = d\bar{\varepsilon}^p \\ d\boldsymbol{\varepsilon}^{\mathrm{p}} = d\bar{\varepsilon}^{\mathrm{p}} \sqrt{\dfrac{3}{2}} \dfrac{\mathbf{s}}{\sqrt{\mathbf{s} : \mathbf{s}}} \end{cases}$$

- So $\displaystyle \int_0^{\boldsymbol{\varepsilon}^p} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^{p\prime} = \int_0^{\boldsymbol{\varepsilon}^p} \boldsymbol{s} : d\boldsymbol{\varepsilon}^{p\prime} = \int_0^{\boldsymbol{\varepsilon}^p} \frac{\sqrt{\frac{3}{2}}\, \boldsymbol{s} : \boldsymbol{s}}{\sqrt{\boldsymbol{s} : \boldsymbol{s}}} d\bar{\varepsilon}^p = \int_0^{\bar{\varepsilon}^p} \sigma_p d\bar{\varepsilon}^p = h(\bar{\varepsilon}^p)$

$$\Longrightarrow U(\boldsymbol{\varepsilon}) = \frac{1}{2} \boldsymbol{\varepsilon}^e(\boldsymbol{\varepsilon}) : \mathcal{H} : \boldsymbol{\varepsilon}^e(\boldsymbol{\varepsilon}) + h\left(\bar{\varepsilon}^p(\boldsymbol{\varepsilon})\right) \quad \& \quad \boldsymbol{\sigma} = \frac{\partial U}{\partial \boldsymbol{\varepsilon}}$$

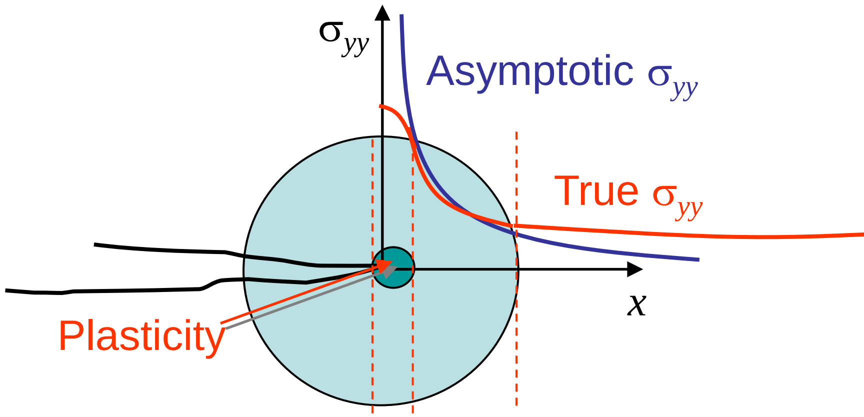
- J2-plasticity without unloading
  - Internal energy
    - $U(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon}^e(\boldsymbol{\varepsilon}) : \mathcal{H} : \boldsymbol{\varepsilon}^e(\boldsymbol{\varepsilon}) + h\left(\bar{\varepsilon}^p(\boldsymbol{\varepsilon})\right)$
    - With $h\left(\bar{\varepsilon}^p(\boldsymbol{\varepsilon})\right) = \int_{\bar{\varepsilon}^p} \sigma_p d\bar{\varepsilon}^p$
  - Example: Perfect plasticity
    - We have $\sigma_p(\bar{\varepsilon}^p) = \sigma_p^0$

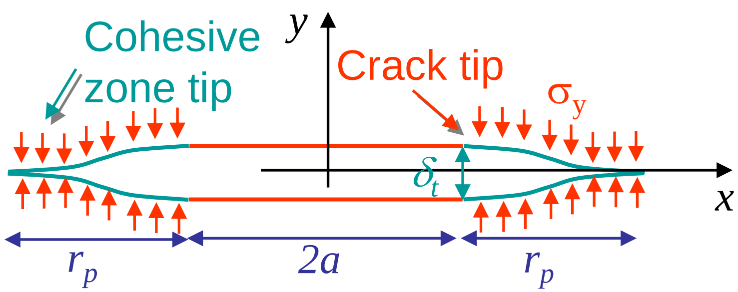
$$\Longrightarrow U = \frac{\boldsymbol{\varepsilon}^e : \mathcal{H} : \boldsymbol{\varepsilon}^e}{2} + \sigma_p^0 \bar{\varepsilon}^p$$

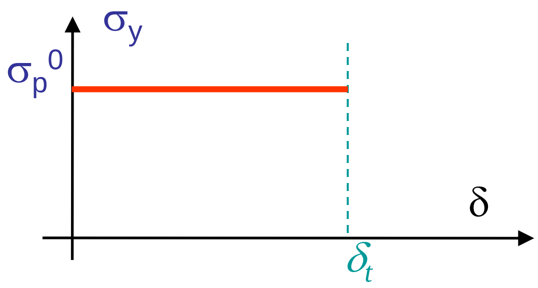


Dugdale (1960) & Barenblatt (1962)'s cohesive model

- Dugdale cohesive zone/Yielding strip
  - Goal
    - Remove the singularity
    - Introduce plasticity at crack tip
  - Assumptions
    - Nonlinearities localized at crack tip
    - Elastic perfectly plastic material ($h^p=0$)
  - Model

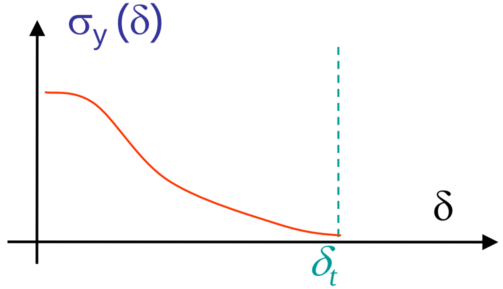






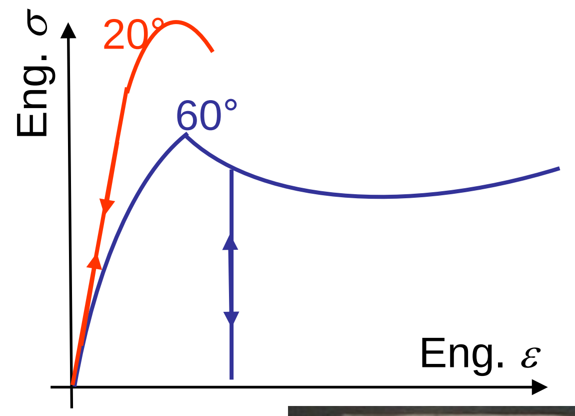
    - Yield prevents stress singularity

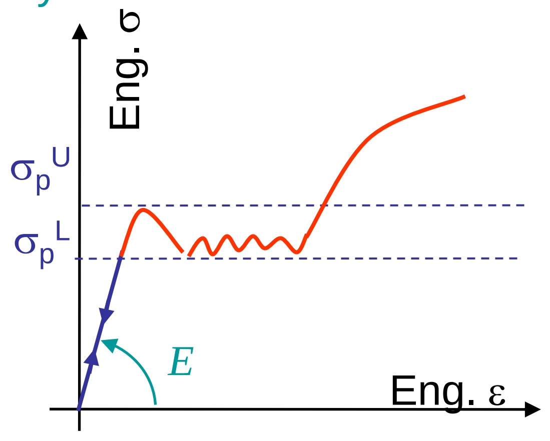
- Barenblatt cohesive model
  - Also avoids singularity at crack tip
  - But this is atomic or molecular attraction that prevents separation
  - Fracture response is uniquely governed by the shape of the decreasing curve

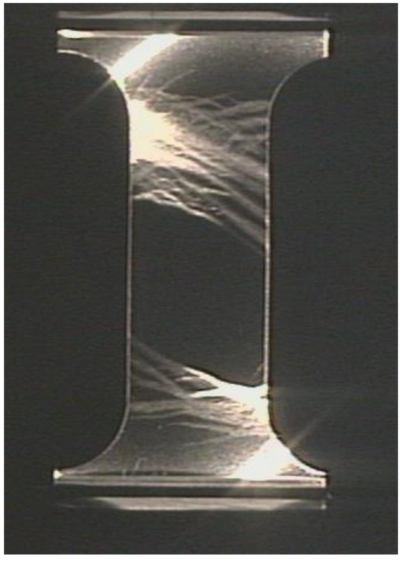


Dugdale (1960) & Barenblatt (1962)'s cohesive model

- Validity of cohesive zone/Yielding strip
  - Validity of Dugdale's model
    - Glassy polymers: PVC, Plexiglas, under transition temperature (from "ductile" to brittle)
    - Thin sheets of elastic perfectly plastic material
    - Low-C steels exhibiting Lüders' bands
      - Dislocations motion is initially blocked by solute atoms
      - Once freed, the yielding point decreases





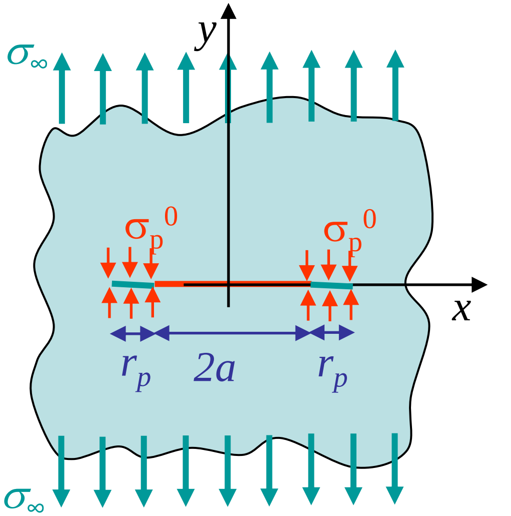


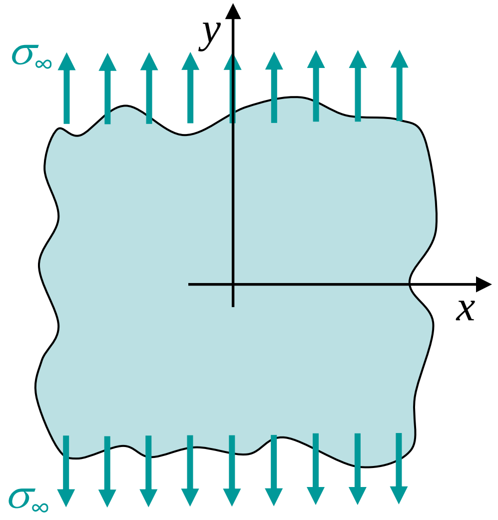
*Lüders bands formation in steel, contributed by Mike Meier, University of California, Davis

  - Validity of Barenblatt's model
    - Brittle metals

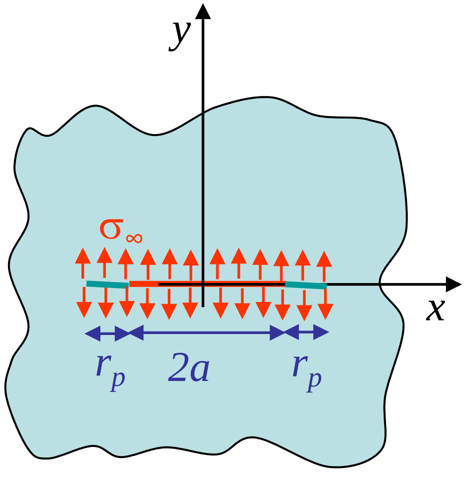
- Description of the model
  - Determine the size r_p of the cohesive zone so that stress remains finite everywhere
  - As non-linearities
    - Are localized in the process zone &
    - Are replaced by boundary conditions

    the problem can be solved using LEFM
  - Therefore the superposition principle holds
    - The solution is the superposition of the three following cases

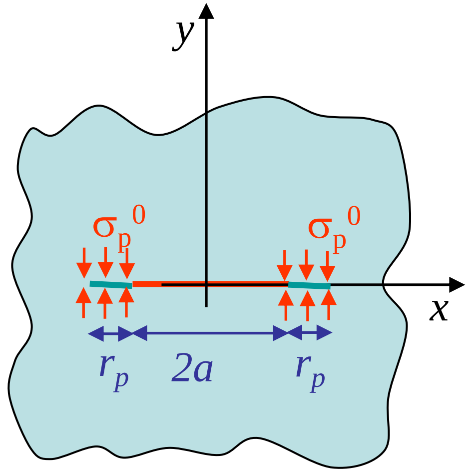




Case 1

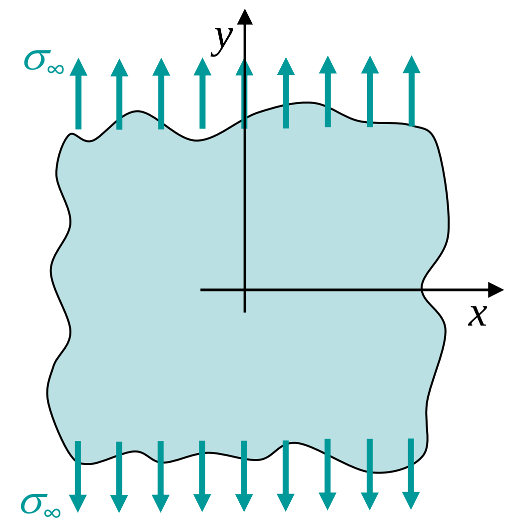


Case 2



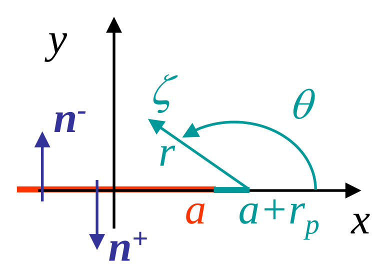
Case 3

- Resolution of case 1
  - Tension



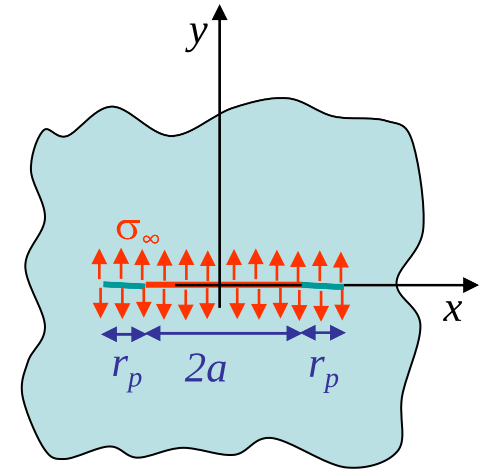
$$\left.\begin{array}{l}\boldsymbol{\sigma}_{yy} = \boldsymbol{\sigma}_\infty \\ \boldsymbol{\sigma}_{xx} = \boldsymbol{\sigma}_{xy} = 0\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\boldsymbol{\varepsilon}_{xx} = \dfrac{(1+\nu)(3-\kappa)}{4E}\boldsymbol{\sigma}_\infty \\ \boldsymbol{\varepsilon}_{yy} = \dfrac{(1+\nu)(1+\kappa)}{4E}\boldsymbol{\sigma}_\infty \\ \boldsymbol{\varepsilon}_{xy} = 0 \\ \boldsymbol{u}_x = \dfrac{(1+\nu)(3-\kappa)}{4E}\boldsymbol{\sigma}_\infty x \\ \boldsymbol{u}_y = \dfrac{(1+\nu)(1+\kappa)}{4E}\boldsymbol{\sigma}_\infty y\end{array}\right.$$

- With for plane σ: $\kappa = \dfrac{3-\nu}{1+\nu}$ & for plane ε: $\kappa = 3-4\nu$



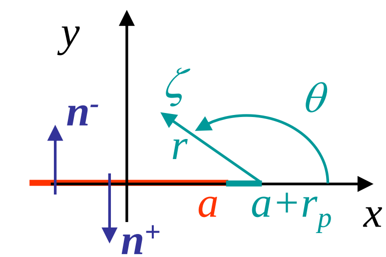
- So beyond crack tip ($\theta = 0$)
  - $\sigma(r, \theta = 0) = \sigma_\infty$
- So on crack lips ($\theta = \pm\pi$)
  - $\boldsymbol{u}_y = \dfrac{(1+\nu)(1+\kappa)}{4E}\boldsymbol{\sigma}_\infty y = 0$

- Resolution of case 2
  - See lecture on SIF, with a replaced by $a+r_p$



- Stress beyond cohesive zone tip

$$\boldsymbol{\sigma}_{yy}(\theta=0) = \sigma_\infty \left(\frac{x}{\sqrt{x^2-(a+r_p)^2}} - 1 \right)$$
$$\rightarrow \sigma_\infty \left(\sqrt{\frac{a+r_p}{2(x-a-r_p)}} - 1 \right)$$



- Displacement of crack and cohesive zone lips

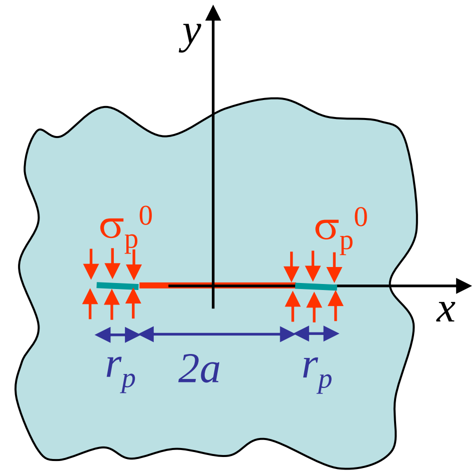
$$\lim_{\theta\rightarrow\pm\pi} \boldsymbol{u}_y = \pm\frac{(1+\nu)(\kappa+1)\sigma_\infty}{2E}\sqrt{(a+r_p)^2-x^2}$$

- Resolution of case 3
  - Analytical solution (reminder):
    - For any Ω & ω

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \Omega' + \bar{\Omega}' - \dfrac{\bar{\zeta}\Omega'' + \omega'' + \zeta\bar{\Omega}'' + \bar{\omega}''}{2} \\ \boldsymbol{\sigma}_{yy} = \Omega' + \bar{\Omega}' + \dfrac{\bar{\zeta}\Omega'' + \omega'' + \zeta\bar{\Omega}'' + \bar{\omega}''}{2} \\ \boldsymbol{\sigma}_{xy} = i\dfrac{\zeta\bar{\Omega}'' + \bar{\omega}'' - \bar{\zeta}\Omega'' - \omega''}{2} \\ \boldsymbol{u} = -\dfrac{1+\nu}{E}\left(\zeta\bar{\Omega}' + \bar{\omega}' - \kappa\Omega\left(\zeta\right)\right) \end{cases}$$

satisfy 2D linear elastic and isotropic equations

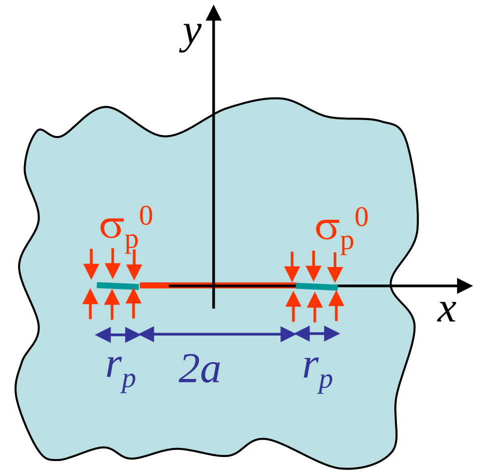
$\Rightarrow$ Choose Ω & ω to satisfy the BCs



  - Mode I (Westergaard):

$\Rightarrow$ $\omega'' = -\zeta\Omega''$ $\Rightarrow$ $\boldsymbol{\sigma}_{xy} = -y2\mathcal{R}\left(\Omega''\right)$ $\Rightarrow$ no shearing for $y = 0$

- Resolution of case 3 (2)
  - Analytical solution (reminder):
    - For any Ω & ω



$$\begin{cases} \boldsymbol{\sigma}_{xx} = \Omega' + \bar{\Omega}' - \dfrac{\bar{\zeta}\Omega'' + \omega'' + \zeta\bar{\Omega}'' + \bar{\omega}''}{2} \\ \boldsymbol{\sigma}_{yy} = \Omega' + \bar{\Omega}' + \dfrac{\bar{\zeta}\Omega'' + \omega'' + \zeta\bar{\Omega}'' + \bar{\omega}''}{2} \\ \boldsymbol{\sigma}_{xy} = i\dfrac{\zeta\bar{\Omega}'' + \bar{\omega}'' - \bar{\zeta}\Omega'' - \omega''}{2} \\ \boldsymbol{u} = -\dfrac{1+\nu}{E}\left(\zeta\bar{\Omega}' + \bar{\omega}' - \kappa\Omega\left(\zeta\right)\right) \end{cases}$$

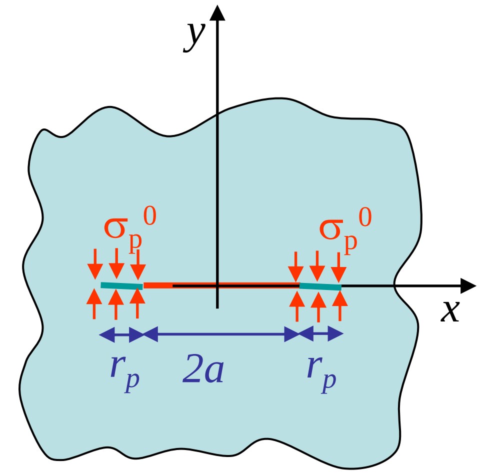
- Mode I (Westergaard): $\omega'' = -\zeta\Omega''$

$$\Rightarrow \begin{cases} \boldsymbol{\sigma}_{xx} = 2\mathcal{R}\left(\Omega'\right) - 2y\mathcal{I}\left(\Omega''\right) \\ \boldsymbol{\sigma}_{yy} = 2\mathcal{R}\left(\Omega'\right) + 2y\mathcal{I}\left(\Omega''\right) \\ \boldsymbol{\sigma}_{xy} = -y2\mathcal{R}\left(\Omega''\right) \\ \boldsymbol{u}_x = \mathcal{R}\left(\boldsymbol{u}\right) = \dfrac{1+\nu}{E}\left[\left(\kappa - 1\right)\mathcal{R}\left(\Omega\right) - 2y\mathcal{I}\left(\Omega'\right) - \mathcal{R}\left(C\right)\right] \\ \boldsymbol{u}_y = \mathcal{I}\left(\boldsymbol{u}\right) = \dfrac{1+\nu}{E}\left[\left(\kappa + 1\right)\mathcal{I}\left(\Omega\right) - 2y\mathcal{R}\left(\Omega'\right) + \mathcal{I}\left(C\right)\right] \end{cases}$$

Only Ω to be defined

- Resolution of case 3 (3)
  - Mode I solution

$$\begin{cases} \boldsymbol{\sigma}_{xx} = 2\mathcal{R}\left(\Omega'\right) - 2y\mathcal{I}\left(\Omega''\right) \\ \boldsymbol{\sigma}_{yy} = 2\mathcal{R}\left(\Omega'\right) + 2y\mathcal{I}\left(\Omega''\right) \\ \boldsymbol{\sigma}_{xy} = -y2\mathcal{R}\left(\Omega''\right) \end{cases}$$



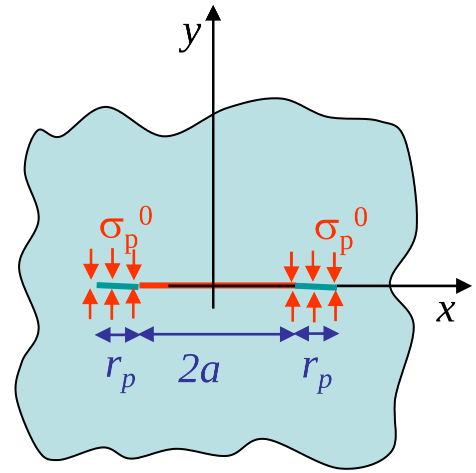
  - Problem is to find Ω' so that BCs are satisfied
    - On crack lips ($x<a$), we want to satisfy

$$\mathbf{0} = \boldsymbol{t} = \boldsymbol{n}\cdot\boldsymbol{\sigma} = \begin{pmatrix} 0 \\ \mp 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} 0 \\ \mp\sigma_{yy} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mp 2\mathcal{R}(\Omega') \\ 0 \end{pmatrix}$$

    - On cohesive zone lips ($a<x<a+r_p$), we want to satisfy

$$\begin{pmatrix} 0 \\ \mp\sigma_p^0 \\ 0 \end{pmatrix} = \boldsymbol{t} = \boldsymbol{n}\cdot\boldsymbol{\sigma} = \begin{pmatrix} 0 \\ \mp 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} 0 \\ \mp\sigma_{yy} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mp 2\mathcal{R}(\Omega') \\ 0 \end{pmatrix}$$

- Resolution of case 3 (4)
  - For $x \geq 0$, let us consider So Ω' -> 0 far away from crack



$$\Omega' = -\frac{\sigma_p^0}{\pi}\frac{\zeta}{\sqrt{\zeta^2-(a+r_p)^2}}\text{arcotan}\frac{a}{\sqrt{r_p^2+2ar_p}} + \frac{\sigma_p^0}{\pi}\text{arcotan}\left(\frac{a}{\zeta}\sqrt{\frac{\zeta^2-(a+r_p)^2}{r_p^2+2ar_p}}\right)$$

Discontinuities in $a+r_p$: For y=0, $\xi = x$

•For $x>a+r_p$, real number → real function

•For $x<a+r_p$?

- Using $\text{arcotan}\,\zeta = \frac{i}{2}\ln\frac{\zeta - i}{\zeta + i}$

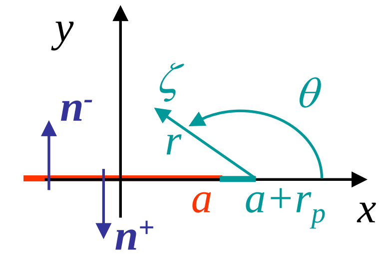
$$\Rightarrow \quad \Omega' = -\frac{\sigma_p^0}{\pi}\frac{\zeta}{\sqrt{\zeta^2-(a+r_p)^2}}\text{arcotan}\frac{a}{\sqrt{r_p^2+2ar_p}} + \frac{i\sigma_p^0}{2\pi}\ln\frac{\frac{a}{\zeta}\sqrt{\frac{\zeta^2-(a+r_p)^2}{r_p^2+2ar_p}} - i}{\frac{a}{\zeta}\sqrt{\frac{\zeta^2-(a+r_p)^2}{r_p^2+2ar_p}} + i}$$

Discontinuities in a : For y=0, $\xi = x$

•For $x>a$, ?

•For $x<a$?

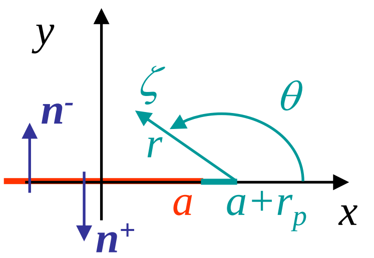
- Resolution of case 3 (5)
  - For $\zeta = x \pm |\varepsilon|\, i$, with $x < a+r_p$,
    - $$\Omega' = -\frac{\sigma_p^0}{\pi}\frac{\zeta}{\sqrt{\zeta^2-(a+r_p)^2}}\text{arcotan}\frac{a}{\sqrt{r_p^2+2ar_p}} + \frac{i\sigma_p^0}{2\pi}\ln\frac{\frac{a}{\zeta}\sqrt{\frac{\zeta^2-(a+r_p)^2}{r_p^2+2ar_p}}-i}{\frac{a}{\zeta}\sqrt{\frac{\zeta^2-(a+r_p)^2}{r_p^2+2ar_p}}+i}$$



- Using $$\lim_{\theta\to\pm\pi}\sqrt{\zeta^2-(a+r_p)^2} = \lim_{\theta\to\pm\pi}\sqrt{\zeta-a-r_p}\sqrt{\zeta+a+r_p}$$
  $$= \pm i\sqrt{a+r_p-x}\sqrt{x+a+r_p} = \pm i\sqrt{(a+r_p)^2-x^2}$$

$$\Rightarrow \quad \Omega'(\theta\to\pm\pi) = \pm i\frac{\sigma_p^0}{\pi}\frac{x}{\sqrt{(a+r_p)^2-x^2}}\text{arcotan}\frac{a}{\sqrt{r_p^2+2ar_p}} + \frac{i\sigma_p^0}{2\pi}\ln\frac{\pm\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}-1}{\pm\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}+1}$$

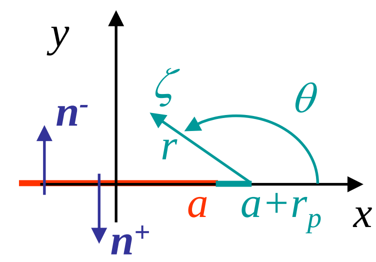
- Resolution of case 3 (6)
  - For $\zeta = x \pm |\varepsilon|\, i$, with $x < a + r_p$ (2)
    - $$\Omega'(\theta \to \pm\pi) = \pm i \frac{\sigma_p^0}{\pi} \frac{x}{\sqrt{(a+r_p)^2 - x^2}} \operatorname{arcotan} \frac{a}{\sqrt{r_p^2 + 2ar_p}} + \frac{i\sigma_p^0}{2\pi} \ln \frac{\pm\frac{a}{x}\sqrt{\frac{(a+r_p)^2 - x^2}{r_p^2 + 2ar_p}} - 1}{\pm\frac{a}{x}\sqrt{\frac{(a+r_p)^2 - x^2}{r_p^2 + 2ar_p}} + 1}$$
    - When is $\ln$ real?
      - This is the case if $\frac{a}{x}\sqrt{\frac{(a+r_p)^2 - x^2}{r_p^2 + 2ar_p}} > 1$
      - Or again if $(a^2 - x^2)(a+r_p)^2 > 0$, so if $x < a$



    - So we have different behaviors for $y = 0$
      - On crack lips ($x<a$)
      - On cohesive zone ($a<x<a+r_p$)
      - Ahead of crack tip ($x>a+r_p$)
  - Full demonstration in Annex 1

- Resolution of case 3 (7)
  - Stress field for $\zeta = x \pm |\varepsilon|\, i$, with $a+r_p < x$ can be directly deduced from Ω'

$$\Omega' = -\frac{\sigma_p^0}{\pi}\frac{\zeta}{\sqrt{\zeta^2-(a+r_p)^2}}\operatorname{arcotan}\frac{a}{\sqrt{r_p^2+2ar_p}} + \frac{\sigma_p^0}{\pi}\operatorname{arcotan}\left(\frac{a}{\zeta}\sqrt{\frac{\zeta^2-(a+r_p)^2}{r_p^2+2ar_p}}\right)$$



$$\boldsymbol{\sigma}_{yy}(\theta=0) = 2\mathcal{R}\Omega'(\theta=0) = -\frac{2\sigma_p^0}{\pi}\frac{x}{\sqrt{x^2-(a+r_p)^2}}\operatorname{arcotan}\frac{a}{\sqrt{r_p^2+2ar_p}} + \frac{2\sigma_p^0}{\pi}\operatorname{arcotan}\left(\frac{a}{x}\sqrt{\frac{x^2-(a+r_p)^2}{r_p^2+2ar_p}}\right)$$

Dugdale's cohesive zone/Yielding strip Model

- Yielding strip Model
  - Case 1, 2 & 3: Stress field
    - Case 1 (infinite plane): $\boldsymbol{\sigma}_{yy} = \boxed{\boldsymbol{\sigma}_\infty}$
    - Case 2 (crack loaded by σ_∞): $\boldsymbol{\sigma}_{yy}(\theta = 0) = \sigma_\infty \left(\dfrac{x}{\sqrt{x^2 - (a + r_p)^2}} \boxed{-1} \right)$

$\rightarrow \infty$ for $x \xrightarrow{+} a + r_p$

    - Case 3 (cohesive zone closed):

$$\boldsymbol{\sigma}_{yy}(\theta = 0) = -\frac{2\sigma_p^0}{\pi} \frac{x}{\sqrt{x^2 - (a + r_p)^2}} \operatorname{arcotan} \frac{a}{\sqrt{r_p^2 + 2ar_p}} + \frac{2\sigma_p^0}{\pi} \operatorname{arcotan} \left(\frac{a}{x} \sqrt{\frac{x^2 - (a + r_p)^2}{r_p^2 + 2ar_p}} \right)$$

$\rightarrow \infty$ for $x \xrightarrow{+} a + r_p$

$\rightarrow \frac{\pi}{2}$ for $x \xrightarrow{+} a + r_p$

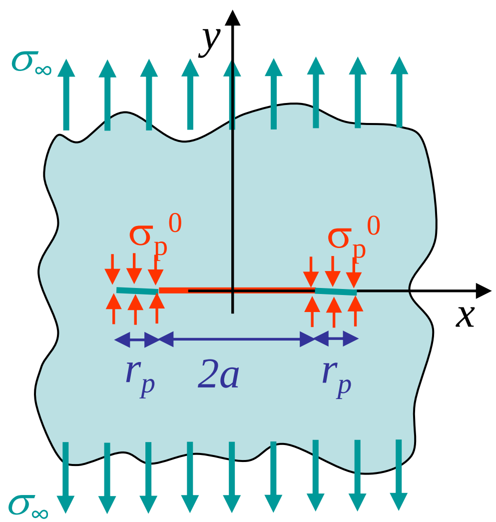
    - The singularity is avoided if $\sigma_\infty = \dfrac{2\sigma_p^0}{\pi} \operatorname{arcotan} \dfrac{a}{\sqrt{r_p^2 + 2ar_p}}$

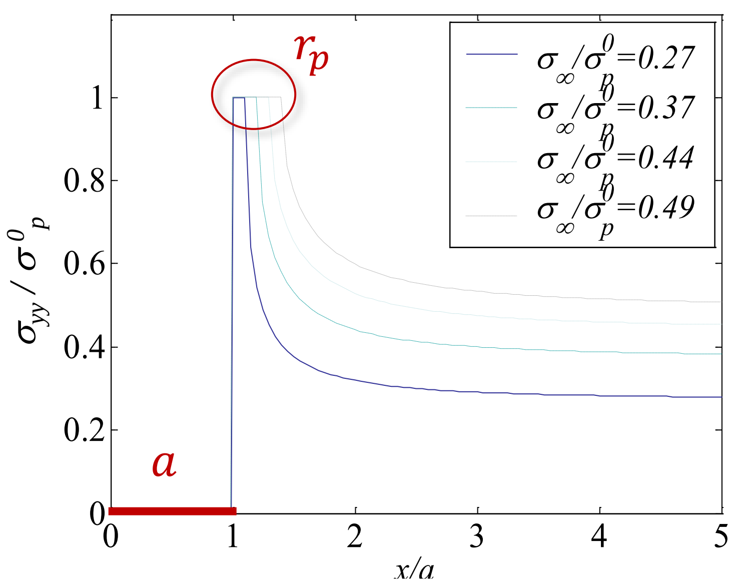
$$\tan\left(\frac{\pi\sigma_\infty}{2\sigma_p^0}\right) = \frac{\sqrt{r_p^2 + 2ar_p}}{a} \Rightarrow \frac{1 - \cos^2\left(\frac{\pi\sigma_\infty}{2\sigma_p^0}\right)}{\cos^2\left(\frac{\pi\sigma_\infty}{2\sigma_p^0}\right)} = \frac{r_p^2 + 2ar_p}{a^2} \Rightarrow \frac{1}{\cos^2\left(\frac{\pi\sigma_\infty}{2\sigma_p^0}\right)} = \frac{(a + r_p)^2}{a^2}$$

$$\Rightarrow \boxed{r_p = a\left(\sec\frac{\sigma_\infty \pi}{2\sigma_p^0} - 1\right)}$$

- Yielding strip Model (2)
  - Case 1, 2 & 3: Stress field (2)
    - The resulting stress field reads

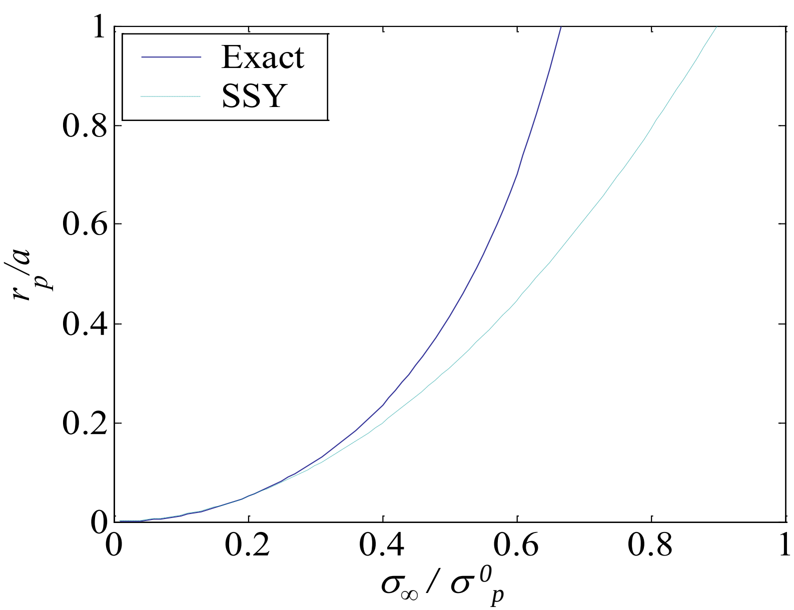
$$\boldsymbol{\sigma}_{yy}\left(y=0\right)=\begin{cases}0 & \text{if } x<a\\ \sigma_p^0 & \text{if } a<x<a+r_p\\ \frac{2\sigma_p^0}{\pi}\,\text{arcotan}\left(\frac{a}{x}\sqrt{\frac{x^2-(a+r_p)^2}{r_p^2+2ar_p}}\right) & \text{if } a+r_p<x\end{cases}$$

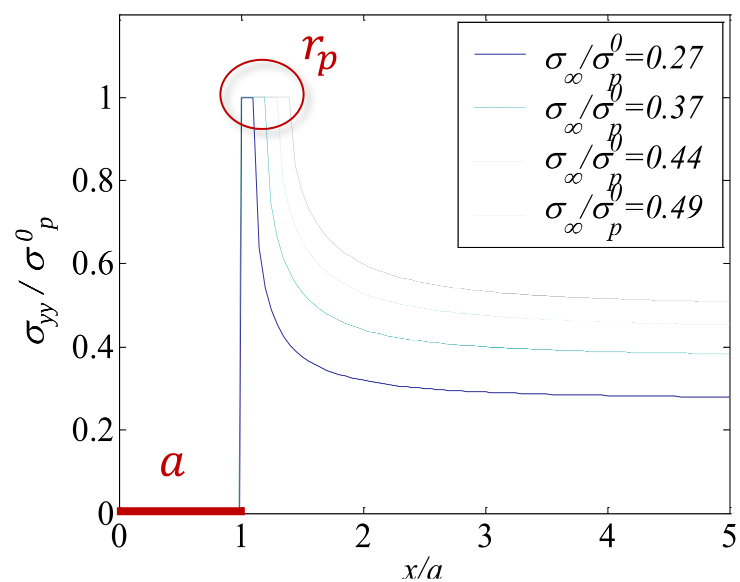




- Yielding strip Model (3)
  - Cohesive zone length
    - The cohesive zone has a size $r_p = a\left(\sec\frac{\sigma_\infty \pi}{2\sigma_p^0} - 1\right)$
    - Since $\frac{1}{\cos x} \simeq \frac{1}{1-\frac{x^2}{2}} \simeq 1 + \frac{x^2}{2}$

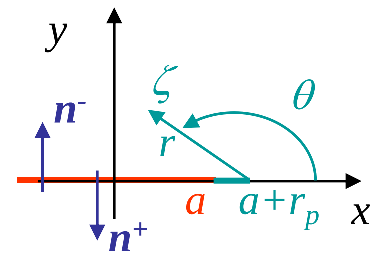
    In Small Scale Yielding we have $r_p \simeq \frac{a\pi^2}{8}\left(\frac{\sigma_\infty}{\sigma_p^0}\right)^2$





Dugdale’s cohesive zone/Yielding strip Model

- Resolution of case 3 (8)
  – Displacement field for $\zeta = x \pm |\varepsilon|\, i$, $a+r_p > x$, can be directly deduced from Ω



- $$\boldsymbol{u}_y\left(\theta \rightarrow \pi\right) = \frac{\left(1+\nu\right)\left(1+\kappa\right)}{E} \mathcal{I}\Omega\left(\theta \rightarrow \pi\right)$$

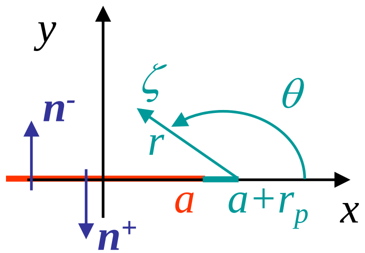
- But we know

$$\Omega' = -\frac{\sigma_p^0}{\pi} \frac{\zeta}{\sqrt{\zeta^2 - \left(a+r_p\right)^2}} \operatorname{arcotan} \frac{a}{\sqrt{r_p^2 + 2ar_p}} + \frac{\sigma_p^0}{\pi} \operatorname{arcotan}\left(\frac{a}{\zeta}\sqrt{\frac{\zeta^2 - \left(a+r_p\right)^2}{r_p^2 + 2ar_p}}\right)$$

- After integration (see appendix 2), one has

$$\Omega = -\frac{\sigma_p^0}{\pi}\sqrt{\zeta^2 - \left(a+r_p\right)^2} \operatorname{arcotan} \frac{a}{\sqrt{r_p^2 + 2ar_p}} - \frac{\sigma_p^0 a}{\pi} \operatorname{arcotan} \sqrt{\frac{\zeta^2 - \left(a+r_p\right)^2}{r_p^2 + 2ar_p}} + \frac{\sigma_p^0 \zeta}{\pi} \operatorname{arcotan}\left(\frac{a}{\zeta}\sqrt{\frac{\zeta^2 - \left(a+r_p\right)^2}{r_p^2 + 2ar_p}}\right)$$

- Resolution of case 3 (9)
  - Displacement field for $\zeta = x \pm |\varepsilon|\, i,\ a + r_p > x$, can be directly deduced from Ω
    - $$\boldsymbol{u}_y\left(\theta \rightarrow \pi\right) = \frac{\left(1+\nu\right)\left(1+\kappa\right)}{E}\mathcal{I}\Omega\left(\theta \rightarrow \pi\right)$$
    - Then we have (see Appendix 2)



$$\begin{aligned}
\boldsymbol{u}_y\left(\theta \rightarrow \pm\pi,\ r < r_p\right) &= \pm\frac{\left(1+\nu\right)\left(1+\kappa\right)\sigma_p^0}{E\pi}\left[-\frac{\sigma_\infty \pi}{2\sigma_p^0}\sqrt{\left(a+r_p\right)^2 - x^2} + \right.\\
&\left. a \operatorname{arcoth}\sqrt{\frac{\left(a+r_p\right)^2 - x^2}{r_p^2 + 2ar_p}} - x\,\boxed{\operatorname{arctanh}}\left(\frac{a}{x}\sqrt{\frac{\left(a+r_p\right)^2 - x^2}{r_p^2 + 2ar_p}}\right)\right]\\
\boldsymbol{u}_y\left(\theta \rightarrow \pm\pi,\ r > r_p\right) &= \pm\frac{\left(1+\nu\right)\left(1+\kappa\right)\sigma_p^0}{E\pi}\left[-\frac{\sigma_\infty \pi}{2\sigma_p^0}\sqrt{\left(a+r_p\right)^2 - x^2} + \right.\\
&\left. a \operatorname{arcoth}\sqrt{\frac{\left(a+r_p\right)^2 - x^2}{r_p^2 + 2ar_p}} - x\,\boxed{\operatorname{arcoth}}\left(\frac{a}{x}\sqrt{\frac{\left(a+r_p\right)^2 - x^2}{r_p^2 + 2ar_p}}\right)\right]
\end{aligned}$$

- Crack Opening Displacement (COD)
  - Case 1, 2 & 3: displacement
    - Case 1 (infinite plane): $\boldsymbol{u}_y = \frac{(1+\nu)(1+\kappa)}{4E}\boldsymbol{\sigma}_\infty y$ → 0 for y→ 0
    - Case 2 (crack loaded by σ_∞): $\lim_{\theta\to\pm\pi} \boldsymbol{u}_y = \boxed{\pm\frac{(1+\nu)(\kappa+1)\sigma_\infty}{2E}\sqrt{(a+r_p)^2 - x^2}}$
    - Case 3 (cohesive zone closed):
      - On cohesive zone lips

$$\boldsymbol{u}_y(\theta\to\pm\pi,\, r<r_p) = \boxed{\pm\frac{(1+\nu)(1+\kappa)\sigma_p^0}{E\pi}\Big[-\frac{\sigma_\infty\pi}{2\sigma_p^0}\sqrt{(a+r_p)^2-x^2}}+ a\,\mathrm{arcoth}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}} - x\,\boxed{\mathrm{arctanh}}\left(\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}\right)\Big]$$

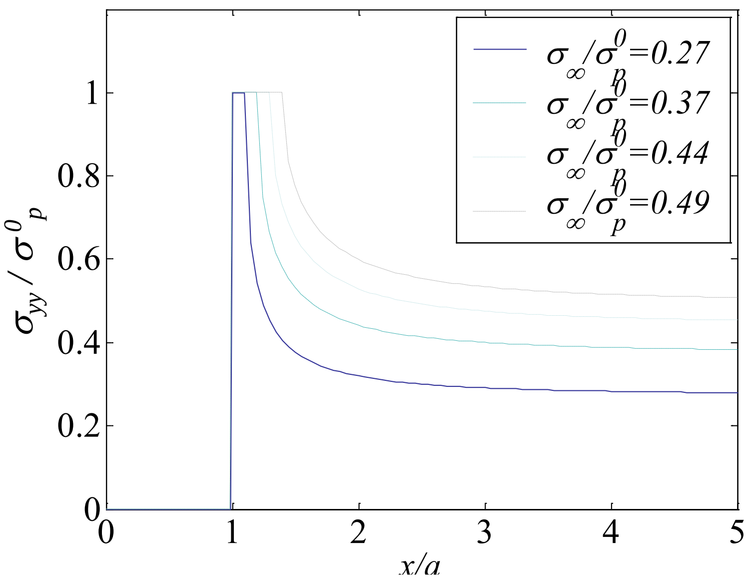
      - On crack lips

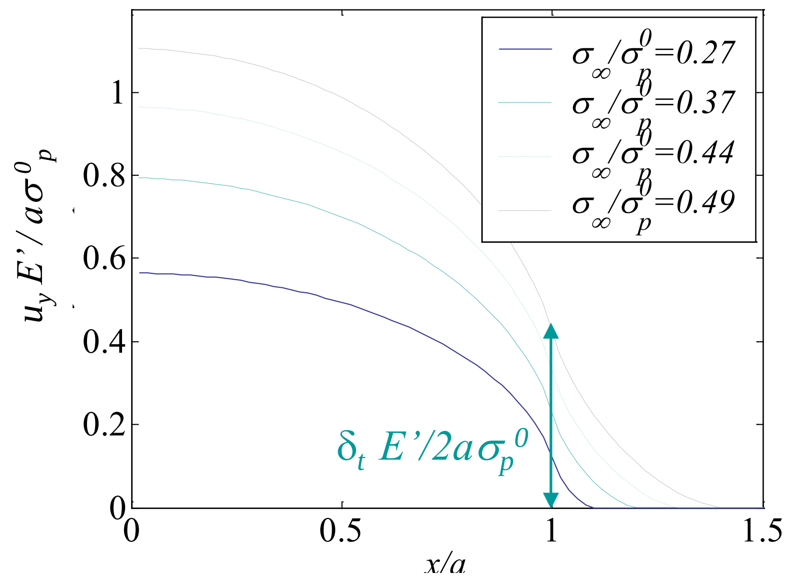
$$\boldsymbol{u}_y(\theta\to\pm\pi,\, r>r_p) = \boxed{\pm\frac{(1+\nu)(1+\kappa)\sigma_p^0}{E\pi}\Big[-\frac{\sigma_\infty\pi}{2\sigma_p^0}\sqrt{(a+r_p)^2-x^2}}+ a\,\mathrm{arcoth}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}} - x\,\boxed{\mathrm{arcoth}}\left(\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}\right)\Big]$$

- Crack Opening Displacement (COD) (2)
  - Crack Displacement Opening is the superposition of the 3 cases

$$[\![\boldsymbol{u}_y]\!] = 2\frac{(1+\nu)(1+\kappa)\sigma_p^0}{E\pi}\left[a \operatorname{arcoth}\left(\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}\right) - xf\left(\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}\right)\right]$$

with f standing for arctanh is $a < x < a+r_p$ and for arcoth is $x < a$





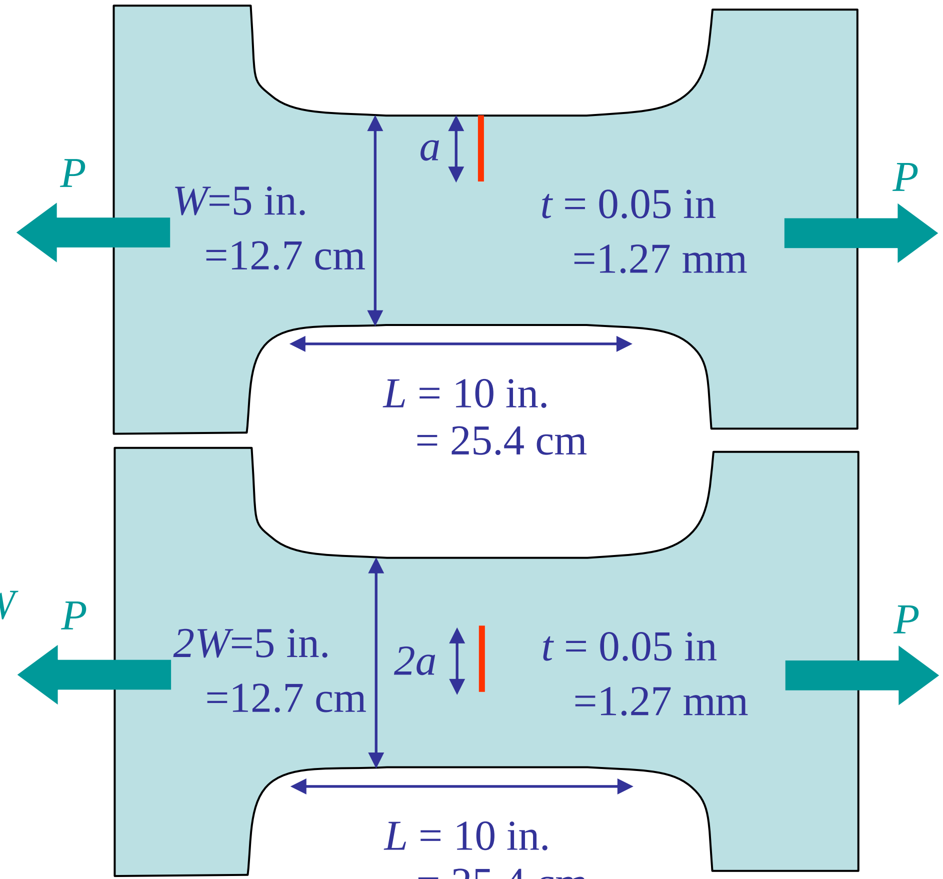
  - Crack Tip Opening Displacement (COD at $x=a$)

$$\delta_t = \lim_{x\to a} [\![\boldsymbol{u}_y]\!] = 8\frac{\sigma_p^0 a}{E'\pi}\ln\left(\sec\frac{\sigma_\infty\pi}{2\sigma_p^0}\right) \quad \text{with } E' = \frac{4E}{(1+\nu)(\kappa+1)} = \begin{cases} E & \mathrm{P}\sigma \\ \frac{E}{1-\nu^2} & \mathrm{P}\varepsilon \end{cases}$$

(see appendix 3 for details)

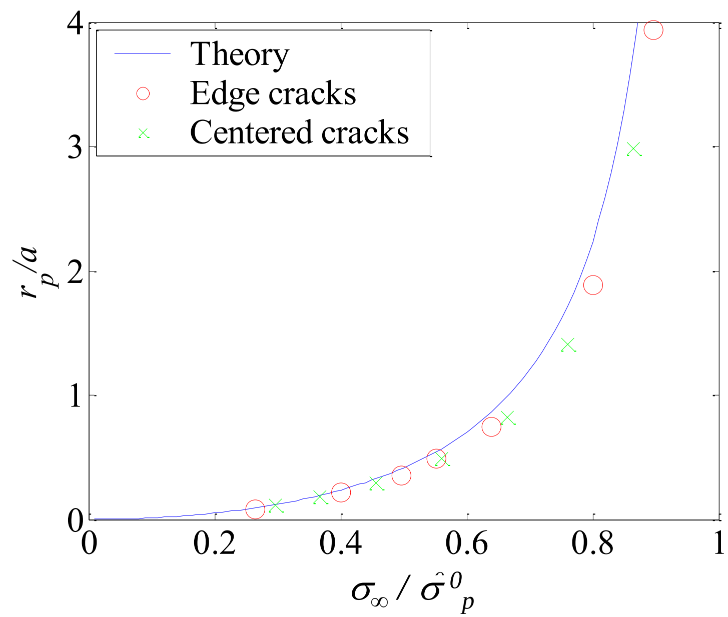
Dugdale's cohesive zone/Yielding strip Model

- Dugdale's experiments
  - Mild steel specimen
    - 0.05% C, 0.4% Mn, 0.013% N
    - Annealed at 900°C after cut
    - Yield: σ_p^0 = 190 MPa
    - Young: E = 210 GPa
  - 2 cases
    - Edge cracks and centered cracks
    - Crack length small compared to W
    - Various P as to approach yield
  - Measure of r_p
    - Specimen cut
    - Cut polished and etched
  - These are not fracture tests



- Dugdale's experiments (2)
  - Results compared to $r_p = a\left(\sec\frac{\sigma_\infty \pi}{2\sigma_p^0} - 1\right)$
    - The formula holds for
      - Centered crack
      - Edge cracks
    - As long as $a+r_p \ll W$ (infinite plate assumption)

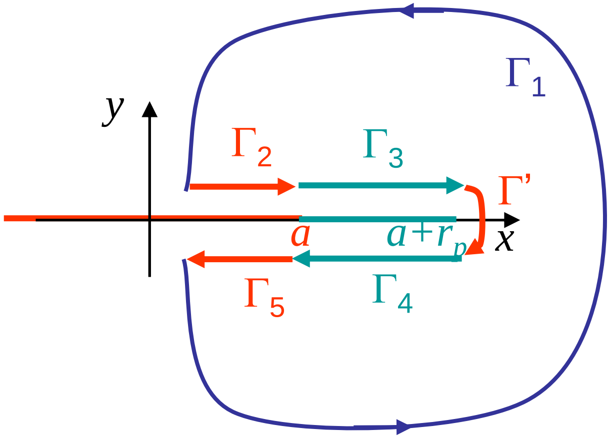
Edge crack			Centered crack		
a (cm)	P/Wt (MPa)	r_p (cm)	a (cm)	P/Wt (MPa)	r_p (cm)
1.27	50.16	0.107	1.27	56.24	0.1448
1.016	76	0.224	1.27	69.92	0.2362
0.63	94.24	0.221	1.016	86.64	0.2946
0.63	104.9	0.31	1.016	106.4	0.5
0.63	121.6	0.47	0.63	126.16	0.5182
0.63	152	1.19	0.63	144.4	0.8890
0.254	170.24	1	0.381	164.16	1.1379

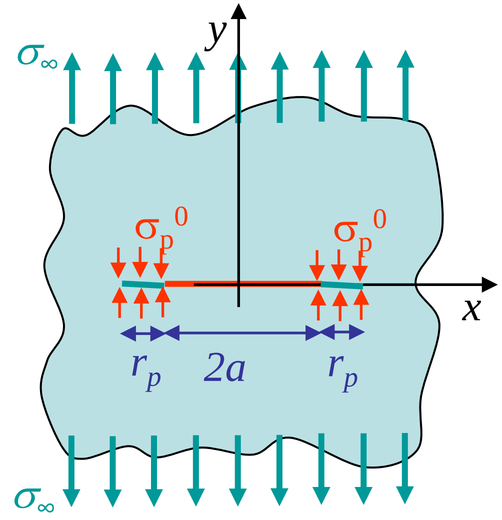


- Relation with J-integral
  - Assuming no unloading, $U(\boldsymbol{\varepsilon})$ exists:
  - J integral is path independent, so
    - $$J = \int_{\Gamma_1} \left[U(\boldsymbol{\varepsilon})\boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}\right] dl$$

      $$\Longrightarrow J = -\int_{\Gamma_{2-5}} \left[U(\boldsymbol{\varepsilon})\boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}\right] dl$$
  - On the different curves:
    - On Γ_2 and Γ_5:
      - $\boldsymbol{n}_x = 0 \Longrightarrow U(\boldsymbol{\varepsilon})\boldsymbol{n}_x = 0$
      - The crack is stress free $\Longrightarrow \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{0}$
    - Integration on Γ'
      - Vanishes as its length tends toward 0
    - On Γ_3 (and Γ_4):
      - $\boldsymbol{n}_x = 0 \Longrightarrow U(\boldsymbol{\varepsilon})\boldsymbol{n}_x = 0$

$$\Longrightarrow J = \int_{\Gamma_{3,4}} \boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \, dl$$





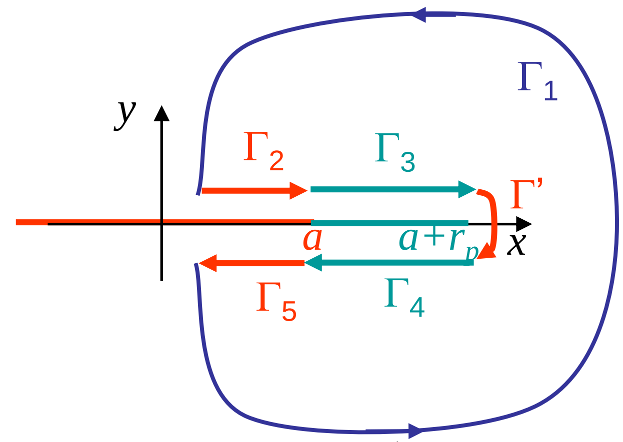
- Relation with J-integral (2)
  - J integral reduces to
    - $J = \int_{\Gamma_{3,4}} \boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \, dl$
  - On the remaining different curves
    - On Γ_3:
      - $\boldsymbol{n} = -\boldsymbol{e}_y \Rightarrow \boldsymbol{\sigma} \cdot \boldsymbol{n} = -\boldsymbol{\sigma}_{yy}(\delta)\, \boldsymbol{e}_y$

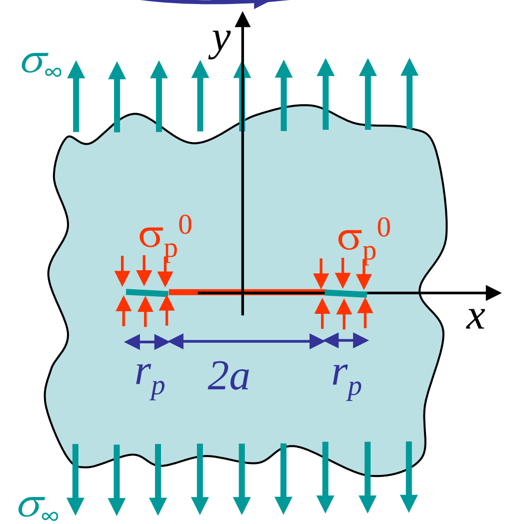
        $\Rightarrow (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{u}_{,x} = -\boldsymbol{\sigma}_{yy(\delta)} \boldsymbol{e}_y \cdot \boldsymbol{u}_{,x} = -\boldsymbol{\sigma}_{yy(\delta)} \boldsymbol{u}_{\boldsymbol{y},\boldsymbol{x}}$
      - $dl = dx$
    - On Γ_4:
      - $\boldsymbol{n} = +\boldsymbol{e}_y \Rightarrow \boldsymbol{\sigma} \cdot \boldsymbol{n} = +\boldsymbol{\sigma}_{yy}(\delta)\, \boldsymbol{e}_y$

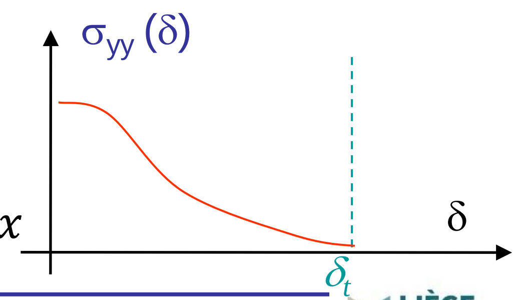
        $\Rightarrow (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{u}_{,x} = +\boldsymbol{\sigma}_{yy(\delta)} \boldsymbol{e}_y \cdot \boldsymbol{u}_{,x} = +\boldsymbol{\sigma}_{yy(\delta)} \boldsymbol{u}_{\boldsymbol{y},\boldsymbol{x}}$
      - $dl = -dx$

$$\Rightarrow J = -\int_{\Gamma_3} \boldsymbol{u}_{y,x} \boldsymbol{\sigma}_{yy}(\delta)\, dx - \int_{\Gamma_4} \boldsymbol{u}_{y,x} \boldsymbol{\sigma}_{yy}(\delta)\, dx$$

$$\Rightarrow J = -\int_a^{a+r_p} [\![\boldsymbol{u}_{y,x}]\!] \boldsymbol{\sigma}_{yy}(\delta)\, dx = -\int_a^{a+r_p} \delta_{,x}\, \boldsymbol{\sigma}_{yy}(\delta)\, dx$$



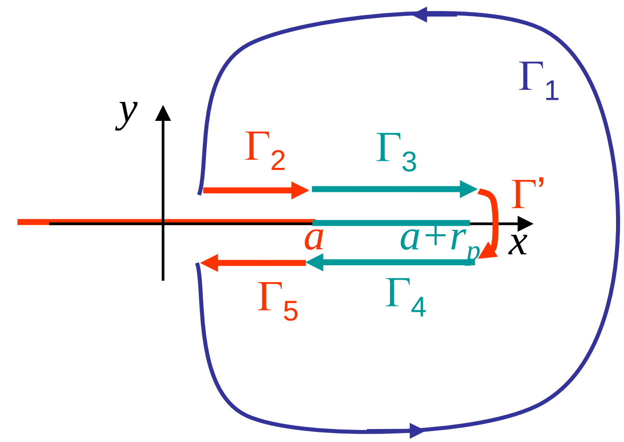


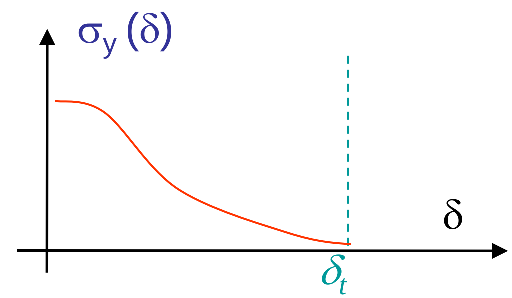


- Relation with J-integral (2)
  - J integral is rewritten as

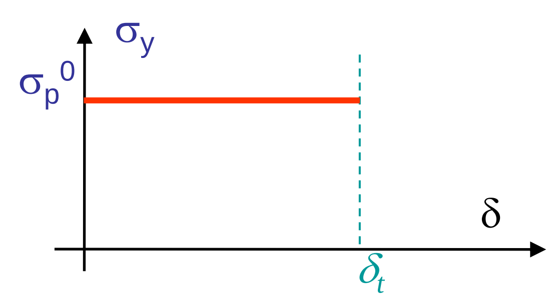
$$J = -\int_a^{a+r_p} \boldsymbol{\sigma}_{yy}(\delta)\delta_{,x}\, dx$$

$$\Rightarrow \quad J = -\int_{\delta_t}^{0} \boldsymbol{\sigma}_{yy}(\delta)\, d\delta = \int_0^{\delta_t} \boldsymbol{\sigma}_{yy}(\delta)\, d\delta$$





- The flow of energy toward the crack tip corresponds to the separation energy of the cohesive zone
- J-integral can be used in plasticity and could be used as a crack growth criterion $J=J_C$
- This criterion could also be related to a critical crack tip opening $\delta_t = \delta_C$



  - In the particular case of Dugdale's models:

$$J_{\text{Dugdale}} = \sigma_p^0 \int_0^{\delta_t} d\delta = \sigma_p^0 \delta_t = 8\frac{\left(\sigma_p^0\right)^2 a}{E'\pi} \ln\left(\sec\frac{\sigma_\infty \pi}{2\sigma_p^0}\right)$$

- In this case, a criterion on J is directly related to a maximal value of δ_t: $J_C = \delta_C\, \sigma_p^0$

- Can LEFM be extended to elastic perfectly plastic analysis?
  - It is possible if there is no extensive plasticity prior to fracture
    - Plastic zone small compared to characteristic lengths
  - A criterion of crack growth might be $J \geq J_C$
  - Are the SIFs still meaningful? If so how do we compute them?
  - The extension is based on the use of an “effective crack length”
    - Effective crack length $a_{\text{eff}} = a + \eta r_p$ with η a factor to be determined
    - As we want to use LEFM, J is rewritten in terms of SIFs
      - Mode I & infinite plane: $J = \frac{K_I^2\left(a_{\text{eff}}\right)}{E'}$ & $K_I = \sigma_\infty \sqrt{\pi a_{\text{eff}}}$

$$\Rightarrow J = 8\frac{\left(\sigma_p^0\right)^2 a}{E'\pi} \ln\left(\sec\frac{\sigma_\infty \pi}{2\sigma_p^0}\right) = \frac{K_I^2\left(a_{\text{eff}}\right)}{E'} = \frac{\pi\sigma_\infty^2 a_{\text{eff}}}{E'}$$

      - So the effective length is $a_{\text{eff}} = \frac{8a}{\pi^2}\left(\frac{\sigma_p^0}{\sigma_\infty}\right)^2 \ln\left(\sec\frac{\sigma_\infty \pi}{2\sigma_p^0}\right)$

- Can LEFM be extended to elastic perfectly plastic analysis (2)?
  - The extension is based on the use of an "effective crack length" (2)
    - Mode I & infinite plane: $a_{\text{eff}} = \frac{8a}{\pi^2}\left(\frac{\sigma_p^0}{\sigma_\infty}\right)^2 \ln\left(\sec\frac{\sigma_\infty \pi}{2\sigma_p^0}\right)$
    - If $\sigma_\infty < \sigma_p^0$
      - Since $\sec(x) \to 1 + \frac{x^2}{2} + \frac{5x^4}{24}$

$$\Longrightarrow a_{\text{eff}} \to \frac{8a}{\pi^2}\left(\frac{\sigma_p^0}{\sigma_\infty}\right)^2 \ln\left(1 + \frac{1}{2}\left(\frac{\sigma_\infty \pi}{2\sigma_p^0}\right)^2 + \frac{5}{24}\left(\frac{\sigma_\infty \pi}{2\sigma_p^0}\right)^4\right)$$

      - Since $\ln(1+x) \to x - \frac{x^2}{2}$

2/(24 x 16)

$$\Longrightarrow a_{\text{eff}} \to \frac{8a}{\pi^2}\left(\frac{\sigma_p^0}{\sigma_\infty}\right)^2 \left(\frac{1}{2}\left(\frac{\sigma_\infty \pi}{2\sigma_p^0}\right)^2 + \frac{5}{24}\left(\frac{\sigma_\infty \pi}{2\sigma_p^0}\right)^4 - \frac{1}{8}\left(\frac{\sigma_\infty \pi}{2\sigma_p^0}\right)^4\right)$$

$$\Longrightarrow a_{\text{eff}} = a + \eta r_p \to a + \frac{a}{24}\left(\frac{\sigma_\infty \pi}{\sigma_p^0}\right)^2 = a + \frac{r_p}{3}$$

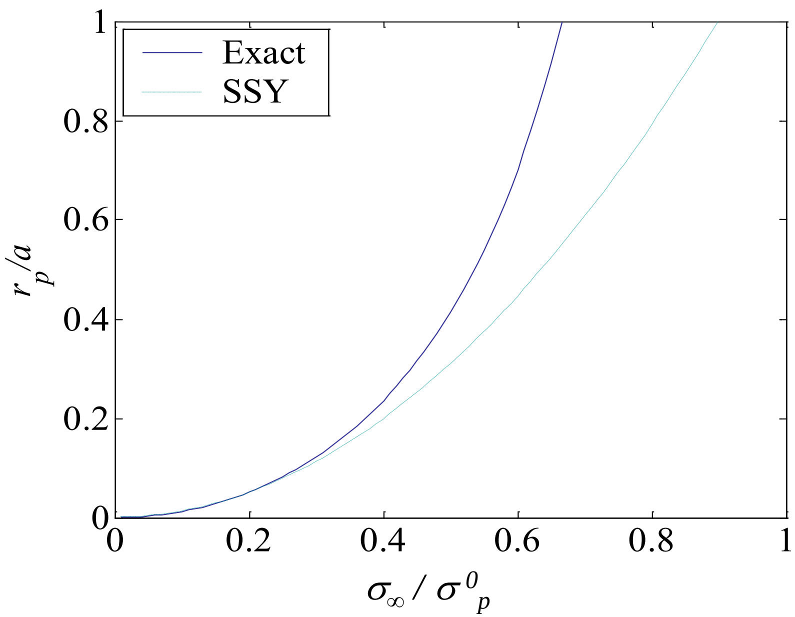
      - Eventually, for SSY, $\eta = 1/3$
    - CTOD can be computed from a_{eff} as $J = \sigma_p^0 \delta_t = \frac{\pi \sigma_\infty^2 a_{\text{eff}}}{E'}$

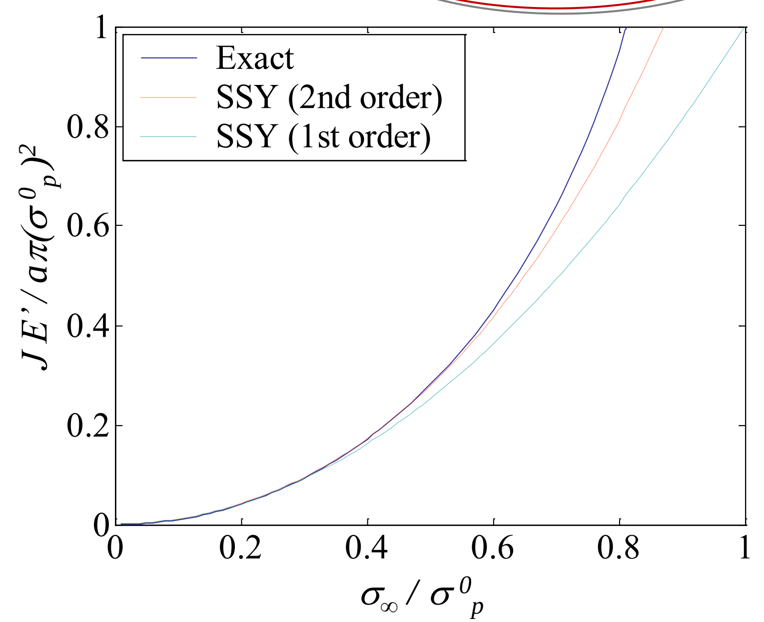
$$\Longrightarrow \delta_t = \frac{\pi \sigma_\infty^2 a_{\text{eff}}}{E' \sigma_p^0} \to \frac{\pi \sigma_\infty^2 a}{E' \sigma_p^0} + \frac{\pi^3 \sigma_\infty^4 a}{24 E' \left(\sigma_p^0\right)^3}$$

- When can we assume SSY?
  - Cohesive zone length $r_p = a\left(\sec\dfrac{\sigma_\infty \pi}{2\sigma_p^0} - 1\right) \rightarrow \dfrac{a\pi^2}{8}\left(\dfrac{\sigma_\infty}{\sigma_p^0}\right)^2$
  - The J-integral:
    - Since $a_{\text{eff}} = \dfrac{8a}{\pi^2}\left(\dfrac{\sigma_p^0}{\sigma_\infty}\right)^2 \ln\left(\sec\dfrac{\sigma_\infty \pi}{2\sigma_p^0}\right) \rightarrow a + \dfrac{a}{24}\left(\dfrac{\sigma_\infty \pi}{\sigma_p^0}\right)^2$

2nd order term

$$J = \frac{\pi \sigma_\infty^2 a_{\text{eff}}}{E'} = \frac{8a\left(\sigma_p^0\right)^2}{\pi E'} \ln\left(\sec\frac{\sigma_\infty \pi}{2\sigma_p^0}\right) \rightarrow \frac{\pi \sigma_\infty^2 a}{E'} + \frac{\pi \sigma_\infty^2 a}{24E'}\left(\frac{\sigma_\infty \pi}{\sigma_p^0}\right)^2$$





  - If $\sigma_\infty < 40\%$ of σ_p^0, then a first order SSY (or simply SSY) assumption holds
  - If $\sigma_\infty < 60\%$ of σ_p^0, LEFM and SSY can still be used if a_{eff} is considered

- SSY (first order) for Dugdale's model
  - If σ_∞< 40% of σ_p^0 then a first order SSY (or simply SSY) assumption holds
    - The cohesive zone is limited (r_p < 20% of a)
    - For a infinite plate in mode I, the J-integral is reduced to $J = \frac{\pi\sigma_\infty^2 a}{E'} = \frac{K_I^2(a)}{E'}$

      ⟹ SIFs concept of LEFM holds (without correction of the crack size)

      – Size of the plastic (cohesive zone) $r_p \simeq \frac{\pi}{8}\left(\frac{K_I}{\sigma_p^0}\right)^2$
    - As SSY requires r_p < 20% of a, elastic fracture criterion can be applied if

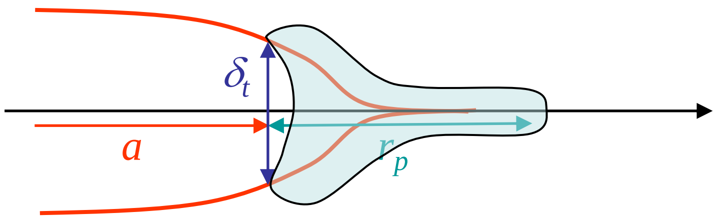
$$a \geq \frac{5\pi}{8}\left(\frac{K_C}{\sigma_p^0}\right)^2 \simeq 2\left(\frac{K_C}{\sigma_p^0}\right)^2$$

      – So crack length has to be large enough compared to the plastic zone
      – The method is actually applicable if the plastic zone is small compared to
        » The crack (a> 5 r_p)
        » The distance from the crack tip to the nearest free surface (L > 5 r_p)
    - If we are well before fracture (beginning of fatigue e.g.), we do not have to use K_C

$$a \geq \frac{5\pi}{8}\left(\frac{K_I}{\sigma_p^0}\right)^2 \simeq 2\left(\frac{K_I}{\sigma_p^0}\right)^2$$

- SSY (first order) for Dugdale's model (2)
  - There is a blunting of the crack tip due to the cohesive zone

$$\delta_t = \frac{J}{\sigma_p^0} = \frac{K_I^2}{E'\sigma_p^0}$$



  - A fracture criterion base on the CTOD can therefore be defined

$$\delta_t = \delta_C = \frac{K_C^2}{E'\sigma_p^0}$$

    - δ_C can be measured experimentally

  - These relations are valid for Dugdale's model only !
    - Thin sheet (plane stress: $r_p >> t$) so $E' = E$
    - Low-C steel (perfectly plastic)
    - Glassy polymers

Effective crack length assumption

- Effective crack length for Dugdale's model
  - If σ_∞< 60% of σ_p^0 then a second order SSY assumption holds
    - The cohesive zone is not limited (exact r_p reached 70% of a)
    - For an infinite plate in mode I, the J-integral is reduced to

⇒ $$J = \frac{K_I^2\left(a_{\text{eff}}\right)}{E'} \quad \text{with} \quad K_I = \sigma_\infty \sqrt{\pi a_{\text{eff}}}$$

⇒ SIFs concept of LEFM holds if corrected by the effective crack size

- As σ_∞< 60% of σ_p^0 & r_p < 70% of a, the effective crack size can be stated as

$$a_{\text{eff}} = a + \frac{r_p}{3} \quad \text{with} \quad \frac{r_p}{3} = \frac{a_{\text{eff}}}{24}\left(\frac{\sigma_\infty \pi}{\sigma_p^0}\right)^2 \simeq \frac{\pi}{24}\left(\frac{K_I\left(a_{\text{eff}}\right)}{\sigma_p^0}\right)^2$$

» Use of $\frac{r_p}{3} = \cancel{\frac{a}{24}\left(\frac{\sigma_\infty \pi}{\sigma_p^0}\right)^2} \simeq \frac{\pi}{24}\left(\frac{K_I\left(a\right)}{\sigma_p^0}\right)^2$ is 1st order accurate

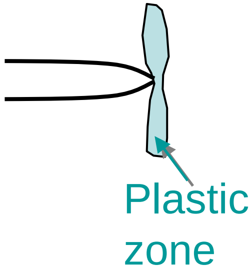
- Expression $\frac{r_p}{3} = \frac{\pi}{24}\left(\frac{K_I\left(a_{\text{eff}}\right)}{\sigma_p^0}\right)^2$ is correct for all cracks in finite plate*

    - So there is an iterative procedure to follow:
      - a) compute K from a
      - b) compute effective crack size
      - c) compute new K from a_{eff} and back to b) if needed
  - These equations are valid for Dugdale's model (see previous slide)

*Edmund & Willis, jmps, 1976, vol. 24, pp. 205 & 225 & 1977, vol. 25, p. 423

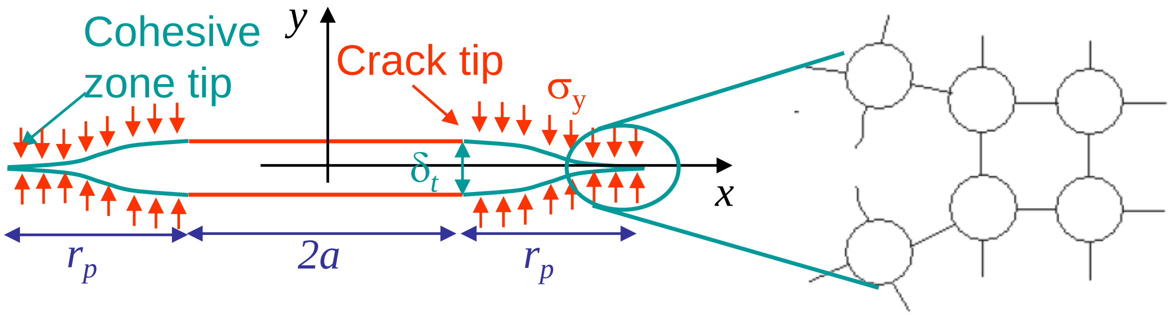
Dugdale's model summary

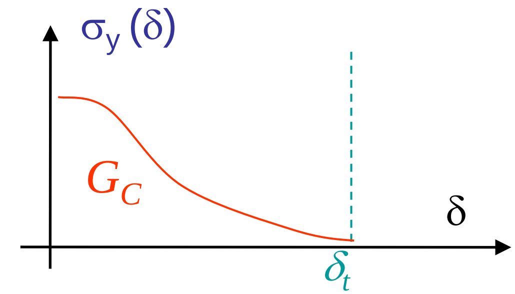
- If σ_∞< 40% of σ_p^0 & $a > 5\, r_p$ then use first order SSY
  - Use classic LEFM
  - Remaining ligament should be > 5 r_p
- If σ_∞< 60% of σ_p^0 & $a > 1.4\, r_p$ then use a second order SSY
  - Use classic LEFM but
    - Correct the crack size to obtain an effective crack size
    - There is an iterative procedure to follow
  - Remaining ligament should be > 1.4 r_p (see lecture on J-integral)
- If σ_∞> 60% of σ_p^0 then use full expression
  - Compute the J –integral (see lecture on J-integral)
    - Ex: for an infinite plane (only) $J = 8\frac{\left(\sigma_p^0\right)^2 a}{E'\pi}\ln\left(\sec\frac{\sigma_\infty \pi}{2\sigma_p^0}\right) \geq J_C$
    - With the new crack growth threshold $J_C = \delta_C\,(\text{Temperature})\,\sigma_p^0$
  - Remaining ligament should be > r_p (see lecture on J-integral)
- These equations are valid for Dugdale's model
  - Thin sheet of low-C steel or glassy polymers
  - For other materials, in plane strain, …
    - The plastic zone has a different shape
    - Another model is required (even for the SSY)



Cohesive elements

- The cohesive method is based on Barenblatt model
  - This model is an idealization of the brittle fracture mechanisms
    - Separation of atoms at crack tips (cleavage)
    - As long as the atoms are not separated by a distance δ_t, there are attractive forces (see overview lecture)

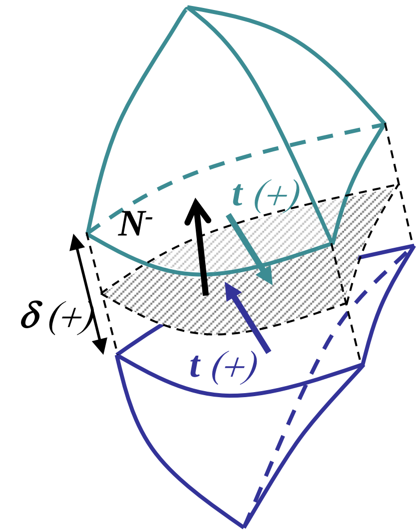


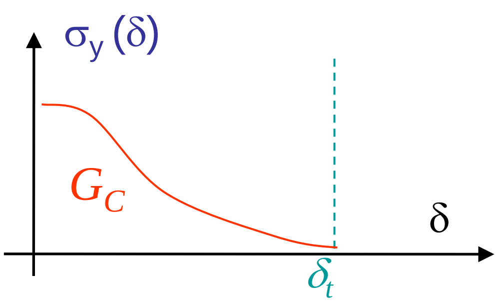


  - For elasticity $J = \int_0^{\delta_t} \sigma_y \left(\delta\right) d\delta$
    - So the area below the σ-δ curve corresponds to G_C if crack grows straight ahead
  - This model requires only 2 parameters
    - Peak cohesive traction σ_{max} (spall strength)
    - Fracture energy G_C
    - Shape of the curves has no importance as long as it is monotonically decreasing

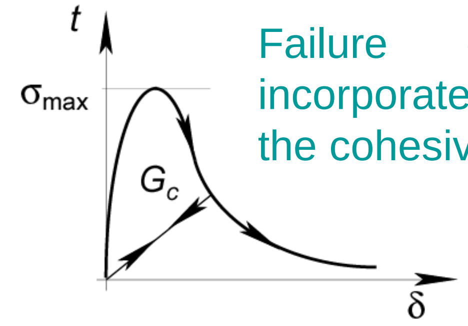
- Insertion of cohesive elements
  - Between 2 volume elements
  - Computation of the opening (cohesive element)
    - Normal to the interface in the deformed configuration $\boldsymbol{N}^-$
    - Normal opening $\delta_n = \max(\llbracket \boldsymbol{u} \rrbracket \cdot \boldsymbol{N}^-,\, 0)$
    - Sliding $\boldsymbol{\delta}_s = \llbracket \boldsymbol{u} \rrbracket - \llbracket \boldsymbol{u} \rrbracket \cdot \boldsymbol{N}^- \boldsymbol{N}^-$
    - Resulting opening $\delta = \sqrt{\delta_n^2 + \beta_c^2 \left\| \boldsymbol{\delta}_s \right\|^2}$

      with β_c the ratio between the shear and normal critical tractions
  - Definition of a potential
    - Potential $\phi = \phi(\delta)$ to match the traction separation law (TSL) curve
    - Traction (in the deformed configuration) derives from this potential $\boldsymbol{t} = \frac{\partial \phi}{\partial \boldsymbol{\delta}} = \frac{\partial \phi}{\partial \delta_n} \boldsymbol{N}^- + \frac{\partial \phi}{\partial \delta_s} \frac{\boldsymbol{\delta_s}}{\delta_s}$



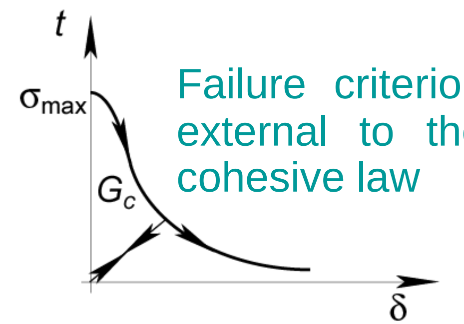


- Computational framework
  - How are the cohesive elements inserted?
  - First method: intrinsic Law
    - Cohesive elements inserted from the beginning
    - So the elastic part prior to crack propagation is accounted for by the TSL
    - Drawbacks:
      - Requires a priori knowledge of the crack path to be efficient
      - Mesh dependency [Xu & Needelman, 1994]
      - Initial slope that modifies the effective elastic modulus
        - » Alteration of a wave propagation
      - This slope should tend to infinity [Klein et al. 2001]
        - » Critical time step is reduced



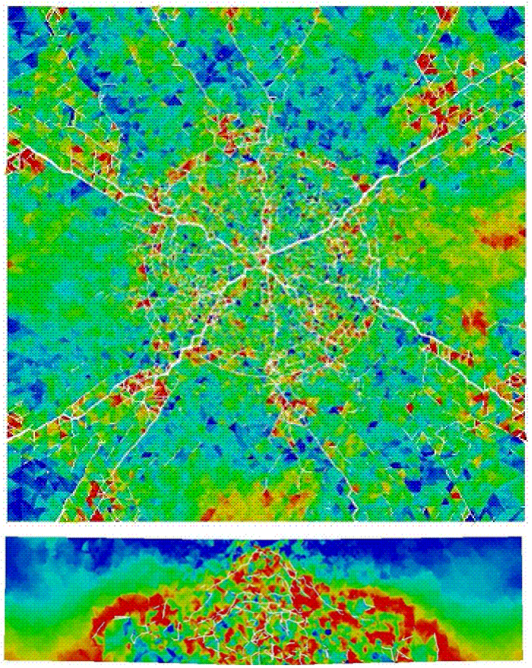
Failure criterion incorporated within the cohesive law

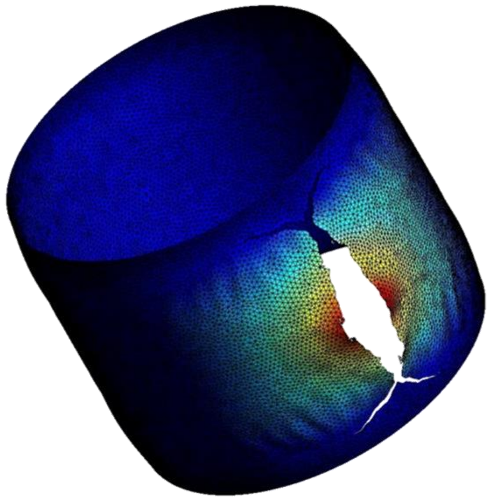
  - Second method: extrinsic law
    - Cohesive elements inserted on the fly when failure criterion ($\sigma > \sigma_{max}$) is verified [Ortiz & Pandolfi 1999]
    - Drawback:
      - Complex implementation in 3D especially for parallelization

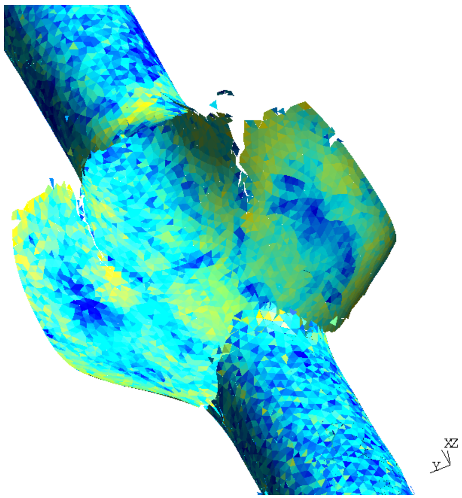


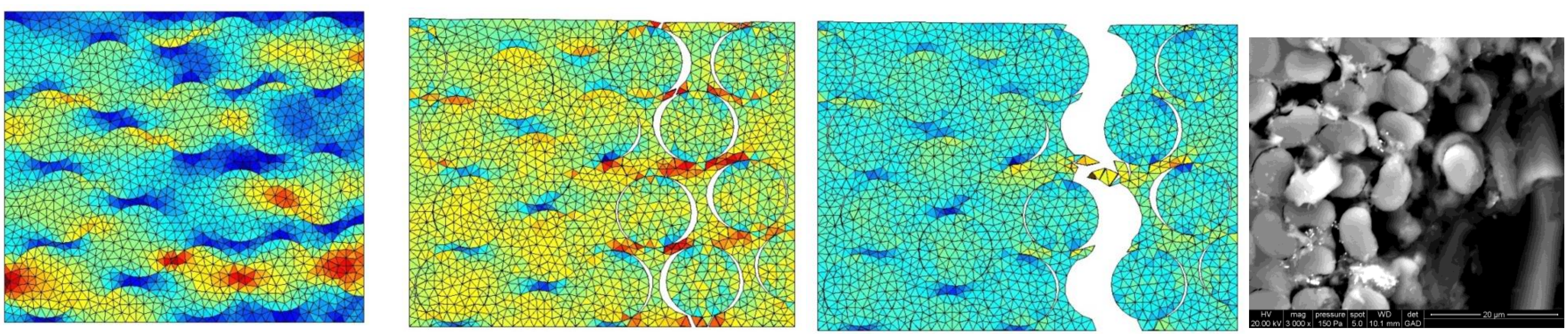
Failure criterion external to the cohesive law

- Examples



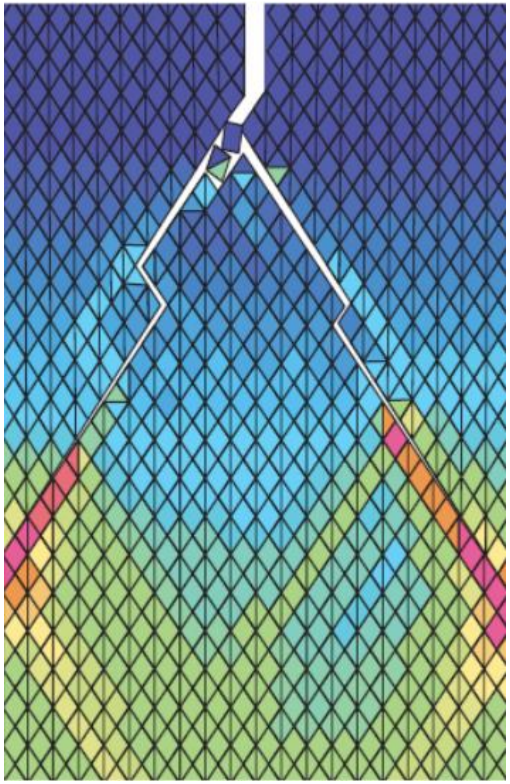




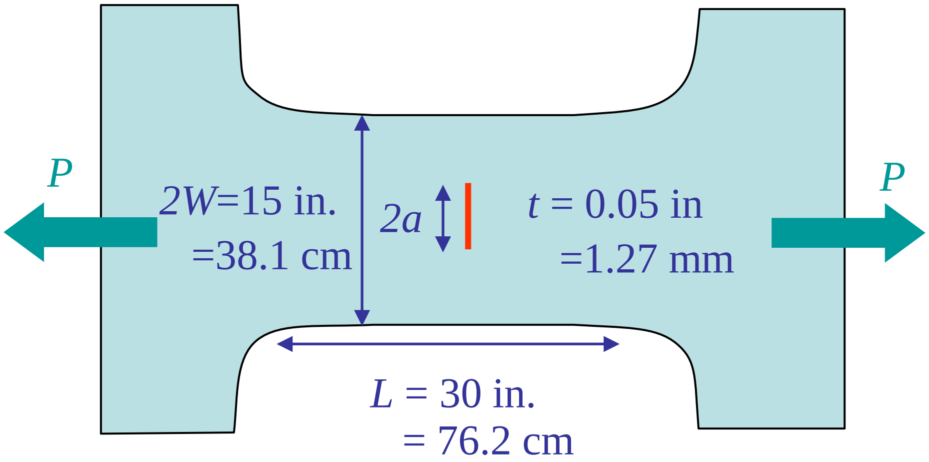


Cohesive elements

- Advantages of the method
  - Can be mesh independent (non regular meshes)
  - Can be used for large problem size
  - Automatically accounts for time scale [Camacho & Ortiz, 1996]
    - Fracture dynamics has not been studied in these classes
  - Really useful when crack path is already known
    - Debonding of fibers
    - Delamination of composite plies
    - …
  - No need for an initial crack
    - The method can detect the initiation of a crack
- Drawbacks
  - Still requires a conforming mesh
  - Requires fine meshes
    - So parallelization is mandatory
  - Could be mesh dependent



- Sheet with centered crack:
  - Mild steel specimen
    - Yield: $\sigma_p^0 = 190$ MPa
    - Young: $E = 210$ GPa



  - What is the limit load in terms of the slit size?
    - Use $\delta_C\ (T_1) = 0.1$ mm as fracture criterion
    - Using Dugdale's mode, compare different methods
      - SSY
      - Effective crack length
      - Full solution
  - What happens if
    - $W = 5$ cm ?
    - $\delta_C\ (T_2) = 0.7$ mm & $\delta_C\ (T_3) = 0.05$ mm ($2W = 15$ in. for these cases) ?

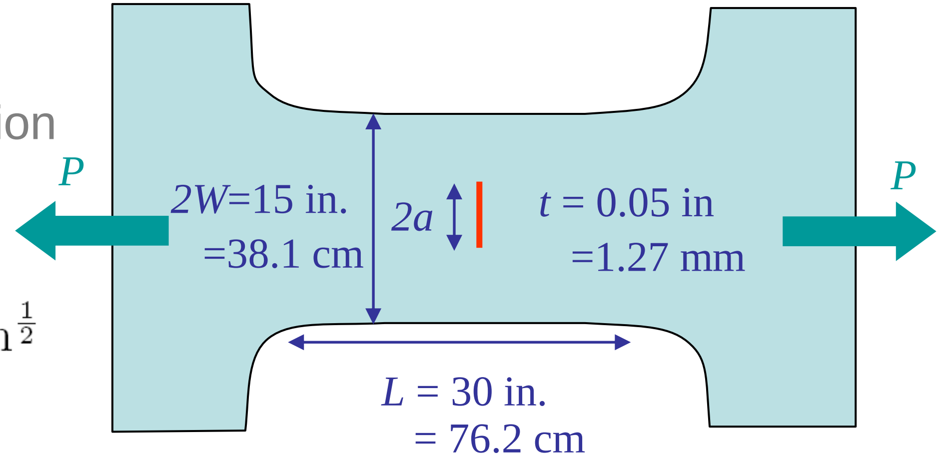
- SSY, $\delta_C\,(T_1) = 0.1$ mm
  - Use $\delta_C = 0.1$ mm as fracture criterion

$$J_C = \delta_C\,(\mathrm{T})\,\sigma_p^0 = 19000\ \mathrm{N}\cdot\mathrm{m}^{-1}$$

  - Plane stress assumption: $E' = E$

$$\Rightarrow K_{I,\,\mathrm{lim}} = \sqrt{E\,J_C} = 63\ \mathrm{MPa}\cdot\mathrm{m}^{\frac{1}{2}}$$

this is not the Plane ε toughness



  - SSY assumption, so use SIF:

$$K_I = \sigma_\infty \sqrt{\pi a} f\left(\frac{a}{W}\right) \text{ with } f\left(\frac{a}{W}\right) = \left[1 - 0.025\left(\frac{a}{W}\right)^2 + 0.06\left(\frac{a}{W}\right)^4\right]\sqrt{\frac{1}{\cos\frac{\pi a}{2W}}}$$

The applied stress leading to failure is therefore

$$\sigma_{\infty,\,\mathrm{lim}} = \frac{K_{I,\,\mathrm{lim}}}{\sqrt{\pi a} f\left(\frac{a}{W}\right)} \text{ with } \sigma_\infty = \frac{P}{2Wt} = 2067\ P\ \mathrm{m}^{-2}$$

N.B. $2Wt = 15 * 0.05 * 0.0254^2\ \mathrm{m}^2 = 1/2067\ \mathrm{m}^2$

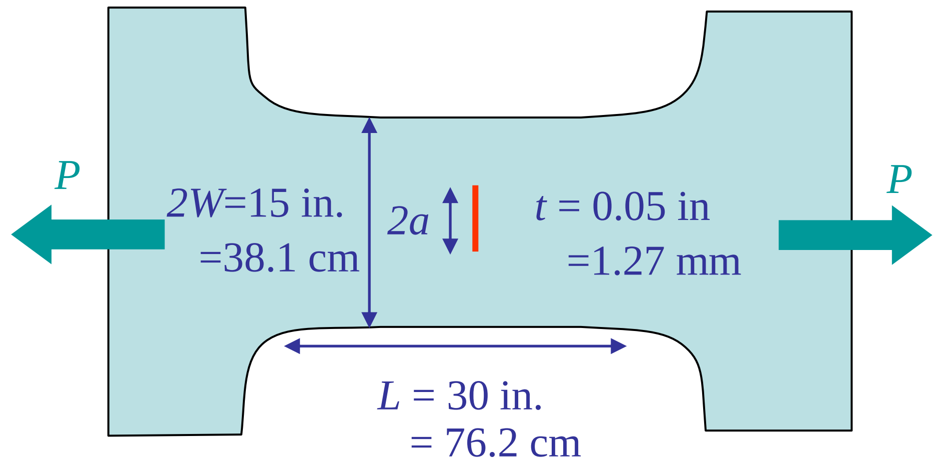
  - Example: limit loading for a semi-crack length of 9 cm

$$P\,(a = 9\,\mathrm{cm}) = \frac{63.10^6}{2067\sqrt{\pi 0.09}\ f\,(0.4724)}\mathrm{N} = \frac{63.10^6}{2067\sqrt{\pi 0.09}\ 1.1618}\mathrm{N} = 49337\,\mathrm{N}$$

- SSY, δ_C (T_1) = 0.1 mm (2)
  - In terms of semi-crack length a:

$$\sigma_{\infty,\,\lim} = \frac{K_{I,\,\lim}}{\sqrt{\pi a} f\left(\frac{a}{W}\right)}$$

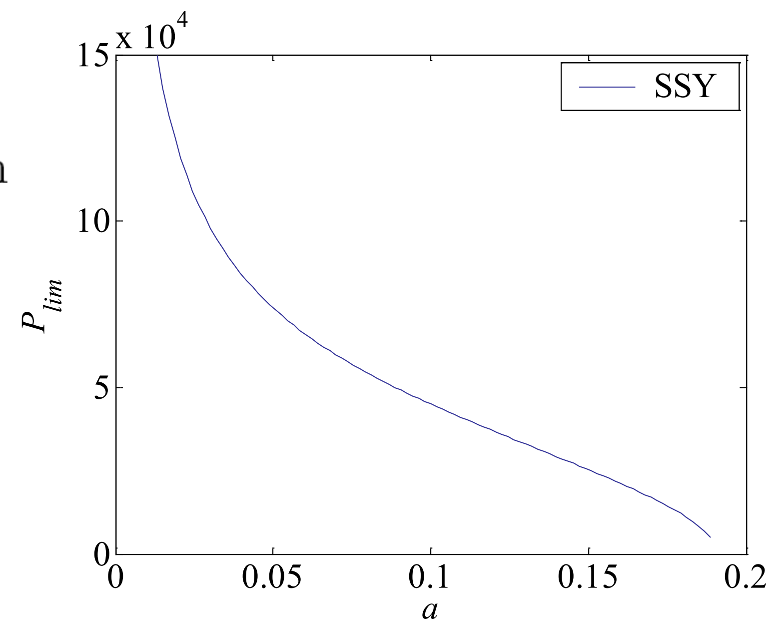
$$\sigma_\infty = \frac{P}{2Wt} = 2067\ P\ \ \mathrm{m}^{-2}$$



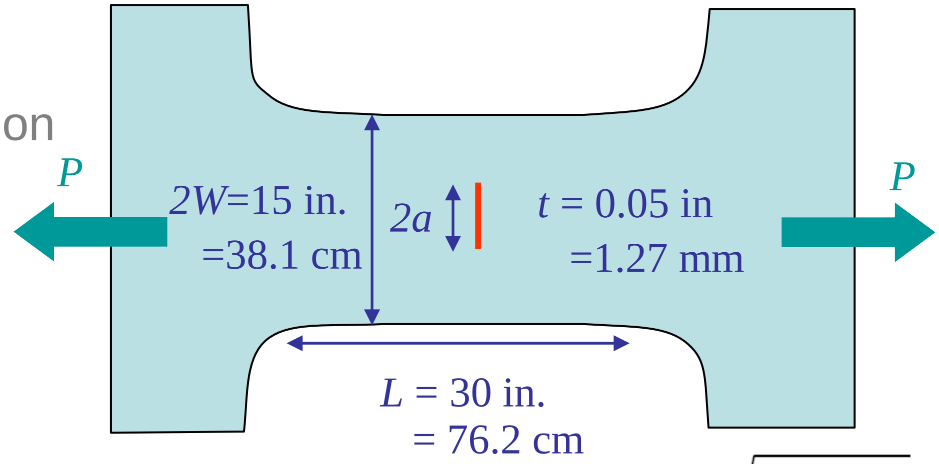
  - Validity ?
    - Plastic zone length at fracture

$$r_p \simeq \frac{\pi}{8}\left(\frac{K_{I,\,\lim}}{\sigma_p^0}\right)^2 = \frac{\pi\ 63^2}{8\ 190^2} = 0.0434\,\mathrm{m}$$

      - Plane σ?
        - » t = 1.3 cm < r_p OK
      - But a & W-a have to be > 5 r_p
        - » Impossible



Exercise: Solution (3)



- Effective length, δ_C (T_1) = 0.1 mm
  - Use δ_C = 0.1 mm as fracture criterion

$$K_{I,\,\mathrm{lim}} = \sqrt{E\, J_C} = 63\ \mathrm{MPa}\cdot\mathrm{m}^{\frac{1}{2}}$$

  - Effective length assumption,
    - So use effective SIF:

$$K_{I\,,\mathrm{eff}} = \sigma_\infty \sqrt{\pi a_{\mathrm{eff}}} f\left(\frac{a_{\mathrm{eff}}}{W}\right) \text{ with}$$

$$f\left(\frac{a}{W}\right) = \left[1 - 0.025\left(\frac{a}{W}\right)^2 + 0.06\left(\frac{a}{W}\right)^4\right]\sqrt{\frac{1}{\cos\frac{\pi a}{2W}}}$$

    - The applied stress leading to failure is therefore

$$\sigma_{\infty,\,\mathrm{lim}} = \frac{K_{I,\,\mathrm{lim}}}{\sqrt{\pi a_{\mathrm{eff}}} f\left(\frac{a_{\mathrm{eff}}}{W}\right)} \text{ with } \sigma_\infty = \frac{P}{2Wt} = 2067\ P\ \ \mathrm{m}^{-2}$$

    - The effective crack length is computed by

$$a_{\mathrm{eff}} = a + \frac{r_p}{3} = a + \frac{\pi}{24}\left(\frac{K_{I,\,\mathrm{lim}}}{\sigma_p^0}\right)^2 = a + \frac{\pi\ 63^2}{24\ 190^2} = a + 0.0145\,\mathrm{m}$$

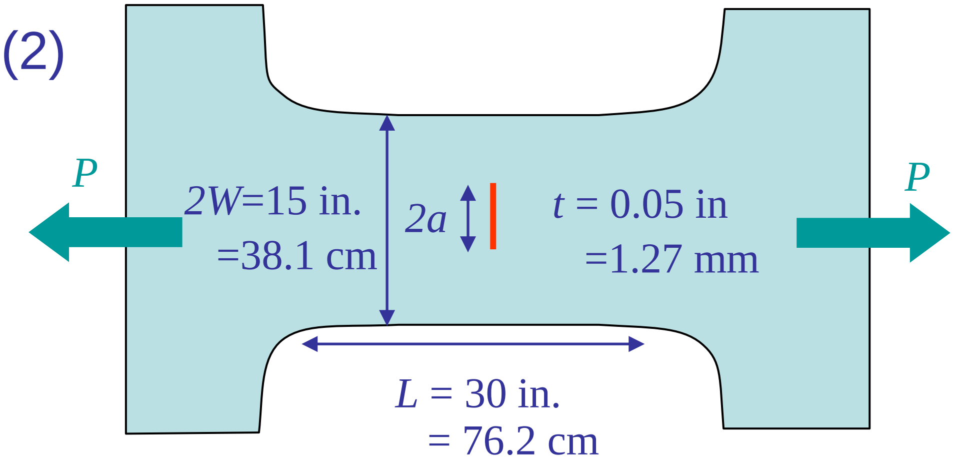
    - Example a = 9 cm

$$P\,(a = 9\,\mathrm{cm}) = \frac{63.10^6}{2067\sqrt{\pi 0.1045}\ f\,(0.5486)}\mathrm{N} = \frac{63.10^6}{2067\sqrt{\pi 0.1045}\ 1.2366}\mathrm{N} = 43017\,\mathrm{N}$$

- Effective length, δ_C (T_1) = 0.1 mm (2)
  - In terms of crack length a:

$$\sigma_{\infty,\,\mathrm{lim}} = \frac{K_{I,\,\mathrm{lim}}}{\sqrt{\pi a_{\mathrm{eff}}} f\left(\frac{a_{\mathrm{eff}}}{W}\right)}$$

$$\sigma_\infty = \frac{P}{2Wt} = 2067\ P\ \ \mathrm{m}^{-2}$$

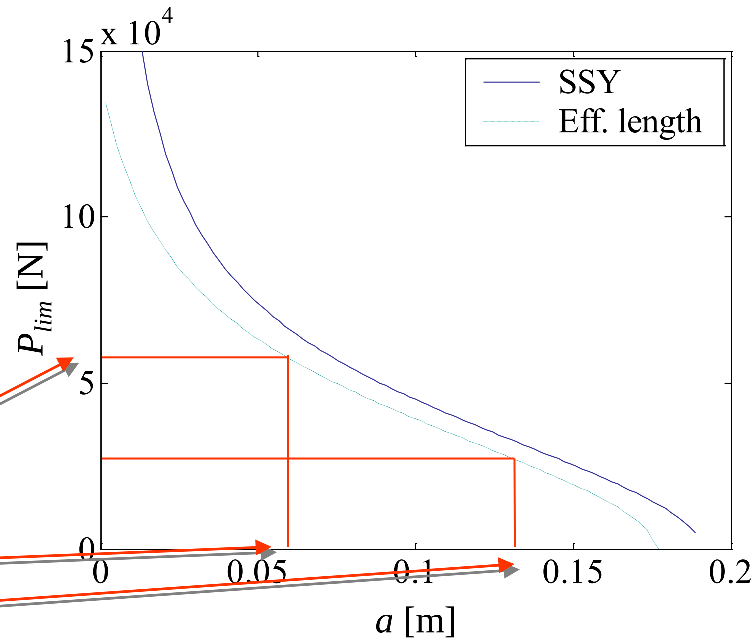


  - Validity ?
    - Limit loading for a crack free sample

$$P_{\mathrm{yield}} = \sigma_p^0 2Wt = 91935\,\mathrm{N}$$

    - Plastic zone length at fracture

$$r_p \simeq \frac{\pi}{8}\left(\frac{K_{I,\,\mathrm{lim}}}{\sigma_p^0}\right)^2 = \frac{\pi\ 63^2}{8\ 190^2} = 0.0434\,\mathrm{m}$$



      - Plane σ?
        - » t = 1.3 cm < r_p OK
      - We want σ_∞<0.6 σ_p^0
      - We want a > 1.4 r_p ~ 6.1 cm
      - We want W-a > 1.4 r_p

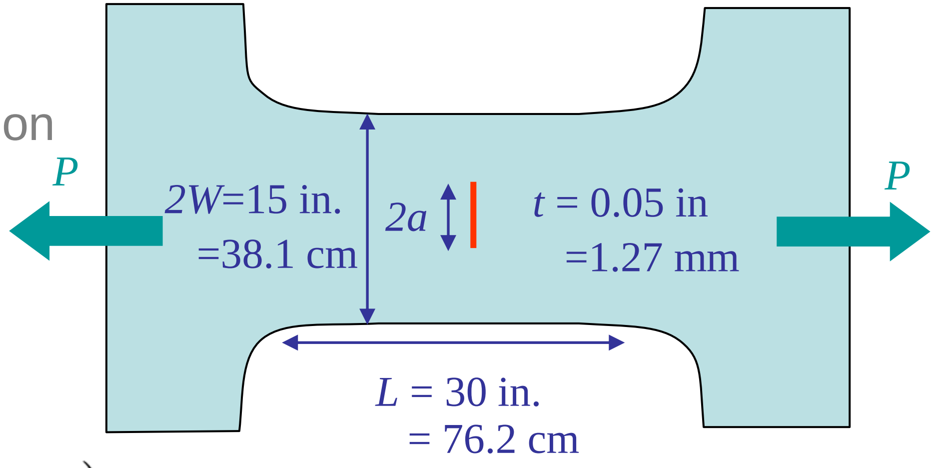
⇒ The solution is accurate only for cracks a little larger than 6 cm

Exercise: Solution (5)

- Full solution, δ_C (T$_1$) = 0.1 mm
  - Use δ_C = 0.1 mm as fracture criterion

$$J_C = \delta_C\left(\mathrm{T}\right)\sigma_p^0 = 19000\ \mathrm{N}\cdot\mathrm{m}^{-1}$$

  - How can we compute J ?
    - Assuming $a << W$, the infinite plate solution holds



$$J_{\mathrm{Dugdale}} = \sigma_p^0\delta_t = 8\frac{\left(\sigma_p^0\right)^2 a}{E\pi}\ln\left(\sec\frac{\sigma_\infty\pi}{2\sigma_p^0}\right)$$

$$\Rightarrow \sigma_{\infty,\,\mathrm{lim}} = \frac{2\sigma_p^0}{\pi}\arccos\left(\exp\frac{-EJ_C\pi}{8a\left(\sigma_p^0\right)^2}\right) \quad \& \quad \boldsymbol{\sigma}_\infty = \frac{P}{2Wt} = 6200\ P\ \ \mathrm{m}^{-2}$$

- Example a = 1 cm

$$P\left(a=1\,\mathrm{cm}\right) = \frac{2\ 190.10^6}{2067\pi}\arccos\left(\exp\frac{-210.10^9\ 19.10^3\pi}{8\ 0.01\left(190.10^6\right)^2}\right) = 91173\mathrm{N}$$

Exercise: Solution (6)

- Full solution, δ_C (T_1) = 0.1 mm (2)
  - In terms of crack length

$$\sigma_{\infty,\,\mathrm{lim}} = \frac{2\sigma_p^0}{\pi}\,\mathrm{arcos}\left(\exp\frac{-EJ_C\pi}{8a\left(\sigma_p^0\right)^2}\right)$$

$$\sigma_\infty = \frac{P}{2Wt} = 2067\ P\ \ \mathrm{m}^{-2}$$

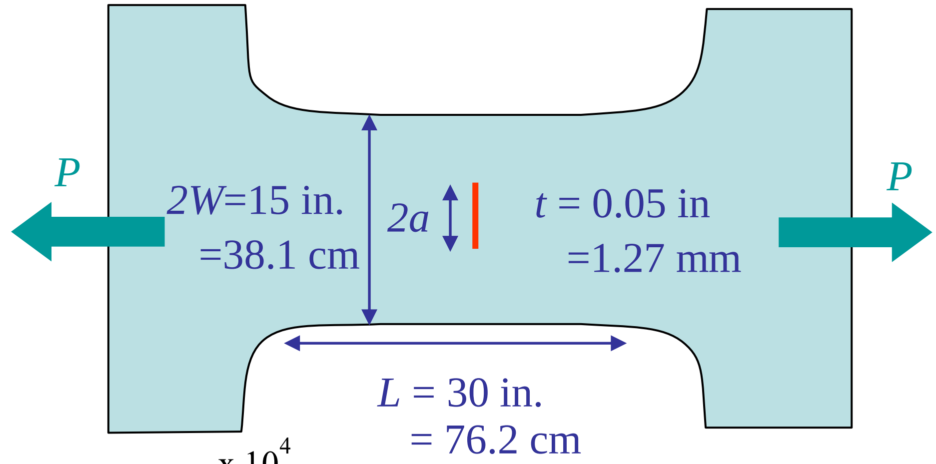
  - Validity ?
    - Limit loading for a crack free sample

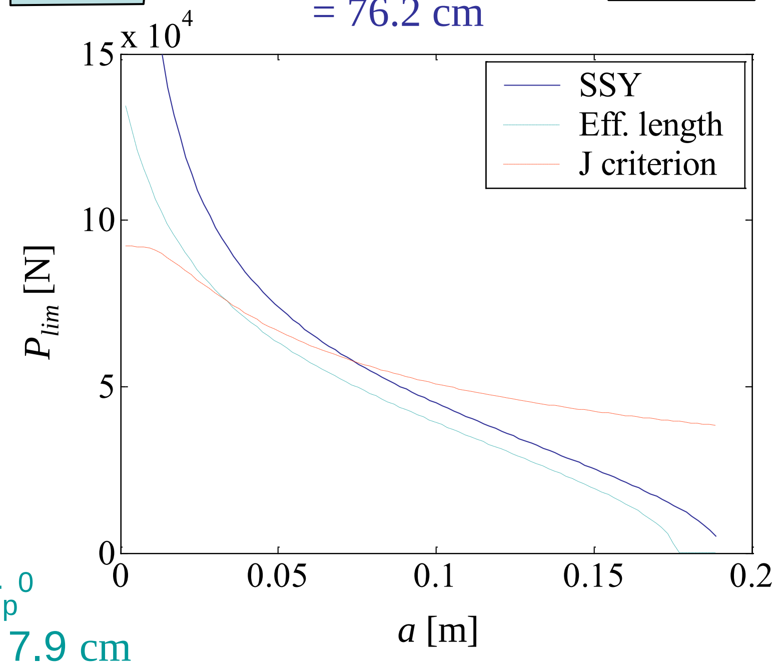
$$P_{\mathrm{yield}} = \sigma_p^0 2Wt = 91935\,\mathrm{N}$$

    - Plastic zone length at fracture

$$r_p = a\left(\sec\frac{\sigma_\infty\pi}{2\sigma_p^0} - 1\right)$$

      - $r_p(a=1\ \mathrm{cm}) \sim 75\ \mathrm{cm}$!
      - This is due to the infinite plate approximation when using $\sigma_\infty \rightarrow \sigma_p^0$
      - For a=4 cm : P_{lim} = 71700 N & r_p = 7.9 cm

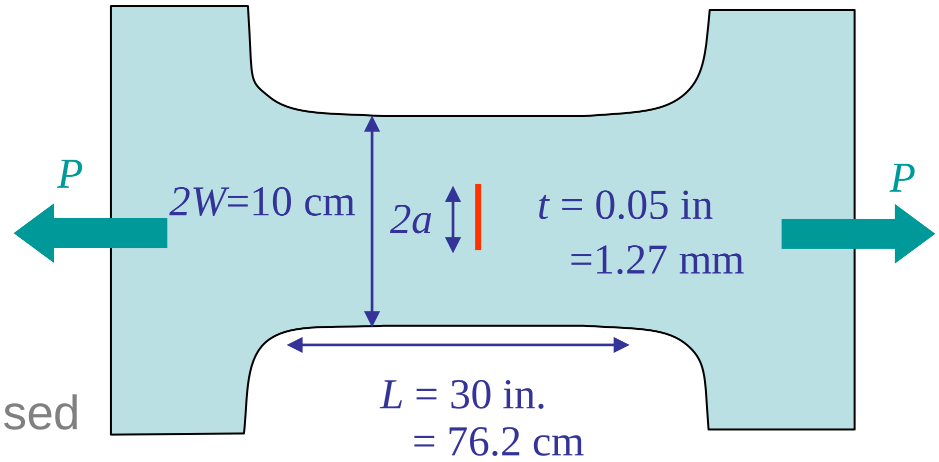




- So the correct J integral and the correct r_p should be evaluated (see lecture on J)
- The expression of an infinite plate can be used only below limit loading

- Case $W = 5$ cm
  - In this case,
    - Neither SSY
    - Nor the effective length method

    is possible as $W \sim$ predicted r_p
  - So the exact J-integral has to be used



Exercise: Solution (8)

- Case W = 19.05 cm & δ_C (T_2) = 0.7 mm
  - SIF limit?

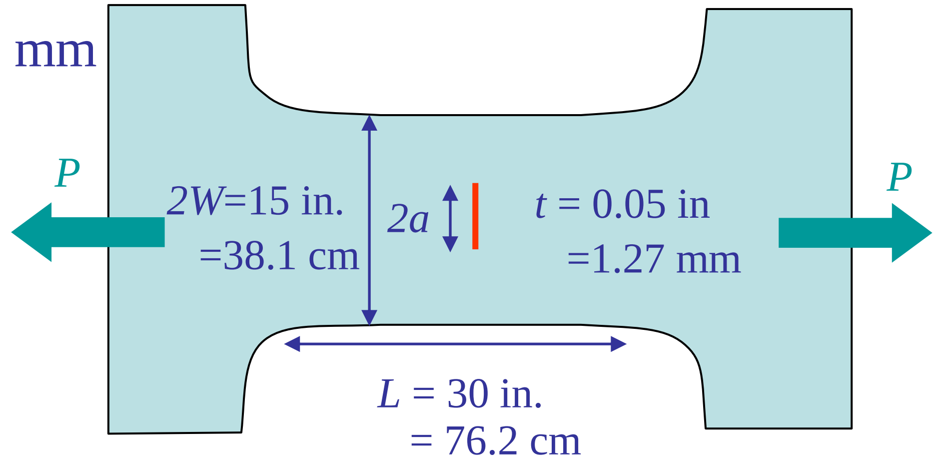
$$J_C = \delta_C\left(\mathrm{T}\right)\sigma_p^0 = 133000\ \mathrm{N}\cdot\mathrm{m}^{-1}$$

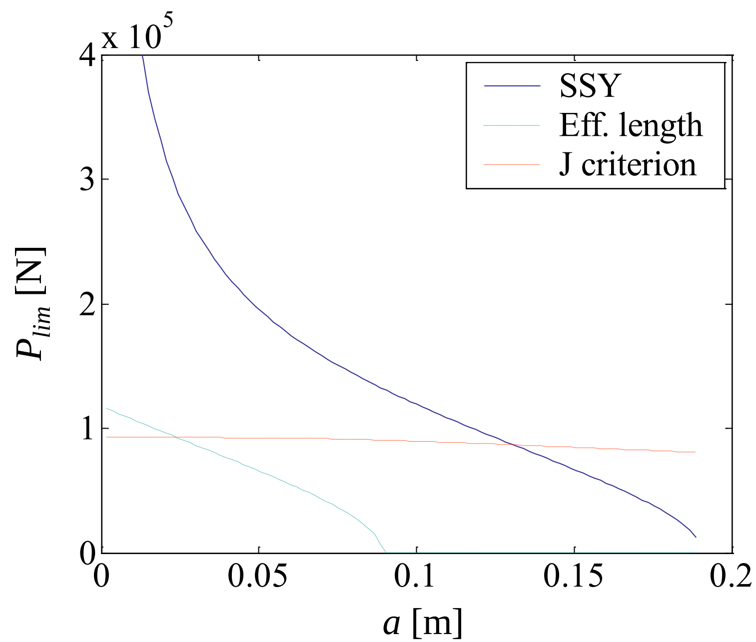
$$K_{I,\,\mathrm{lim}} = \sqrt{E\,J_C} = 167\ \mathrm{MPa}\cdot\mathrm{m}^{\frac{1}{2}}$$

$$r_p \simeq \frac{\pi}{8}\left(\frac{K_{I,\,\mathrm{lim}}}{\sigma_p^0}\right)^2 = 0.3\,\mathrm{m}$$

  - In this case,
    - Neither SSY
    - Nor the effective length method

    is possible as W ~ predicted r_p
  - So the exact J-integral would have to be used



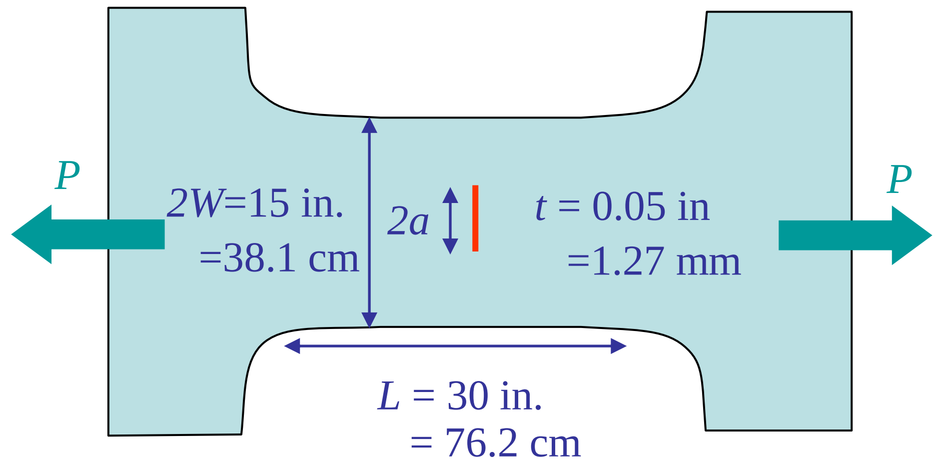


- Case W = 19.05 cm & δ_C (T_3) = 0.05 mm
  - SIF limit?

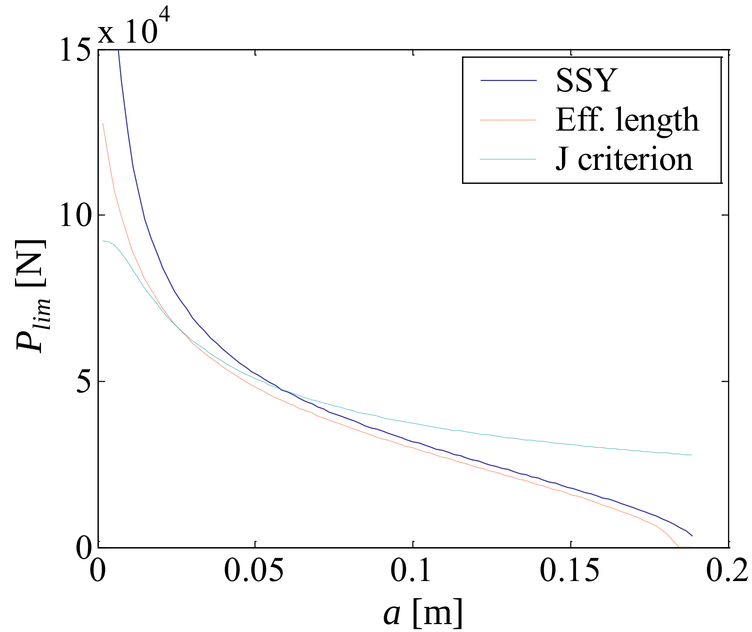
$$J_C = \delta_C\,(\mathrm{T})\,\sigma_p^0 = 9500\ \mathrm{N}\cdot\mathrm{m}^{-1}$$

$$K_{I,\,\mathrm{lim}} = \sqrt{E\,J_C} = 45\ \mathrm{MPa}\cdot\mathrm{m}^{\frac{1}{2}}$$

$$r_p \simeq \frac{\pi}{8}\left(\frac{K_{I,\,\mathrm{lim}}}{\sigma_p^0}\right)^2 = 0.021\,\mathrm{m}$$



  - In this case,
    - Effective length method is very accurate for cracks between 2.9 and ~ 14 cm (as it is required W-a>$1.4\ r_p$)
    - Compared to effective length method, SSY overestimates the limit loading
      - For a=10 cm: by less than 10%
      - For a=3 cm: by about 17%



Exercise: Summary

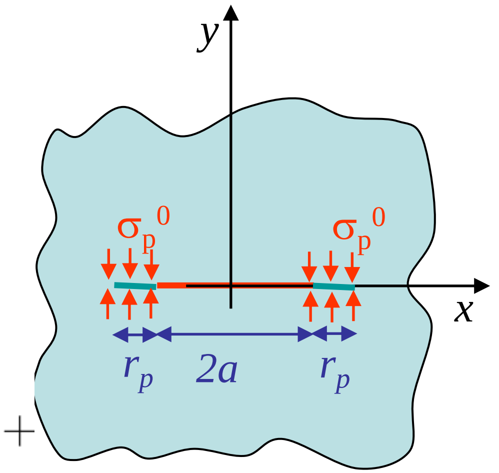
- For ductile materials
  - SSY is almost never a good approximation
    - The plastic zone is too large compared to crack size and remaining ligament
    - Except in very low loading
  - Effective length scale
    - Can be used for
      - Critical loading estimation
        - » For large specimen and
        - » If crack length is in a specific range
      - Computation of SIF for fatigue
        - » As K is reduced, the plastic zone size is reduced
        - » So the crack length validity range is increased
  - J-integral can be used, but
    - The infinite plane approximation is a good approximation only
      - If loading < 90% of yield
      - Crack size and plastic zone size << W
    - Exact solution can be computed (see lecture on J-integral)

References

- Lecture notes
  - Lecture Notes on Fracture Mechanics, Alan T. Zehnder, Cornell University, Ithaca, http://hdl.handle.net/1813/3075
- Other references
  - « on-line »
    - Fracture Mechanics, Piet Schreurs, TUe, http://www.mate.tue.nl/~piet/edu/frm/sht/bmsht.html
  - Book
    - Fracture Mechanics: Fundamentals and applications, D. T. Anderson. CRC press, 1991.
    - Fatigue of Materials, S. Suresh, Cambridge press, 1998.

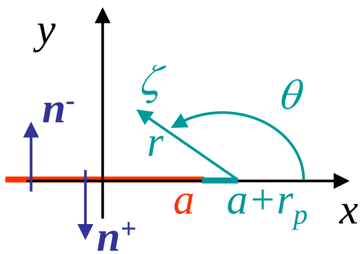
Appendix 1: BCs of Dugdale's cohesive zone/Yielding strip Model

- Case 3: Check boundary conditions
  - For $\zeta = x \pm |\varepsilon|\, i$, with $x < a + r_p$



    - Using $\lim_{\theta\to\pm\pi} \sqrt{\zeta - a - r_p} = \pm i\sqrt{r} = \pm i\sqrt{a + r_p - x}$
    - The function

$$\Omega' = -\frac{\sigma_p^0}{\pi}\frac{\zeta}{\sqrt{\zeta^2 - (a+r_p)^2}}\operatorname{arcotan}\frac{a}{\sqrt{r_p^2 + 2ar_p}} + \frac{i\sigma_p^0}{2\pi}\ln\frac{\frac{a}{\zeta}\sqrt{\frac{\zeta^2-(a+r_p)^2}{r_p^2+2ar_p}} - i}{\frac{a}{\zeta}\sqrt{\frac{\zeta^2-(a+r_p)^2}{r_p^2+2ar_p}} + i}$$

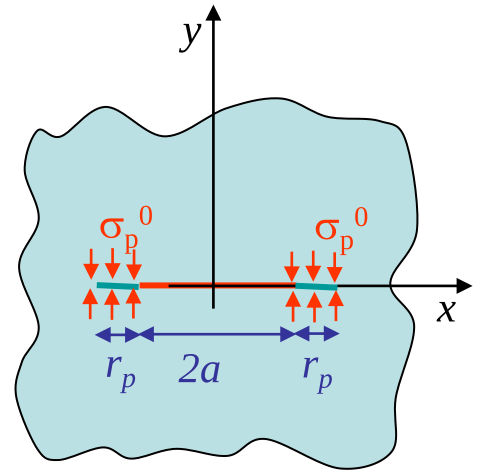


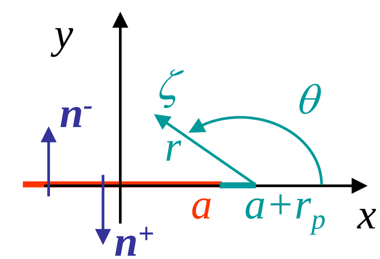
is rewritten

$$\Omega'(\theta \to \pm\pi) = \pm i\frac{\sigma_p^0}{\pi}\frac{x}{\sqrt{(a+r_p)^2 - x^2}}\operatorname{arcotan}\frac{a}{\sqrt{r_p^2 + 2ar_p}} + \frac{i\sigma_p^0}{2\pi}\ln\frac{\pm\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}} - 1}{\pm\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}} + 1}$$

- Case 3: Check boundary conditions (2)
  - For $\zeta = x \pm |\varepsilon|\, i$, with $x < a + r_p$ (2)
    - Let us assume $y > 0$, (solution for $y < 0$ is symmetric)

$$\Omega'(\theta \to \pi) = i\frac{\sigma_p^0}{\pi}\frac{x}{\sqrt{(a+r_p)^2 - x^2}}\operatorname{arcotan}\frac{a}{\sqrt{r_p^2 + 2ar_p}} + \frac{i\sigma_p^0}{2\pi}\ln\frac{\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}} - 1}{\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}} + 1}$$





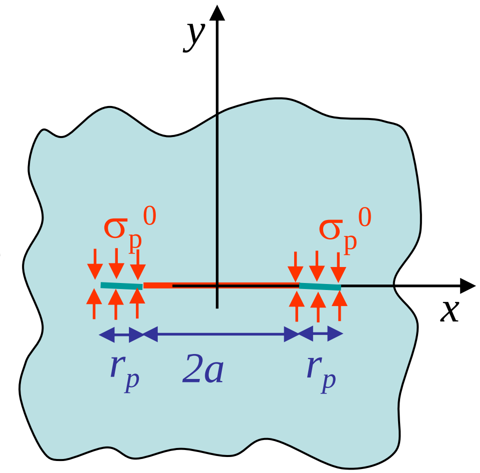
- When is $\ln$ is real?
  - This is the case if $\frac{a}{x}\sqrt{\frac{(a+r_p)^2 - x^2}{r_p^2 + 2ar_p}} > 1,$

    or again if $\left(a^2 - x^2\right)\left(a + r_p\right)^2 > 0$, so if $x < a$
  - Then on the crack lips, the boundary condition

$$\mathbf{0} = \boldsymbol{t} = \boldsymbol{n}\cdot\boldsymbol{\sigma} = \begin{pmatrix} 0 \\ \mp 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} 0 \\ \mp\sigma_{yy} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mp 2\mathcal{R}(\Omega') \\ 0 \end{pmatrix}$$

    is satisfied

- Case 3: Check boundary conditions (3)
  - For $\zeta = x \pm |\varepsilon|\, i$, with $a < x < a+r_p$
    - $\ln$ is complex, and



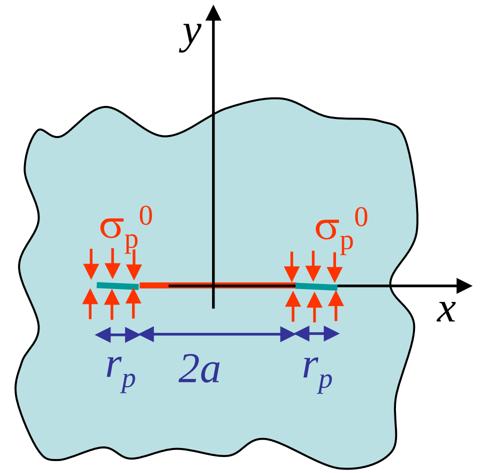
$$\Omega'(\theta \to \pi) = i\frac{\sigma_p^0}{\pi}\frac{x}{\sqrt{(a+r_p)^2 - x^2}}\,\mathrm{arcotan}\,\frac{a}{\sqrt{r_p^2 + 2ar_p}} + \frac{i\sigma_p^0}{2\pi}\ln\left(a\sqrt{\frac{(a+r_p)^2 - x^2}{r_p^2 + 2ar_p}} - x\right) - \frac{i\sigma_p^0}{2\pi}\ln\left(a\sqrt{\frac{(a+r_p)^2 - x^2}{r_p^2 + 2ar_p}} + x\right)$$

- But on upper lip $\ln z = \ln|z| + i\arg z$ & $\arg\left(a\sqrt{\frac{(a+r_p)^2 - \zeta^2}{r_p^2 + 2ar_p}} - \zeta\right) = -\pi$

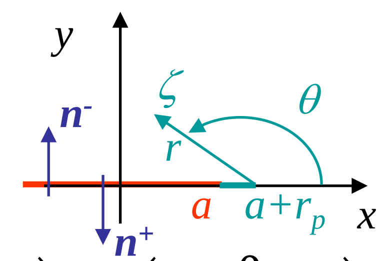
then $$\Omega'(\theta \to \pi) = i\frac{\sigma_p^0}{\pi}\frac{x}{\sqrt{(a+r_p)^2 - x^2}}\,\mathrm{arcotan}\,\frac{a}{\sqrt{r_p^2 + 2ar_p}} + \frac{\sigma_p^0}{2} + \frac{i\sigma_p^0}{2\pi}\ln\frac{1 - \frac{a}{x}\sqrt{\frac{(a+r_p)^2 - x^2}{r_p^2 + 2ar_p}}}{\frac{a}{x}\sqrt{\frac{(a+r_p)^2 - x^2}{r_p^2 + 2ar_p}} + 1}$$

Appendix 1: BCs of Dugdale's cohesive zone/Yielding strip Model

- Case 3: Check boundary conditions (4)
  - For $\zeta = x \pm |\varepsilon|\, i$, with $a < x < a + r_p$ (2)



$$\Omega'(\theta \to \pm\pi) = \pm i \frac{\sigma_p^0}{\pi} \frac{x}{\sqrt{(a+r_p)^2 - x^2}} \operatorname{arcotan} \frac{a}{\sqrt{r_p^2 + 2ar_p}} \pm \frac{\sigma_p^0}{2} \pm \frac{i\sigma_p^0}{2\pi} \ln \frac{1 - \frac{a}{x}\sqrt{\frac{(a+r_p)^2 - x^2}{r_p^2 + 2ar_p}}}{\frac{a}{x}\sqrt{\frac{(a+r_p)^2 - x^2}{r_p^2 + 2ar_p}} + 1}$$



- As the $\ln$ is real, the boundary condition on the crack lips

$$\begin{pmatrix} 0 \\ \mp\sigma_p^0 \\ 0 \end{pmatrix} = \boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{\sigma} = \begin{pmatrix} 0 \\ \mp 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} 0 \\ \mp\sigma_{yy} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mp 2\mathcal{R}(\Omega') \\ 0 \end{pmatrix}$$

is satisfied

- Case 3: Check boundary conditions (5)
  - Far away from the crack:

$$\lim_{\zeta\to\infty} \Omega' \quad = \quad -\frac{\sigma_p^0}{\pi} \operatorname{arcotan} \frac{a}{\sqrt{r_p^2 + 2ar_p}} + \frac{\sigma_p^0}{\pi} \operatorname{arcotan} \frac{a}{\sqrt{r_p^2 + 2ar_p}} = 0$$

& so for $\lim_{|\zeta|\to\infty} \Omega'' = 0$

$\Longrightarrow$ $\boldsymbol{\sigma}_{xx}(\zeta\to\infty) \to 0$ & $\boldsymbol{\sigma}_{yy}(\zeta\to\infty) \to 0$ satisfied

  - Symmetry with respect to Ox?

$$\left.\begin{array}{l} \Omega\left(\bar{\zeta}\right) = \bar{\Omega} \\ \Omega'\left(\bar{\zeta}\right) = \bar{\Omega}' \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \mathcal{R}\left(\Omega\left(x - iy\right)\right) = \mathcal{R}\left(\Omega\left(x + iy\right)\right) \\ \mathcal{R}\left(\Omega'\left(x - iy\right)\right) = \mathcal{R}\left(\Omega'\left(x + iy\right)\right) \\ \mathcal{I}\left(\Omega\left(x - iy\right)\right) = -\mathcal{I}\left(\Omega\left(x + iy\right)\right) \\ \mathcal{I}\left(\Omega'\left(x - iy\right)\right) = -\mathcal{I}\left(\Omega'\left(x + iy\right)\right) \end{array}\right.$$

$\Longrightarrow$ $\boldsymbol{u}_x(-y) = \boldsymbol{u}_x(y)$ & $\boldsymbol{u}_y(-y) = -\boldsymbol{u}_y(y)$ satisfied

- Case 3: Check boundary conditions (6)
  - Symmetry with respect to *Oy*?
    - For $x=0$

$$
\begin{cases}
\Omega\left(x=0\right) = \mp\dfrac{i\sigma_p^0}{\pi}\sqrt{y^2+\left(a+r_p\right)^2}\,\text{arcotan}\,\dfrac{a}{\sqrt{r_p^2+2ar_p}} - \\
\dfrac{\sigma_p^0 a}{\pi}\,\text{arcotan}\left(\pm i\sqrt{\dfrac{y^2+\left(a+r_p\right)^2}{r_p^2+2ar_p}}\right) + \dfrac{\sigma_p^0 iy}{\pi}\,\text{arcotan}\left(\pm\dfrac{a}{y}\sqrt{\dfrac{y^2+\left(a+r_p\right)^2}{r_p^2+2ar_p}}\right) \\
\Omega'\left(x=0\right) \quad = \quad \dfrac{\pm\sigma_p^0}{\pi}\dfrac{y}{\sqrt{y^2+\left(a+r_p\right)^2}}\,\text{arcotan}\,\dfrac{a}{\sqrt{r_p^2+2ar_p}} + \\
\qquad\qquad \dfrac{\sigma_p^0}{\pi}\,\text{arcotan}\left(\pm\dfrac{a}{y}\sqrt{\dfrac{y^2+\left(a+r_p\right)^2}{r_p^2+2ar_p}}\right)
\end{cases}
$$

- $\mathcal{R}(\Omega) = 0$ and $\mathcal{I}(\Omega')=0$ $\Longrightarrow$ $\boldsymbol{u}_x(x=0) = 0$ satisfied

- Case 3: Determine Ω form Ω'
  - The integration of

$$\Omega' = -\frac{\sigma_p^0}{\pi}\frac{\zeta}{\sqrt{\zeta^2-(a+r_p)^2}}\operatorname{arcotan}\frac{a}{\sqrt{r_p^2+2ar_p}}+\frac{\sigma_p^0}{\pi}\operatorname{arcotan}\left(\frac{a}{\zeta}\sqrt{\frac{\zeta^2-(a+r_p)^2}{r_p^2+2ar_p}}\right)$$

leads to
$$\Omega = -\frac{\sigma_p^0}{\pi}\sqrt{\zeta^2-(a+r_p)^2}\operatorname{arcotan}\frac{a}{\sqrt{r_p^2+2ar_p}}-$$
$$\frac{\sigma_p^0 a}{\pi}\operatorname{arcotan}\sqrt{\frac{\zeta^2-(a+r_p)^2}{r_p^2+2ar_p}}+\frac{\sigma_p^0\zeta}{\pi}\operatorname{arcotan}\left(\frac{a}{\zeta}\sqrt{\frac{\zeta^2-(a+r_p)^2}{r_p^2+2ar_p}}\right)$$

  - This result follows from direct derivation, using $d_\zeta \operatorname{arcotan}\zeta = -\frac{1}{1+\zeta^2}$,

which yields
$$\Omega' = -\frac{\sigma_p^0}{\pi}\frac{\zeta}{\sqrt{\zeta^2-(a+r_p)^2}}\operatorname{arcotan}\frac{a}{\sqrt{r_p^2+2ar_p}}+\cancel{\frac{\sigma_p^0 a\zeta}{\pi\left(\zeta^2-a^2\right)}\sqrt{\frac{r_p^2+2ar_p}{\zeta^2-\left(a+r_p^2\right)^2}}}+$$
$$\frac{\sigma_p^0}{\pi}\operatorname{arcotan}\left(\frac{a}{\zeta}\sqrt{\frac{\zeta^2-(a+r_p)^2}{r_p^2+2ar_p}}\right)-\cancel{\frac{\sigma_p^0}{\pi}\frac{\zeta^3\left(r_p^2+2ar_p\right)}{\zeta^2\left(r_p^2+2ar_p\right)-a^2\left(a+r_p\right)^2+a^2\zeta^2}\left[-\frac{a}{\zeta^2}\sqrt{\frac{\zeta^2-(a+r_p)^2}{r_p^2+2ar_p}}+\frac{a}{\sqrt{r_p^2+2ar_p}\sqrt{\zeta^2-(a+r_p)^2}}\right]}$$

- Displacement field of case 3
  - Displacement field on crack lips for $\zeta = x+|\varepsilon|\, i$, with $x < a+r_p$
    - Displacement field on upper crack lips is obtained from

$$\boldsymbol{u}_y\left(\theta\rightarrow\pi\right)=\frac{\left(1+\nu\right)\left(1+\kappa\right)}{E}\mathcal{I}\Omega\left(\theta\rightarrow\pi\right)$$

with $$\Omega\left(\theta\rightarrow\pi\right)=-\frac{\sigma_p^0 i}{\pi}\sqrt{\left(a+r_p\right)^2-x^2}\,\mathrm{arcotan}\,\frac{a}{\sqrt{r_p^2+2ar_p}}-$$

$$\frac{\sigma_p^0 a}{\pi}\,\mathrm{arcotan}\left(i\sqrt{\frac{\left(a+r_p\right)^2-x^2}{r_p^2+2ar_p}}\right)+\frac{\sigma_p^0 x}{\pi}\,\mathrm{arcotan}\left(\frac{ai}{x}\sqrt{\frac{\left(a+r_p\right)^2-x^2}{r_p^2+2ar_p}}\right)$$

or again $$\Omega\left(\theta\rightarrow\pi\right)=-\frac{\sigma_p^0 i}{\pi}\sqrt{\left(a+r_p\right)^2-x^2}\,\mathrm{arcotan}\,\frac{a}{\sqrt{r_p^2+2ar_p}}-$$

$$\frac{\sigma_p^0 ai}{2\pi}\ln\frac{\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}-1}{\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}+1}+\frac{\sigma_p^0 xi}{2\pi}\ln\frac{\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}-1}{\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}+1}$$

- Displacement field of case 3 (2)
  - Displacement field for $\zeta = x + |\varepsilon|\, i$, with $x < a + r_p$ (2)
    - If all $\ln$ are real, which means if $x < a$, then
      - As $\operatorname{arcoth} x = \frac{1}{2} \ln \frac{x+1}{x-1}$,

$$\Omega\left(\theta \rightarrow \pi\right) = -\frac{\sigma_p^0 i}{\pi}\sqrt{\left(a + r_p\right)^2 - x^2}\,\operatorname{arcotan}\frac{a}{\sqrt{r_p^2 + 2ar_p}} +$$

$$\frac{\sigma_p^0 a i}{\pi}\operatorname{arcoth}\sqrt{\frac{\left(a + r_p\right)^2 - x^2}{r_p^2 + 2ar_p}} - \frac{\sigma_p^0 x i}{\pi}\operatorname{arcoth}\left(\frac{a}{x}\sqrt{\frac{\left(a + r_p\right)^2 - x^2}{r_p^2 + 2ar_p}}\right)$$

      - And the displacement field for $x < a$, is

$$\boldsymbol{u}_y\left(\theta \rightarrow \pi\right) = \frac{\left(1+\nu\right)\left(1+\kappa\right)\sigma_p^0}{E\pi}\left[-\sqrt{\left(a + r_p\right)^2 - x^2}\,\operatorname{arcotan}\frac{a}{\sqrt{r_p^2 + 2ar_p}} + \right.$$

$$\left. a\operatorname{arcoth}\sqrt{\frac{\left(a + r_p\right)^2 - x^2}{r_p^2 + 2ar_p}} - x\operatorname{arcoth}\left(\frac{a}{x}\sqrt{\frac{\left(a + r_p\right)^2 - x^2}{r_p^2 + 2ar_p}}\right)\right]$$

- Displacement field of case 3 (3)
  - Displacement field for $\zeta = x+|\varepsilon|\, i$, with $a < x < a+r_p$
    - If $x > a$, the second $\ln$ is not real, and using $\ln z = \ln|z| + i \arg z$ leads to

$$\Omega\left(\theta\to\pi\right) = -\frac{\sigma_p^0 i}{\pi}\sqrt{\left(a+r_p\right)^2 - x^2}\,\text{arcotan}\,\frac{a}{\sqrt{r_p^2+2ar_p}} - \frac{\sigma_p^0 ai}{2\pi}\ln\left(\frac{\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}-1}{\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}+1}\right) + \frac{\sigma_p^0 x}{2} + \frac{\sigma_p^0 xi}{2\pi}\ln\left(\frac{1-\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}}{\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}+1}\right)$$

  - As $\text{arctanh}\, x = \frac{1}{2}\ln\frac{x+1}{1-x}$, it yields

$$\Omega\left(\theta\to\pi\right) = -\frac{\sigma_p^0 i}{\pi}\sqrt{\left(a+r_p\right)^2 - x^2}\,\text{arcotan}\,\frac{a}{\sqrt{r_p^2+2ar_p}} + \frac{\sigma_p^0 x}{2} + \frac{\sigma_p^0 ai}{\pi}\,\text{arcoth}\sqrt{\frac{\left(a+r_p\right)^2-x^2}{r_p^2+2ar_p}} - \frac{\sigma_p^0 xi}{\pi}\,\text{arctanh}\left(\frac{a}{x}\sqrt{\frac{\left(a+r_p\right)^2-x^2}{r_p^2+2ar_p}}\right)$$

- Displacement field of case 3 (4)
  - Displacement field on crack and cohesive zone lips
    - The cohesive zone size is such that

$$\sigma_\infty = \frac{2\sigma_p^0}{\pi} \operatorname{arcotan} \frac{a}{\sqrt{r_p^2 + 2ar_p}}$$

    - If $x > a$, then the displacement field becomes

$$\begin{aligned} \boldsymbol{u}_y\left(\theta \rightarrow \pm\pi,\, r < r_p\right) \quad = \quad & \pm\frac{\left(1+\nu\right)\left(1+\kappa\right)\sigma_p^0}{E\pi}\left[-\frac{\sigma_\infty \pi}{2\sigma_p^0}\sqrt{\left(a+r_p\right)^2 - x^2} + \right. \\ & \left. a \operatorname{arcoth}\sqrt{\frac{\left(a+r_p\right)^2 - x^2}{r_p^2 + 2ar_p}} - x \operatorname{arctanh}\left(\frac{a}{x}\sqrt{\frac{\left(a+r_p\right)^2 - x^2}{r_p^2+2ar_p}}\right)\right] \end{aligned}$$

    - If $x < a$, we previously found

$$\begin{aligned} \boldsymbol{u}_y\left(\theta \rightarrow \pm\pi,\, r > r_p\right) \quad = \quad & \pm\frac{\left(1+\nu\right)\left(1+\kappa\right)\sigma_p^0}{E\pi}\left[-\frac{\sigma_\infty \pi}{2\sigma_p^0}\sqrt{\left(a+r_p\right)^2 - x^2} + \right. \\ & \left. a \operatorname{arcoth}\sqrt{\frac{\left(a+r_p\right)^2 - x^2}{r_p^2 + 2ar_p}} - x \operatorname{arcoth}\left(\frac{a}{x}\sqrt{\frac{\left(a+r_p\right)^2 - x^2}{r_p^2+2ar_p}}\right)\right] \end{aligned}$$

Appendix 2: COD of Dugdale's cohesive zone/Yielding strip Model

- Crack Opening Displacement (COD)
  - Case 1, 2 & 3: displacement
    - Case 1 (infinite plane): $\boldsymbol{u}_y = \cancel{\frac{(1+\nu)(1+\kappa)}{4E}}\boldsymbol{\sigma}_\infty y$
    - Case 2 (crack loaded by σ∞): $\lim\limits_{\theta\to\pm\pi} \boldsymbol{u}_y = \pm\cancel{\frac{(1+\nu)(\kappa+1)\sigma_\infty}{2E}}\sqrt{(a+r_p)^2 - x^2}$
    - Case 3 (cohesive zone closed):
      - On cohesive zone lips

$$\boldsymbol{u}_y(\theta\to\pm\pi,\, r<r_p) = \pm\frac{(1+\nu)(1+\kappa)\sigma_p^0}{E\pi}\left[-\frac{\sigma_\infty\pi}{2\sigma_p^0}\cancel{\sqrt{(a+r_p)^2 - x^2}}+\right.$$
$$\left. a\,\mathrm{arcoth}\sqrt{\frac{(a+r_p)^2 - x^2}{r_p^2 + 2ar_p}} - x\,\mathrm{arctanh}\left(\frac{a}{x}\sqrt{\frac{(a+r_p)^2 - x^2}{r_p^2 + 2ar_p}}\right)\right]$$

      - On crack lips

$$\boldsymbol{u}_y(\theta\to\pm\pi,\, r>r_p) = \pm\frac{(1+\nu)(1+\kappa)\sigma_p^0}{E\pi}\left[-\frac{\sigma_\infty\pi}{2\sigma_p^0}\cancel{\sqrt{(a+r_p)^2 - x^2}}+\right.$$
$$\left. a\,\mathrm{arcoth}\sqrt{\frac{(a+r_p)^2 - x^2}{r_p^2 + 2ar_p}} - x\,\mathrm{arcoth}\left(\frac{a}{x}\sqrt{\frac{(a+r_p)^2 - x^2}{r_p^2 + 2ar_p}}\right)\right]$$

- Case 3: Crack Tip Opening Displacement
  - The upper limit becomes

$$
\begin{aligned}
\lim_{x\to a^+} [\![\boldsymbol{u}_y]\!] &= \lim_{x\to a^+} 8\frac{\sigma_p^0}{E'\pi}\left[a \operatorname{arcoth}\left(\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}\right) - x \operatorname{arcoth}\left(\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}\right)\right] \\
&= \lim_{x\to a^+} 4\frac{\sigma_p^0}{E'\pi}\left[\cancelto{0}{a \ln\left(\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}+1\right)} - a \ln\left(\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}-1\right) - \right. \\
&\qquad \left. \cancelto{0}{x \ln\left(\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}+1\right)} + x \ln\left(\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}-1\right)\right]
\end{aligned}
$$

  - Rearranging the terms leads to

$$
\begin{aligned}
\lim_{x\to a^+} [\![\boldsymbol{u}_y]\!] &= \lim_{x\to a^+} 4\frac{\sigma_p^0}{E'\pi}\left[-a \ln\left(\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}-1\right) + \right. \\
&\qquad \left. (x-a) \ln\left(\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}-1\right) + a \ln\left(\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}-1\right)\right] \\
&= \lim_{x\to a^+} 4\frac{\sigma_p^0}{E'\pi}\left[a \ln\left(\frac{\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}-1}{\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}-1}\right) + \cancelto{0}{(x-a) \ln\left(\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}-1\right)}\right]
\end{aligned}
$$

Polynomial "wins" on $\ln$

- Case 3: Crack Tip Opening Displacement (2)
  - Hospital's theorem yields

$$\lim_{x\to a^+} \frac{\frac{a}{x}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}-1}{\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}-1} = \lim_{x\to a^+} \frac{-\frac{a}{x^2}\sqrt{\frac{(a+r_p)^2-x^2}{r_p^2+2ar_p}}-\frac{a}{\sqrt{r_p^2+2ar_p}\sqrt{(a+r_p)^2-x^2}}}{\frac{-x}{\sqrt{r_p^2+2ar_p}\sqrt{(a+r_p)^2-x^2}}}$$

$$\lim_{x\to a^+} \frac{-\frac{a}{x^2}\left((a+r_p)^2-x^2\right)-a}{-x} = \frac{(a+r_p)^2}{a^2}$$

  - And eventually

$$\lim_{x\to a^+} [\![\boldsymbol{u}_y]\!] \;=\; 4\frac{\sigma_p^0}{E'\pi}a\ln\frac{(a+r_p)^2}{a^2} = 8\frac{\sigma_p^0}{E'\pi}a\ln\frac{a+r_p}{a} = 8\frac{\sigma_p^0}{E'\pi}a\ln\sec\frac{\sigma_\infty\pi}{2\sigma_p^0}$$

    - Where the size of the cohesive zone $r_p = a\left(\sec\frac{\sigma_\infty\pi}{2\sigma_p^0}-1\right)$ has been used