University of Liège
Aerospace & Mechanical Engineering

Fracture Mechanics, Damage and Fatigue
Linear Elastic Fracture Mechanics – Stress Intensity Factor (SIF)

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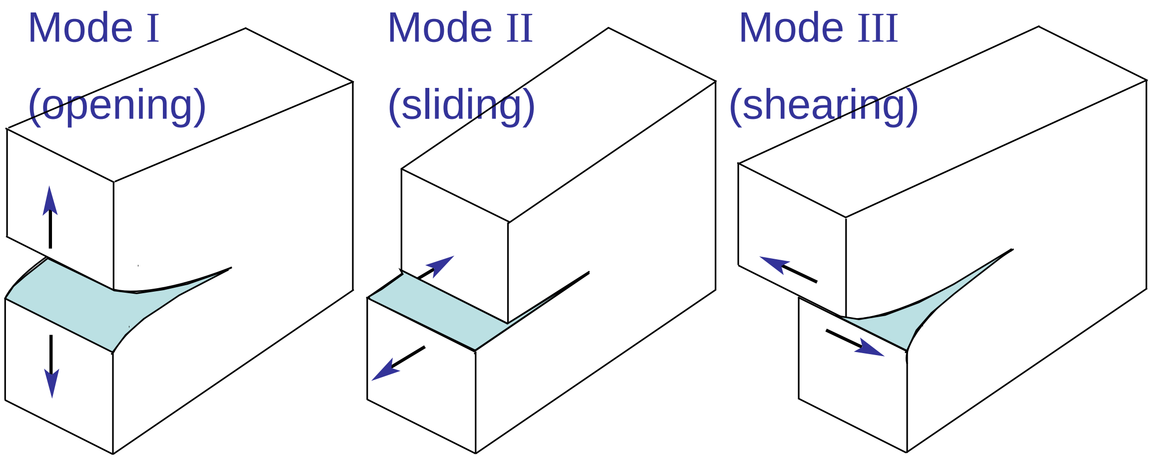
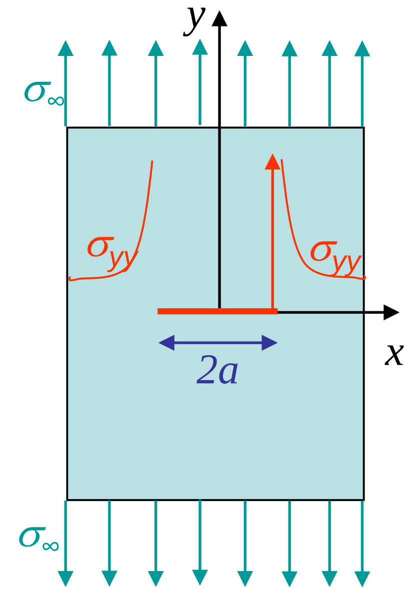
- Cracked body: summary
  - 3 failure modes

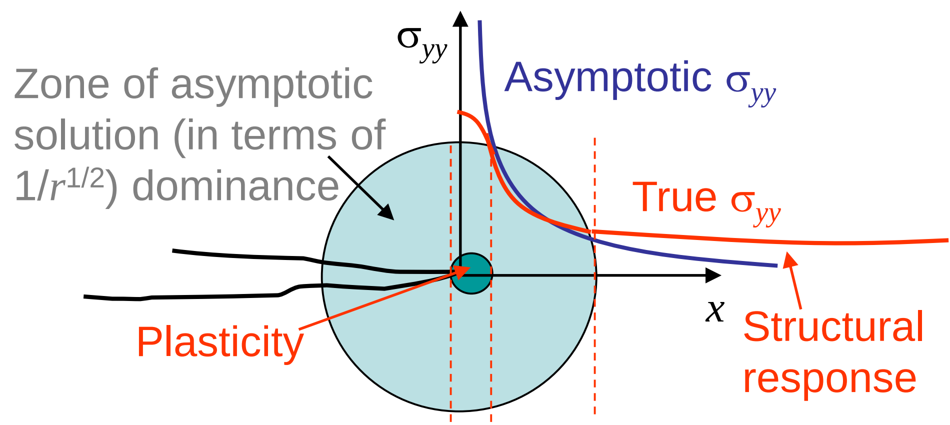




  - Asymptotic solution governed by stress intensity factors

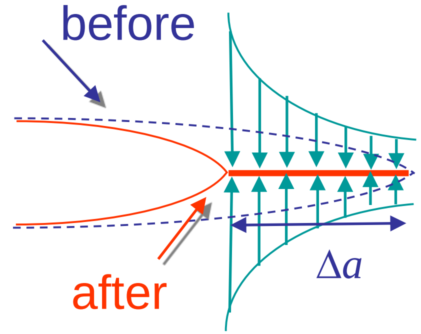
$$\boxed{\boldsymbol{\sigma}^{\text{mode } i} = \frac{K_i}{\sqrt{2\pi r}} f^{\text{mode } i}(\theta) + o(r^0)}$$

$$\boxed{\boldsymbol{u}^{\text{mode } i} = K_i \sqrt{\frac{r}{2\pi}}\, g^{\text{mode } i}(\theta) + o(r^0)}$$



- Cracked body: summary
  - Potential energy $\Pi_T = E_{\text{int}} - Qu$
  - Crack closure integral
    - Energy required to close crack tip

$$\Delta\Pi_T = \int_{\Delta A} \int_{\boldsymbol{u}}^{\boldsymbol{u}+\boldsymbol{\Delta u}} \boldsymbol{t} \cdot [\![\boldsymbol{u}']\!] d\boldsymbol{u}' \; dA$$

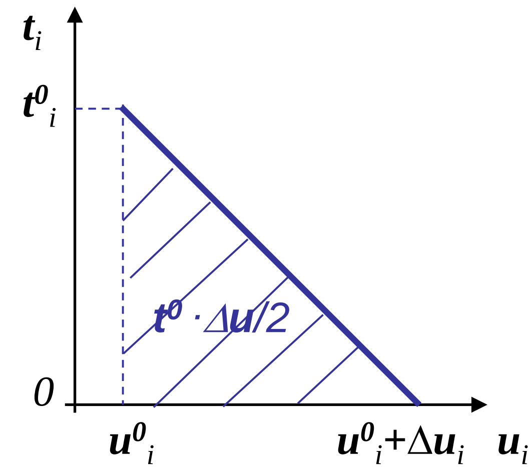


  - Energy release rate
    - Variation of potential energy in case of crack growth

$$G = -\partial_{\text{A}} \left(E_{\text{int}} - W_{\text{ext}}\right) = -\partial_A \Delta\Pi_T$$

    - In linear elasticity

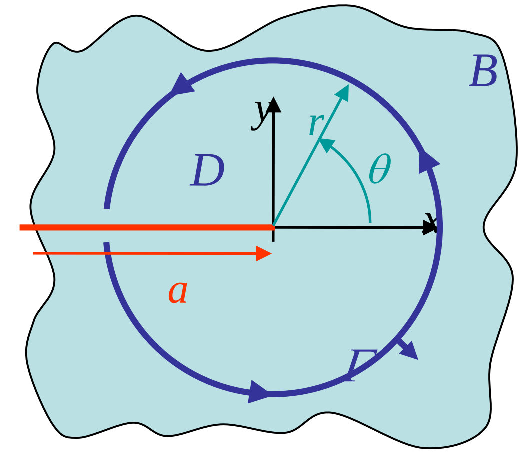
$$G = -\partial_A \Delta\Pi_T = -\lim_{\Delta A \to 0} \frac{1}{\Delta A} \int_{\Delta A} \frac{1}{2} \boldsymbol{t}^{\mathbf{0}} \cdot [\![\boldsymbol{\Delta u}]\!] \, dA$$



  - In linear elasticity & if crack grows straight ahead

$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \qquad E' = \frac{4E}{(1+\nu)(\kappa+1)} = \begin{cases} E & \text{P}\sigma \\ \frac{E}{1-\nu^2} & \text{P}\varepsilon \end{cases}$$

- Cracked body: summary
  - J-integral
    - Strain energy flow



$$J = \int_{\Gamma} \left[U(\boldsymbol{\varepsilon})\, \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \right] dl$$

  - Exists if an internal potential exists
    - Is path independent if the contour Γ embeds a straight crack tip
    - No assumption on subsequent growth direction
    - Can be extended to plasticity if no unloading (see later)
  - If crack grows straight ahead ⟹ $G=J$
  - In linear elasticity (independently of crack growth direction):

$$J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \qquad E' = \frac{4E}{(1+\nu)(\kappa+1)} = \begin{cases} E & \mathrm{P}\sigma \\ \frac{E}{1-\nu^2} & \mathrm{P}\varepsilon \end{cases}$$

- Cracked body: different concepts

- Structural properties
  - Depend on geometry
  - Depend on loading conditions
  - Different related concepts
    - Stress intensity factors
    - Energy release rate
    - J-integral
  - How to evaluate them?

- Material properties
  - Depend on environmental conditions
  - Different related concepts
    - Toughness
    - Fracture energy
    - Resistance curves
  - How to measure them?

- Crack propagation criterion
  - Structural property < Material property: OK
  - Structural property > Material property: What happens?

LEFM: Computation of SIF

- Computation of the SIFs
  - Analytical methods (for LEFM)
    - Full field solution (see next slides)
      - Limited to infinite planes
      - SIFs obtained from asymptotic limit
    - Superposition
      - Of existing solutions
    - Energetic approach (see previous lecture)
      - Related to Griffith's work
  - Numerical
    - Collocation method (no more used)
    - FEM
      - Capture asymptotic solutions
      - Energetic approach
      - J integral
  - Experimental
    - Normalized experiments
    - Strain Gauge Method
    - DIC
  - Use of SIF handbook (http://ebooks.asmedigitalcollection.asme.org/book.aspx?bookid=230)

- Reminder: method in linear elasticity
  - Problem is governed by the bi-harmonic equation $\nabla^2\nabla^2\Phi\left(\zeta,\,\bar{\zeta}\right)=0$
  - One solution of this equation has the form $\Phi=\dfrac{\bar{\zeta}\Omega+\zeta\bar{\Omega}+\omega+\bar{\omega}}{2}$

    where $\omega(\zeta)$ and $\Omega(\zeta)$ are functions to be defined so that
    - The stress field

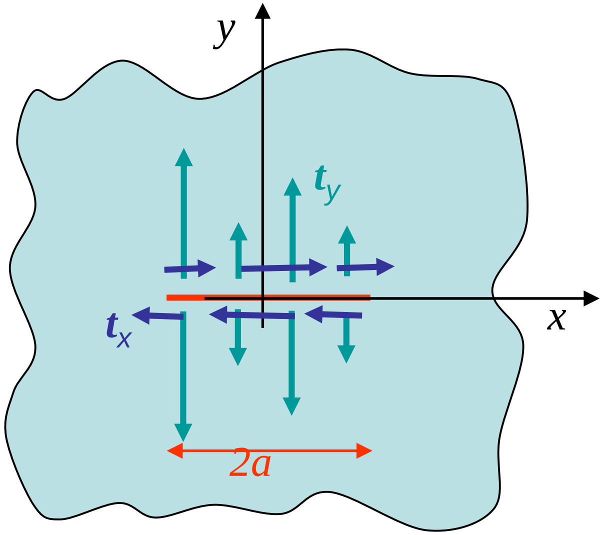
$$\begin{cases}\boldsymbol{\sigma}_{xx}=\Omega'+\bar{\Omega}'-\dfrac{\bar{\zeta}\Omega''+\omega''+\zeta\bar{\Omega}''+\bar{\omega}''}{2}\\ \boldsymbol{\sigma}_{yy}=\Omega'+\bar{\Omega}'+\dfrac{\bar{\zeta}\Omega''+\omega''+\zeta\bar{\Omega}''+\bar{\omega}''}{2}\\ \boldsymbol{\sigma}_{xy}=i\dfrac{\zeta\bar{\Omega}''+\bar{\omega}''-\bar{\zeta}\Omega''-\omega''}{2}\end{cases}$$

    - The displacement field

$$\boldsymbol{u}=-\frac{1+\nu}{E}\left(\zeta\bar{\Omega}'+\bar{\omega}'-\kappa\Omega\left(\zeta\right)\right)\qquad\begin{cases}\kappa=\dfrac{3-\nu}{1+\nu} & \text{Plane }\sigma\\ \kappa=3-4\nu & \text{Plane }\varepsilon\end{cases}$$

    satisfy the BCs

- Westergaard solution
  - Full field solution of a crack submitted to traction
    - Infinite plane
    - Mode I: $\boldsymbol{t} = \boldsymbol{t}_y \boldsymbol{e_2}$, and $t_y(-x) = t_y(x)$, $t_y(-y) = -t_y(y)$



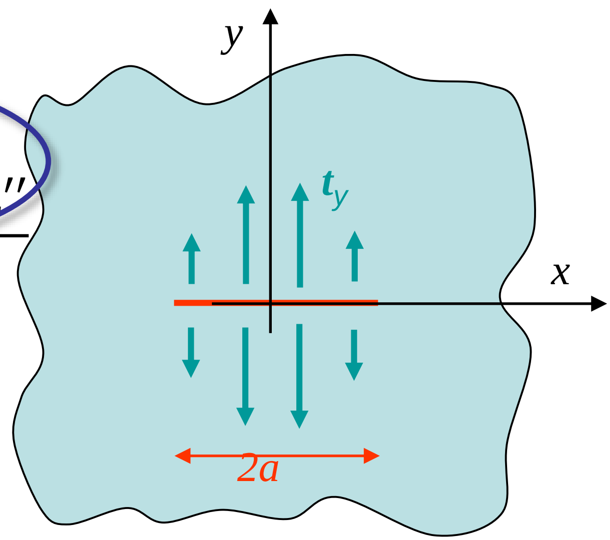
  - Westergaard approach for mode I
    - Since $\boldsymbol{\sigma}_{xy} = i\frac{\zeta\bar{\Omega}'' + \bar{\omega}'' - \bar{\zeta}\Omega'' - \omega''}{2}$

      and since for mode I, one should have

      $\boldsymbol{\sigma}_{xy} = 0$ for $y = 0$,

      the solution is

      $$\omega'' = -\zeta\Omega'' \Longrightarrow \boldsymbol{\sigma}_{xy} = i\frac{\overbrace{\underbrace{(\zeta-\bar{\zeta})}_{2iy}\bar{\Omega}'' - \underbrace{(\bar{\zeta}-\zeta)}_{-2iy}\Omega''}^{4iy\,\mathcal{R}(\Omega'')}}{2}$$



    - Indeed with this choice

$$\begin{cases} \boldsymbol{\sigma}_{xy} = -y2\mathcal{R}(\Omega'') \\ \omega' = -\zeta\Omega' + \Omega + C \end{cases}$$

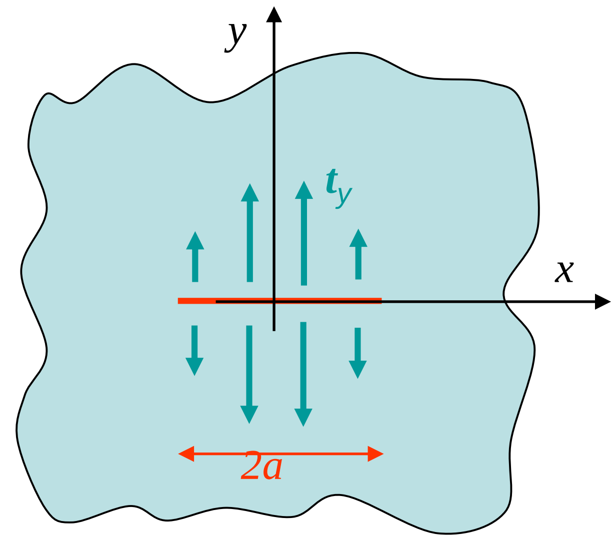
- Westergaard solution (2)
  - Westergaard solution for mode I (2)
    - The general stress/displacement fields read

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \Omega' + \bar{\Omega}' - \dfrac{\bar{\zeta}\Omega'' + \omega'' + \zeta\bar{\Omega}'' + \bar{\omega}''}{2} \\ \boldsymbol{\sigma}_{yy} = \Omega' + \bar{\Omega}' + \dfrac{\bar{\zeta}\Omega'' + \omega'' + \zeta\bar{\Omega}'' + \bar{\omega}''}{2} \\ \boldsymbol{\sigma}_{xy} = i\dfrac{\zeta\bar{\Omega}'' + \bar{\omega}'' - \bar{\zeta}\Omega'' - \omega''}{2} \\ \boldsymbol{u} = -\dfrac{1+\nu}{E}\left(\zeta\bar{\Omega}' + \bar{\omega}' - \kappa\Omega\left(\zeta\right)\right) \end{cases}$$

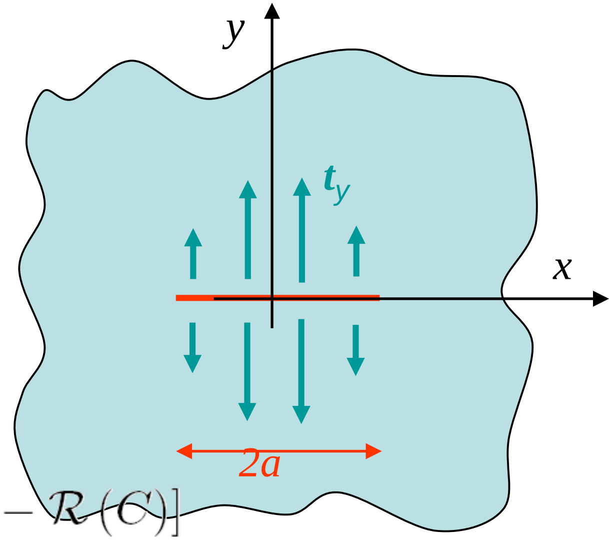
    - Using $\omega'' = -\zeta\Omega''$ they become

$$\begin{cases} \boldsymbol{\sigma}_{xx} = 2\mathcal{R}\left(\Omega'\right) - 2y\mathcal{I}\left(\Omega''\right) \\ \boldsymbol{\sigma}_{yy} = 2\mathcal{R}\left(\Omega'\right) + 2y\mathcal{I}\left(\Omega''\right) \\ \boldsymbol{\sigma}_{xy} = -y2\mathcal{R}\left(\Omega''\right) \\ \boldsymbol{u}_x = \mathcal{R}\left(\boldsymbol{u}\right) = \dfrac{1+\nu}{E}\left[\left(\kappa-1\right)\mathcal{R}\left(\Omega\right) - 2y\mathcal{I}\left(\Omega'\right) - \mathcal{R}\left(C\right)\right] \\ \boldsymbol{u}_y = \mathcal{I}\left(\boldsymbol{u}\right) = \dfrac{1+\nu}{E}\left[\left(\kappa+1\right)\mathcal{I}\left(\Omega\right) - 2y\mathcal{R}\left(\Omega'\right) + \mathcal{I}\left(C\right)\right] \end{cases}$$

Only Ω to be defined



- Westergaard solution (3)
  - Full field solution of a crack submitted to traction
    - Westergaard solution for mode I



$$\begin{cases} \boldsymbol{\sigma}_{xx} = 2\mathcal{R}\left(\Omega'\right) - 2y\mathcal{I}\left(\Omega''\right) \\ \boldsymbol{\sigma}_{yy} = 2\mathcal{R}\left(\Omega'\right) + 2y\mathcal{I}\left(\Omega''\right) \\ \boldsymbol{\sigma}_{xy} = -y2\mathcal{R}\left(\Omega''\right) \\ \boldsymbol{u}_x = \mathcal{R}\left(\boldsymbol{u}\right) = \dfrac{1+\nu}{E}\left[\left(\kappa-1\right)\mathcal{R}\left(\Omega\right) - 2y\mathcal{I}\left(\Omega'\right) - \mathcal{R}\left(C\right)\right] \\ \boldsymbol{u}_y = \mathcal{I}\left(\boldsymbol{u}\right) = \dfrac{1+\nu}{E}\left[\left(\kappa+1\right)\mathcal{I}\left(\Omega\right) - 2y\mathcal{R}\left(\Omega'\right) + \mathcal{I}\left(C\right)\right] \end{cases}$$

- Solution $\Omega(x\geq 0)$ satisfying the BCs for a uniform traction (see next slides)

$$\Omega' = \frac{\boldsymbol{t}_y}{2}\left(\frac{\zeta}{\sqrt{\zeta^2-a^2}} - 1\right) \Rightarrow \begin{cases} \Omega = \dfrac{\boldsymbol{t}_y}{2}\left(\sqrt{\zeta^2-a^2} - \zeta + C_1\right) \\ \Omega'' = -\dfrac{\boldsymbol{t}_y}{2}\left(\dfrac{a^2}{\sqrt{\zeta^2-a^2}^3}\right) \end{cases}$$

- NB: General solution (Sedov, 1972): $\Omega' = \dfrac{1}{2\pi\sqrt{\zeta^2-a^2}}\displaystyle\int_{-a}^{a} \boldsymbol{t}_y\left(\xi\right)\frac{\sqrt{a^2-\xi^2}}{\zeta-\xi}d\xi$

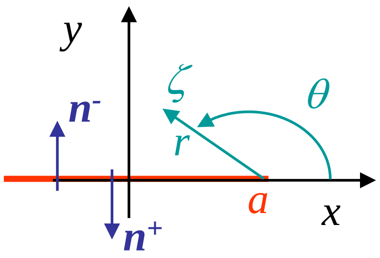
- Westergaard solution (4)
  - Check solution

$$\begin{cases} \boldsymbol{u}_x = \mathcal{R}(\boldsymbol{u}) = \dfrac{1+\nu}{E}\left[(\kappa-1)\,\mathcal{R}(\Omega) - 2y\mathcal{I}(\Omega') - \mathcal{R}(C)\right] \\ \boldsymbol{u}_y = \mathcal{I}(\boldsymbol{u}) = \dfrac{1+\nu}{E}\left[(\kappa+1)\,\mathcal{I}(\Omega) - 2y\mathcal{R}(\Omega') + \mathcal{I}(C)\right] \\ \boldsymbol{\sigma}_{xx} = 2\mathcal{R}(\Omega') - 2y\mathcal{I}(\Omega'') \\ \boldsymbol{\sigma}_{yy} = 2\mathcal{R}(\Omega') + 2y\mathcal{I}(\Omega'') \\ \boldsymbol{\sigma}_{xy} = -y2\mathcal{R}(\Omega'') \end{cases}$$

  - with $\Omega' = \dfrac{\boldsymbol{t}_y}{2}\left(\dfrac{\zeta}{\sqrt{\zeta^2-a^2}} - 1\right) \Longrightarrow \begin{cases} \Omega = \dfrac{\boldsymbol{t}_y}{2}\left(\sqrt{\zeta^2-a^2} - \zeta + C_1\right) \\ \Omega'' = -\dfrac{\boldsymbol{t}_y}{2}\left(\dfrac{a^2}{\sqrt{\zeta^2-a^2}^{\,3}}\right) \end{cases}$

  - Symmetry with respect to Ox and Oy

$\Longrightarrow \mathcal{R}(C) = \mathcal{R}(C_1) = \mathcal{I}(C) = \mathcal{I}(C_1) = 0$

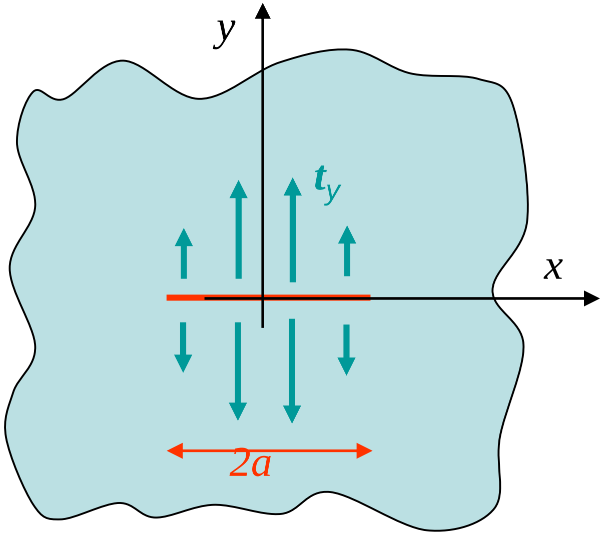


SIF: Analytical methods

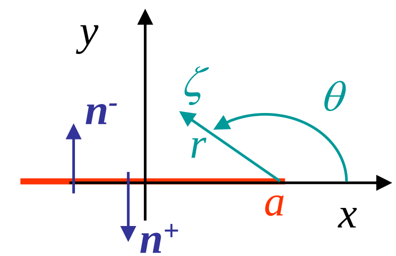
- Westergaard solution (5)
  - Check solution (2)

$$\left\{\begin{array}{l}\boldsymbol{\sigma}_{yy} = 2\mathcal{R}\left(\Omega'\right) + 2y\mathcal{I}\left(\Omega''\right)\\ \boldsymbol{\sigma}_{xy} = -y2\mathcal{R}\left(\Omega''\right)\end{array}\right.$$

with $\Omega' = \frac{\boldsymbol{t}_y}{2}\left(\frac{\zeta}{\sqrt{\zeta^2 - a^2}} - 1\right)$



- On crack lips $\zeta = x \pm i\,|\varepsilon|$, $\varepsilon \to 0$, $|x| < a$
  - $\frac{\zeta}{\sqrt{\zeta^2 - a^2}} = \frac{\zeta}{\sqrt{\zeta - a}\sqrt{\zeta + a}}$ $\to x$ $\to \sqrt{x+a}$
  - $\sqrt{\zeta - a} = \sqrt{r\exp\left(i\theta\right)} = \sqrt{r}\cos\frac{\theta}{2} + i\sqrt{r}\sin\frac{\theta}{2}$

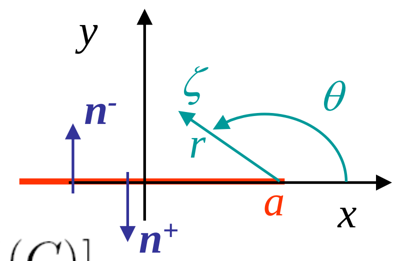


$$\Longrightarrow \lim_{\theta\to\pm\pi}\sqrt{\zeta - a} = \pm i\sqrt{r} = \pm i\sqrt{(a-x)}$$

$$\Longrightarrow \lim_{\theta\to\pm\pi}\Omega' = \pm\frac{\boldsymbol{t}_y}{2}\frac{x}{i\sqrt{a+x}\sqrt{a-x}} - \frac{\boldsymbol{t}_y}{2} = \mp\frac{ix\boldsymbol{t}_y}{2\sqrt{a^2 - x^2}} - \frac{\boldsymbol{t}_y}{2}$$

- On crack lips: $\boldsymbol{t} = \boldsymbol{n}\cdot\boldsymbol{\sigma} = \mp\boldsymbol{\sigma}_{yy}\,(y=0) = \mp 2\,\mathcal{R}(\Omega')$ $\Longrightarrow$ $\boldsymbol{t} = \pm\,\boldsymbol{t}_y$ is satisfied

- Westergaard solution (6)
  - Asymptotic field on crack lips



$$\begin{cases} \Omega = \dfrac{\boldsymbol{t}_y}{2}\left(\sqrt{\zeta^2 - a^2} - \zeta + C_1\right) \\ \boldsymbol{u}_y = \mathcal{I}(\boldsymbol{u}) = \dfrac{1+\nu}{E}\left[(\kappa+1)\mathcal{I}(\Omega) - 2y\mathcal{R}(\Omega') + \mathcal{I}(C)\right] \end{cases}$$

- On crack lips: $\zeta = a\text{-}r \pm i\,|\varepsilon|$, $\varepsilon \longrightarrow 0$
  - $\sqrt{\zeta^2 - a^2} - \zeta = \sqrt{\zeta - a}\,\underbrace{\sqrt{\zeta + a}}_{\rightarrow \sqrt{x+a}} - \overset{\rightarrow x}{\zeta}$
  - $\sqrt{\zeta - a} = \sqrt{r\exp(i\theta)} = \sqrt{r}\cos\dfrac{\theta}{2} + i\sqrt{r}\sin\dfrac{\theta}{2}$

$$\Longrightarrow \lim_{\theta\rightarrow\pm\pi} \sqrt{\zeta - a} = \pm i\sqrt{r} = \pm i\sqrt{(a-x)}$$

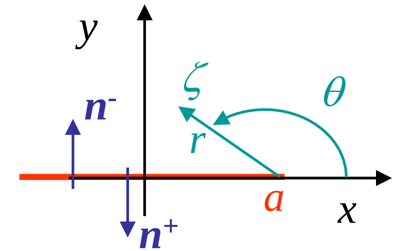
$$\Longrightarrow \lim_{\theta\rightarrow\pm\pi} \Omega = \frac{\boldsymbol{t}_y}{2}\left(\pm i\sqrt{a^2 - x^2} - x\right)$$

  - Crack Opening Displacement (COD)

$$\lim_{\theta\rightarrow\pm\pi} \boldsymbol{u}_y = \pm\frac{(1+\nu)(\kappa+1)\boldsymbol{t_y}}{2E}\sqrt{a^2 - x^2}$$

$$\Longrightarrow [\![\boldsymbol{u}_y]\!] = \frac{(1+\nu)(\kappa+1)\boldsymbol{t_y}}{E}\sqrt{a^2 - x^2}$$

- Westergaard solution (7)
  - Asymptotic field ahead of crack

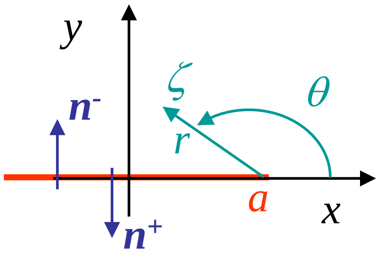


$$\begin{cases} \Omega = \frac{\boldsymbol{t}_y}{2}\left(\sqrt{\zeta^2 - a^2} - \zeta + C_1\right) \\ \Omega' = \frac{\boldsymbol{t}_y}{2}\left(\frac{\zeta}{\sqrt{\zeta^2 - a^2}} - 1\right) \\ \Omega'' = -\frac{\boldsymbol{t}_y}{2}\left(\frac{a^2}{\sqrt{\zeta^2 - a^2}^3}\right) \end{cases}$$

- For $\zeta = a + r \pm i\,|\varepsilon|$, $\varepsilon \to 0$, $r \to 0^+$
  - $\sqrt{\zeta^2 - a^2} = \sqrt{x^2 - a^2} = \sqrt{(a+r)^2 - a^2} = \sqrt{2ar + r^2} \to \sqrt{2ar}$

$$\Rightarrow \begin{cases} \Omega\left(\theta = 0\right) = \frac{\boldsymbol{t}_y}{2}\left(\sqrt{x^2 - a^2} - x + C_1\right) \to \frac{\boldsymbol{t}_y}{2}\left(\sqrt{2ar} - a + C_1\right) \\ \Omega'\left(\theta = 0\right) = \frac{\boldsymbol{t}_y}{2}\left(\frac{x}{\sqrt{x^2 - a^2}} - 1\right) \to \frac{\boldsymbol{t}_y}{2}\left(\sqrt{\frac{a}{2r}} - 1\right) \\ \Omega''\left(\theta = 0\right) = -\frac{\boldsymbol{t}_y}{2}\left(\frac{a^2}{\sqrt{x^2 - a^2}^3}\right) \to -\frac{\boldsymbol{t}_y}{4r}\sqrt{\frac{a}{2r}} \end{cases}$$

- Westergaard solution (8)
  - Asymptotic stress field ahead of crack



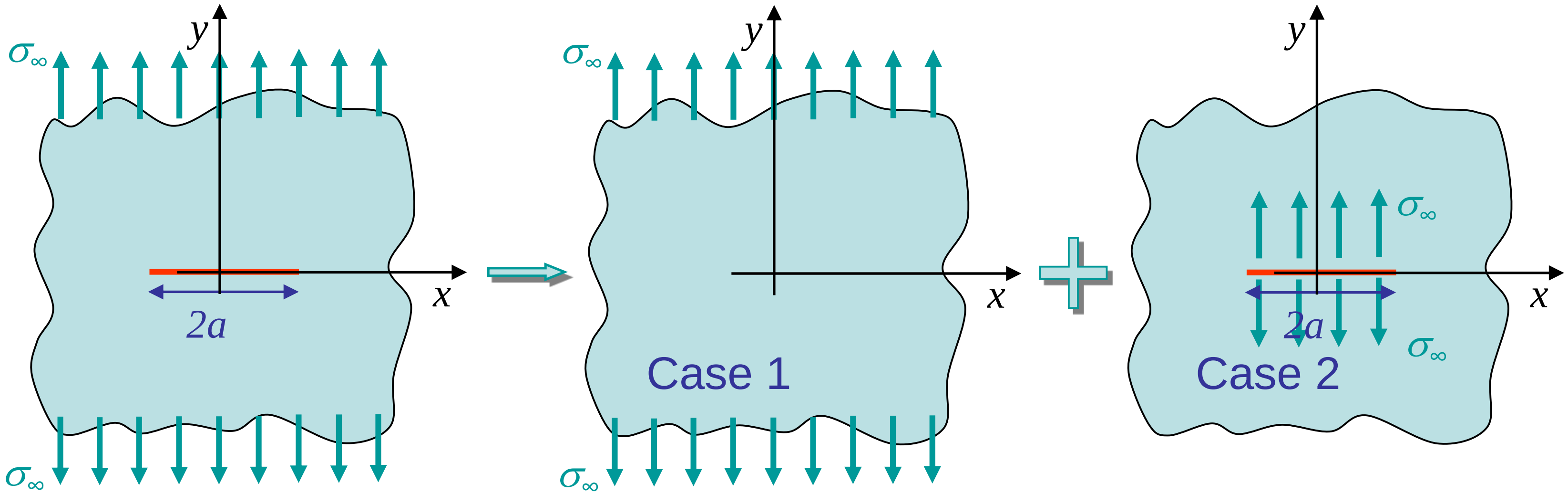
$$\begin{cases} \boldsymbol{\sigma}_{xx} = 2\mathcal{R}\left(\Omega'\right) - 2y\mathcal{I}\left(\Omega''\right) \\ \boldsymbol{\sigma}_{yy} = 2\mathcal{R}\left(\Omega'\right) + 2y\mathcal{I}\left(\Omega''\right) \\ \boldsymbol{\sigma}_{xy} = -y2\mathcal{R}\left(\Omega''\right) \end{cases}$$

- For $\zeta = a+r \pm i\,|\varepsilon|$, $\varepsilon \to 0$, $r \to 0^+$

$$\begin{cases} \Omega\left(\theta=0\right) = \dfrac{\boldsymbol{t}_y}{2}\left(\sqrt{x^2-a^2} - x + C_1\right) \to \dfrac{\boldsymbol{t}_y}{2}\left(\sqrt{2ar} - a + C_1\right) \\ \Omega'\left(\theta=0\right) = \dfrac{\boldsymbol{t}_y}{2}\left(\dfrac{x}{\sqrt{x^2-a^2}} - 1\right) \to \dfrac{\boldsymbol{t}_y}{2}\left(\sqrt{\dfrac{a}{2r}} - 1\right) \\ \Omega''\left(\theta=0\right) = -\dfrac{\boldsymbol{t}_y}{2}\left(\dfrac{a^2}{\sqrt{x^2-a^2}^3}\right) \to -\dfrac{\boldsymbol{t}_y}{4r}\sqrt{\dfrac{a}{2r}} \end{cases}$$

$$\Rightarrow \begin{cases} \boldsymbol{\sigma}_{xx}\left(\theta=0\right) = \boldsymbol{\sigma}_{yy}\left(\theta=0\right) = 2\mathcal{R}\left(\Omega'\right) = \boldsymbol{t}_y\left(\dfrac{x}{\sqrt{x^2-a^2}} - 1\right) \to \boldsymbol{t}_y\left(\sqrt{\dfrac{a}{2r}} - 1\right) \\ \boldsymbol{\sigma}_{xy}\left(\theta=0\right) = 0 \end{cases}$$

- Computation of SIF by analytical method (LEFM)
  - Crack in an infinite plate under perpendicular loading
    - Elasticity ⟹ Superposition OK

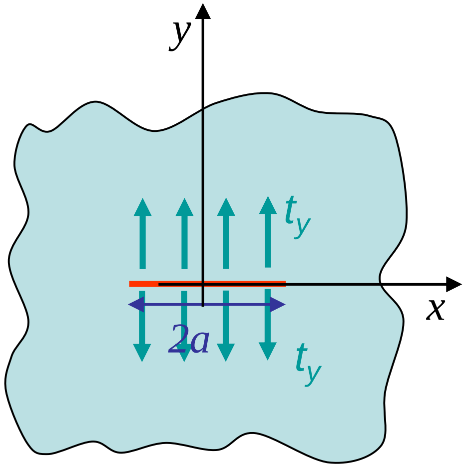


- Case 1:

$$\left.\begin{array}{l}\boldsymbol{\sigma}_{yy} = \boldsymbol{\sigma}_\infty \\ \boldsymbol{\sigma}_{xx} = \boldsymbol{\sigma}_{xy} = 0\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\boldsymbol{\varepsilon}_{xx} = \dfrac{(1+\nu)(3-\kappa)}{4E}\boldsymbol{\sigma}_\infty \;\; \& \;\; \boldsymbol{u}_x = \dfrac{(1+\nu)(3-\kappa)}{4E}\boldsymbol{\sigma}_\infty x \\ \boldsymbol{\varepsilon}_{yy} = \dfrac{(1+\nu)(1+\kappa)}{4E}\boldsymbol{\sigma}_\infty \;\; \& \;\; \boldsymbol{u}_y = \dfrac{(1+\nu)(1+\kappa)}{4E}\boldsymbol{\sigma}_\infty y \\ \boldsymbol{\varepsilon}_{xy} = 0\end{array}\right.$$

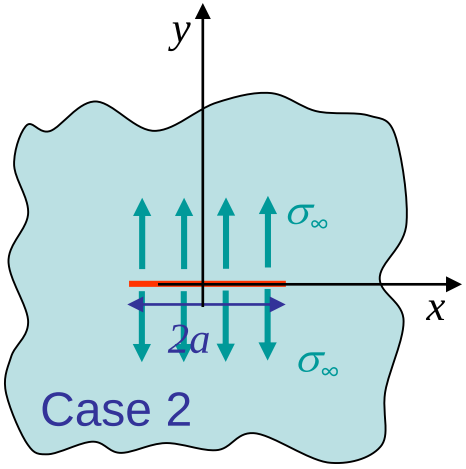
With for plane σ: $\kappa = \dfrac{3-\nu}{1+\nu}$ & for plane ε: $\kappa = 3 - 4\nu$

- Computation of SIF by analytical method (LEFM) (2)
  - Case 2: Westergaard solution in mode I:



$$\begin{cases} \boldsymbol{\sigma}_{xx}\left(\theta=0\right)=\boldsymbol{\sigma}_{yy}\left(\theta=0\right)=2\mathcal{R}\left(\Omega'\right) \\ \qquad =\boldsymbol{t}_y\left(\frac{x}{\sqrt{x^2-a^2}}-1\right)\rightarrow \boldsymbol{t}_y\left(\sqrt{\frac{a}{2r}}-1\right) \\ \boldsymbol{\sigma}_{xy}\left(\theta=0\right)=0 \end{cases}$$

  - Here use $t_y = \sigma_\infty$



- Computation of SIF by analytical method (LEFM) (3)
  - Crack in an infinite plate under perpendicular loading
    - Full field solution from Ω and case 1 along $y = 0$, for $x > a$

$$\boldsymbol{\sigma}_{yy} = \boldsymbol{\sigma}_{yy}^{\text{case I}} + \boldsymbol{\sigma}_{yy}^{\text{case II}} = \boldsymbol{\sigma}_{\infty} \frac{x}{\sqrt{x^2 - a^2}}$$

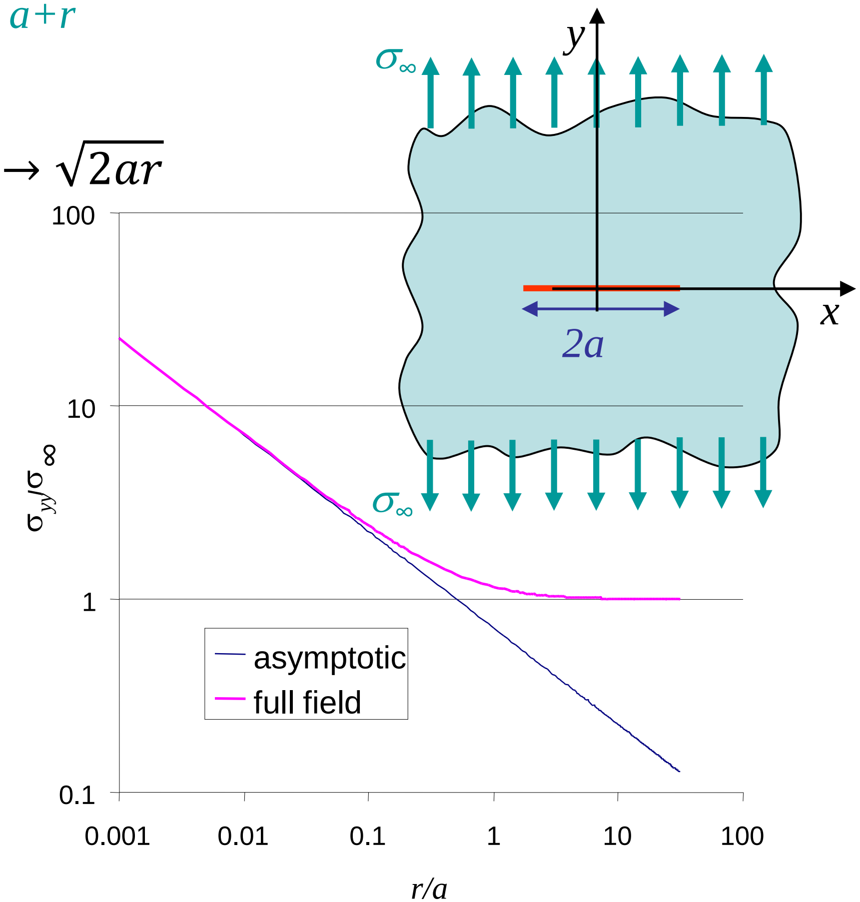
    - Asymptotic solution along $y = 0$, for $x = a+r$

$$\begin{cases} x \rightarrow a \\ \sqrt{x^2 - a^2} = \sqrt{a^2 + 2ar - r^2 - a^2} \rightarrow \sqrt{2ar} \end{cases}$$

$$\Rightarrow \quad \sigma_{yy} \rightarrow \sigma_{\infty} \sqrt{\frac{a}{2r}}$$

    - SIF:

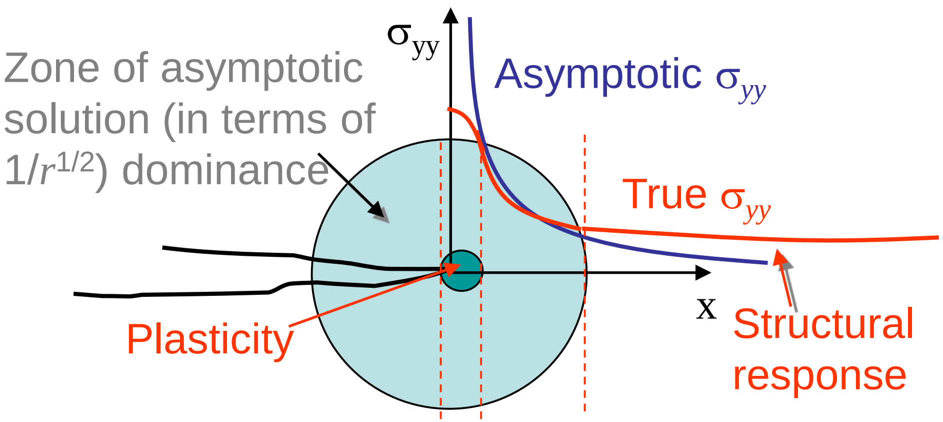
$$\Rightarrow \quad K_I = \lim_{r \rightarrow 0} \left(\sqrt{2\pi r}\, \boldsymbol{\sigma}_{yy}^{\text{mode I}} \Big|_{\theta=0} \right) = \boldsymbol{\sigma}_{\infty} \sqrt{a\pi}$$

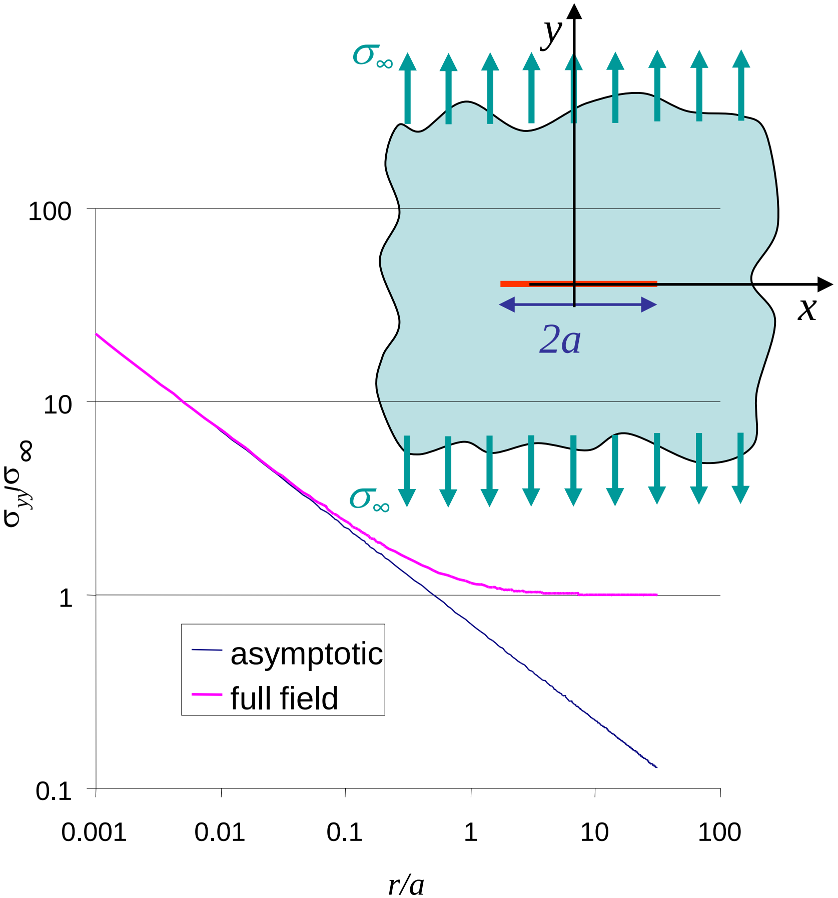


- Computation of SIF by analytical method (LEFM) (4)
  - Crack in an infinite plate under perpendicular loading
    - Full field & asymptotic solution along $y = 0$, for $x > a$

$$\boldsymbol{\sigma}_{yy} = \boldsymbol{\sigma}_{\infty} \frac{x}{\sqrt{x^2 - a^2}} \rightarrow \boldsymbol{\sigma}_{\infty} \sqrt{\frac{a}{2r}}$$

    - In reality also limited by irreversibility





- Computation of SIF by analytical method (LEFM) (5)
  - Crack in an infinite plate under sliding
    - Westergaard approach for mode II
      - For mode II, $\boldsymbol{\sigma}_{yy} = 0$ for $y = 0$

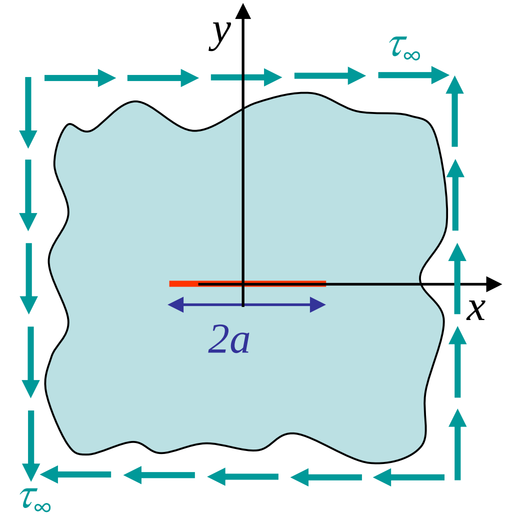
$$\Rightarrow \omega'' = -2\Omega' - \zeta\Omega''$$

      - Crack subjected to a shearing $\boldsymbol{t}_x$

$$\Omega' = \frac{-i}{2\pi\sqrt{\zeta^2 - a^2}} \int_{-a}^{a} \boldsymbol{t}_x(\xi) \frac{\sqrt{a^2 - \xi^2}}{\zeta - \xi} d\xi$$

    - Applying superposition principle and $\boldsymbol{t}_x = \boldsymbol{\tau}_\infty$

$$K_{II} = \lim_{r \to 0} \left(\sqrt{2\pi r}\, \boldsymbol{\sigma}_{xy}^{\text{mode II}} \Big|_{\theta=0} \right) = \tau_\infty \sqrt{a\pi}$$



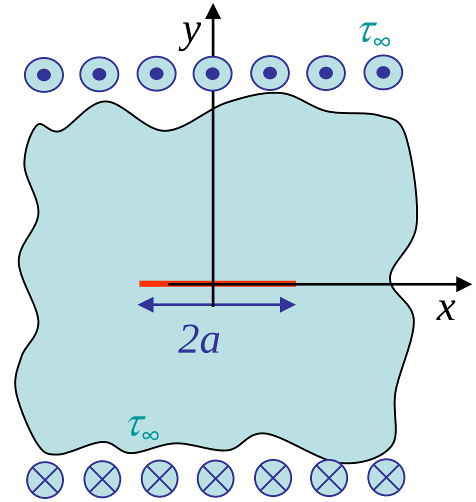
- Computation of SIF by analytical method (LEFM) (6)
  - Crack in an infinite plate under shearing
    - Stress field (see Annex 1)

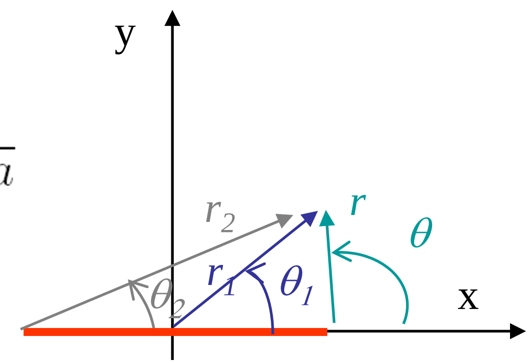
$$\boldsymbol{\sigma}_{yz} = \tau_\infty \mathcal{R}\left(\frac{\zeta}{\sqrt{\zeta^2 - a^2}}\right)$$

    - Ahead of crack tip ($y = 0$ and $x = a + r$)

$$\boldsymbol{\sigma}_{yz} = \tau_\infty \mathcal{R}\left(\frac{a+r}{\sqrt{r^2 + 2ar}}\right) \rightarrow \tau_\infty \sqrt{\frac{a}{2r}}$$

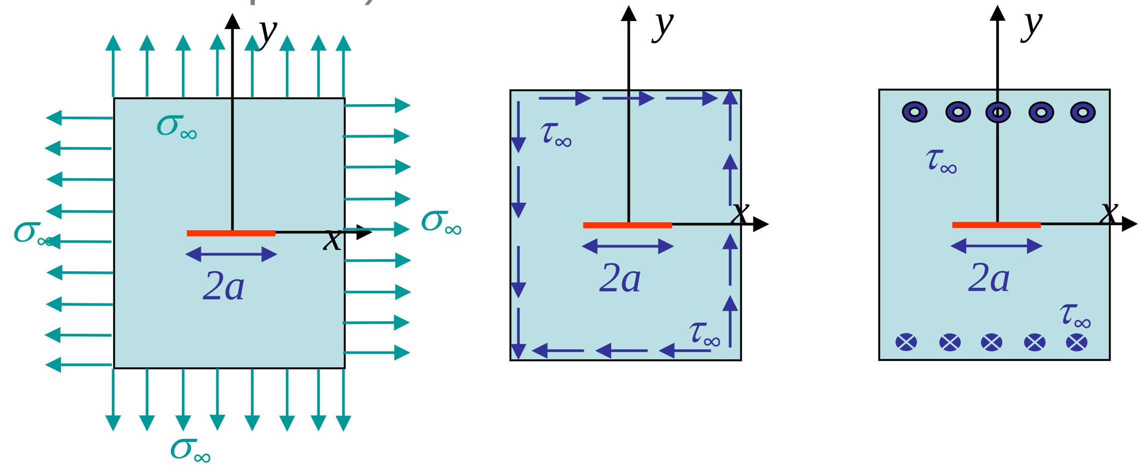
$$\Rightarrow \quad K_{III} = \lim_{r \rightarrow 0}\left(\sqrt{2\pi r}\, \boldsymbol{\sigma}_{yz}^{\text{mode III}}\Big|_{\theta=0}\right) = \tau_\infty \sqrt{\pi a}$$





- Evaluation of the stress Intensity Factor (SIF)
  - Analytical (crack $2a$ in an infinite plane)

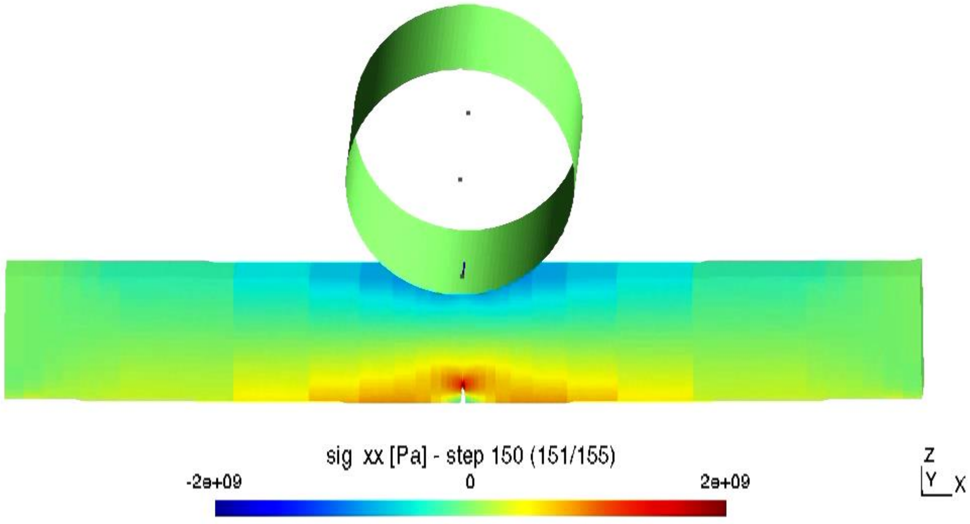
$$\Rightarrow \begin{cases} K_I = \sigma_\infty \sqrt{\pi a} \\ K_{II} = \tau_\infty \sqrt{\pi a} \\ K_{III} = \tau_\infty \sqrt{\pi a} \end{cases}$$



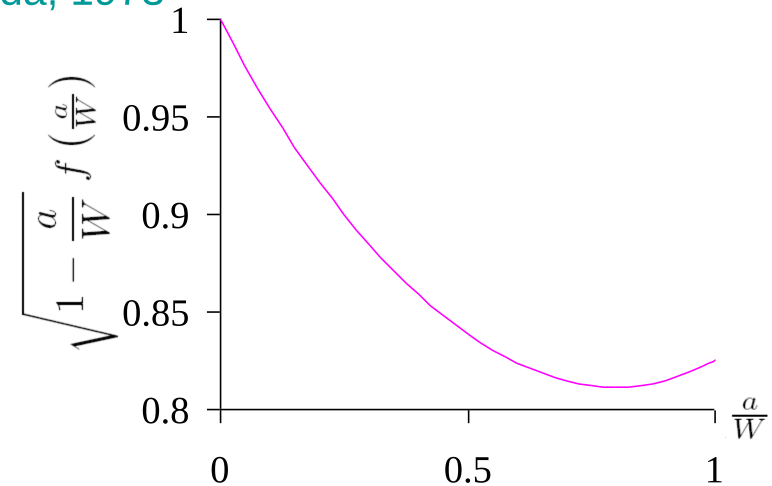
- Computation of SIF by analytical method (LEFM) (7)
  - Here we have obtained the SIF using the full field solution so
    - Why did we develop the asymptotic solution last time instead of using this full field solution directly?
    - Full field solution only for particular cases such as infinite plates
  - General case

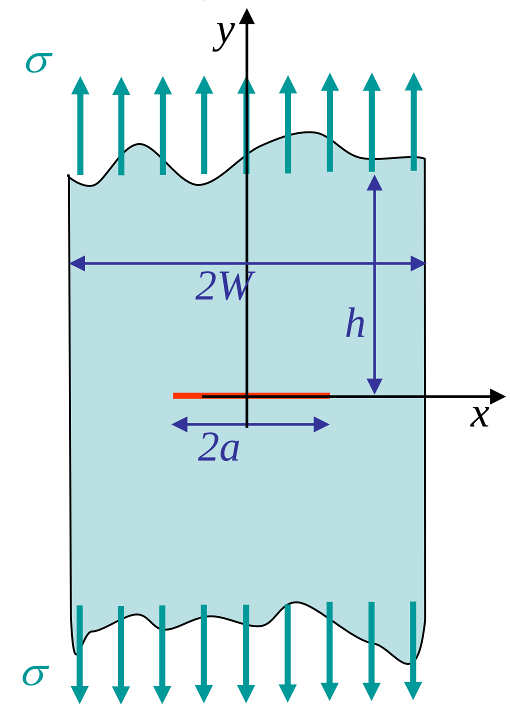
$$\begin{cases} K_I = \beta_I \sigma_\infty \sqrt{\pi a} \\ K_{II} = \beta_{II} \tau_\infty \sqrt{\pi a} \\ K_{III} = \beta_{III} \tau_\infty \sqrt{\pi a} \end{cases}$$

    - β_i depends on
      - Geometry
      - Crack length



- Solutions from SIF handbooks (see references)
  - Obtained by using a variety of methods (analytical, numerical, …)
    - Example: mode I for a crack in a finite plate ($h/W > 3$)
      - General formula $K_I = \sigma\sqrt{\pi a}\, f\left(\frac{a}{W}\right)$
      - Numerical results based on Laurent expansion
        - » Isida, 1973





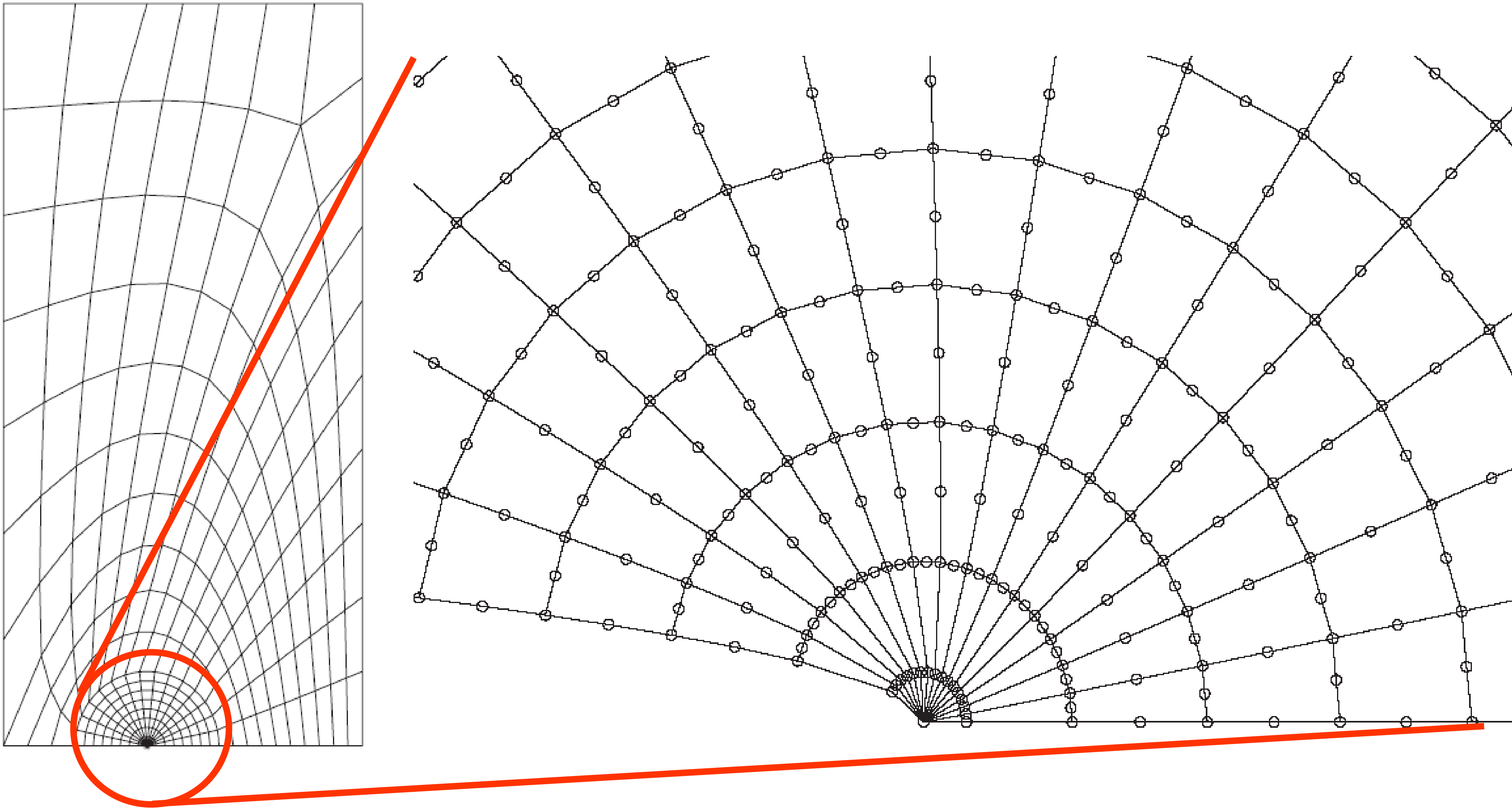
      - Periodic crack approximation
        - » <5% error for $a/W < 0.5$
        - » Irwin, 1957

$$f\left(\frac{a}{W}\right) = \sqrt{\frac{2W}{\pi a}\tan\frac{\pi a}{2W}}$$

      - Fit of Isida's values
        - » <0.1% error
        - » Tada, 1973

$$f\left(\frac{a}{W}\right) = \left[1 - 0.025\left(\frac{a}{W}\right)^2 + 0.06\left(\frac{a}{W}\right)^4\right]\sqrt{\frac{1}{\cos\frac{\pi a}{2W}}}$$

- Finite element model: extraction from stress and displacement fields
  - Here Barsoum elements with quarter-point, see later



- Finite element model: extraction from stress and displacement fields (2)
  - Asymptotic crack tip stress and displacement fields determined by the SIF

$$\boldsymbol{\sigma}^{\text{mode i}} = \frac{K_i}{\sqrt{2\pi r}}\mathbf{f}^{\text{mode i}}(\theta) \quad \& \quad \boldsymbol{u}^{\text{mode i}} = K_i\sqrt{\frac{r}{2\pi}}\boldsymbol{g}^{\text{mode i}}(\theta)$$

  - Stress correlation
    - FEM computation
    - Extract stress field for θ=0

$$\left\{\begin{array}{l}
\boldsymbol{\sigma}_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]\\
\boldsymbol{\sigma}_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\underbrace{\left[1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]}_{1}\\
\boldsymbol{\sigma}_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\underbrace{\sin\frac{\theta}{2}\cos\frac{3\theta}{2}}_{0}
\end{array}\right.
\left\{\begin{array}{l}
\boldsymbol{\sigma}_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left[2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right]\\
\boldsymbol{\sigma}_{yy} = \frac{K_{II}}{\sqrt{2\pi r}}\underbrace{\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}}_{0}\\
\boldsymbol{\sigma}_{xy} = \frac{K_{II}}{\sqrt{2\pi r}}\underbrace{\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]}_{1}
\end{array}\right.
\left\{\begin{array}{l}
\boldsymbol{\sigma}_{xz} = -\frac{K_{III}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\\
\boldsymbol{\sigma}_{yz} = \frac{K_{III}}{\sqrt{2\pi r}}\underbrace{\cos\frac{\theta}{2}}_{1}
\end{array}\right.$$

    - Extrapolate to get the SIFs

$$\begin{pmatrix} K_I \\ K_{II} \\ K_{III} \end{pmatrix} = \lim_{r\to 0}\sqrt{2\pi r}\begin{pmatrix} \boldsymbol{\sigma}_{yy}(r,0) \\ \boldsymbol{\sigma}_{xy}(r,0) \\ \boldsymbol{\sigma}_{zy}(r,0) \end{pmatrix}$$

- Finite element model: extraction from stress and displacement fields (3)
  - Asymptotic crack tip stress and displacement fields determined by the SIF

$$\boldsymbol{\sigma}^{\text{mode i}} = \frac{K_i}{\sqrt{2\pi r}}\mathbf{f}^{\text{mode i}}(\theta) \quad \& \quad \boldsymbol{u}^{\text{mode i}} = K_i\sqrt{\frac{r}{2\pi}}\boldsymbol{g}^{\text{mode i}}(\theta)$$

  - Displacement correlation
    - FEM computation
    - Extract stress field for $\theta = \pi$

$$\left\{\begin{array}{l} \boldsymbol{u}_x = K_I\dfrac{1+\nu}{E}\sqrt{\dfrac{r}{2\pi}}\underbrace{\cos\dfrac{\theta}{2}\left[\kappa - 1 + 2\sin^2\dfrac{\theta}{2}\right]}_{0} \\ \boldsymbol{u}_y = K_I\dfrac{1+\nu}{E}\sqrt{\dfrac{r}{2\pi}}\underbrace{\sin\underset{-}{\dfrac{\theta}{2}}\left[\kappa + 1 - 2\cos^2\dfrac{\theta}{2}\right]}_{\kappa+1} \end{array}\right. \quad \left\{\begin{array}{l} \boldsymbol{u}_x = K_{II}\dfrac{1+\nu}{E}\sqrt{\dfrac{r}{2\pi}}\underbrace{\sin\dfrac{\theta}{2}\left[\kappa + 1 + 2\cos^2\dfrac{\theta}{2}\right]}_{\kappa+1} \\ \boldsymbol{u}_y = K_{II}\dfrac{1+\nu}{E}\sqrt{\dfrac{r}{2\pi}}\underbrace{\cos\dfrac{\theta}{2}\left[1 - \kappa + 2\sin^2\dfrac{\theta}{2}\right]}_{0} \end{array}\right.$$

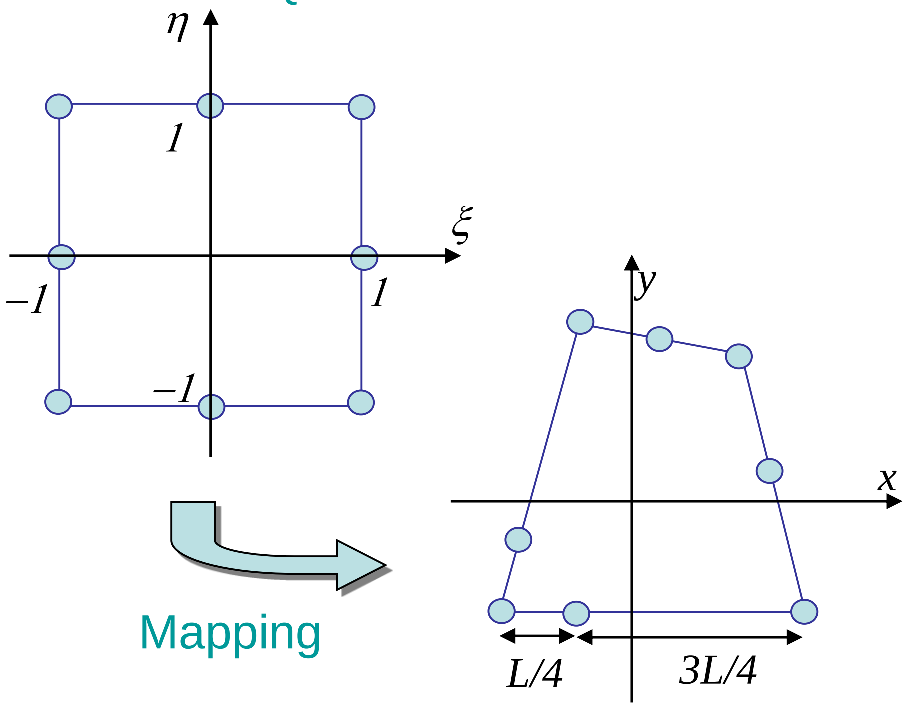
$$\left\{\ \boldsymbol{u}_z = \frac{2K_{III}(1+\nu)}{E}\sqrt{\frac{2r}{\pi}}\underbrace{\sin\frac{\theta}{2}}_{1}\right.$$

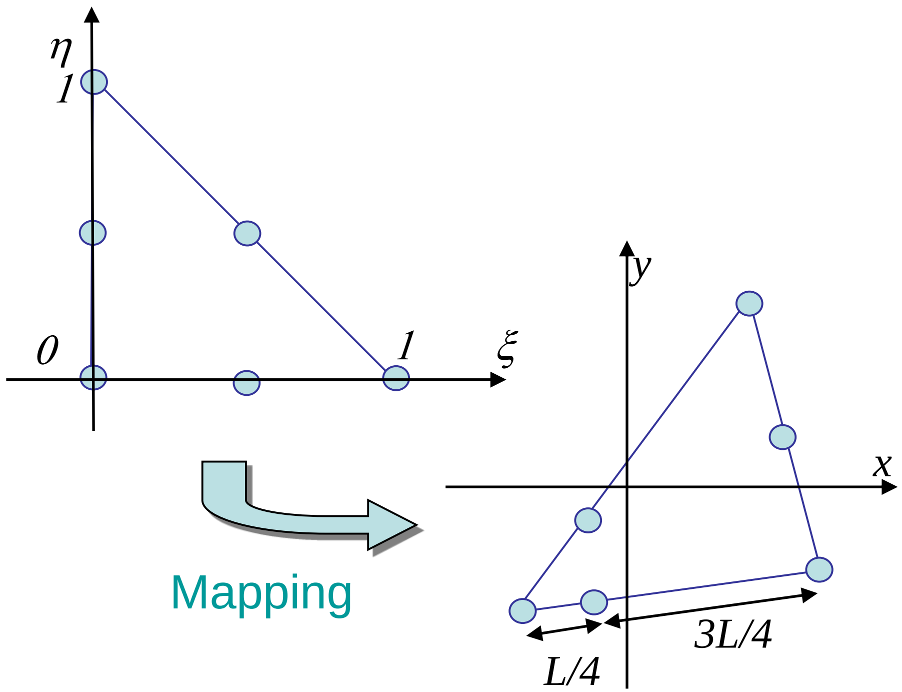
    - Extrapolate to get the SIFs

$$\begin{pmatrix} \frac{4K_I}{E'} \\ \frac{4K_{II}}{E'} \\ \frac{K_{III}}{\mu} \end{pmatrix} = \lim_{r\to 0}\sqrt{\frac{2\pi}{r}}\begin{pmatrix} \boldsymbol{u}_y(r,\pi) \\ \boldsymbol{u}_x(r,\pi) \\ \boldsymbol{u}_z(r,\pi) \end{pmatrix} \qquad E' = \frac{4E}{(1+\nu)(\kappa+1)} = \left\{\begin{array}{ll} E & \text{P}\sigma \\ \frac{E}{1-\nu^2} & \text{P}\varepsilon \end{array}\right.$$

- Finite element model: extraction from stress and displacement fields (4)
  - Advantages of the method
    - Simple
    - Can be used with any FE software
    - Only one post-processing to determine the 3 SIFs (if suitable loading)
    - Can be used along a crack front in 3D
    - When using displacement correlation, the field is the primary solution
  - Drawbacks
    - The accuracy is strongly dependent on the mesh refinement
    - The mesh refinement required depends on the element ability to capture the singularity at crack tip (e.g. Barsoum elements)

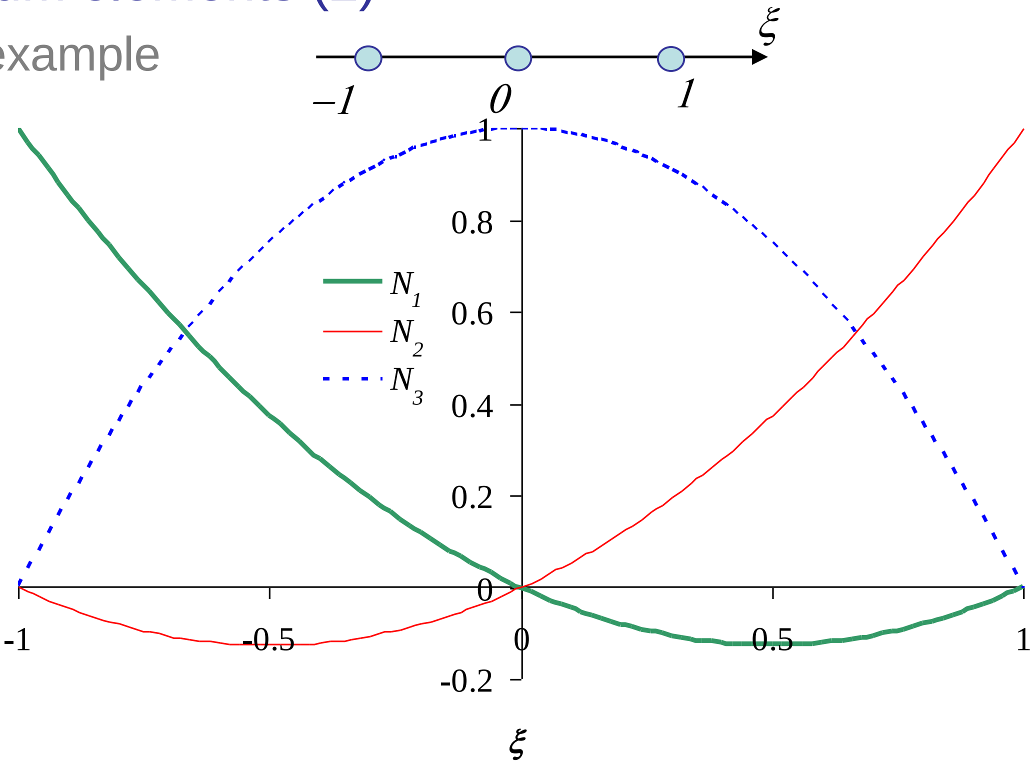
- Finite element model: Barsoum elements
  - Previous method requires a fine mesh since a singularity in $r^{½}$ is not naturally captured by usual FE
  - This singularity can be captured
    - By enriching elements (as in XFEM for LEFM, see later)
    - By using Barsoum (quarter-point) elements
  - Quarter-point elements
    - Quadratic elements with some mid-nodes located at a quarter of the edge





- Finite element model: Barsoum elements (2)
  - Quarter-point element: 1D example
    - Shape functions

$$\begin{cases} N_1 = \dfrac{\xi}{2}\left(\xi - 1\right) \\ N_2 = \dfrac{\xi}{2}\left(\xi + 1\right) \\ N_3 = \left(1 - \xi^2\right) \end{cases}$$



    - Mapping

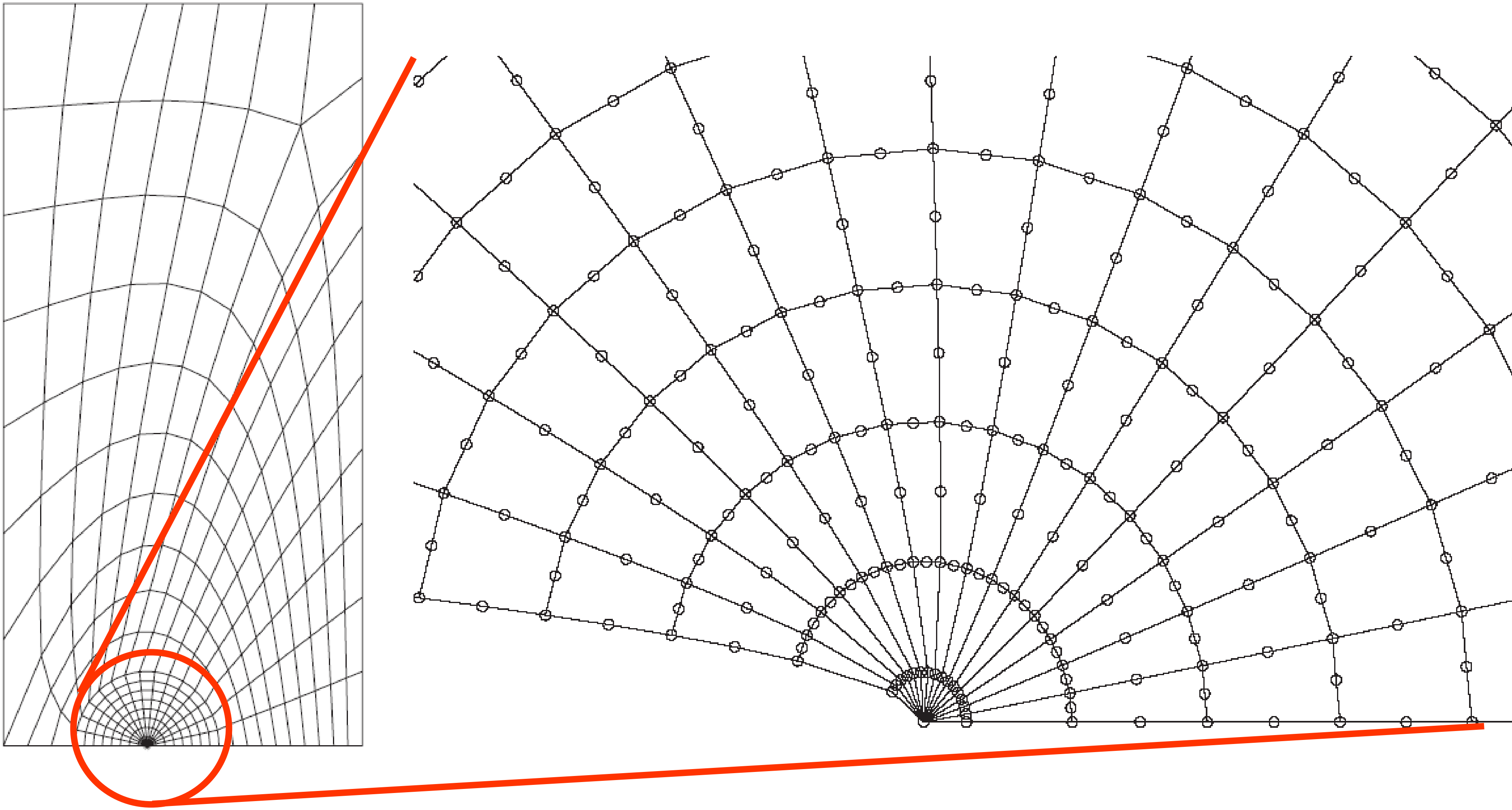
$$x\left(\xi\right) = N_i x_i = \frac{\xi L}{2}\left(\xi + 1\right) + \frac{L}{4}\left(1 - \xi^2\right) = \frac{L}{4}\left(1 + \xi\right)^2 \Longrightarrow \xi\left(x\right) = \sqrt{\frac{4x}{L}} - 1$$

    - Strain field

$$u = N_i\left(\xi\right) u_i \Longrightarrow \varepsilon = \partial_x u = \partial_\xi N_i\left(\xi\right) \partial_x \xi u_i = \frac{1}{\sqrt{Lx}} N_{i,\xi} u_i$$

    - So a strain field (and so for stress) in $1/r^{½}$ can be captured

- Finite element model: Barsoum elements (3)
  - Quarter-point element: 2D example
    - Fine mesh at crack tip: mid-nodes of first ring are moved to quarter points



- Finite element model: global energy & compliance
  - For cracks growing straight ahead (if only one mode is involved)
    - $G=J$, and in linear elasticity $J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$
    - Prescribed loading: $G = \partial_A E_{\text{int}}\big|_Q$ (in linear elasticity)
    - Prescribed displacement: $G = -\, \partial_A E_{\text{int}}\big|_u$
  - Perform 2 computations
    - With the same loading (displacement) but with two different crack lengths

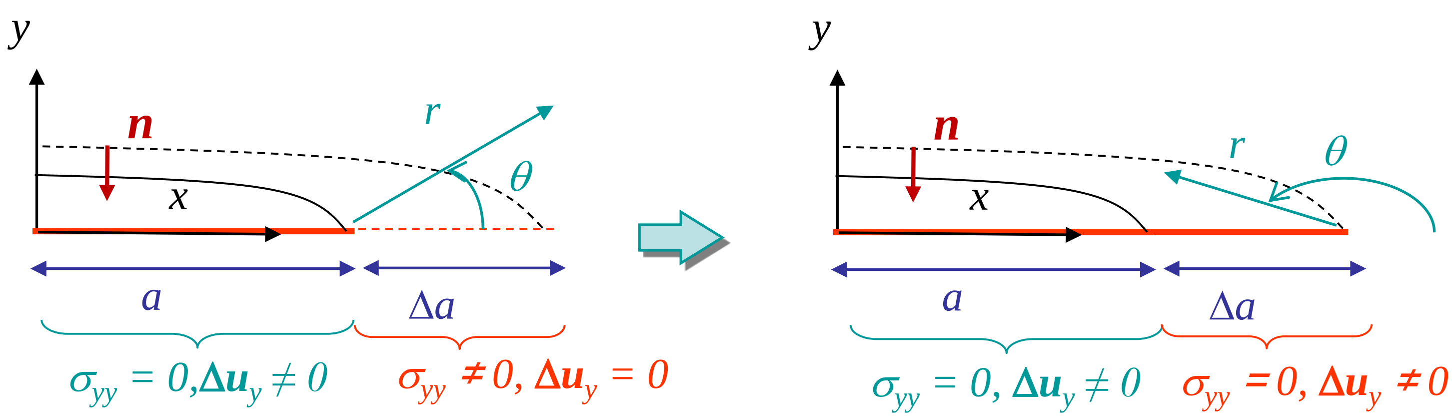
$$G \simeq \pm \frac{E_{\text{int}}(a+\Delta a) - E_{\text{int}}(a)}{t\Delta a}$$

    - Could be rewritten using compliance C in terms of the generalized loading Q

$$G \simeq Q^2 \frac{C(a+\Delta a) - C(a)}{2t\Delta a}$$ (linear elasticity)

  - Advantages
    - Simple
    - Can be used with any FE software and requires only post-processing
    - Less mesh sensitive than correlation methods as a global variable is used
  - Drawbacks
    - 2 computations needed
    - Only one mode (so one SIF) can be considered at a time
    - In 3D, SIF variation along crack front cannot be determined

- Finite element model: crack closure integral
  - Cracks growing straight ahead



  - For such cracks, in linear elasticity

$$G = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_{\Delta a} \frac{1}{2} \boldsymbol{\sigma}^{\mathbf{0}}_{\boldsymbol{iy}}(r, \theta = 0) \cdot [\![\boldsymbol{\Delta u_i}(\Delta a - r, \theta = \pi)]\!]\, dr$$

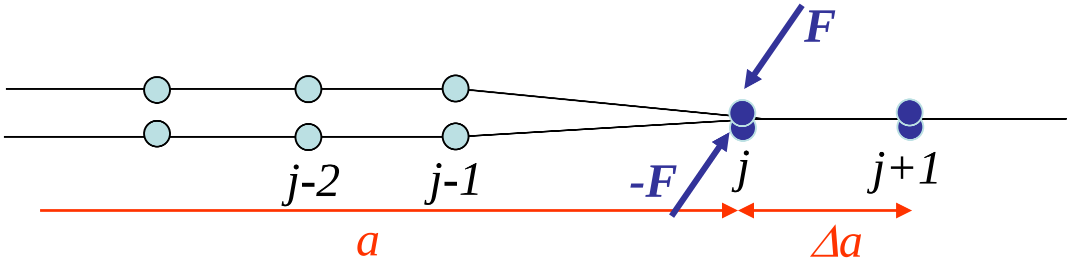
Stress before crack growth

Displacement jump after crack growth

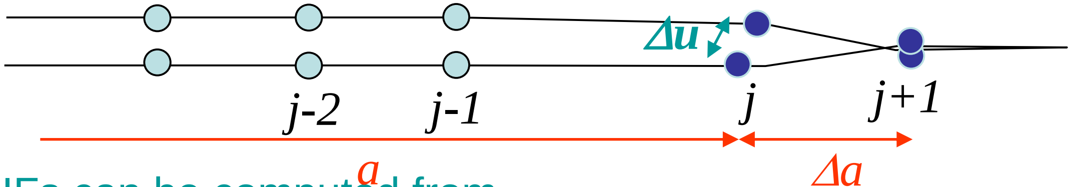
- Finite element model: crack closure integral (2)
  - This integral can be computed using FE software

$$G = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_{\Delta a} \frac{1}{2} \boldsymbol{\sigma}_{iy}^{\mathbf{0}}(r, \theta = 0) \cdot [\![\boldsymbol{\Delta u_i}(\Delta a - r, \theta = \pi)]\!] \, dr$$

- Nodal release
  - First computation: nodes *j* on both crack lips are constrained together



  - Second computation: constrain on node *j* is released



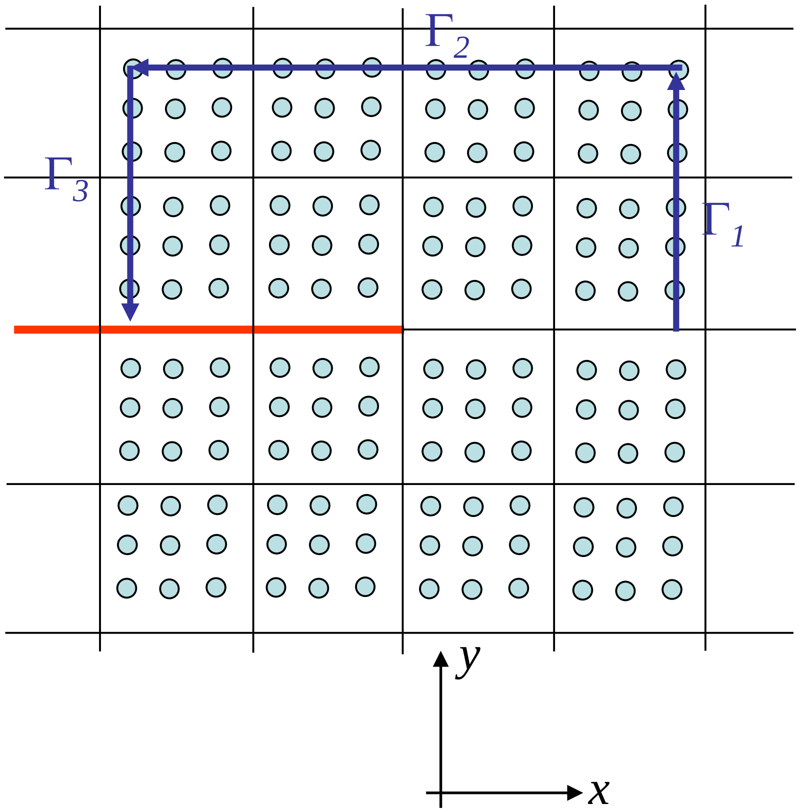
- SIFs can be computed from
  - Reaction forces of first computation &
  - Displacement jumps of second computation
  - $\frac{K_I^2}{E'} = \frac{\boldsymbol{F}_y \Delta \boldsymbol{u}_y}{2\Delta a}$, $\frac{K_{II}^2}{E'} = \frac{\boldsymbol{F}_x \Delta \boldsymbol{u}_x}{2\Delta a}$ & $\frac{K_{III}^2}{2\mu} = \frac{\boldsymbol{F}_z \Delta \boldsymbol{u}_z}{2\Delta a}$

- Finite element model: crack closure integral (3)
  - Advantage
    - Requires only post processing
  - Drawbacks
    - If the crack is not a line of symmetry, the node displacements have to be constrained (using Lagrange multipliers e.g.)
    - Two computations are needed (can be improved using modified nodal release)

- Direct computation of J-integral
  - For linear elasticity

$$J = \int_{\Gamma} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \right] dl \;\; \& \;\; J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

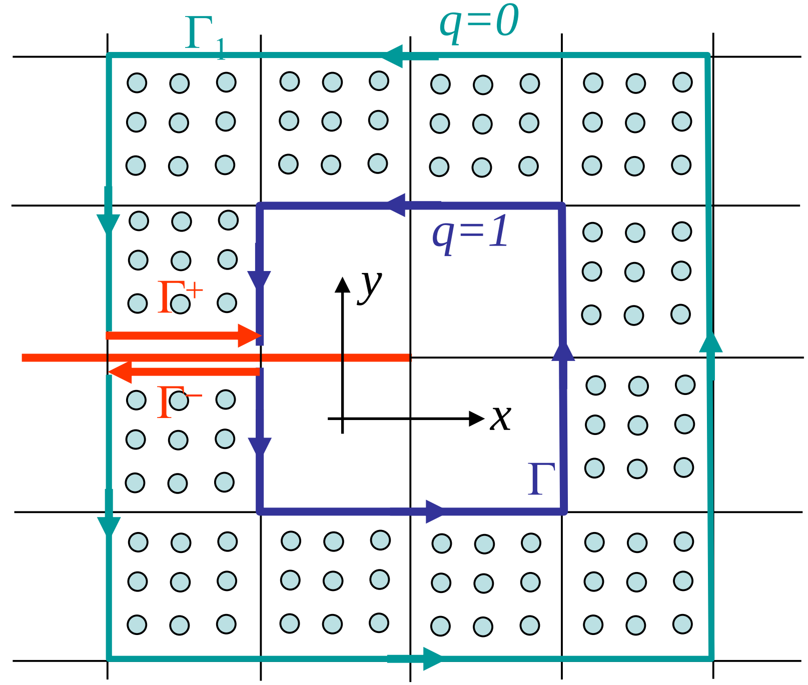
  - Contour Γ embedding the crack tip
    - Path independent ⟹ choose a contour through Gauss points
    - Values are computed on Gauss points ⟹ integration is direct
  - Not always accurate
    - We would prefer to define a contour from elements and not Gauss points



- *J*-integral by domain integration
  - For linear elasticity and for any contour Γ embedding a straight crack tip

$$J = \int_{\Gamma} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \right] dl$$

  - Let us define a contour $C=\Gamma_1+\Gamma^-+\Gamma^+-\Gamma$
    - Interior of this contour is the region D
    - Define $q(x, y)$ such that
      - $q = 1$ on Γ
      - $q = 0$ on Γ_1



$$\Longrightarrow J = \int_{\Gamma} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x \overset{1}{q} - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \overset{1}{q} \right] dl - \int_{\Gamma_1} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x \overset{0}{q} - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \overset{0}{q} \right] dl$$

    - As crack is stress free

$$\int_{\Gamma^{\pm}} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x q - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} q \right] dl = 0$$

$$\Longrightarrow J = - \int_{C} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x q - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} q \right] dl$$

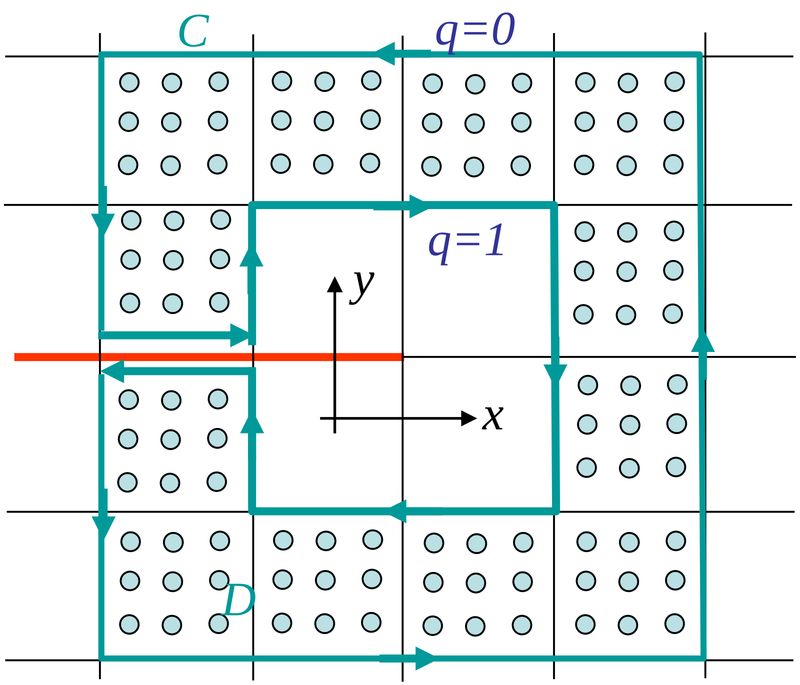
- J-integral by domain integration (2)
  - Computation of $J = -\int_C \left[U\left(\boldsymbol{\varepsilon}\right)\boldsymbol{n}_x q - \boldsymbol{u}_{,x}\cdot\boldsymbol{T}q\right]dl$
    - C is closed ⟹ divergence theorem
    - First term, with $\boldsymbol{\sigma}$ symmetric

$$\int_C U(\boldsymbol{\varepsilon})\, q\, \boldsymbol{n}_x\, dl = \int_D [U(\boldsymbol{\varepsilon})\, q\,]_{,x} dA$$

$$\Longrightarrow \int_C U(\boldsymbol{\varepsilon})\, q\, \boldsymbol{n}_x\, dl = \int_D \left[\partial_{\boldsymbol{\varepsilon}} U(\boldsymbol{\varepsilon}) : \boldsymbol{\varepsilon}_{,x}\, q + U(\boldsymbol{\varepsilon})\, q_{,x}\right] dA$$

$$\Longrightarrow \int_C U(\boldsymbol{\varepsilon})\, q\, \boldsymbol{n}_x\, dl = \int_D \left[\boldsymbol{\sigma}_{ij}\frac{\boldsymbol{u}_{i,jx} + \boldsymbol{u}_{j,ix}}{2} q + U(\boldsymbol{\varepsilon})\, q_{,x}\right] dA$$

$$\Longrightarrow \int_C U(\boldsymbol{\varepsilon})\, q\, \boldsymbol{n}_x\, dl = \int_D \left[\boldsymbol{\sigma}_{ij}\boldsymbol{u}_{i,jx} q + U(\boldsymbol{\varepsilon})\, q_{,x}\right] dA$$

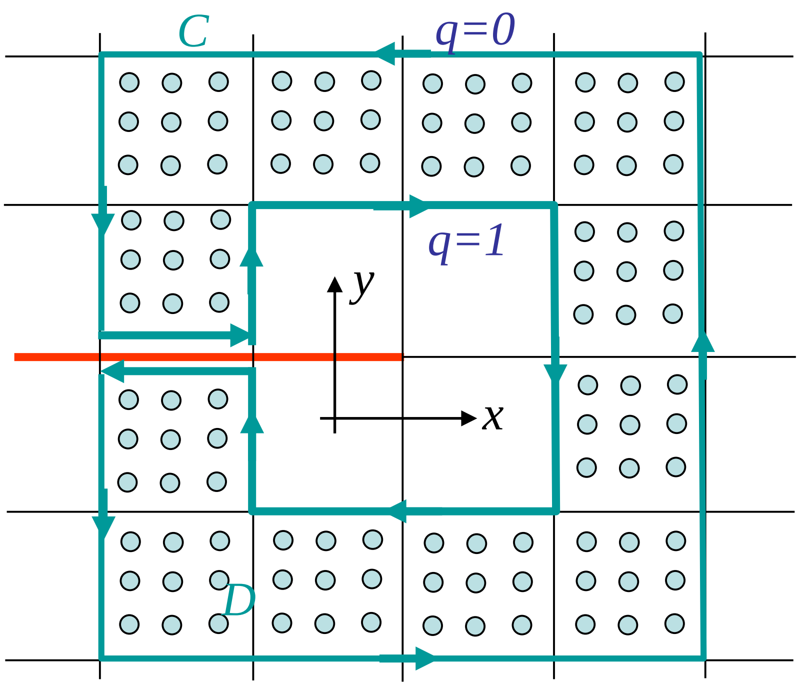


- *J*-integral by domain integration (3)
  - Computation of $J = -\int_C \left[U(\boldsymbol{\varepsilon})\, \boldsymbol{n}_x q - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} q \right] dl$
    - C is closed ⇒ divergence theorem
    - Second term, with $\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}$

$$\int_C q\boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}\, dl = \int_D \nabla \cdot [q\boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma}] dA$$

⇒
$$\int_C q\boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}\, dl = \int_D \nabla q \cdot [\boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma}] dA + \int_D q\ [\nabla \cdot \boldsymbol{\sigma}] \cdot \boldsymbol{u}_{,x} dA + \int_D q\boldsymbol{\sigma} : \nabla \boldsymbol{u}_{,x} dA$$

⇒
$$\int_C q\boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}\, dl = \int_D \nabla q \cdot [\boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma}] dA + \int_D \boldsymbol{\sigma}_{ij} \boldsymbol{u}_{i,jx} q dA$$

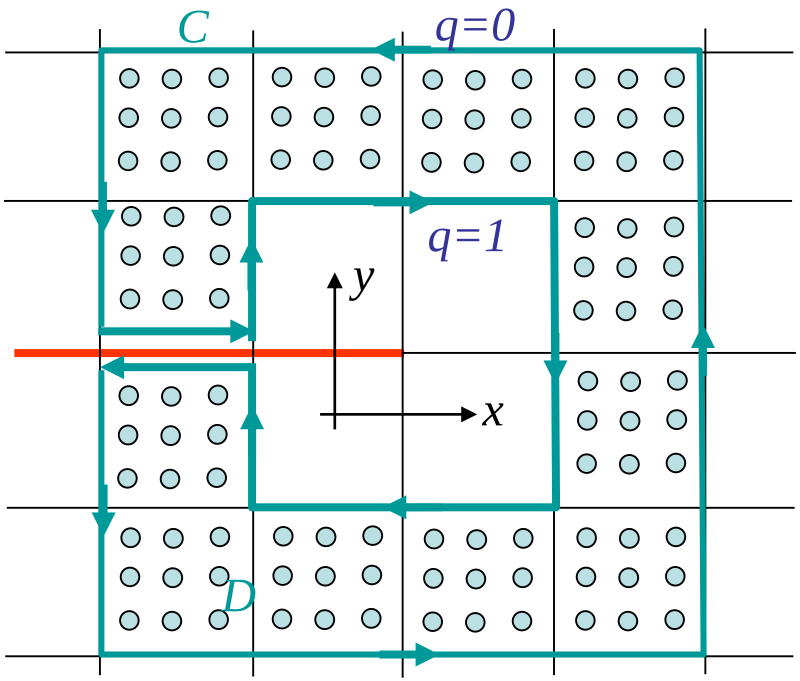


- *J*-integral by domain integration (4)
  - Computation of $J = -\int_C \left[U\left(\boldsymbol{\varepsilon}\right)\boldsymbol{n}_x q - \boldsymbol{u}_{,x}\cdot\boldsymbol{T}q\right] dl$
    - With divergence theorem:

$$\Rightarrow \int_C U(\boldsymbol{\varepsilon})\, q\, \boldsymbol{n}_x\, dl = \int_D \left[\boxed{\boldsymbol{\sigma}_{ij}\boldsymbol{u}_{i,jx}q} + U(\boldsymbol{\varepsilon})\, q_{,x}\right] dA$$

$$\Rightarrow \int_C q\boldsymbol{u}_{,x}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\, dl = \int_D \nabla q\cdot\left[\boldsymbol{u}_{,x}\cdot\boldsymbol{\sigma}\right] dA + \int_D \boxed{\boldsymbol{\sigma}_{ij}\boldsymbol{u}_{i,jx}q}\, dA$$

$$\Rightarrow J = \int_D \left[\boldsymbol{\sigma}_{ij}\boldsymbol{u}_{i,x}q_{,j} - Uq_{,x}\right] dA$$



  - This integral is valid for any annular region around the crack tip
    - q is discretized using the same shape functions than the elements
    - As long as the crack lips are straight

- Finite element model: J-integral
  - Advantages
    - Accurate (especially using domain integration)
    - Does not require Barsoum elements
  - Drawbacks
    - Only one mode at the time
    - Either modifying the FE code in order to have easy post processing or
    - Post processing difficult when using a standard FE software
  - Since this method is really accurate and computationally efficient, can it be modified in order to extract the SIF of each mode?

- Finite element model: *J*-integral (2)
  - How to extract the different SIFs?
    - Compute J for the solution $\boldsymbol{u}$ of the considered problem $\Longrightarrow J$

$$J = \int_{\Gamma} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \right] dl = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

    - Compute J for another field $\boldsymbol{u}^{aux}$ to be specialized $\Longrightarrow J^{aux}$

$$J^{aux} = \int_{\Gamma} \left[U\left(\boldsymbol{\varepsilon}^{aux}\right) \boldsymbol{n}_x - \boldsymbol{u}^{aux}_{,x} \cdot \boldsymbol{T}\left(\boldsymbol{u}^{aux}\right) \right] dl = \frac{K_I^{aux\,2}}{E'} + \frac{K_{II}^{aux\,2}}{E'} + \frac{K_{III}^{aux\,2}}{2\mu}$$

    - Compute J for the sum of $\boldsymbol{u}$ & $\boldsymbol{u}^{aux}$ to be specialized $\Longrightarrow J^s$

$$J^s = \int_{\Gamma} \left[U\left(\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^{aux}\right) \boldsymbol{n}_x - \left(\boldsymbol{u}_{,x} + \boldsymbol{u}^{aux}_{,x}\right) \cdot \boldsymbol{T}\left(\boldsymbol{u} + \boldsymbol{u}^{aux}\right) \right] dl$$

    - Relation between the different Js?

- Finite element model: J-integral (3)
  - How to extract the different SIFs (2)?
    - Relation between the different Js can be deduced from
      - $$U\left(\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}^{aux}\right)=\frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\nabla}\boldsymbol{u}^{aux}\right):\mathcal{H}:\left(\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\nabla}\boldsymbol{u}^{aux}\right)$$
        $$\Longrightarrow U\left(\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}^{aux}\right)=U\left(\boldsymbol{\varepsilon}\right)+U\left(\boldsymbol{\varepsilon}^{aux}\right)+\boldsymbol{\nabla}\boldsymbol{u}:\mathcal{H}:\boldsymbol{\nabla}\boldsymbol{u}^{aux}$$
      - $$\left(\boldsymbol{u}_{,x}+\boldsymbol{u}_{,x}^{aux}\right)\cdot\boldsymbol{T}\left(\boldsymbol{u}+\boldsymbol{u}^{aux}\right)=\boldsymbol{u}_{,x}\cdot\boldsymbol{T}\left(\boldsymbol{u}\right)+\boldsymbol{u}_{,x}^{aux}\cdot\boldsymbol{T}\left(\boldsymbol{u}^{aux}\right)+\boldsymbol{u}_{,x}\cdot\boldsymbol{T}\left(\boldsymbol{u}^{aux}\right)+\boldsymbol{u}_{,x}^{aux}\cdot\boldsymbol{T}\left(\boldsymbol{u}\right)$$
    - So $J^s=\int_\Gamma\left[U\left(\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}^{aux}\right)\boldsymbol{n}_x-\left(\boldsymbol{u}_{,x}+\boldsymbol{u}_{,x}^{aux}\right)\cdot\boldsymbol{T}\left(\boldsymbol{u}+\boldsymbol{u}^{aux}\right)\right]dl$ is rewritten

$$J^s-J-J^{aux}=\int_\Gamma\left[\boldsymbol{\nabla}\boldsymbol{u}:\mathcal{H}:\boldsymbol{\nabla}\boldsymbol{u}^{aux}\boldsymbol{n}_x-\boldsymbol{u}_{,x}\cdot\boldsymbol{T}\left(\boldsymbol{u}^{aux}\right)-\boldsymbol{u}_{,x}^{aux}\cdot\boldsymbol{T}\left(\boldsymbol{u}\right)\right]dl$$

  - The right term is called the interaction integral
    - » What is its expression in terms of the SIFs ?

- Finite element model: *J*-integral (4)
  - How to extract the different SIFs (3)?
    - It has been found that

$$J^s - J - J^{aux} = \int_\Gamma \left[\nabla \boldsymbol{u} : \mathcal{H} : \nabla \boldsymbol{u}^{aux} \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T}\left(\boldsymbol{u}^{aux}\right) - \boldsymbol{u}^{aux}_{,x} \cdot \boldsymbol{T}\left(\boldsymbol{u}\right)\right] dl$$

    - With $\begin{cases} \boldsymbol{\sigma} = \sum_{i=I}^{III} \frac{K_i}{\sqrt{2\pi r}} \mathbf{f}^{\text{mode i}}(\theta) \\ \boldsymbol{\sigma}\left(\boldsymbol{u}^{aux}\right) = \sum_{i=I}^{III} \frac{K_i^{aux}}{\sqrt{2\pi r}} \mathbf{f}^{\text{mode i}}(\theta) \end{cases}$ & $\begin{cases} \boldsymbol{u} = \sum_{i=I}^{III} K_i \sqrt{\frac{r}{2\pi}} \boldsymbol{g}^{\text{mode i}}(\theta) \\ \boldsymbol{u}^{aux} = \sum_{i=I}^{III} K_i^{aux} \sqrt{\frac{r}{2\pi}} \boldsymbol{g}^{\text{mode i}}(\theta) \end{cases}$

    - Direct substitution leads to
      - $\nabla \boldsymbol{u} : \mathcal{H} : \nabla \boldsymbol{u}^{aux} = \sum_i \sum_j K_i K_j^{aux} \nabla \frac{\sqrt{r}\boldsymbol{g}^{\text{mode i}}}{\sqrt{2\pi}} : \mathcal{H} : \nabla \frac{\sqrt{r}\boldsymbol{g}^{\text{mode j}}}{\sqrt{2\pi}}$
      - $\boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma}\left(\boldsymbol{u}^{aux}\right) + \boldsymbol{u}^{aux}_{,x} \cdot \boldsymbol{\sigma}\left(\boldsymbol{u}\right) =$

$$\sum_i \sum_j \left[K_i K_j^{aux} \partial_x \frac{\sqrt{r}\boldsymbol{g}^{\text{mode i}}}{\sqrt{2\pi}} \cdot \frac{\mathbf{f}^{\text{mode j}}}{\sqrt{2\pi r}} + K_j K_i^{aux} \partial_x \frac{\sqrt{r}\boldsymbol{g}^{\text{mode i}}}{\sqrt{2\pi}} \cdot \frac{\mathbf{f}^{\text{mode j}}}{\sqrt{2\pi r}}\right]$$

- Finite element model: *J*-integral (5)
  - How to extract the different SIFs (4)?
    - Since

$$J^s - J - J^{aux} = \int_{\Gamma} \left[\nabla \boldsymbol{u} : \mathcal{H} : \nabla \boldsymbol{u}^{aux} \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T}\left(\boldsymbol{u}^{aux}\right) - \boldsymbol{u}^{aux}_{,x} \cdot \boldsymbol{T}\left(\boldsymbol{u}\right) \right] dl$$

      - With

$$\begin{cases} \nabla \boldsymbol{u} : \mathcal{H} : \nabla \boldsymbol{u}^{aux} = \sum_i \sum_j K_i K_j^{aux} \nabla \frac{\sqrt{r} \boldsymbol{g}^{\text{mode i}}}{\sqrt{2\pi}} : \mathcal{H} : \nabla \frac{\sqrt{r} \boldsymbol{g}^{\text{mode j}}}{\sqrt{2\pi}} \\ \boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma}\left(\boldsymbol{u}^{aux}\right) + \boldsymbol{u}^{aux}_{,x} \cdot \boldsymbol{\sigma}\left(\boldsymbol{u}\right) = \\ \sum_i \sum_j \left[K_i K_j^{aux} \partial_x \frac{\sqrt{r} \boldsymbol{g}^{\text{mode i}}}{\sqrt{2\pi}} \cdot \frac{\mathbf{f}^{\text{mode j}}}{\sqrt{2\pi r}} + K_j K_i^{aux} \partial_x \frac{\sqrt{r} \boldsymbol{g}^{\text{mode i}}}{\sqrt{2\pi}} \cdot \frac{\mathbf{f}^{\text{mode j}}}{\sqrt{2\pi r}} \right] \end{cases}$$

    - And since these last two relations are symmetric in K and K^{aux}
      - By analogy with $J = \int_{\Gamma} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \right] dl = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$
      - One can feel that $\boxed{\frac{J^s - J - J^{aux}}{2} = \frac{K_I K_I^{aux}}{E'} + \frac{K_{II} K_{ii}^{aux}}{E'} + \frac{K_{III} K_{III}^{aux}}{2\mu}}$
    - This is obtained explicitly after substitution of $\mathbf{f}$ and $\boldsymbol{g}$ by their closed forms

- Finite element model: *J*-integral (6)
  - The SIFs are deduced from the so-called interaction integral
    - $I^s = J^s - J - J^{aux} = \frac{2}{E'}\left(K_I K_I^{aux} + K_{II} K_{II}^{aux}\right) + \frac{1}{\mu} K_{III} K_{III}^{aux}$
    - Indeed, if u^{aux} is chosen such that only $K_i^{aux} \neq 0$, K_i is obtained directly
  - This method
    - Is very accurate compared to correlation methods
    - But it requires
      - More computations
      - Extensive modifications of the FE code

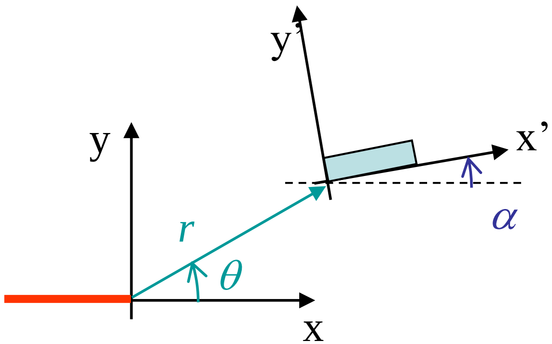
- Strain gauge method
  - Strain gauge can measure the strain $\Longrightarrow$ deduce the stress profile
  - SIF from the stress profile (like in FEM)
    - Asymptotic solution for mode I

$$\begin{cases} \sigma_{xx} = \dfrac{K_I}{\sqrt{2\pi r}} \cos\dfrac{\theta}{2}\left[1 - \sin\dfrac{\theta}{2}\sin\dfrac{3\theta}{2}\right] + A(\theta)r^0 + B(\theta)\sqrt{r}\ \ldots \\ \sigma_{yy} = \dfrac{K_I}{\sqrt{2\pi r}} \cos\dfrac{\theta}{2}\left[1 + \sin\dfrac{\theta}{2}\sin\dfrac{3\theta}{2}\right] + C(\theta)r^0 + D(\theta)\sqrt{r}\ \ldots \\ \sigma_{xy} = \dfrac{K_I}{\sqrt{2\pi r}} \cos\dfrac{\theta}{2}\sin\dfrac{\theta}{2}\cos\dfrac{3\theta}{2} + E(\theta)r^0 + F(\theta)\sqrt{r}\ \ldots \end{cases}$$

    - But the asymptotic solution in $\sqrt{r}$ cannot be matched accurately using a gauge
    - So consider up to the 3rd order

$$\Longrightarrow \begin{pmatrix} \boldsymbol{\sigma}_{rr} \\ \boldsymbol{\sigma}_{\theta\theta} \\ \boldsymbol{\sigma}_{r\theta} \end{pmatrix} = \frac{K_I}{\sqrt{2\pi r}} \boldsymbol{f}_{-1/2}(\theta) + C_0 \boldsymbol{f}_0(\theta) + C_{1/2}\sqrt{r}\boldsymbol{f}_{1/2}(\theta)$$

- Strain gauge method (2)
  - Position of the strain gauge
    - Located at (r, θ) of the crack tip
    - With an orientation α



  - One** can show that in the referential $O'x'y'$

$$2\mu\varepsilon_{x'x'} = \frac{K_I}{\sqrt{2\pi r}} F(\nu,\,\theta,\,\alpha) + C_0\left(\frac{1-\nu}{1+\nu} + \cos 2\alpha\right) + C_{1/2}\sqrt{r}\left(\frac{1-\nu}{1+\nu} + \sin^2\frac{\theta}{2}\cos 2\alpha - \frac{1}{2}\sin\theta\sin 2\alpha\right)$$

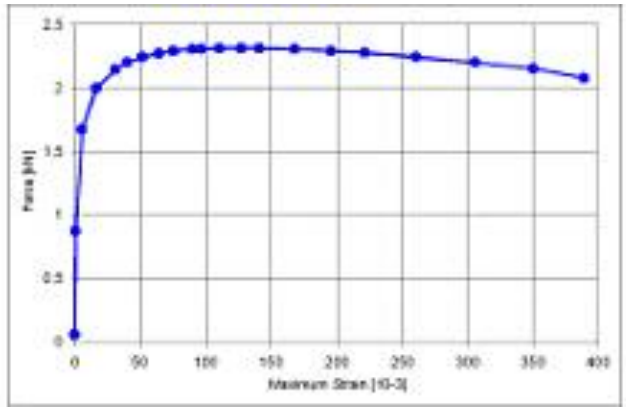
with $F(\nu,\,\theta,\,\alpha) = \frac{1-\nu}{1+\nu}\cos\frac{\theta}{2} - \frac{1}{2}\sin\theta\sin\frac{3\theta}{2}\cos 2\alpha + \frac{1}{2}\sin\theta\cos\frac{3\theta}{2}\sin 2\alpha$

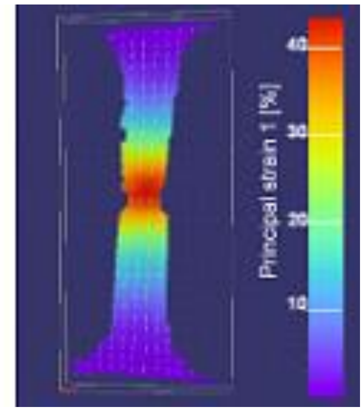
  - The gauge is located at (r, θ^*, α^*) such that
    - $\cos 2\alpha^* = -\frac{1-\nu}{1+\nu}$ ⟹ Term in $C_0\, r^0$ vanishes
    - $-\cot 2\alpha^* = \tan\frac{\theta^*}{2}$ ⟹ Term in $C_{1/2}\sqrt{r}$ vanishes

⟹ $K_I = \frac{2\mu\varepsilon_{x'x'}\sqrt{2\pi r}}{F(\nu,\,\theta^*,\,\alpha^*)}$

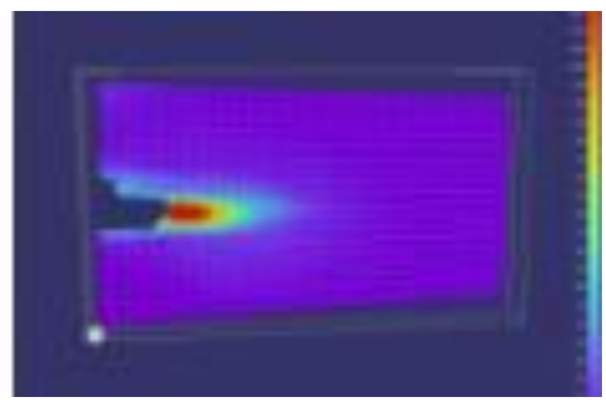
**Dally JW and Sanford RJ (1988), Strain gage methods for measuring the opening mode stress intensity factor KI , Experimental Mechanics 27, 381–388.

- Strain gauge method (3)
  - Strain gauges are now replaced by Digital Image Correlation (DIC)
    - Markers on the surface are tracked optically
    - Displacements and strains can be computed

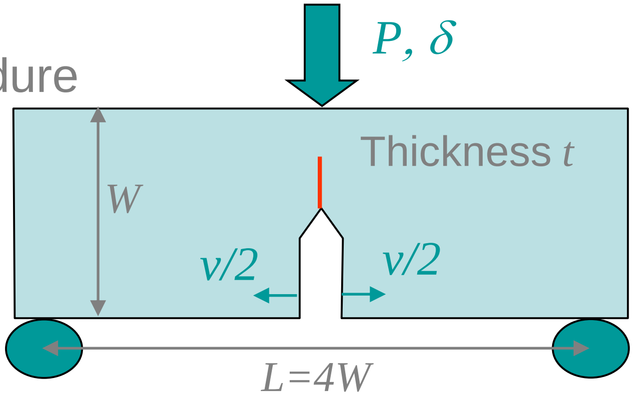


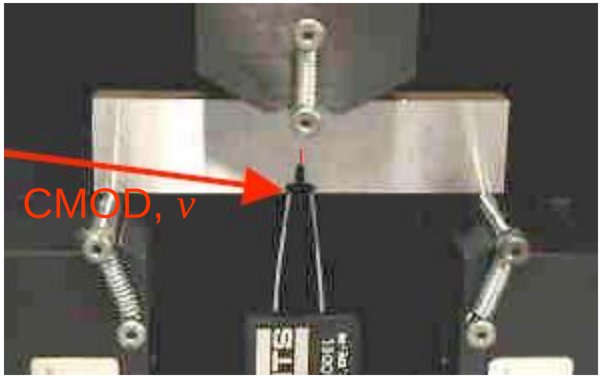


  - These strains can be used to deduce the SIFs

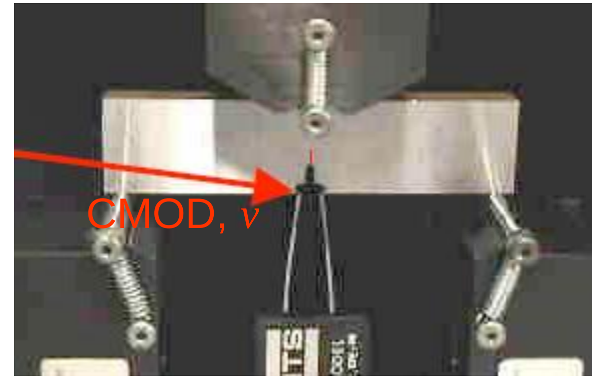


- Measuring K_{Ic}
  - Done by strictly following the ASTM E399 procedure
  - Specimen
    - Normalized, e.g. Single Notched Bend (SENB)
    - Plane strain constraint (thick enough)
      - ⟹ conservative (see next slide)
    - Specimen machined with a V-notch
  - Crack initiation
    - Cyclic loading to initiate a fatigue crack
    - Crack length from compliance
      - Crack Mouth Opening Displacement (CMOD=v) measured with a clipped gauge
      - Calibrated using FEM

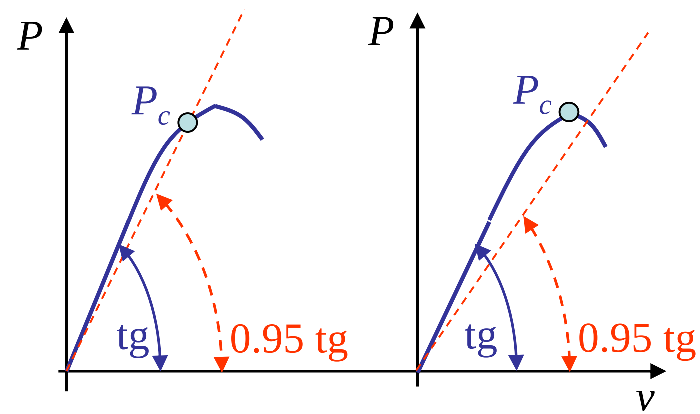




- Measuring K_{ic} (2)
  - Done by strictly following the ASTM E399 procedure
  - Toughness test
    - Calibrated P, δ recording equipment
    - Crack Mouth Opening Displacement (CMOD=v) measured with a clipped gauge



    - P_c is obtained on $P-v$ curves
      - Either the 95% offset value or
      - The maximal value reached before



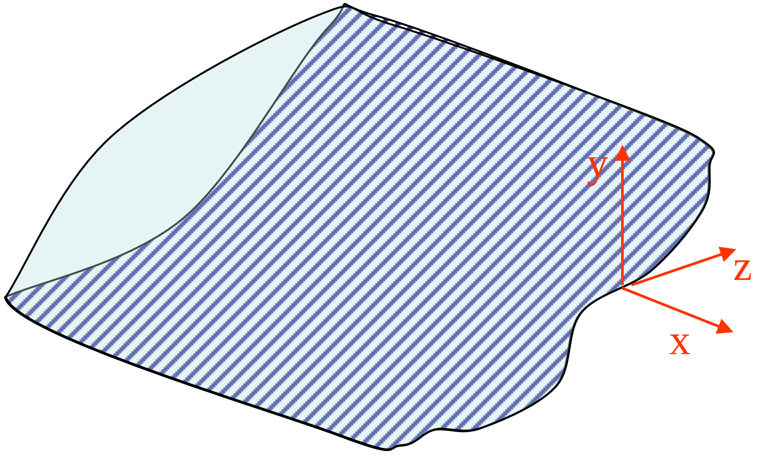
    - K_{Ic} is deduced from P_c using

$$K_I = \frac{PL}{tW^{\frac{3}{2}}} f\left(\frac{a}{W}\right)$$

      - *f(a/W)* depends on the test (SENB, …)
      - *f(a/W)* calibrated using FEM etc, in the norm
  - Check the constraint once you have K_{Ic}
    - Plane strain constraint (thick enough)

3D problems

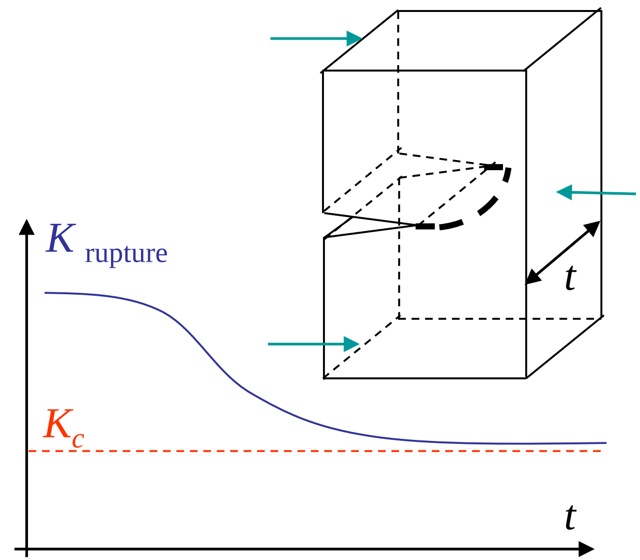
- Only 2D solutions have been considered but a real crack is clearly 3D
  - At any point of the crack line
    - A local referential can be defined
    - Since the asymptotic solutions hold for $r \rightarrow 0$, at this distance the crack line seems straight, and the problem is locally 2D
    - The crack tip field can be broken into 3 2D problems (3 2D modes)



- 3D effects
  - Near the border of a specimen the problem is plane σ, while it is plane ε near the center
    - ⟹ at the center there is a triaxial state
    - The SIF is larger at the center as no lateral deformations are possible (see next lecture)
    - 2 consequences
      - The front will first propagate at the center
      - The toughness decreases with the thickness
        - » Crude approximation

$$K_C\left(t\right) \simeq K_C\left(t\rightarrow\infty\right)\left[1+\frac{1.4}{t^2}\left(\frac{K_C\left(t\rightarrow\infty\right)}{\sigma_p^0}\right)^4\right]^{\frac{1}{2}}$$

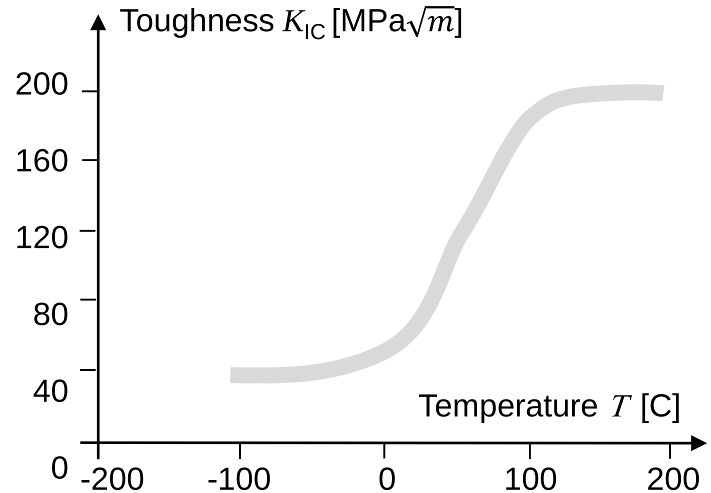
        - » Later on we will see how to evaluate this effect



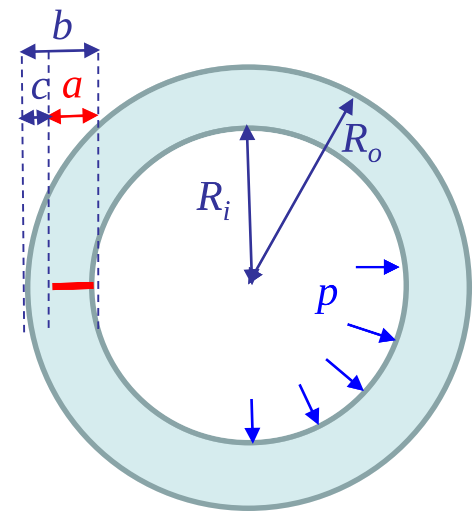
  - The practical toughness K_c is the plane strain one

Exercise 1

- Safety of a pressurized vessel
  - Extended internal axial crack
  - Plane strain condition along its axis
  - Made of steel with a DBTT

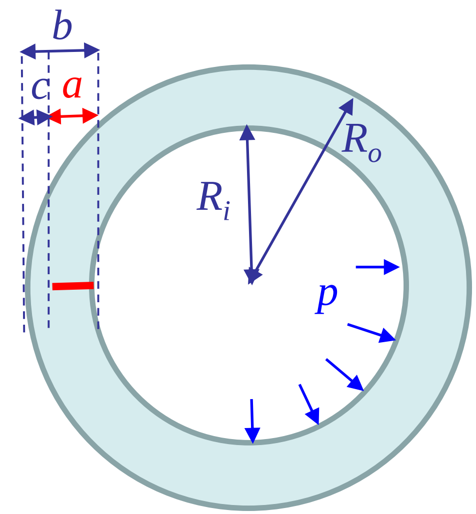


  - For a given pressure what is the maximal crack length that the flawed cylinder can sustain
    - LEFM: only for brittle (at low temperature)
    - At high temperature: non-linear analysis needed



Properties	Values
Internal radius R_i	1.5 m
External radius Ro	1.65 m
Young E	210 GPa
Yield σ_p^0	250 MPa
Poisson ν [-]	0.3
Hardening exponent n	10
Hardening parameter α	1

- Non-linear handbook, but elastic solution
  - Stress intensity factor
    - $$K_I = \frac{2pR_o^2\sqrt{\pi a}}{\left(R_o^2 - R_i^2\right)} F\left(\frac{a}{b}, \frac{R_i}{R_o}\right)$$
  - Mouth opening displacement
    - $$v_m = \frac{8pR_o^2 a}{\left(R_o^2 - R_i^2\right)E'} V_1\left(\frac{a}{b}, \frac{R_i}{R_o}\right)$$

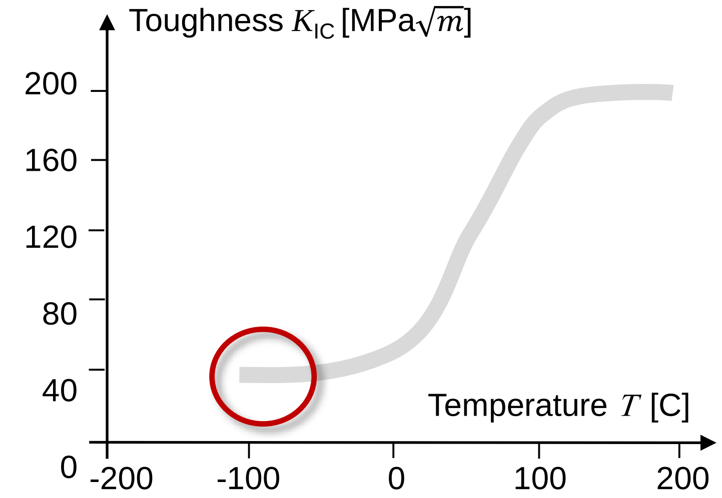


		$a/b=1/8$	$a/b=1/4$	$a/b=1/2$	$a/b=3/4$
$b/R_i = 1/5$	F	1.19	1.38	2.10	3.30
	V_1	1.51	1.83	3.44	7.50
$b/R_i = 1/10$	F	1.20	1.44	2.36	4.23
	V_1	1.54	1.91	3.96	10.4
$b/R_i = 1/20$	F	1.20	1.45	2.51	5.25
	V_1	1.54	1.92	4.23	13.5

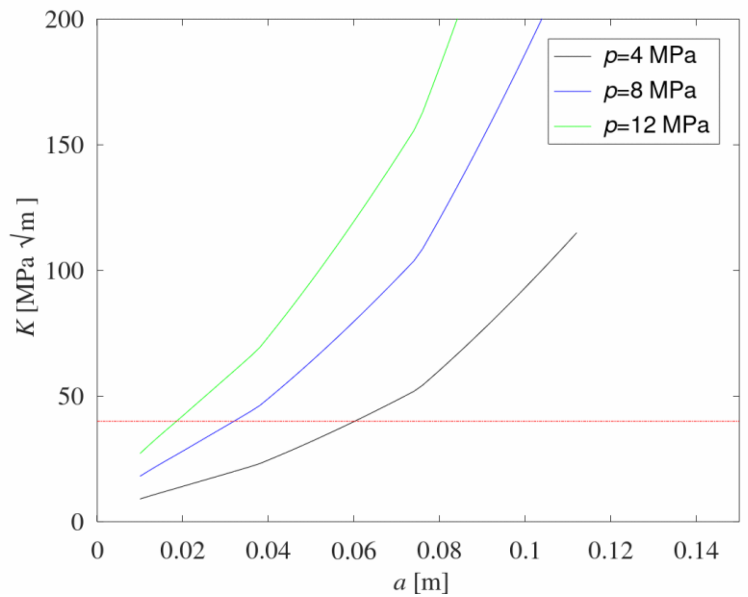
Properties	Values
Internal radius R_i	1.5 m
External radius Ro	1.65 m
Young E	210 GPa
Yield σ_p^0	250 MPa
Poisson ν [-]	0.3
Hardening exponent n	10
Hardening parameter α	1

- Critical crack length at low temperature
  - Toughness: $K_C = 40 \text{ MPa}\sqrt{\text{m}}$
  - Critical crack length a_c
    - $K_I = \frac{2pR_o^2\sqrt{\pi a}}{\left(R_o^2 - R_i^2\right)} F\left(\frac{a}{b}, \frac{R_i}{R_o}\right)$

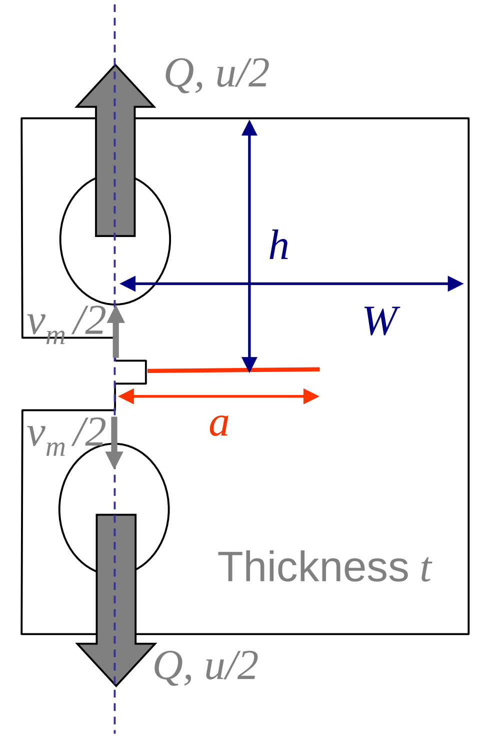
$$\Rightarrow \sqrt{a_c}\, F\left(\frac{a_c}{b}, \frac{R_i}{R_o}\right) = K_C \frac{\left(R_o^2 - R_i^2\right)}{2pR_o^2\sqrt{\pi}}$$

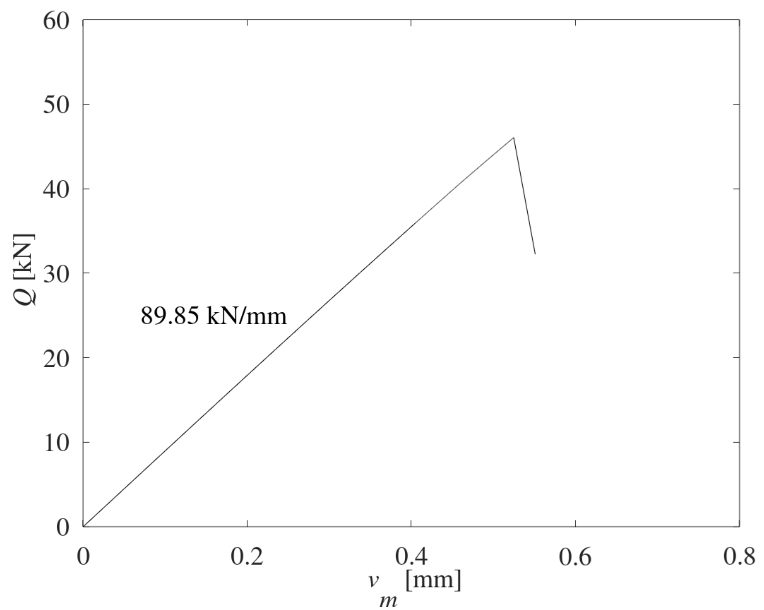


  - Has to be solved
    - Iteratively
    - Graphically



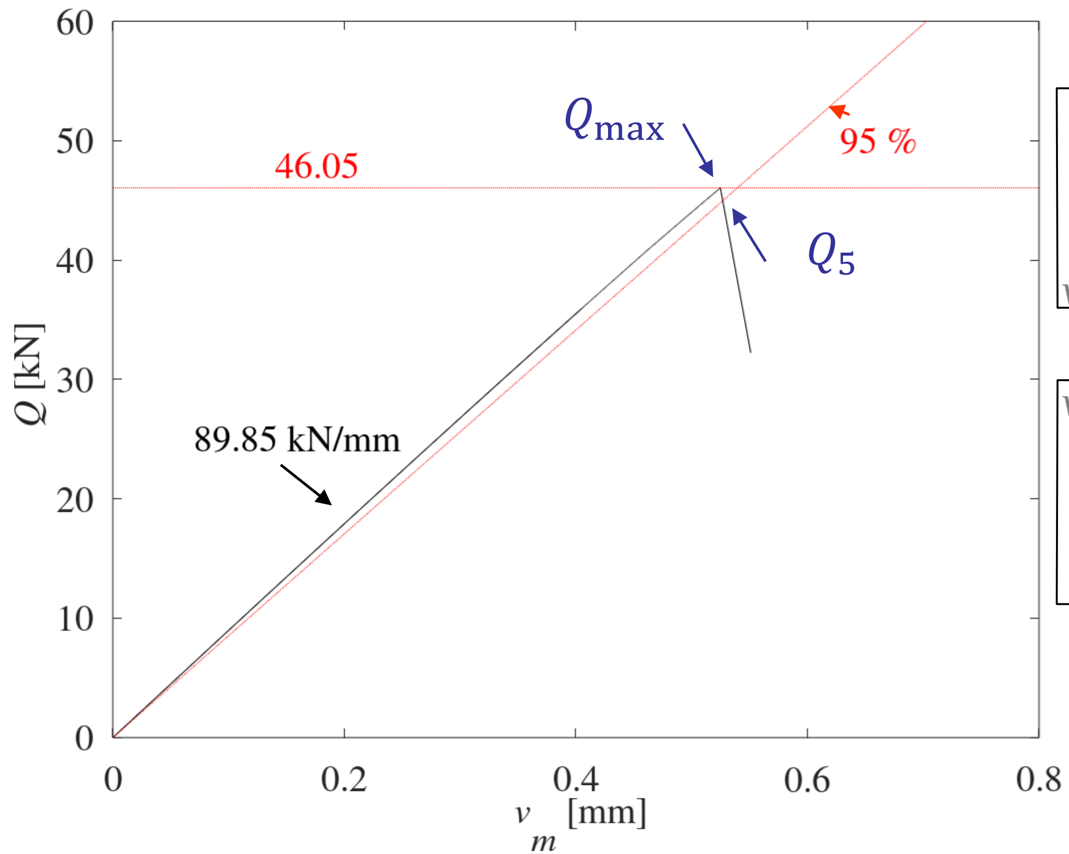
- Toughness evaluation
  - Brittle material: Aluminium alloy
    - For ductile, this requires non-linear analysis
  - Follow the norm ASTM E399
  - Normalized specimen
    - Compact Tension Specimen
    - Thick enough $a, t > 2.5\left(\frac{K_C}{\sigma_p^0}\right)^2$
    - Fracture test

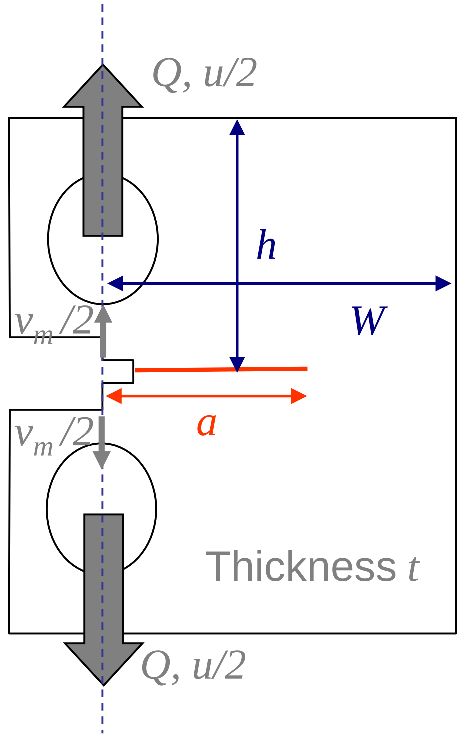




Properties	Values
Half height h	0.045 m
Width W	0.075 m
Thickness t	0.025 m
Young E	71 Gpa
Yield σ_p^0	430 Mpa
Poisson ν [-]	0.33

- Step 1: analyze fracture curve
  - In this case $Q_{\max} > Q_5$
    - Failure for $Q = Q_{\max} = 46.05\ \text{kN}$
  - Compliance
    - $\frac{v_{\text{m}}}{Q} = \frac{1}{89.85} = 0.01113\ \text{mm/kN}$

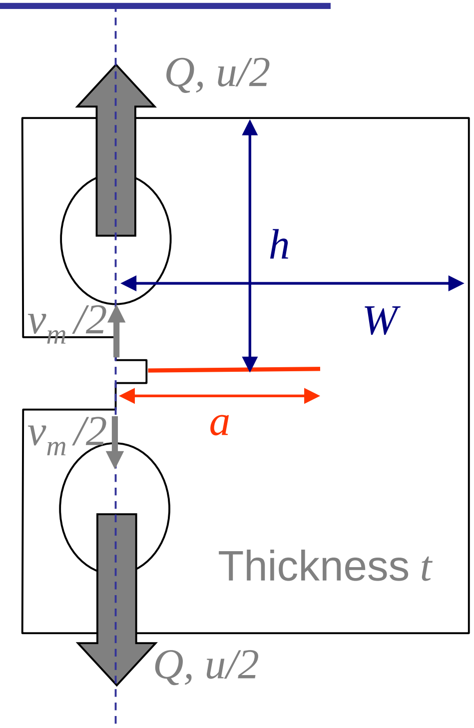




Exercise 2: Solution

- Step 2: evaluate crack length
  - Results from cyclic loading (fatigue)
  - Evaluated from the compliance
    - Calibration of the geometry following the norm

$$\begin{cases} U = \dfrac{1}{1 + \sqrt{\dfrac{E' t v_m}{Q}}} \\ \dfrac{a}{W} = 1 - 4.5U + 13.157U^2 - 172.551U^3 + \\ \qquad 879.944U^4 - 1514.671U^5 \end{cases}$$



- For this test

$$\frac{v_\mathrm{m}}{Q} = \frac{1}{89.85} = 0.01113\ \mathrm{mm/kN}$$

$$\Rightarrow \quad E' = \frac{E}{1 - \nu^2} = 79.68\ \mathrm{GPa}$$

$$\Rightarrow \quad U = 0.17518$$

$$\Rightarrow \quad \frac{a}{W} = 0.267$$

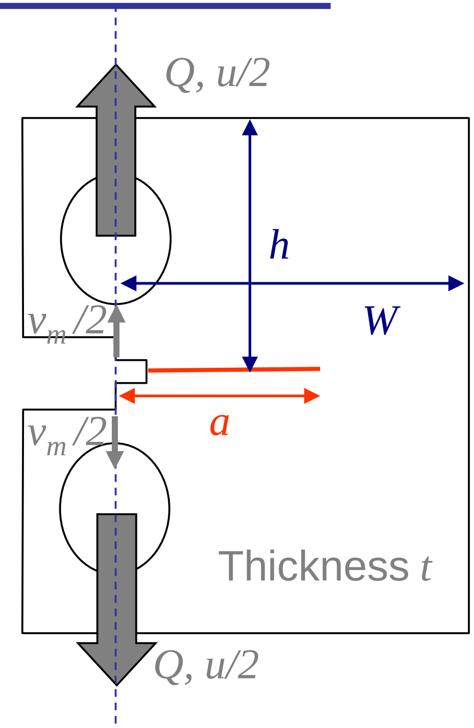
Properties	Values
Half height h	0.045 m
Width W	0.075 m
Thickness t	0.025 m
Young E	71 Gpa
Yield σ_p^0	430 Mpa
Poisson ν [-]	0.33

- Step 3: evaluate toughness
  - Stress intensity factor
    - Calibration of the geometry following the norm

$$K_I = \frac{Q}{tW^{\frac{1}{2}}} \frac{\left(2+\frac{a}{W}\right)\left(\begin{array}{c}0.886 + 4.64\frac{a}{W} - 13.32\left(\frac{a}{W}\right)^2 + \\ 14.72\left(\frac{a}{W}\right)^3 - 5.6\left(\frac{a}{W}\right)^4\end{array}\right)}{\left(1-\frac{a}{W}\right)^{\frac{3}{2}}}$$

  - At failure
    - Failure for $Q = Q_{\max} = 46.05\ \mathrm{kN}$
    - With $\frac{a}{W} = 0.267$

$$\Rightarrow K_C = 34.8\ \mathrm{MPa}\sqrt{\mathrm{m}}$$



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Half height h	0.045 m
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Thickness t	0.025 m
Young E	71 Gpa
Yield σ_p^0	430 Mpa
Poisson ν [-]	0.33

- Step 4: check validity
  - Norm requires
    - Plain strain specimen for toughness definition

$$t > 2.5\left(\frac{K_C}{\sigma_p^0}\right)^2 = 0.016\ \text{m}$$

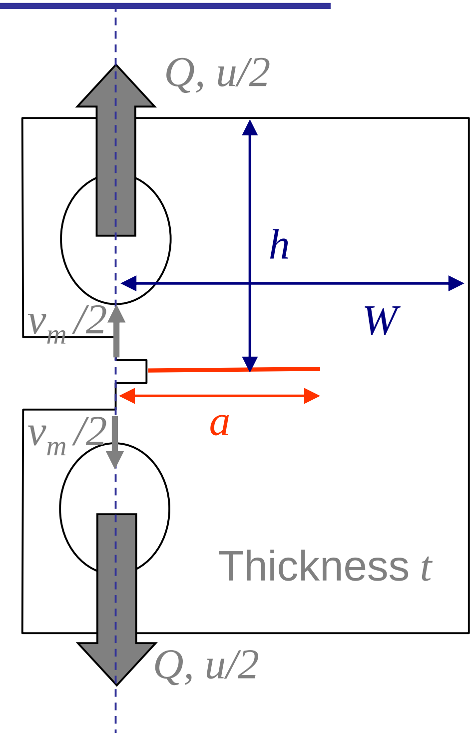
    - LEFM to hold

$$0.02\ \text{m} = a > 2.5\left(\frac{K_C}{\sigma_p^0}\right)^2 = 0.016\ \text{m}$$

    - Deep enough crack for calibrated formula to hold

$$\frac{a}{W} = 0.267 > 0.2$$

All conditions satisfied



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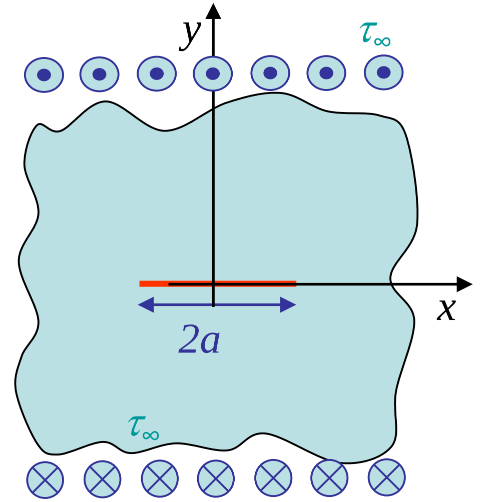
References

- Lecture notes
  - Lecture Notes on Fracture Mechanics, Alan T. Zehnder, Cornell University, Ithaca, http://hdl.handle.net/1813/3075
  - Fracture Mechanics Online Class, L. Noels, ULg, http://www.ltas-cm3.ulg.ac.be/FractureMechanics
- Book
  - Fracture Mechanics: Fundamentals and applications, D. T. Anderson. CRC press, 1991.
  - S. Suresh, Fatigue of Materials, Cambridge University Press, 2001

- Computation of SIF by analytical method (LEFM)
  - Crack in an infinite plate under shearing
    - Mode III:

$$\begin{cases} \boldsymbol{\varepsilon}_{\alpha z} = \frac{1}{2}\boldsymbol{u}_{z,\alpha} & \Longrightarrow \quad \boldsymbol{\sigma}_{\alpha z} = \frac{E}{1+\nu}\boldsymbol{\varepsilon}_{\alpha z} = \frac{E}{2(1+\nu)}\boldsymbol{u}_{z,\alpha} \\ \boldsymbol{\sigma}_{\alpha z,\alpha} = 0 & \Longrightarrow \quad \nabla^2 \boldsymbol{u}_z = 0 \end{cases}$$

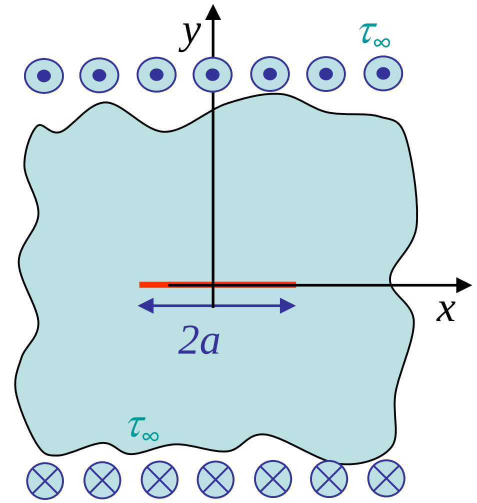
$\Longrightarrow$ $\boldsymbol{u}_z$ is the imaginary part of a function $z(\zeta)$



- This function has to be found to satisfy the BCs
- Stress field
  - As $\partial_\zeta z = \partial_x z = -i\partial_y z$
  - One has $\boldsymbol{\sigma}_{yz} = \frac{E}{2(1+\nu)}\boldsymbol{u}_{z,y} = \frac{E}{2(1+\nu)}\partial_y \mathcal{I}(z) = \frac{E}{2(1+\nu)}\mathcal{I}(iz')$
  - And $\boldsymbol{\sigma}_{xz} = \frac{E}{2(1+\nu)}\boldsymbol{u}_{z,x} = \frac{E}{2(1+\nu)}\partial_x \mathcal{I}(z) = \frac{E}{2(1+\nu)}\mathcal{I}(z')$

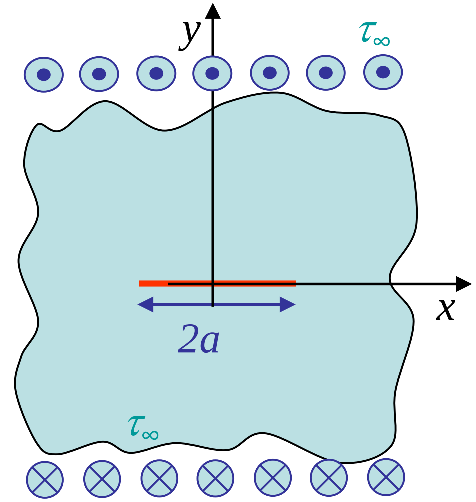
- Computation of SIF by analytical method (LEFM) (2)
  - Crack in an infinite plate under shearing (2)
    - Solution of the problem?
      - $z(\zeta) = \frac{2\tau_\infty (1+\nu)}{E} \sqrt{\zeta^2 - a^2}$
      - with

$$\begin{cases} \boldsymbol{\sigma}_{yz} = \frac{E}{2(1+\nu)} \mathcal{I}(iz') \\ \Rightarrow \boldsymbol{\sigma}_{yz} = \tau_\infty \mathcal{R}\left(\frac{\zeta}{\sqrt{\zeta^2 - a^2}}\right) \\ \boldsymbol{u}_z = \mathcal{I}(z) \end{cases}$$



- Symmetry?
  - $z(\bar{\zeta}) = \bar{z} \Rightarrow$ $\boldsymbol{u}_z(-y) = -\boldsymbol{u}_z(y)$ satisfied
- Far away field?
  - For $y \to \pm\infty$: $\boldsymbol{\sigma}_{yz} \to \pm\boldsymbol{\tau}_\infty$ satisfied

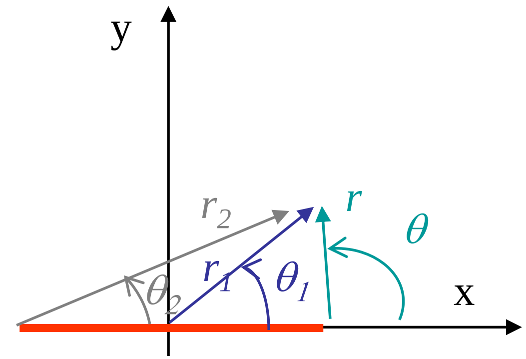
- Computation of SIF by analytical method (LEFM) (3)
  - Crack in an infinite plate under shearing (3)



- Solution of the problem?
  - $z(\zeta) = \dfrac{2\tau_\infty(1+\nu)}{E}\sqrt{\zeta^2 - a^2}$
  - with

$$\boldsymbol{\sigma}_{yz} = \tau_\infty \mathcal{R}\left(\frac{\zeta}{\sqrt{\zeta^2 - a^2}}\right)$$

- Crack lips stress free?



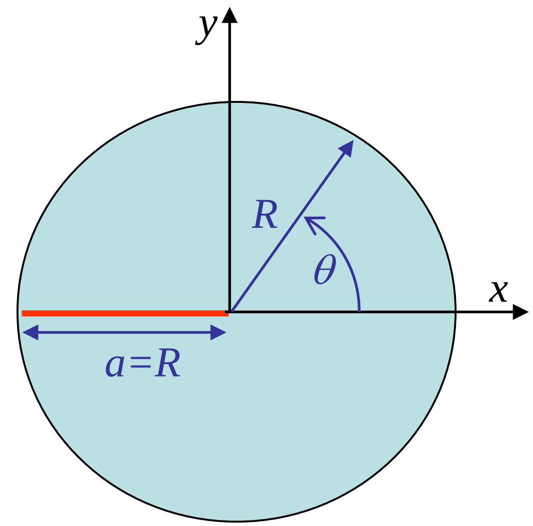
$$\begin{cases} \lim\limits_{\theta\to\pm\pi} \zeta = x \\ \lim\limits_{\theta\to\pm\pi} \sqrt{\zeta - a} = \lim\limits_{\theta\to\pm\pi} \sqrt{r}\exp\left(i\dfrac{\theta}{2}\right) = \pm i\sqrt{a-x} \\ \lim\limits_{\theta_2\to 0^\pm} \sqrt{\zeta + a} = \lim\limits_{\theta_2\to 0^\pm} \sqrt{r_2}\exp\left(i\dfrac{\theta_2}{2}\right) = \sqrt{x+a} \end{cases}$$

$$\Rightarrow \lim_{\theta\to\pm\pi,\,\theta_2\to 0} \boldsymbol{\sigma}_{yz} = \lim_{\theta\to\pm\pi,\,\theta_2\to 0} \tau_\infty \mathcal{R}\left(\frac{\zeta}{\sqrt{\zeta^2 - a^2}}\right) \to \tau_\infty \mathcal{R}\left(\frac{\mp i x}{\sqrt{a^2 - x^2}}\right) = 0$$

satisfied

- Computation of SIF by Boundary Collocation Method (LEFM)
  - Example: crack in a finite circular plate
    - Considering mode III (can be done for I and II)

$$\begin{cases} \tau = \boldsymbol{\sigma}_{yz} + i\boldsymbol{\sigma}_{xz} = \dfrac{E}{2\left(1+\nu\right)} z' \\ \boldsymbol{n} = \boldsymbol{n}_x + i\boldsymbol{n}_y \\ \bar{\boldsymbol{T}}_z = \boldsymbol{\sigma}_{xz}\boldsymbol{n}_x + \boldsymbol{\sigma}_{yz}\boldsymbol{n}_y = \mathcal{I}\left(\boldsymbol{\tau}\boldsymbol{n}\right) \end{cases}$$



$\overline{T}_z = f(\theta)$

- Laurent series $z\left(\zeta\right) = \sum_{n=-\infty}^{\infty} a_n \left(\zeta - \zeta_0\right)^n$

with $a_n = \dfrac{1}{2i\pi} \displaystyle\int_{\mathcal{C}} \dfrac{z\left(\xi\right)}{\left(\xi - \zeta_0\right)^{n+1}} d\xi$

- But for the crack lies in θ=π, so we use $\tau\left(\zeta\right) = \sum_{n=-\infty}^{\infty} a_n \zeta^{n-\frac{1}{2}}$

$$\Longrightarrow \lim_{\theta\to\pi} \boldsymbol{\sigma}_{yz} = \lim_{\theta\to\pi} \mathcal{R}\left(\tau\right) = \lim_{\theta\to\pi} \sum_{n=-\infty}^{\infty} a_n r^{n-\frac{1}{2}} \cos\left(\frac{2n-1}{2}\pi\right) = 0$$ is satisfied

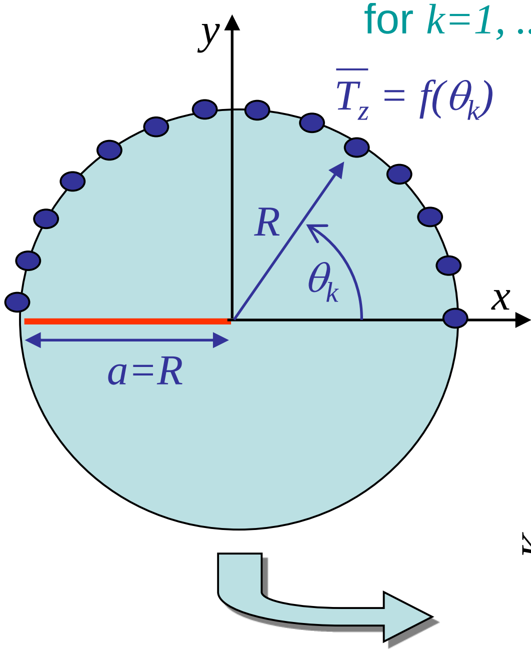
– Finite displacement $\Longrightarrow$ n = 0, 1, …

$$\Longrightarrow \tau\left(\zeta\right) = \sum_{n=0}^{\infty} a_n \zeta^{n-\frac{1}{2}} \Longrightarrow K_{III} = \lim_{r\to 0}\left(\sqrt{2\pi r}\, \boldsymbol{\sigma}_{yz}|_{\theta=0}\right) = \sqrt{2\pi} a_0$$

- Computation of SIF by Boundary Collocation Method (LEFM) (2)
  - Example: crack in a finite circular plate (2)
    - Unknowns a_n are obtained by defining m collocation points on the boundary
      - At these points

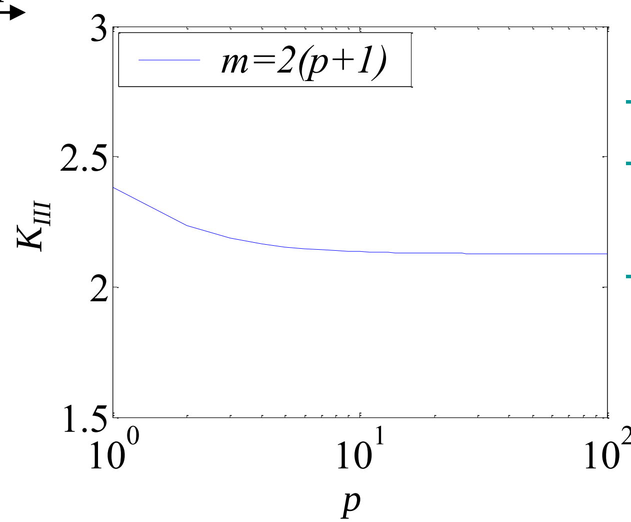
$$\bar{\boldsymbol{T}}_z\left(\theta_k\right)=\mathcal{I}\left(\boldsymbol{\tau}\left(\theta_k\right)\boldsymbol{n}\left(\theta_k\right)\right)=\mathcal{I}\left(\sum_{n=0}^{p}a_n r_k^{n-\frac{1}{2}}\exp\left(i\frac{2n-1}{2}\theta_k\right)\boldsymbol{n}\left(\theta_k\right)\right)$$

for $k=1, .., m$, & assuming an expansion up to order p, $p+1 \leq m$



- Example: $R=1$, & loading $\bar{\boldsymbol{T}}_z = \sin\theta$

$$\Longrightarrow \sin\theta_k = \sum_{n=0}^{p} a_n \sin\left(\frac{2n+1}{2}\theta_k\right)$$



  - For accuracy $m = 2p+2$
  - Collocation points only on upper side (avoid trivial solution)
  - Least squares resolution