University of Liège
Aerospace & Mechanical Engineering

Fracture Mechanics, Damage and Fatigue
Linear Elastic Fracture Mechanics - Energetic Approach

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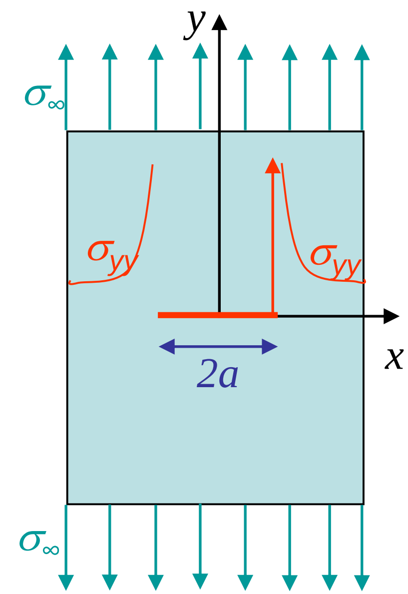
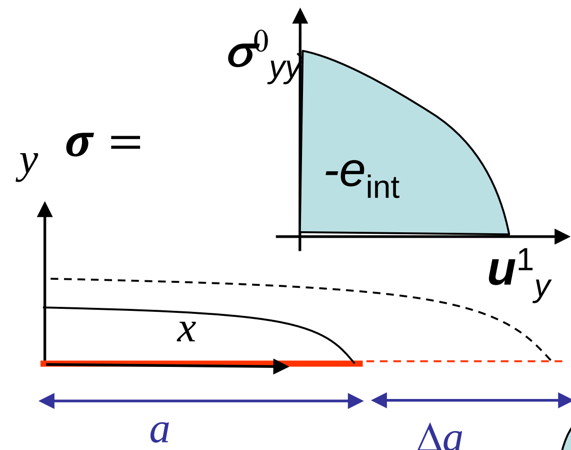
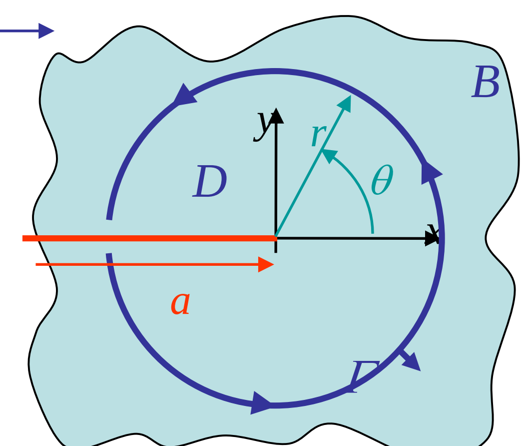
- Cracked body: different concepts
  - Stress intensity factor
    - Governing the asymptotic solution
    - $$\boldsymbol{\sigma}_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + o(r^0)$$



  - Energy release rate
    - Variation of strain energy if crack grows
    - $$G = -\frac{\partial \text{Energy}}{\partial A}$$



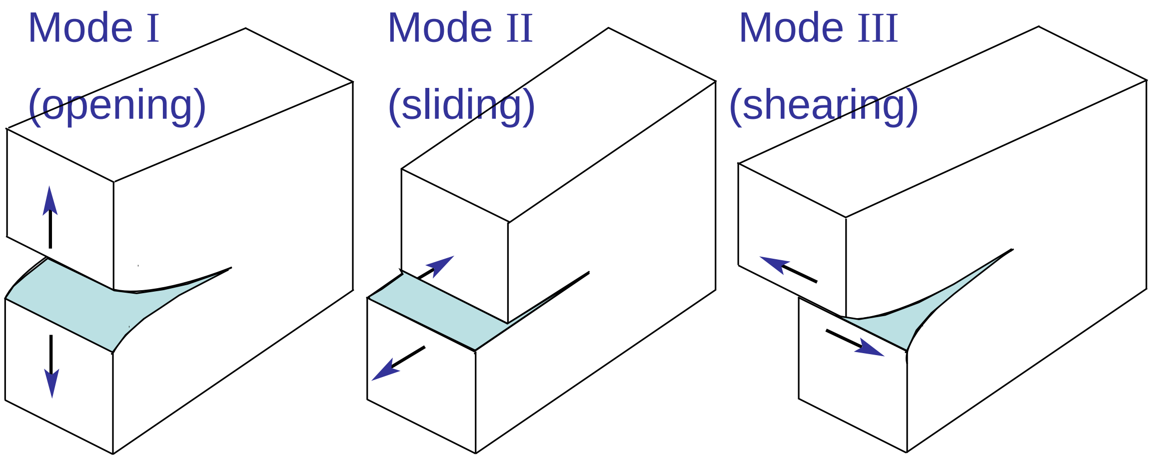
  - J-integral
    - Strain energy flow
    - $$J = \int_\Gamma \left[U(\boldsymbol{\varepsilon})\, \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T}\right] dl$$

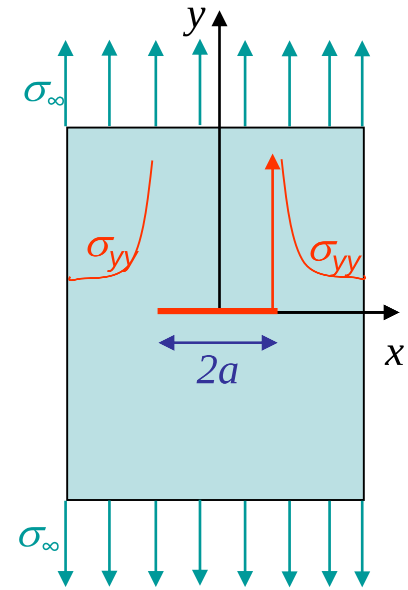


- More in depth
  - How are they defined?
  - How are they related?

Linear Elastic Fracture Mechanics (LEFM)

- Cracked body: summary
  - 3 failure modes

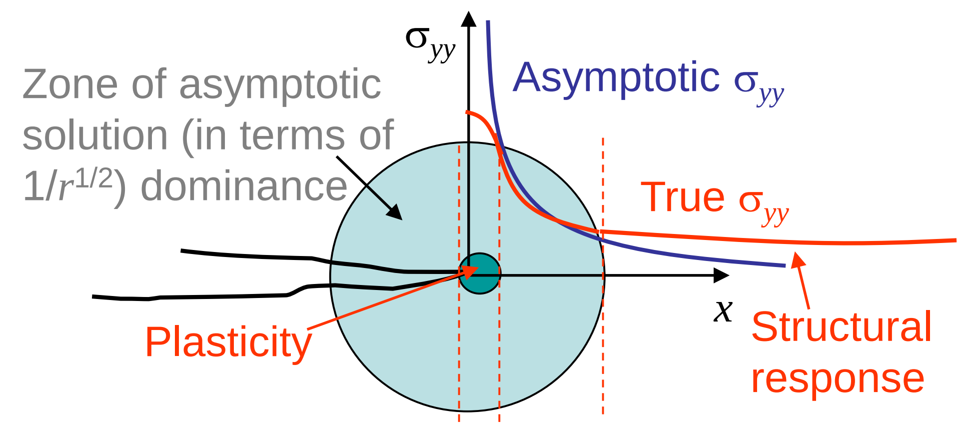




  - Asymptotic solution governed by stress intensity factors

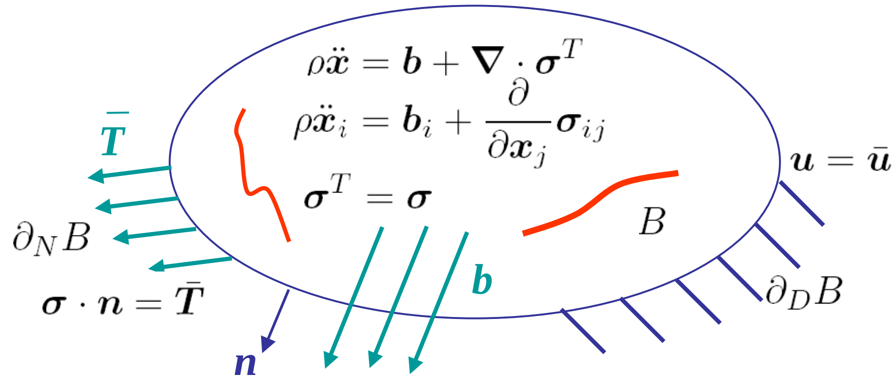
$$\boldsymbol{\sigma}^{\text{mode } i} = \frac{K_i}{\sqrt{2\pi r}} f^{\text{mode } i}(\theta) + o(r^0)$$

$$\boldsymbol{u}^{\text{mode } i} = K_i \sqrt{\frac{r}{2\pi}}\, g^{\text{mode } i}(\theta) + o(r^0)$$



- Relation with energy
  - Tensile strength for materials $\sigma_{\text{TS}}\sqrt{a} \div \sqrt{E\left(2\gamma_s + W_{\text{pl}}\right)}$
    - Involve crack size and fracture energy $\Longrightarrow$ K should be related to energy

- Virtual energy of body B
  - Existence of (stress free) cracks
  - Virtual displacement $\delta\boldsymbol{u}$

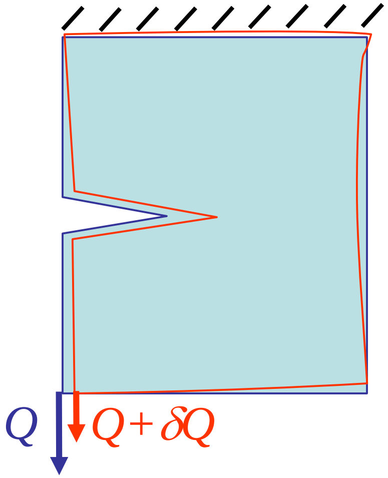


$$\delta E_{\text{int}} = \delta \int_B U dB = \int_{\partial_N B} \bar{\boldsymbol{T}}\cdot\delta\boldsymbol{u} d\partial B + \int_B \boldsymbol{b}\cdot\delta\boldsymbol{u} dB = \delta W_{\text{ext}}$$

- Where we assume that the stress derives from an internal potential: $\boldsymbol{\sigma} = \partial_{\boldsymbol{\varepsilon}} U$
- Example: linear elasticity $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon} = \partial_{\boldsymbol{\varepsilon}} \frac{\boldsymbol{\varepsilon} : \mathcal{H} : \boldsymbol{\varepsilon}}{2} = \partial_{\boldsymbol{\varepsilon}} U$
- So we assume reversibility

- Prescribed displacements
  - Assuming a body with constant displacement field u & subjected to loading Q
  - The crack propagates & the load then decreases
    - Example: body subjected to u constant
    - As the crack grows, the work exerted by Q is constant

$$\Longrightarrow \delta W_{\text{ext}} = 0$$

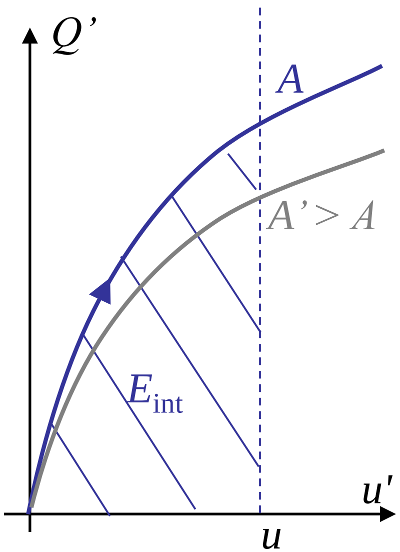


  - Energy release rate G for u constant
    - Energy change related to a crack growth δA

$$\delta E_{\text{int}} = -G\delta A \quad \Longrightarrow G = -\partial_A E_{\text{int}}$$

    - The internal (elastic) energy is therefore function of the displacement and of the crack surface

$$E_{\text{int}} = E_{\text{int}}\left(u, A\right)$$

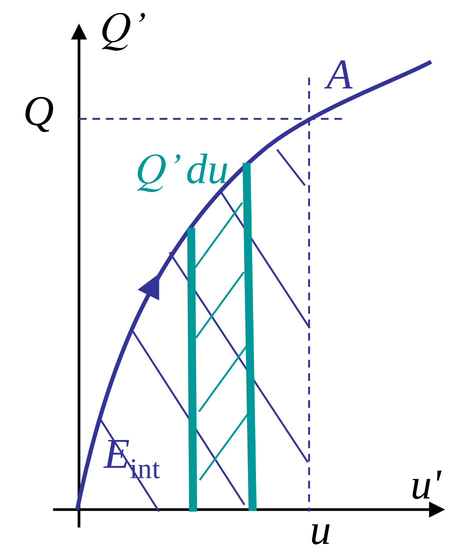


Energy of cracked bodies

- Prescribed displacements (2)
  - Computation of the energy release rate G
    - For a constant crack size: $\delta E_{\text{int}} = Q \delta u$

      $\Longrightarrow \partial_u E_{\text{int}}(u) = Q$
    - Energy release rate: $G = -\partial_A E_{\text{int}}$

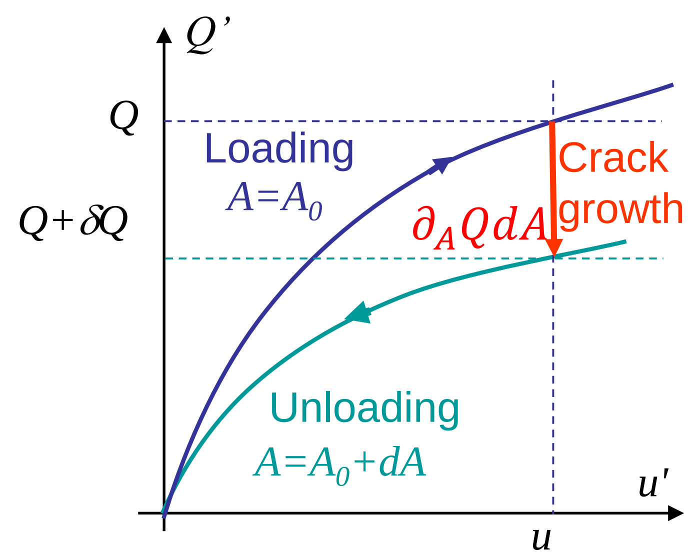
$$-\partial_u G = \partial_A Q \Longrightarrow G = -\int_0^u \partial_A Q(u', A)\, du'$$



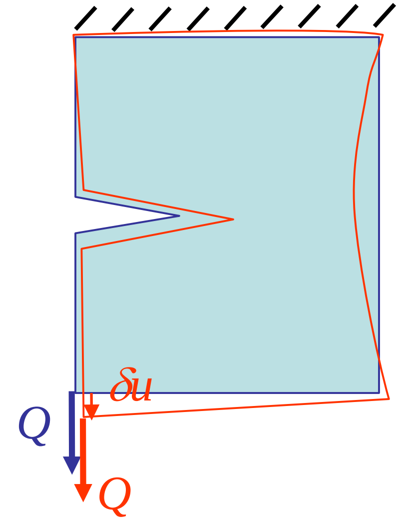
  - Can be measured by conducting (virtual) experiments
    - Body with crack surface A_0 loaded up to a displacement u
    - Crack assumed to grow by dA at constant displacement

      $\Longrightarrow$ the specimen becomes more flexible

      $\Longrightarrow$ so the load decreases by $\partial_A Q dA$
    - Unload to zero
    - The area between the 2 curves is then - $G\, dA$



- Prescribed loading
  - The crack propagates & there is a displacement field $\delta \boldsymbol{u}$
    - Example: body subjected to Q constant
    - As the crack grows, there is a displacement δu

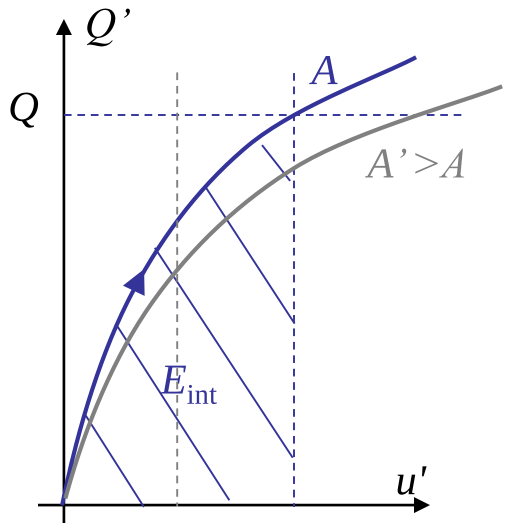


$$\Longrightarrow \delta W_{\text{ext}} = Q \delta u$$

  - Energy release rate G for Q constant
    - Energy change related a crack growth δA

$$\delta E_{\text{int}} = Q \delta u - G \delta A = \delta \left(Q u \right) - G \delta A$$

$$\Longrightarrow G = -\partial_A \left(E_{\text{int}} - Q u \right)$$

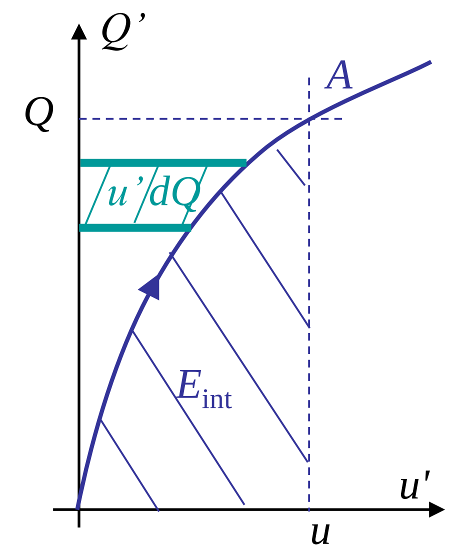


    - The internal (elastic) energy is therefore function of the loading and of the crack surface

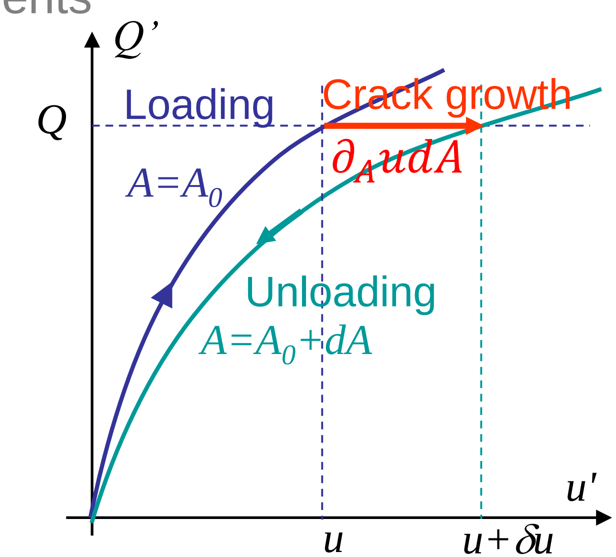
$$E_{\text{int}} = E_{\text{int}} \left(Q, A \right)$$

- Prescribed loading (2)
  - Computation of the energy release rate G
    - Complementary energy $Qu - E_{\text{int}} = \int_0^Q u(Q',\, A)\, dQ'$
    - Derivation $u(Q,\, A) = -\partial_Q (E_{\text{int}} - Qu)$
    - Energy release rate: $G = -\partial_A (E_{\text{int}} - Qu)$

$$\partial_Q G = \partial_A u \Longrightarrow G = \int_0^Q \partial_A u(Q',\, A)\, dQ'$$



  - Can be measured by conducting (virtual) experiments
    - Body with crack surface A_0 loaded up to Q
    - Crack assumed to grow by dA at constant load
      - The specimen becomes more flexible
      - $\Longrightarrow$ displacement increment $\partial_A u dA$
    - Unload to zero
    - The area between the 2 curves is then $G\, dA$



- General loading
  - If $\Pi_T = E_{\text{int}} - Qu$ is the potential energy of the specimen
    - $G = -\partial_A (E_{\text{int}} - W_{\text{ext}}) = -\partial_A \Pi_T$
    - Which reduces to
      - Prescribed displacements $G = -\partial_A E_{\text{int}}$
      - Prescribed loading $G = -\partial_A (E_{\text{int}} - Qu)$
  - Total energy $E = \Pi_T + \Gamma$ is the sum of
    - Potential energy Π_T of the structure
    - The atomistic bond energy Γ where a crack possibly propagates
    - Assuming a crack does propagate by a surface ΔA:

$$\boxed{0 = \frac{\partial \Pi_T}{\partial A} + \frac{\partial \Gamma}{\partial A} = -G + G_c}$$

G_c is a material property

G depends on geometry and loading

- General loading (2)
  - If $\Pi_T = E_{\text{int}} - Qu$ is the potential energy of the specimen

$$G = -\partial_A \left(E_{\text{int}} - W_{\text{ext}} \right) = -\partial_A \Pi_T$$

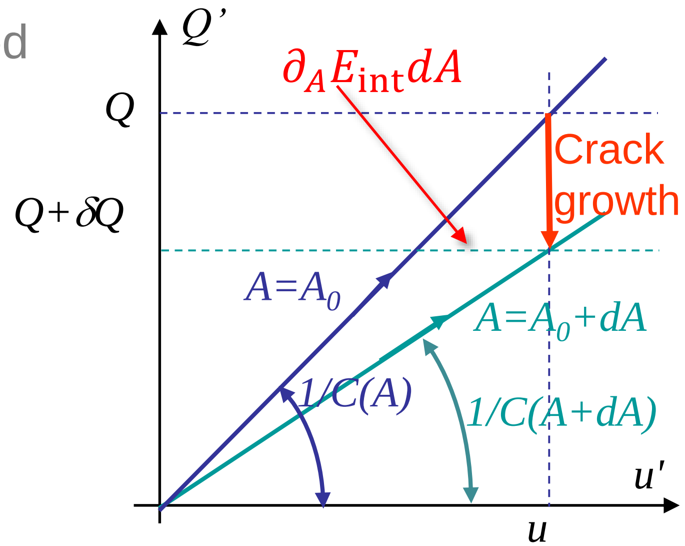
  - We have a crack propagation if $G = G_c = \partial_A \Gamma$

G_c is a material property

G depends on geometry and loading

    - Brittle materials:
      - $G_c = 2\gamma_s$
      - γ_s is the surface energy, a crack creates 2 surfaces
    - For other materials:
      - $G_c = 2\gamma_s + W_{\text{pl}}$
      - Ductility, composites, polymers, …
      - Depends on the failure process (void coalescence, debonding, …)

- Linear case & compliance
  - In linear elasticity, G analysis can be unified
    - Linear response ⟹ Q linear with u
    - The compliance is defined by $C = \dfrac{u}{Q}$
    - Energies
      - $E_{\text{int}} = \dfrac{1}{2}Qu = \dfrac{u^2}{2C} = \dfrac{Q^2C}{2}$
      - $W_{\text{ext}} = Qu = \dfrac{u^2}{C} = Q^2C$

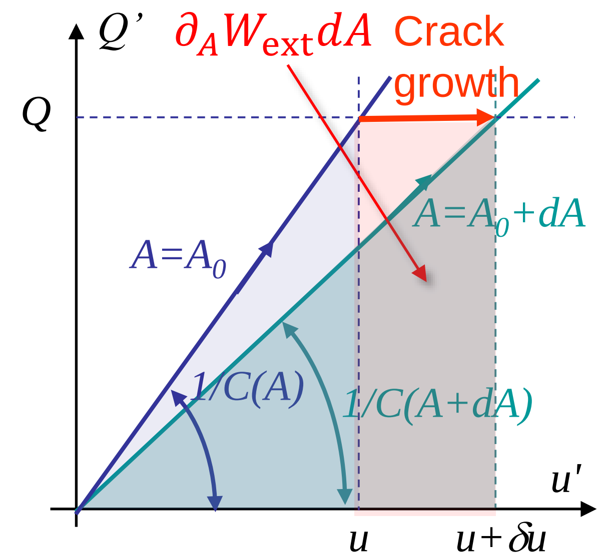


  - Prescribed displacements
    - $G = -\partial_A E_{\text{int}}$

$$\Longrightarrow \quad G = -\left.\partial_A E_{\text{int}}\right|_u = -\left.\partial_A\left(\frac{u^2}{2C}\right)\right|_u = \frac{u^2}{2C^2}\partial_A C = \frac{Q^2}{2}\partial_A C$$

    - For the crack to grow, all the energy required comes from the elastic energy
    - The internal energy decreases with the crack growth

- Linear case & compliance
  - In linear elasticity, G analysis can be unified
    - Linear response ⟹ Q linear with u
    - The compliance is defined by $C = \frac{u}{Q}$
    - Energies
      - $E_{\text{int}} = \frac{1}{2} Q u = \frac{u^2}{2C} = \frac{Q^2 C}{2}$
      - $W_{\text{ext}} = Q u = \frac{u^2}{C} = Q^2 C$



  - Prescribed loading
    - $G = -\partial_A \left(E_{\text{int}} - W_{\text{ext}} \right) = -\partial_A \Pi_T$

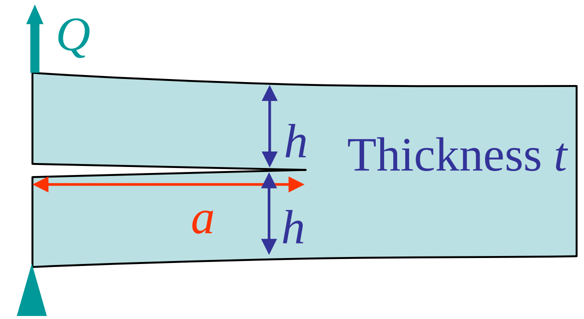
$$\Longrightarrow G = -\left.\partial_A \left(E_{\text{int}} - Qu \right)\right|_Q = -\left.\partial_A \left(\frac{Q^2 C}{2} - Q^2 C \right)\right|_Q = \frac{Q^2}{2} \partial_A C$$

    - Same expression as prescribed displacement but

$$G + \left.\partial_A E_{\text{int}}\right|_Q = Q \partial_A u = Q^2 \partial_A C = 2G$$

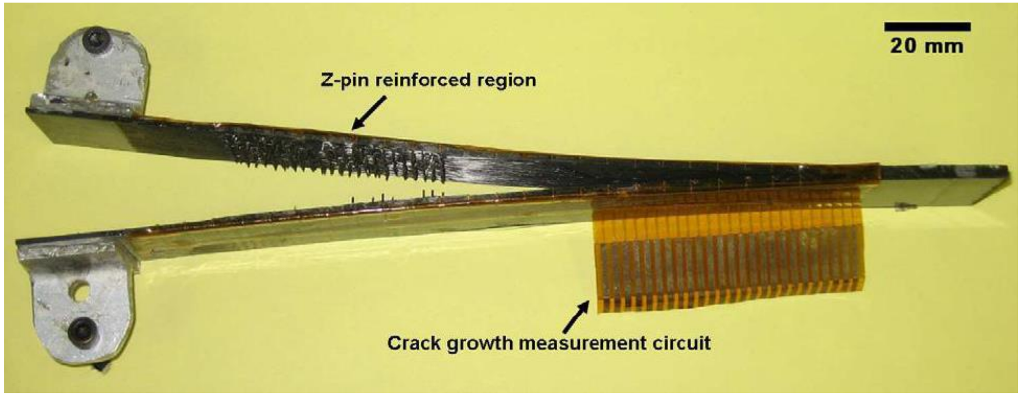
    - For the crack to grow by dA,
      - External load produces a work of $2G$ as
      - The internal energy is also increased by G

- Delamination of composites
  - Assuming $a >> h$: Double Cantilever Beam
    - The parts on the left are 2 cantilever beams
    - The right part is stress free
    - Flexion of one cantilever beam



$$v_{\max} = \frac{Qa^3}{3EI} \text{ with } I = \frac{th^3}{12} \Longrightarrow \text{displacement at loading } u = 2v_{\max} = \frac{8Qa^3}{Eth^3}$$

- Compliance $C = \frac{u}{Q} = \frac{8a^3}{Eth^3} \Longrightarrow \partial_A C = \frac{1}{t}\partial_a \frac{8a^3}{Eth^3} = \frac{24a^2}{Et^2h^3}$
- Energy release rate $G = \frac{Q^2}{2}\partial_A C = \frac{12Q^2a^2}{Et^2h^3}$

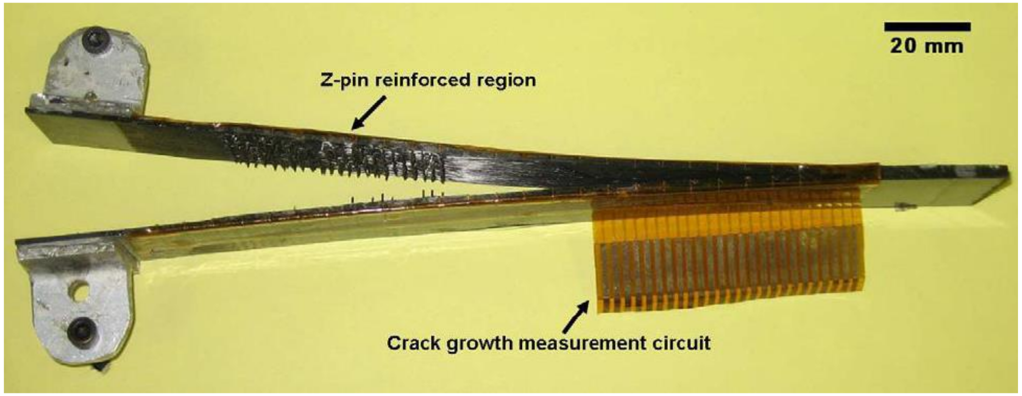


Paul Tihon, coexpair

Applications of the compliance method

- Delamination of composites (2)
  - Since

$$G = \frac{Q^2}{2}\partial_A C = \frac{12Q^2a^2}{Et^2h^3}$$

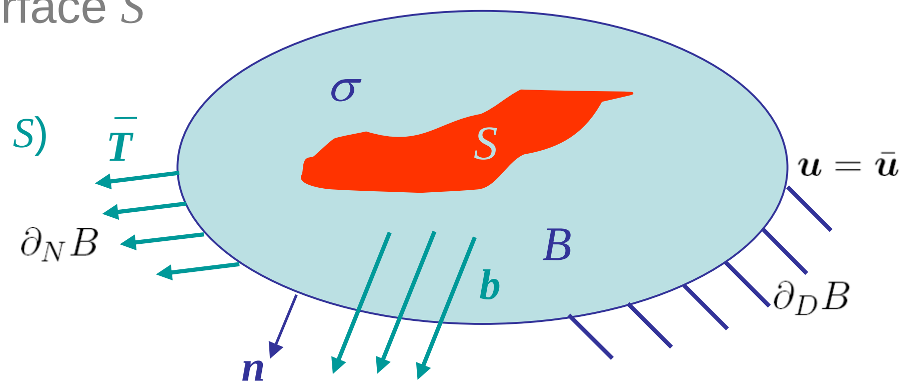


  - Experimental application: measure of G_c
    - G_c mode I for composite
  - Experimental application: crack length determination
    - An existing crack will grow under cyclic loading
    - If $C(A)$ has been determined
      - Analytically (as above for composite)
      - Numerically or
      - Experimentally

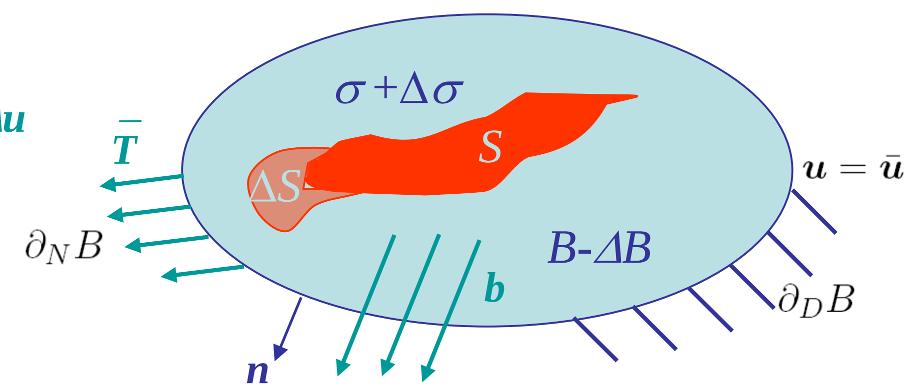
      then the crack length can be determined by measuring the compliance
    - Compliance is obtained by measuring load and load point displacement simultaneously

- Relation between the energy release rate and the SIFs
  - G is a variation of the potential energy with respect of the crack size
  - In linear elasticity, the stress state near crack tip is characterized by K_{mode}
  - How can we relate both concepts?

- 1957, Irwin, crack closure integral
  - Consider a body B with a cavity of surface S
    - The stress state is σ
    - The displacement field is $\boldsymbol{u}$ (in B & on S)
      - Constrained to $\bar{\boldsymbol{u}}$ on $\partial_D B$
    - The surface traction is $\boldsymbol{T}$
      - Constrained to $\bar{\boldsymbol{T}}$ on $\partial_N B$
      - Constrained to 0 on S



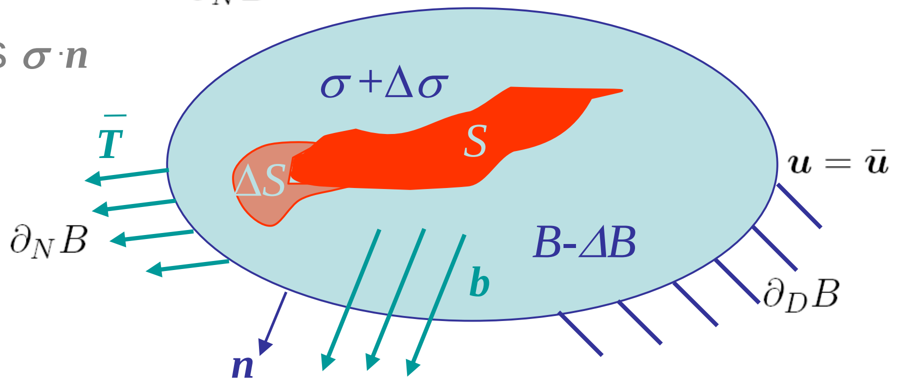
  - The cavity grows to $S+\Delta S$
    - The volume lost is ΔB
    - The stress state becomes $\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma}$
    - The displacement field becomes $\boldsymbol{u}+\Delta\boldsymbol{u}$
      - $\Delta\boldsymbol{u} = 0$ on $\partial_D B$
    - The surface traction becomes $\boldsymbol{T}+\Delta\boldsymbol{T}$
      - $\Delta\boldsymbol{T} = 0$ on $\partial_N B$
      - Constrained to 0 on $S+\Delta S$



- In elasticity (linear or not) and if $\boldsymbol{b}$ assumed equal to 0
  - Potential energy variation

$$\begin{cases} \Pi_T = \int_B U(\nabla \boldsymbol{u})\, dB - \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \boldsymbol{u} d\partial B \\ \Pi_T + \Delta\Pi_T = \int_{B-\Delta B} U(\nabla \boldsymbol{u} + \nabla \Delta \boldsymbol{u})\, dB - \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot (\boldsymbol{u} + \Delta \boldsymbol{u})\, d\partial B \end{cases}$$

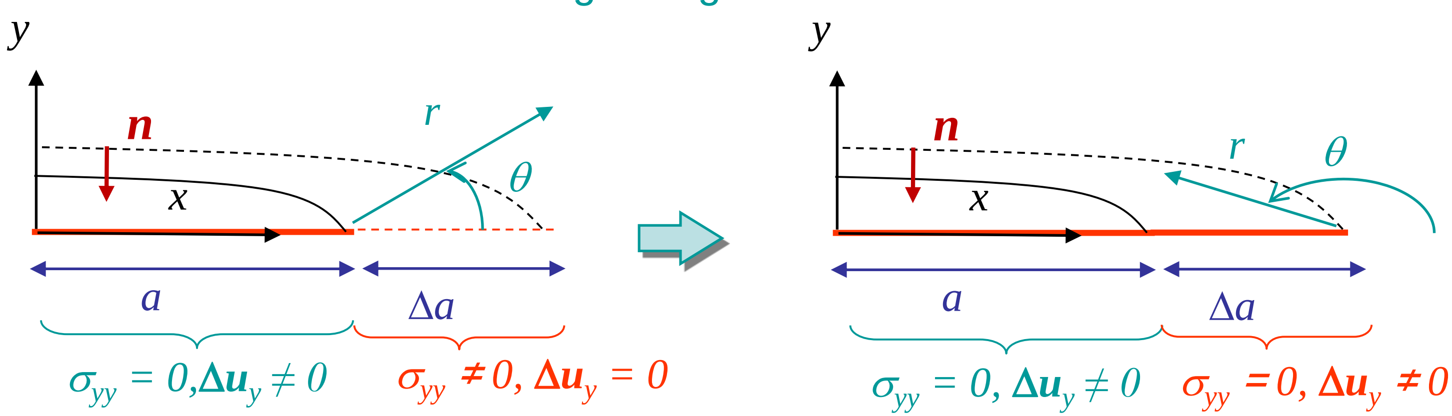
  - On the cavity surface S: $\boldsymbol{t}$ is defined as $\boldsymbol{\sigma} \cdot \boldsymbol{n}$
    - Be careful: $S+\Delta S$ is stress free, but only in the final configuration
      - $(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})\cdot\boldsymbol{n} = 0$ on $S+\Delta S$ but,
      - $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}') \cdot \boldsymbol{n} \neq 0$ on $S+\Delta S$
      - Surface traction $\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n} \neq \boldsymbol{0}$ on ΔS during the growth



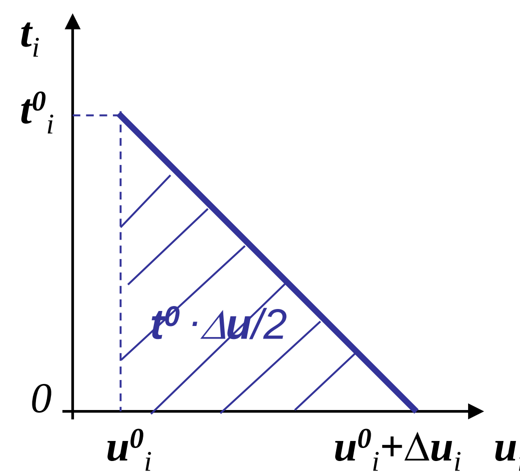
  - For a hole tending to a crack, see annex 1, one has

$$\Delta\Pi_T = \int_{\Delta S} \int_{\boldsymbol{u}}^{\boldsymbol{u}+\Delta\boldsymbol{u}} \boldsymbol{t}(\boldsymbol{u}') \cdot d\boldsymbol{u}' \; dS$$

- Change of potential for a crack growth in elasticity (linear or not)
  - General expression $\Delta\Pi_T = \int_{\Delta S}\int_{\boldsymbol{u}}^{\boldsymbol{u}+\Delta\boldsymbol{u}} \boldsymbol{t}(\boldsymbol{u}')\cdot d\boldsymbol{u}' \; dS$
    - Physical explanation for mode I
      - Let us assume a crack growing from a to $a+\Delta a$

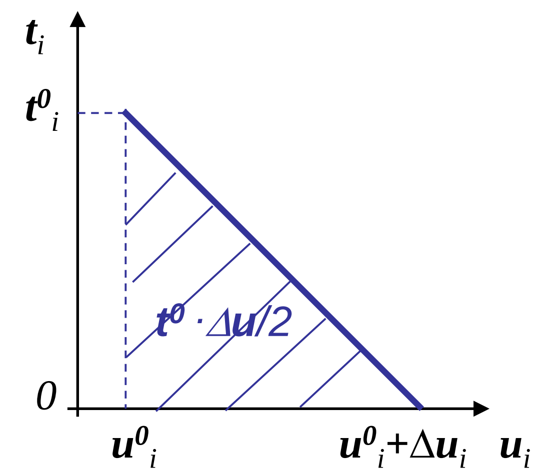


      - σ_{yy} produces a work on Δa



- If the response is elastic AND linear
  - $\boldsymbol{t}$ is decreasing linearly with $\Delta\boldsymbol{u}$
  - The work is then $\boldsymbol{t}^0 \cdot \Delta\boldsymbol{u}/2$

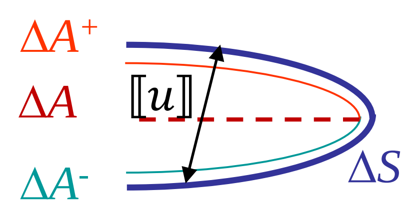
- Change of potential for a crack growth in LINEAR elasticity
  - Variation of potential in LINEAR elasticity



$$\Delta\Pi_T = \int_{\Delta S}\int_{\boldsymbol{u}}^{\boldsymbol{u}+\boldsymbol{\Delta u}} \boldsymbol{t}(\boldsymbol{u}')\cdot d\boldsymbol{u}'\ dS$$

$$\Rightarrow \quad \Delta\Pi_T = \int_{\Delta S}\frac{1}{2}\boldsymbol{t}^0\cdot\boldsymbol{\Delta u}\, dS$$

    - Where $\boldsymbol{t}^0$ is the tension before crack propagation
    - Where $\Delta\boldsymbol{u}$ is the opening after crack propagation
  - The surface created ΔS has actually two sides
    - An upper side ΔA^+
    - A lower side ΔA^-
    - With $\boldsymbol{t}|_{\Delta A^+} = -\,\boldsymbol{t}|_{\Delta A^-}$ & $\Delta\boldsymbol{u}|_{\Delta A^\pm} = \Delta\boldsymbol{u}^\pm$



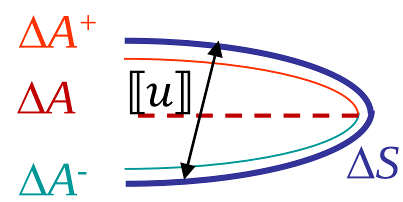
$$\Rightarrow \quad \Delta\Pi_T = \int_{\Delta A^+}\frac{1}{2}\boldsymbol{t}^0\cdot\left[\boldsymbol{\Delta u}^+ - \boldsymbol{\Delta u}^-\right]\ dS \quad = \int_{\Delta A}\frac{1}{2}\boldsymbol{t}^0\cdot[\![\boldsymbol{\Delta u}]\!]\, dS$$

- Change of potential for a crack growth in LINEAR elasticity (2)
  - Variation of potential in linear elasticity

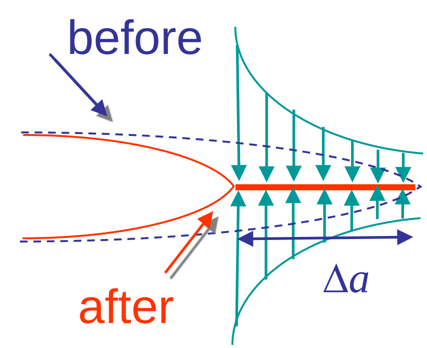
$$\Delta\Pi_T = \int_{\Delta A} \frac{1}{2} \boldsymbol{t}^0 \cdot [\![\boldsymbol{\Delta u}]\!]\, dS$$

  - Energy release rate
    - The increment of fracture area ΔA corresponds to ΔA^+

$$\Rightarrow \quad G = -\partial_A \Delta\Pi_T = -\lim_{\Delta A \to 0} \frac{1}{\Delta A} \int_{\Delta A} \frac{1}{2} \boldsymbol{t}^{\boldsymbol{0}} \cdot [\![\boldsymbol{\Delta u}]\!]\, dA$$



    - Valid for any linear elastic material
    - Valid for any direction of crack growth (mode I, II & III)
  - Tensile mode I: $G > 0$ for a crack growth
    - Π_T decreases $\Rightarrow$ crack growth requires energy
    - G corresponds to the work needed to close the crack by Δa

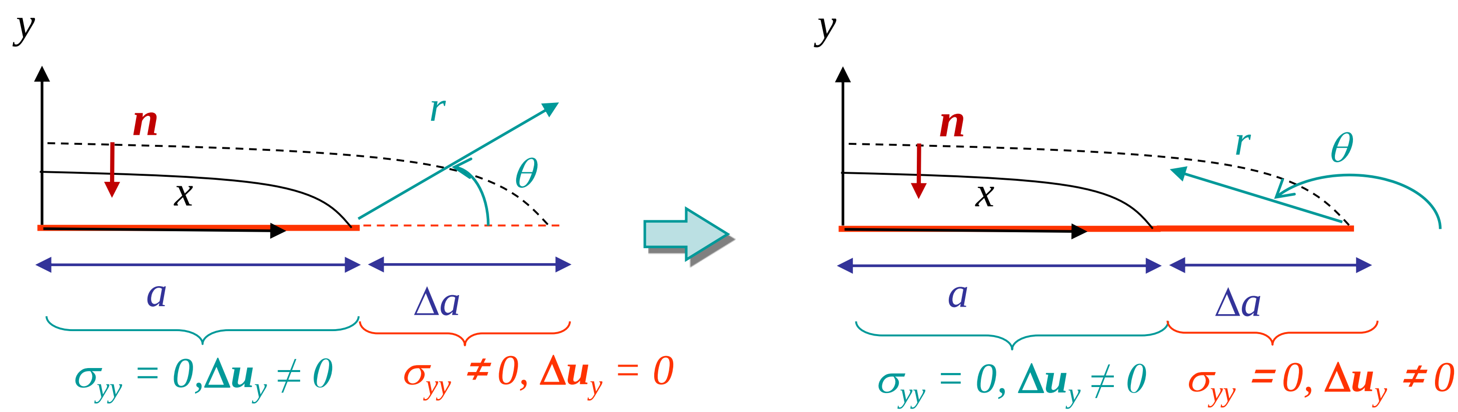


Crack that grows straight ahead

- Energy release rate for LINEAR elasticity

$$G = -\partial_A \Delta\Pi_T = -\lim_{\Delta A \to 0} \frac{1}{\Delta A} \int_{\Delta A} \frac{1}{2} \boldsymbol{t}^{\mathbf{0}} \cdot [\![\boldsymbol{\Delta u}]\!] \, dA$$

- Can be simplified if crack grows straight ahead



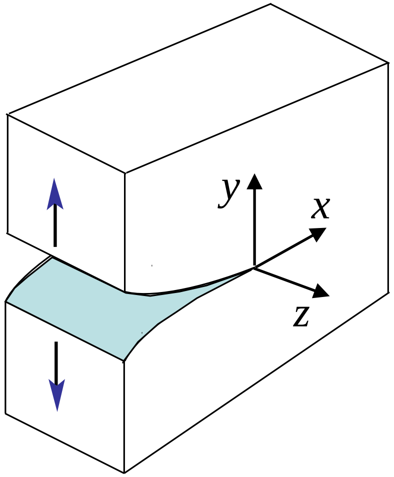
- Increment of fracture area: $\Delta A = t\,\Delta a$ (t = thickness)
- Since ΔA has been chosen equal to ΔA^+: $\boldsymbol{t}_i = \boldsymbol{\sigma}_{ij} \cdot \boldsymbol{n}_j = -\boldsymbol{\sigma}_{iy}$

$$G = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_{\Delta a} \frac{1}{2} \boldsymbol{\sigma}_{\boldsymbol{iy}}^{\mathbf{0}}(r, \theta = 0) \cdot [\![\boldsymbol{\Delta u_i}(\Delta a - r, \theta = \pi)]\!] \, dr$$

Stress before crack growth

Displacement jump after crack growth

- Energy release rate in mode I (LEFM & crack growing straight ahead)
  - Expression in 2D:
    - $G = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_{\Delta a} \frac{1}{2} \boldsymbol{\sigma}^{\mathbf{0}}_{\boldsymbol{iy}}(r, \theta = 0) \cdot [\![\boldsymbol{\Delta u_i}(\Delta a - r, \theta = \pi)]\!] \, dr$
    - Mode I: only term in i=y since $\boldsymbol{\sigma}_{xy}(r', 0) = 0$



  - Asymptotic solution

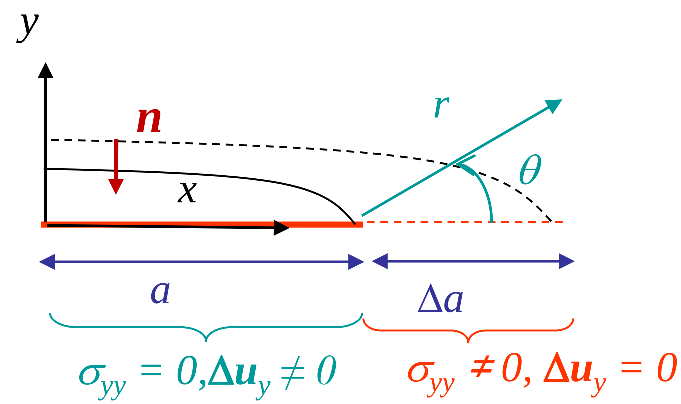
$$\begin{cases} \boldsymbol{\sigma}_{yy} = \dfrac{K_I}{\sqrt{2\pi r}} \cos\dfrac{\theta}{2}\left[1 + \sin\dfrac{\theta}{2}\sin\dfrac{3\theta}{2}\right] \\ \boldsymbol{u}_y = K_I \dfrac{1+\nu}{E}\sqrt{\dfrac{r}{2\pi}} \sin\dfrac{\theta}{2}\left[\kappa + 1 - 2\cos^2\dfrac{\theta}{2}\right] \end{cases}$$

Crack that grows straight ahead

- Energy release rate in mode I (2)
  - Asymptotic solution before crack growth

$$\boldsymbol{\sigma}_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

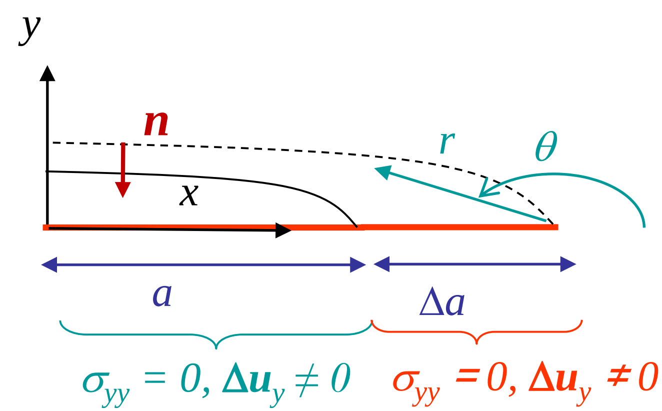
$$\Rightarrow \quad \boldsymbol{\sigma}^{\mathbf{0}}_{yy}(r, \theta = 0) = \frac{K_I}{\sqrt{2\pi r}}$$



  - Asymptotic solution after crack growth

$$\boldsymbol{u}_y = K_I \frac{1+\nu}{E}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left[\kappa + 1 - 2\cos^2\frac{\theta}{2}\right]$$

$$\Rightarrow \quad \boldsymbol{\Delta u}_y(\Delta a - r, \theta = \pm\pi) = \pm K_I \frac{(1+\nu)(1+\kappa)}{E}\sqrt{\frac{\Delta a - r}{2\pi}}$$



  - Energy

$$\boldsymbol{\sigma}^{\mathbf{0}}_{yy}(r, \theta = 0)\left[\!\left[\boldsymbol{\Delta u}_y(\Delta a - r, \theta = \pm\pi)\right]\!\right] = K_I^2 \frac{(1+\nu)(1+\kappa)}{\pi E}\sqrt{\frac{\Delta a - r}{r}}$$

- Energy release rate in mode I (3)
  - After substitution

$$G = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_0^{\Delta a} K_I^2 \frac{(1+\nu)(\kappa+1)}{2\pi E} \sqrt{\frac{\Delta a - r'}{r'}} dr'$$

- Change of variable $r' = \Delta a \cos^2 x$

$$\Longrightarrow G = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_0^{\frac{\pi}{2}} K_I^2 \frac{(1+\nu)(\kappa+1)}{\pi E} \sqrt{\frac{\sin^2 x}{\cos^2 x}} \Delta a \cos x \sin x dx$$

$$\Longrightarrow G = \int_0^{\frac{\pi}{2}} K_I^2 \frac{(1+\nu)(\kappa+1)}{2\pi E} (1 - \cos 2x)\, dx$$

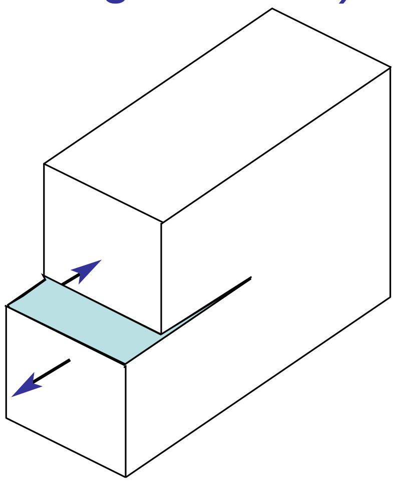
- Plane σ $\kappa = \frac{3-\nu}{1+\nu}$ & plane ε $\kappa = 3 - 4\nu$

$$\Longrightarrow G = K_I^2 \frac{(1+\nu)(\kappa+1)}{4E} = \frac{K_I^2}{E'}$$

with $E' = \frac{4E}{(1+\nu)(\kappa+1)} = \begin{cases} E & \text{P}\sigma \\ \frac{E}{1-\nu^2} & \text{P}\varepsilon \end{cases}$

Crack that grows straight ahead

- Energy release rate in mode II (LEFM & crack growing straight ahead)
  - Asymptotic solution



$$\begin{cases} \boldsymbol{u}_x = K_{II}\dfrac{1+\nu}{E}\sqrt{\dfrac{r}{2\pi}}\sin\dfrac{\theta}{2}\left[\kappa+1+2\cos^2\dfrac{\theta}{2}\right] \\ \boldsymbol{u}_y = K_{II}\dfrac{1+\nu}{E}\sqrt{\dfrac{r}{2\pi}}\cos\dfrac{\theta}{2}\left[1-\kappa+2\sin^2\dfrac{\theta}{2}\right] \end{cases}$$

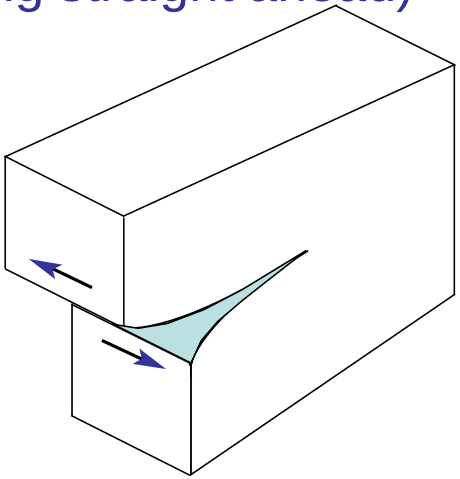
$$\begin{cases} \boldsymbol{\sigma}_{yy} = \dfrac{K_{II}}{\sqrt{2\pi r}}\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\cos\dfrac{3\theta}{2} \\ \boldsymbol{\sigma}_{xy} = \dfrac{K_{II}}{\sqrt{2\pi r}}\cos\dfrac{\theta}{2}\left[1-\sin\dfrac{\theta}{2}\sin\dfrac{3\theta}{2}\right] \end{cases}$$

  - Proceeding as for mode I
    - This time $\boldsymbol{\sigma}^0_{xy}(r, 0)\ \Delta\boldsymbol{u}_x(\Delta a\text{-}r, \pm\pi)$ is the non zero term

$$\Longrightarrow G = \frac{K_{II}^2}{E'}$$

- Energy release rate in mode III (LEFM & crack growing straight ahead)
  - Expression in 2D:

$$G = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_{\Delta a} \frac{1}{2} \boldsymbol{\sigma}^{\mathbf{0}}_{\boldsymbol{iy}}(r, \theta = 0) \cdot [\![\boldsymbol{\Delta u_i}(\Delta a - r, \theta = \pi)]\!] \, dr$$



  - Asymptotic solution

$$\begin{cases} \boldsymbol{u}_z = \dfrac{2K_{III}\,(1+\nu)}{E}\sqrt{\dfrac{2r}{\pi}}\sin\dfrac{\theta}{2} \\ \boldsymbol{\sigma}_{yz} = \dfrac{K_{III}}{\sqrt{2\pi r}}\cos\dfrac{\theta}{2} \end{cases}$$

- So $\boldsymbol{\sigma}^{\mathbf{0}}_{yz}(r, \theta = 0) = \dfrac{K_{III}}{\sqrt{2\pi r}}$ & $\boldsymbol{\Delta u}_z(\Delta a - r, \theta = \pm\pi) = \pm 2K_{III}\dfrac{(1+\nu)}{E}\sqrt{2\dfrac{\Delta a - r}{\pi}}$

$$\Rightarrow \boldsymbol{\sigma}^{\mathbf{0}}_{yz}(r, \theta = 0)[\![\boldsymbol{\Delta u}_z(\Delta a - r, \theta = \pm\pi)]\!] = 4K_{III}^2\frac{(1+\nu)}{\pi E}\sqrt{\frac{\Delta a - r}{r}}$$

- Energy release rate

$$G = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_0^{\Delta a} 2K_{III}^2 \frac{(1+\nu)}{\pi E}\sqrt{\frac{\Delta a - r'}{r'}} dr' = \frac{K_{III}^2\,(1+\nu)}{E} = \frac{K_{III}^2}{2\mu}$$

- Energy release rate (LEFM & crack growing straight ahead)
  - Quadratic field ⟹ superposition ?
  - But when analyzing $G = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_{\Delta a} \frac{1}{2} \boldsymbol{\sigma}_{iy}^{\mathbf{0}}(r, \theta = 0) \cdot [\![\boldsymbol{\Delta u_i}(\Delta a - r, \theta = \pi)]\!] \, dr$
    - $i = x$: $\boldsymbol{\sigma}^0_{xy}(r, 0) \neq 0$ & $\Delta \boldsymbol{u}_x(\Delta a\text{-}r, \pm\pi) \neq 0$ only for mode II
    - $i = y$: $\boldsymbol{\sigma}^0_{yy}(r, 0) \neq 0$ & $\Delta \boldsymbol{u}_y(\Delta a\text{-}r, \pm\pi) \neq 0$ only for mode I
    - $i = z$: $\boldsymbol{\sigma}^0_{yz}(r, 0) \neq 0$ & $\Delta \boldsymbol{u}_z(\Delta a\text{-}r, \pm\pi) \neq 0$ only for mode III

    ⟹ Energies can be added:

$$\left\{ \begin{array}{l} \boxed{G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}} \\ E' = \frac{4E}{(1+\nu)(\kappa+1)} = \left\{ \begin{array}{ll} E & \text{P}\sigma \\ \frac{E}{1-\nu^2} & \text{P}\varepsilon \end{array} \right. \end{array} \right.$$

  - Some remarks
    - This formula is valid for
      - Elastic linear material ONLY
      - Crack that grows straight ahead ONLY
    - So usefulness is questionable in the general case as
      - If more than one mode at work, the crack will not grow straight ahead
      - What if material is not linear?

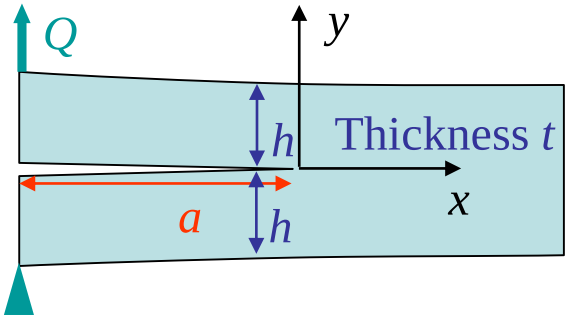
- Delamination of composites
  - Energy release rate

$$G = \frac{Q^2}{2}\partial_A C = \frac{12Q^2a^2}{Et^2h^3}$$

  - Pure mode I since
    - $\boldsymbol{u}_y(\text{-y}) = \text{-}\boldsymbol{u}_y(\text{y})$ & $\boldsymbol{u}_x(\text{-y}) = \boldsymbol{u}_x(\text{y})$
  - Crack is growing straight ahead

$$\Longrightarrow K_I = \sqrt{E'G} = \frac{2Qa}{th}\sqrt{\frac{E'}{E}}\sqrt{\frac{3}{h}}$$

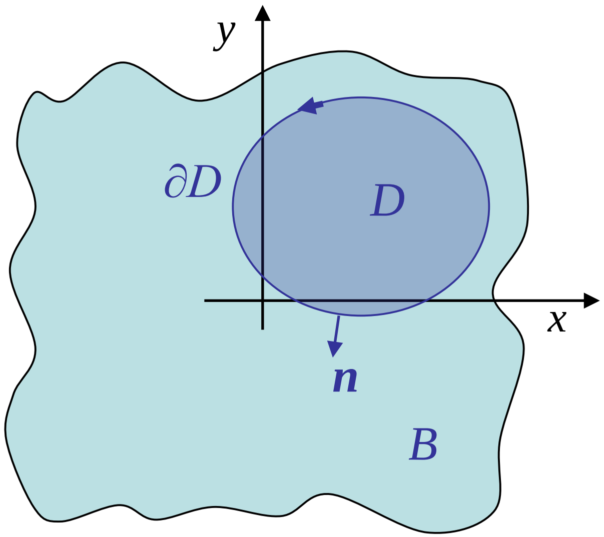
- Plane σ $K_I = \frac{2Qa}{th}\sqrt{\frac{3}{h}}$
- Plane ε $K_I = \frac{2Qa}{th\sqrt{1-\nu^2}}\sqrt{\frac{3}{h}}$



J integral

- The crack closure integral has some limitations
  - Elastic materials
  - Useful only when crack grows straight ahead
- More general energy-related concept?

- Rice (1968) proposed to compute the energy that flows to the crack tip
  - Given a homogeneous uncracked body B
    - D is a subvolume of boundary ∂D
    - The stress tensor derives from a potential U
    - On ∂D traction $\boldsymbol{T}$ is defined as $\boldsymbol{\sigma} \cdot \boldsymbol{n}$
    - Static assumption ⟹ $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \mathbf{0}$
    - We assume the existence of an internal potential U ⟹ elasticity?



- The $\boldsymbol{J}$-integral is the vector defined by

$$\boldsymbol{J} = \int_{\partial D} \left[U\left(\boldsymbol{\nabla}\boldsymbol{u}\right)\boldsymbol{n} - \left(\boldsymbol{\nabla}\boldsymbol{u}\right)^T \boldsymbol{T} \right] dA$$

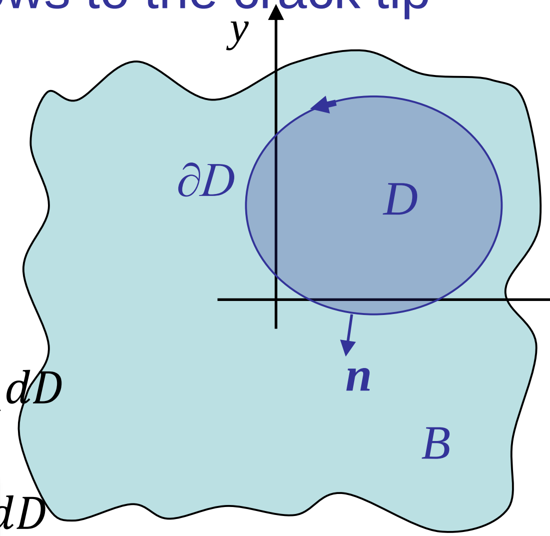
or, along $\boldsymbol{x}_i$

$$\boldsymbol{J}_i = \int_{\partial D} \left[U\left(\boldsymbol{\varepsilon}\right)\boldsymbol{n}_i - \boldsymbol{\nabla}_i \boldsymbol{u}_k \boldsymbol{\sigma}_{km} \boldsymbol{n}_m \right] dA$$

- Rice (1968) proposed to compute the energy that flows to the crack tip
  - Given an homogeneous uncracked body B (2)
    - $$\boldsymbol{J}_i = \int_{\partial D} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_i - \boldsymbol{\nabla}_i \boldsymbol{u}_k \boldsymbol{\sigma}_{km} \boldsymbol{n}_m \right] dA$$
    - First term with $\boldsymbol{\sigma}_{km} = \boldsymbol{\sigma}_{mk} = \partial_{\boldsymbol{\varepsilon}_{km}} U$

$$\int_{\partial D} U(\boldsymbol{\varepsilon})\boldsymbol{n}_i \, dA = \int_D \partial_{\boldsymbol{x}_i} U(\boldsymbol{\varepsilon}) \, dD = \int_D \partial_{\boldsymbol{\varepsilon}_{km}} U(\boldsymbol{\varepsilon}) \, \partial_{\boldsymbol{x}_i} \boldsymbol{\varepsilon}_{km} dD$$

$$= \int_D \boldsymbol{\sigma}_{km} \frac{\partial_{x_i x_m} \boldsymbol{u}_k + \partial_{x_i x_k} \boldsymbol{u}_m}{2} dD = \int_D \boxed{\boldsymbol{\sigma}_{km} \, \partial_{x_i x_m} \boldsymbol{u}_k} dD$$



- Second term with $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \boldsymbol{0}$

$$\int_{\partial D} \partial_{\boldsymbol{x}_i} \boldsymbol{u}_k \boldsymbol{\sigma}_{km} \boldsymbol{n}_m \, dA = \int_D \partial_{\boldsymbol{x}_m} \left(\partial_{\boldsymbol{x}_i} \boldsymbol{u}_k \boldsymbol{\sigma}_{km} \right) \, dD$$

$$= \int_D \left(\boxed{\partial_{\boldsymbol{x}_i \boldsymbol{x}_m} \boldsymbol{u}_k \boldsymbol{\sigma}_{km}} + \partial_{\boldsymbol{x}_i} \boldsymbol{u}_k \partial_{\boldsymbol{x}_m} \boldsymbol{\sigma}_{km} \right) dD$$

=0 (balance eq.)

$$\Rightarrow \quad \boldsymbol{J}_i = \int_{\partial D} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_i - \boldsymbol{\nabla}_i \boldsymbol{u}_k \boldsymbol{\sigma}_{km} \boldsymbol{n}_m \right] dA = 0$$

  - The flow of energy through a closed surface is equal to zero

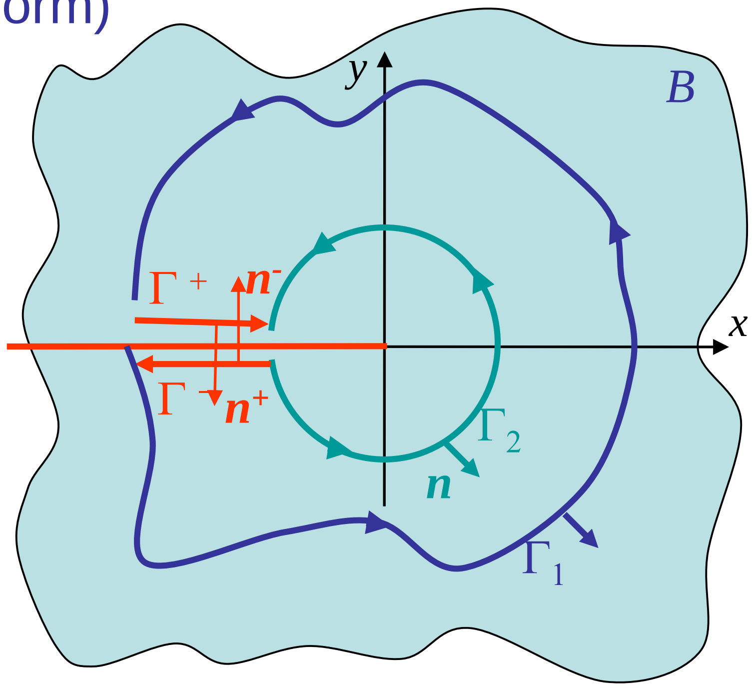
- For heterogeneous materials

$$U\left(\boldsymbol{\nabla u}, \boldsymbol{X}\right) \Longrightarrow \frac{DU\left(\boldsymbol{\nabla u}, \boldsymbol{X}\right)}{D\boldsymbol{X}_i} = \frac{\partial U}{\partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}_{,i} + \frac{\partial U}{\partial \boldsymbol{X}_i} \Longrightarrow \boldsymbol{J}_i \neq 0$$

- For homogeneous cracked materials (2D form)
  - Of practical interest is the flow // crack tip

$$\boldsymbol{J}_x = \oint_{\Gamma} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \right] dl = 0$$

with $\oint_{\Gamma} = \int_{\Gamma_1} + \int_{\Gamma+} + \int_{\Gamma-} - \int_{\Gamma_2}$



- Along Γ^- and Γ^+ :
  - $\boldsymbol{n}_x = 0$, $\boldsymbol{n}_y = \pm 1$
  - Crack is stress free: $\boldsymbol{T}_\alpha = \boldsymbol{\sigma}_{\alpha y}\, \boldsymbol{n}_y = 0$
    - » If there is no friction at the crack

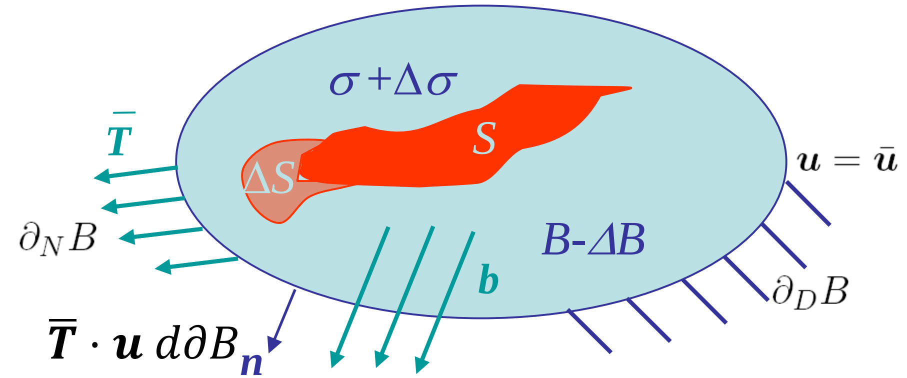
$$\int_{\Gamma-} = \int_{\Gamma+} = 0$$

  - So one can compute the energy that flows toward the crack tip by

$$J = \int_{\Gamma_1} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \right] dl = \int_{\Gamma_2} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \right] dl$$

- It is path independent
- No assumption on linearity has been made (only existence of U)
- Does not depend on subsequent crack growth direction

- The J integral can be specialized
  - Back to crack closure integral



    - Potential energy

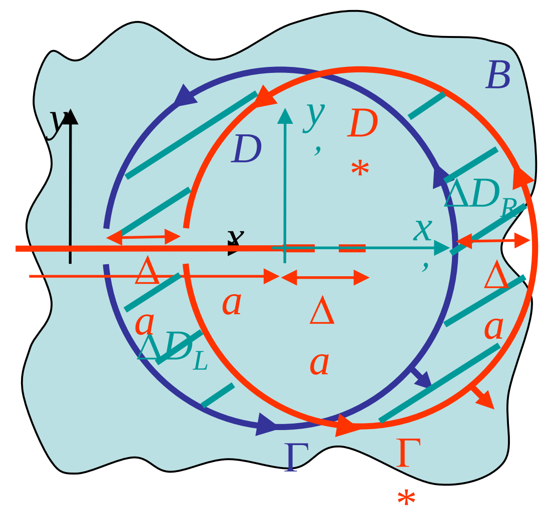
$$\Pi_T = E_{int} - Qu = \int_B U(\varepsilon)\, dB - \int_{\partial_N B} \overline{\boldsymbol{T}} \cdot \boldsymbol{u}\, d\partial B$$

    - Energy release rate

$$G = -\partial_A \Delta\Pi_T = -\lim_{\Delta A \to 0} \Delta \left[\int_B U(\varepsilon)\, dB - \int_{\partial_N B} \overline{\boldsymbol{T}} \cdot \boldsymbol{u}\, d\partial B \right]$$

    - If the crack grows straight ahead, considering a domain moving with the crack tip, it can be shown that (see annex 2)

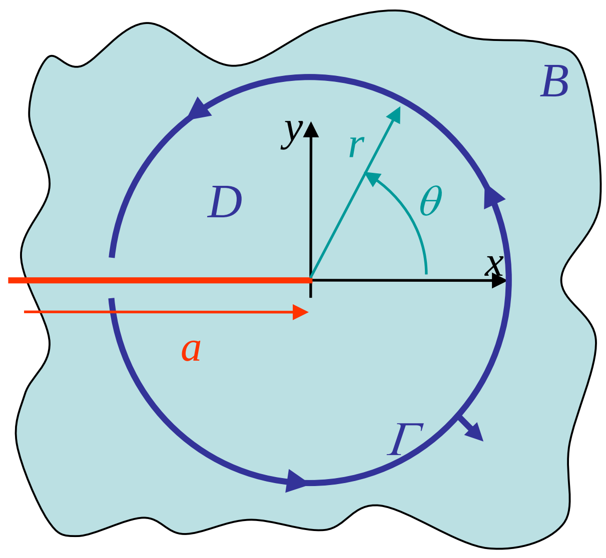
$$G = \int_\Gamma U\left(\boldsymbol{\varepsilon}\left(a\right)\right) \boldsymbol{n}_x dl - \int_\Gamma \partial_x \boldsymbol{u} \cdot \boldsymbol{T} dl = J$$



  - So G=J
    - For materials defined by an internal potential (linear response or not)
    - AND if the crack grows straight ahead

- The *J* integral can be specialized
  - For linear elasticity
    - General expression

$$J=\int_{\Gamma}\left[U\left(\boldsymbol{\varepsilon}\right)\boldsymbol{n}_x-\boldsymbol{u}_{,x}\cdot\boldsymbol{T}\right]dl$$



    - Linear elasticity $U=\frac{\sigma:\varepsilon}{2}$

$$\Longrightarrow J=\int_{\Gamma}\left[\frac{\boldsymbol{\sigma}_{ij}\boldsymbol{u}_{i,j}}{2}\delta_{xk}-\boldsymbol{u}_{i,x}\boldsymbol{\sigma}_{ik}\right]\boldsymbol{n}_k dl$$

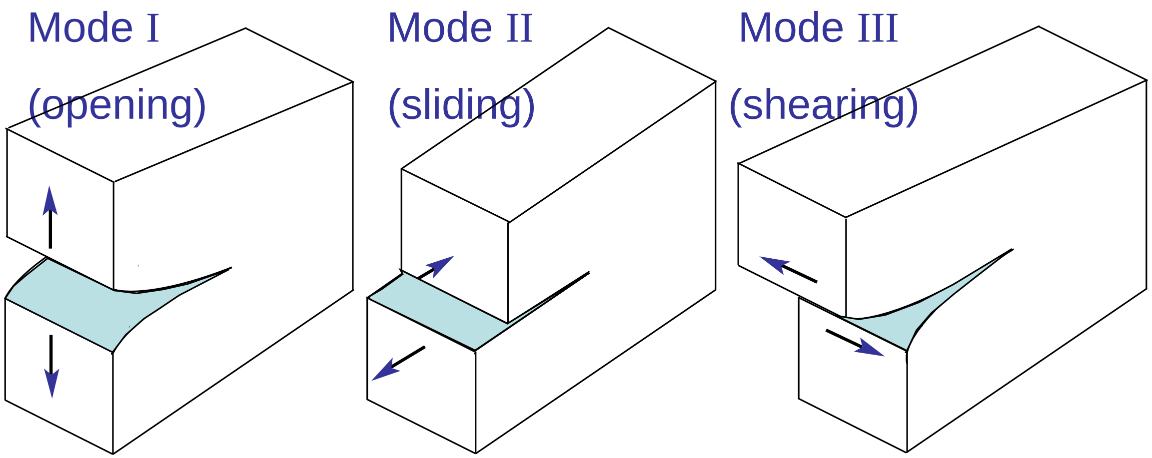
    - Arbitrary path $\Longrightarrow$ choose a circle with $r\rightarrow 0$

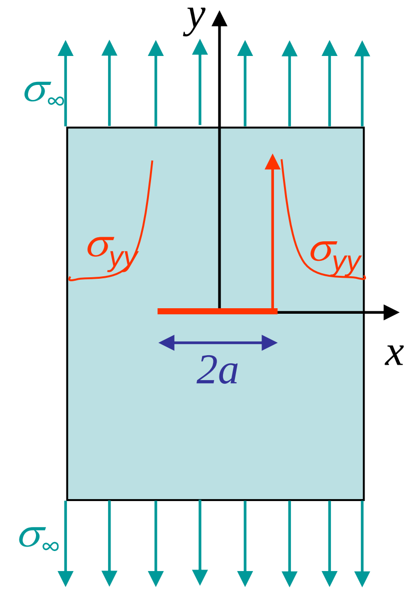
$\Longrightarrow$ asymptotic solution holds and plugged in J

$$\left\{\begin{aligned}\boldsymbol{u}&=\sqrt{\frac{r}{2\pi}}\sum_{i=I}^{III}K_i\boldsymbol{g}^{\text{mode i}}\left(\theta\right)\\ \boldsymbol{\sigma}&=\frac{1}{\sqrt{2\pi r}}\sum_{i=I}^{III}K_i\mathbf{f}^{\text{mode i}}\left(\theta\right)\end{aligned}\right.$$

$$J=\frac{K_I^2}{E'}+\frac{K_{II}^2}{E'}+\frac{K_{III}^2}{2\mu}$$

- Cracked body: summary
  - 3 failure modes

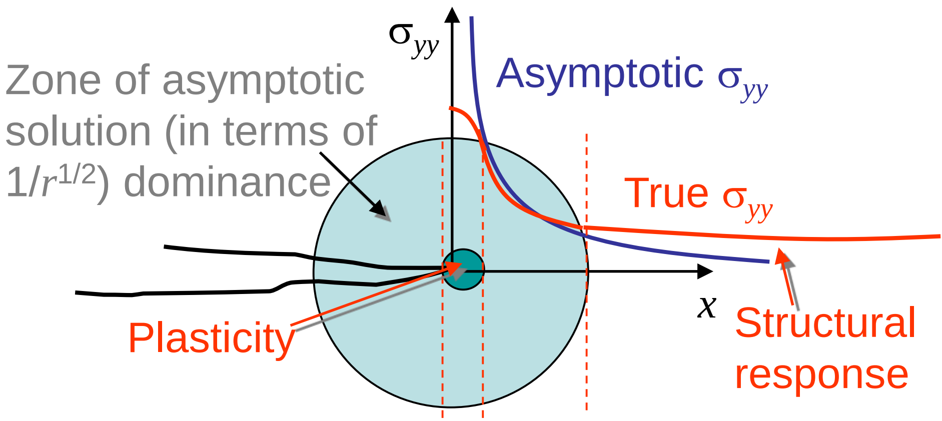




  - Asymptotic solution governed by stress intensity factors

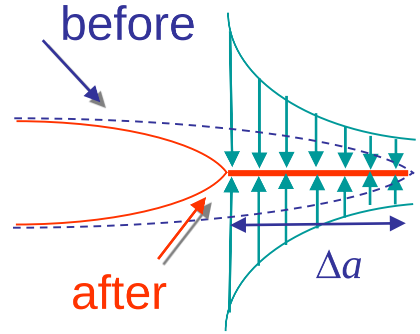
$$\boldsymbol{\sigma}^{\text{mode } i} = \frac{K_i}{\sqrt{2\pi r}} f^{\text{mode } i}(\theta) + o(r^0)$$

$$\boldsymbol{u}^{\text{mode } i} = K_i \sqrt{\frac{r}{2\pi}}\, g^{\text{mode } i}(\theta) + o(r^0)$$



- Cracked body: summary
  - Potential energy $\Pi_T = E_{\text{int}} - Qu$
  - Crack closure integral
    - Energy required to close crack tip

$$\Delta\Pi_T = \int_{\Delta A} \int_{\boldsymbol{u}}^{\boldsymbol{u}+\Delta\boldsymbol{u}} \boldsymbol{t} \cdot [\![\boldsymbol{u}']\!] d\boldsymbol{u}' \; dA$$

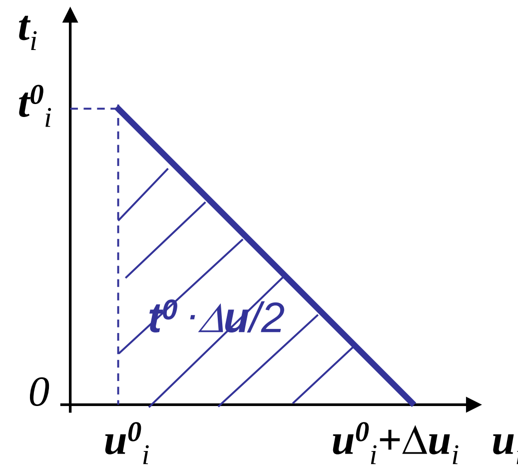


  - Energy release rate
    - Variation of potential energy in case of crack growth

$$G = -\partial_{\text{A}} \left(E_{\text{int}} - W_{\text{ext}}\right) = -\partial_A \Delta\Pi_T$$

    - In linear elasticity

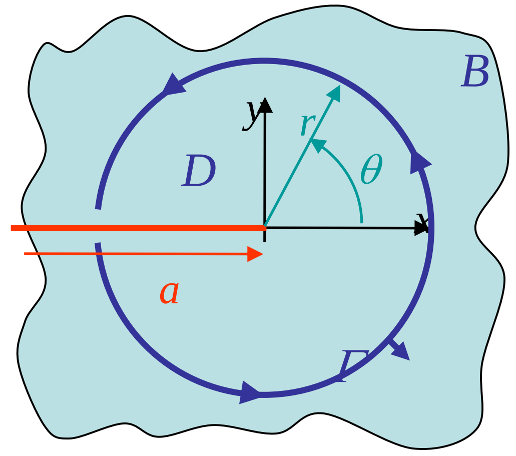
$$G = -\partial_A \Delta\Pi_T = -\lim_{\Delta A \to 0} \frac{1}{\Delta A} \int_{\Delta A} \frac{1}{2} \boldsymbol{t}^{\mathbf{0}} \cdot [\![\boldsymbol{\Delta u}]\!] \, dA$$



  - In linear elasticity & if crack grows straight ahead

$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \qquad E' = \frac{4E}{(1+\nu)(\kappa+1)} = \begin{cases} E & \text{P}\sigma \\ \frac{E}{1-\nu^2} & \text{P}\varepsilon \end{cases}$$

- Cracked body: summary



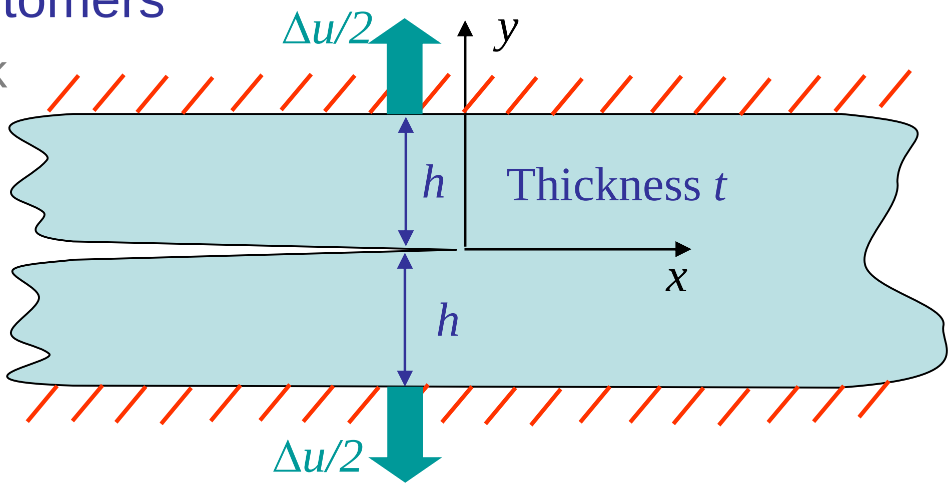
  - J-integral
    - Strain energy flow

$$J = \int_{\Gamma} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \right] dl$$

  - Exists if an internal potential exists
    - Is path independent if the contour Γ embeds a straight crack tip
    - No assumption on subsequent growth direction
    - Can be extended to plasticity if no unloading (see later)
  - If crack grows straight ahead $\Longrightarrow$ $G=J$
  - In linear elasticity (independently of crack growth direction):

$$J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \qquad E' = \frac{4E}{\left(1+\nu\right)\left(\kappa+1\right)} = \begin{cases} E & \text{P}\sigma \\ \frac{E}{1-\nu^2} & \text{P}\varepsilon \end{cases}$$

- Exercise 1: Fracture testing of elastomers
  - Infinite strip with semi-infinite crack
  - Plane σ ($t << h$)
  - Questions
    - 1) Compute J integral
      - What are the assumptions?
    - 2) Compute G
      - Why is it equal to J ?
    - 3) When can we deduce the SIF from there?
      - What is the value of K_I ?



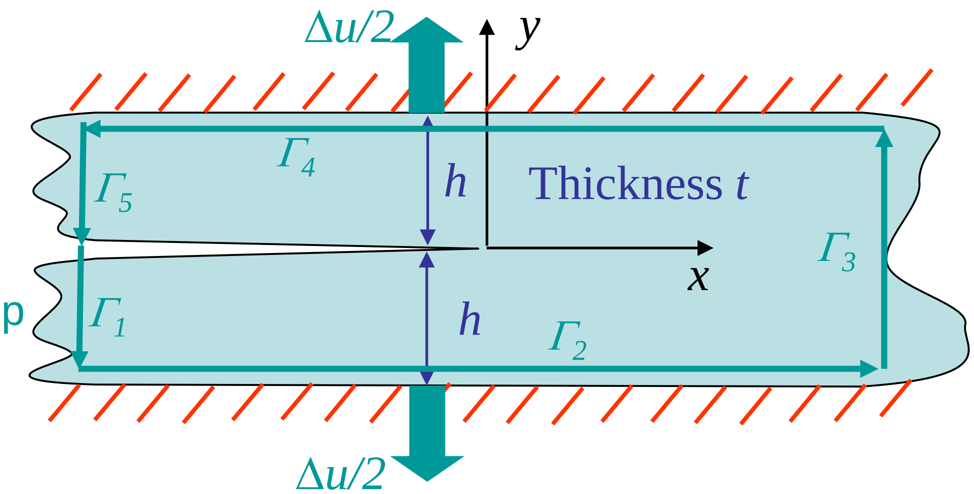
Exercise 1: Solution

- J integral
  - Assuming an internal potential U
  - J is path independent

    ⟹ Choose an easy one

    ⟹ Γ_1, Γ_5, & Γ_3 far away from crack tip

    ⟹ $$J = \sum_{i=1}^{5} \int_{\Gamma_i} \left[U\, \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T}\right] dl$$



  - Evaluate the different contributions
    - On Γ_1 & Γ_5: material unloaded

      $\boldsymbol{\sigma} = 0 \Longrightarrow U = 0$ & $\boldsymbol{T} = \boldsymbol{\sigma} \cdot \boldsymbol{n} = 0 \Longrightarrow 0 = \sum_{i=1,5} \int_{\Gamma_i} \left[U\, \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T}\right] dl$

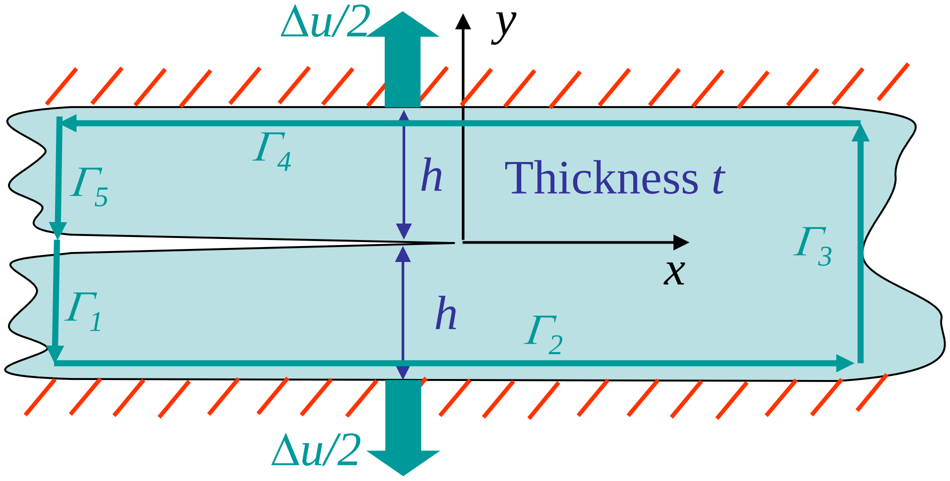
    - On Γ_2 & Γ_4: $\boldsymbol{n}_x = 0$ and $\boldsymbol{u}_{,x} = 0$ because of clamping

      ⟹ $0 = \sum_{i=2,4} \int_{\Gamma_i} \left[U\, \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T}\right] dl$

    - What remains is

$$J = \int_{\Gamma_3} \left[U\, \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T}\right] dl$$

- J integral (2)
  - What remains iw
    - $J = \int_{\Gamma_3} \left[U\, \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \right] dl$
  - On Γ_3:
    - $\boldsymbol{n}_x = 1$
    - Far away from the crack: displacement along y (and z)



$$\left.\begin{array}{l} \boldsymbol{u}_x = 0 \\ \boldsymbol{u}_y = \dfrac{\Delta u}{2}\dfrac{y}{h} \\ \boldsymbol{u}_z = u_z(y, z) \end{array}\right\} \Longrightarrow \left\{\begin{array}{ll} \boldsymbol{u}_{x,x} = 0 & \boldsymbol{u}_{x,y} = 0 \\ \boldsymbol{u}_{y,x} = 0 & \boldsymbol{u}_{y,y} = \dfrac{\Delta u}{2h} \\ \boldsymbol{u}_{z,x} = 0 & \end{array}\right.$$

$$\Longrightarrow \quad \boldsymbol{u}_{,x} \cdot \boldsymbol{T} = \boldsymbol{u}_{,x} \cdot \boldsymbol{\sigma} \cdot [1\ 0\ 0]^{\boldsymbol{T}} = \boldsymbol{u}_{x,x}\, \boldsymbol{\sigma}_{xx} + \boldsymbol{u}_{y,x}\, \boldsymbol{\sigma}_{xy} + \boldsymbol{u}_{z,x}\, \boldsymbol{\sigma}_{xz} = \boldsymbol{0}$$

- Since $\boldsymbol{\varepsilon}_{yy} = \boldsymbol{u}_{y,y}$ is uniform along y, U is also uniform along y

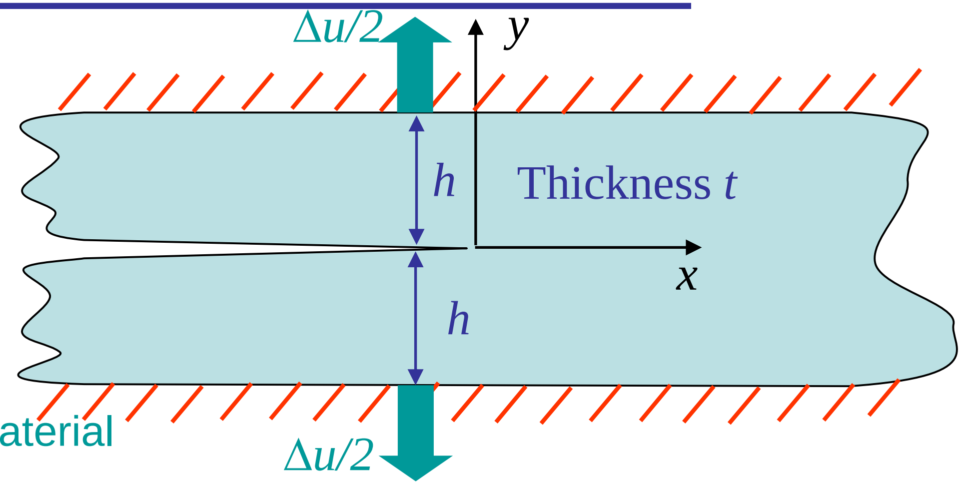
$$\Longrightarrow \quad J = \int_{\Gamma_3} U\, dl \ = 2Uh$$

- Energy release rate G
  - If an internal potential exists

$$G = -\partial_A \left(E_{\text{int}} - W_{\text{ext}}\right) = -\partial_A \Pi_T$$

  - Displacements are prescribed

$$\Longrightarrow G = -\partial_A E_{\text{int}}$$



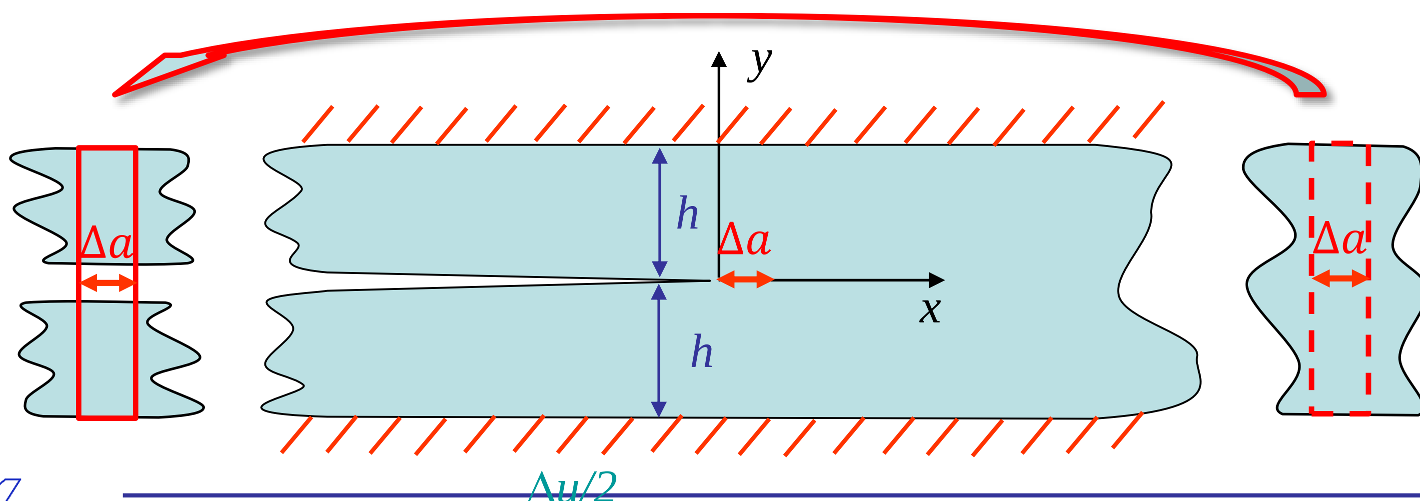
- Far behind the crack: unloaded material

$$\Longrightarrow E_{\text{int}} = 0$$

- Far ahead of the crack, U was found to be uniform

$$\Longrightarrow E_{\text{int}} = 2Uth\Delta a$$

- if the crack growth by Δa the change of energy is $-2\ U\ t\ h\ \Delta a$



Exercise 1: Solution

- Energy release rate G (2)
  - Displacements are prescribed

$$\Rightarrow \quad G = -\partial_A E_{\text{int}}$$

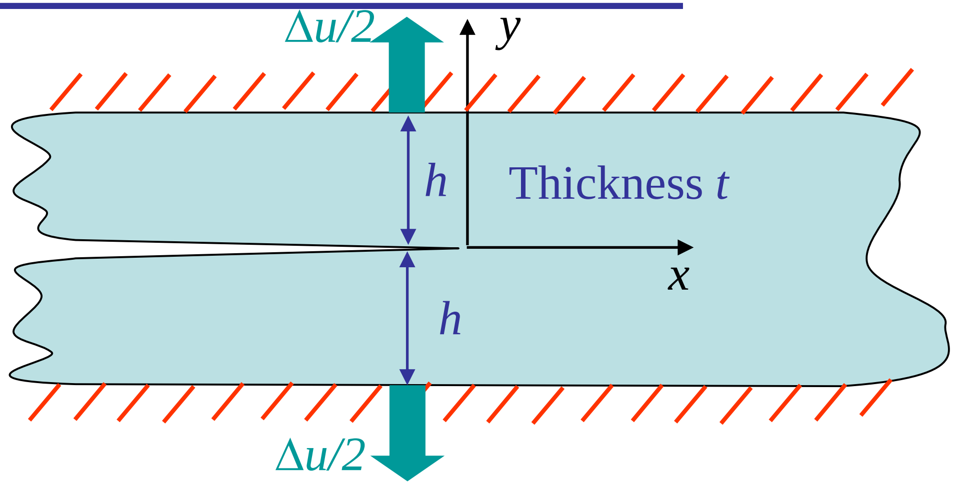
  - Crack grows by Δa

$$\Rightarrow \quad \Delta E_{\text{int}} = -2Uh\,\Delta a\,t$$

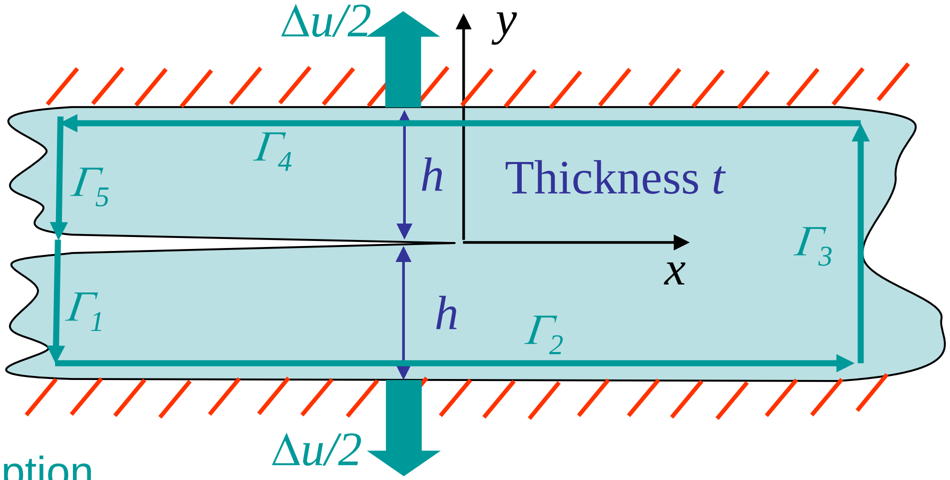
$$\Rightarrow \quad G = -\partial_A E_{\text{int}} = -\frac{1}{t}\partial_a E_{\text{int}}$$

$$\Rightarrow \quad G \simeq \lim_{\Delta a \to 0} \frac{1}{\Delta a} 2Uh\Delta a = 2Uh$$

  - G = J as the crack grows straight ahead (by symmetry)



- Stress intensity factor
  - We have
    - $J = G = 2Uh$
  - Linear elasticity:
    - $U = \frac{1}{2}\sum_{i,j} \boldsymbol{\sigma}_{ij}\, \boldsymbol{\varepsilon}_{ij}$



- But we have a plane-stress assumption ($\boldsymbol{\sigma}_{zz} = 0$) and

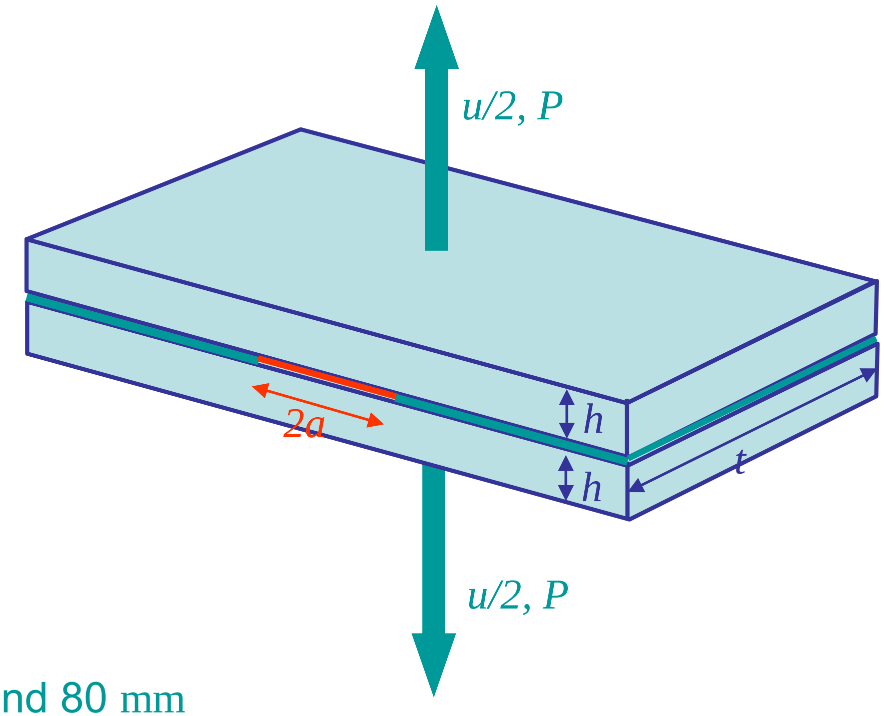
$$\left.\begin{aligned} \boldsymbol{u}_x &= 0 \\ \boldsymbol{u}_y &= \frac{\Delta u}{2}\frac{y}{h} \\ \boldsymbol{u}_z &= u_z(y,z) \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} \boldsymbol{u}_{x,x} &= 0 \quad & \boldsymbol{u}_{x,y} &= 0 \\ \boldsymbol{u}_{y,x} &= 0 \quad & \boldsymbol{u}_{y,y} &= \frac{\Delta u}{2h} \\ \boldsymbol{u}_{z,x} &= 0 \end{aligned}\right.$$

- Plane-strain assumption ($\boldsymbol{\varepsilon}_{xx} = 0$) $\Longrightarrow \boldsymbol{\sigma}_{yy} = \frac{E}{1-\nu^2}\boldsymbol{\varepsilon}_{yy} = \frac{E}{1-\nu^2}\frac{\Delta u}{2h}$

$$\Longrightarrow U = \frac{1}{2}\sum_{i,j} \boldsymbol{\sigma}_{ij}\, \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\boldsymbol{\sigma}_{yy}\boldsymbol{\varepsilon}_{yy} = \frac{E}{2(1-\nu^2)}\left(\frac{\Delta u}{2h}\right)^2$$

- SIF $\quad J = 2Uh = \frac{E}{1-\nu^2}\frac{\Delta u^2}{4h} \Longrightarrow K_I = \sqrt{EJ} = \frac{E\Delta u}{2\sqrt{h\left(1-\nu^2\right)}}$

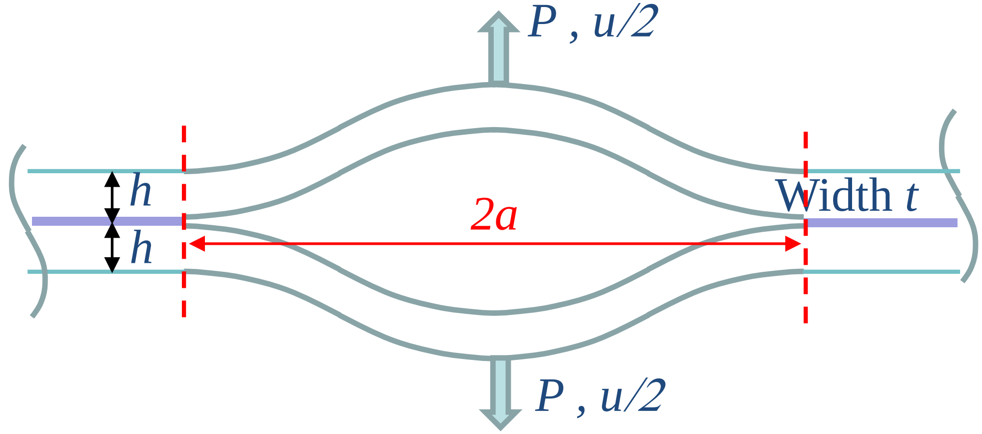
- Exercise 2: Laminated composite
  - 2 long thin strips of steel
    - E = 200 GPa
    - h = 0.97 mm
    - t = 10.1 mm
  - Bonded with epoxy
    - G_c = 300 Pa·m
  - Central crack 2a
  - Questions
    - 1) Critical load for 2a = 60 mm
    - 2) Apply same method for 2a = 70 and 80 mm
      - Report on a P vs u graph the toughness locus
    - 3) Determine the critical energy release rate from that graph



- Compliance method
  - Beam theory

$$\frac{u}{2} = \frac{Pa^3}{24EI} = \frac{Pa^3}{2Eth^3}$$

$$\Rightarrow C = \frac{u}{P} = \frac{a^3}{Eth^3}$$



  - Energy release rate in linear elasticity

$$G = \frac{P^2}{2}\frac{\partial C}{\partial A} = \frac{P^2}{2}\frac{1}{2t}\frac{\partial C}{\partial a} \Rightarrow G = \frac{3P^2a^2}{4Et^2h^3}$$

2 fronts

  - Critical load for $2a$ = 60 mm

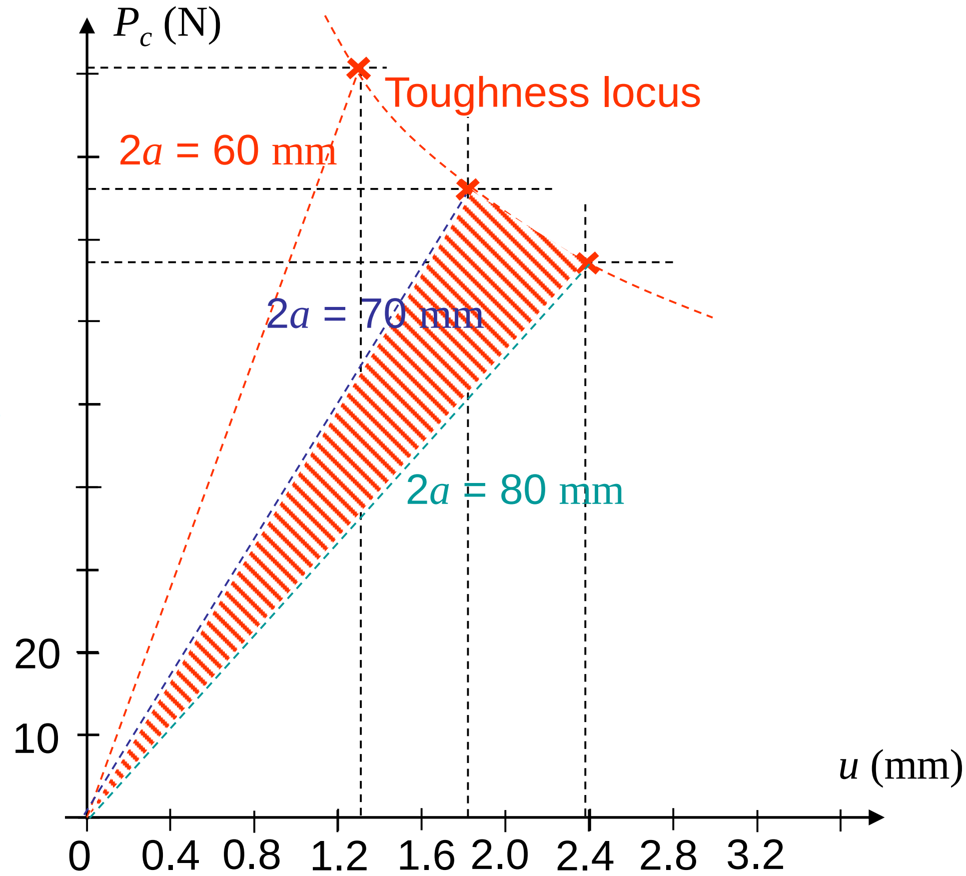
$$P_c(2a = 60\,\text{mm}) = \sqrt{\frac{4EG_ct^2h^3}{3a^2}} = \sqrt{\frac{4\ 200\ 10^9\ 300\ 0.0101^2\ 0.00097^3}{3\ 0.03^3}} = 90.97\,\text{N}$$

$$\Rightarrow u_c(2a = 60\,\text{mm}) = \frac{P_ca^3}{Eth^3} = \frac{90.97\ 0.03^3}{200\ 10^9\ 0.0101\ 0.00097^3} = 0.00133\ \text{m}$$

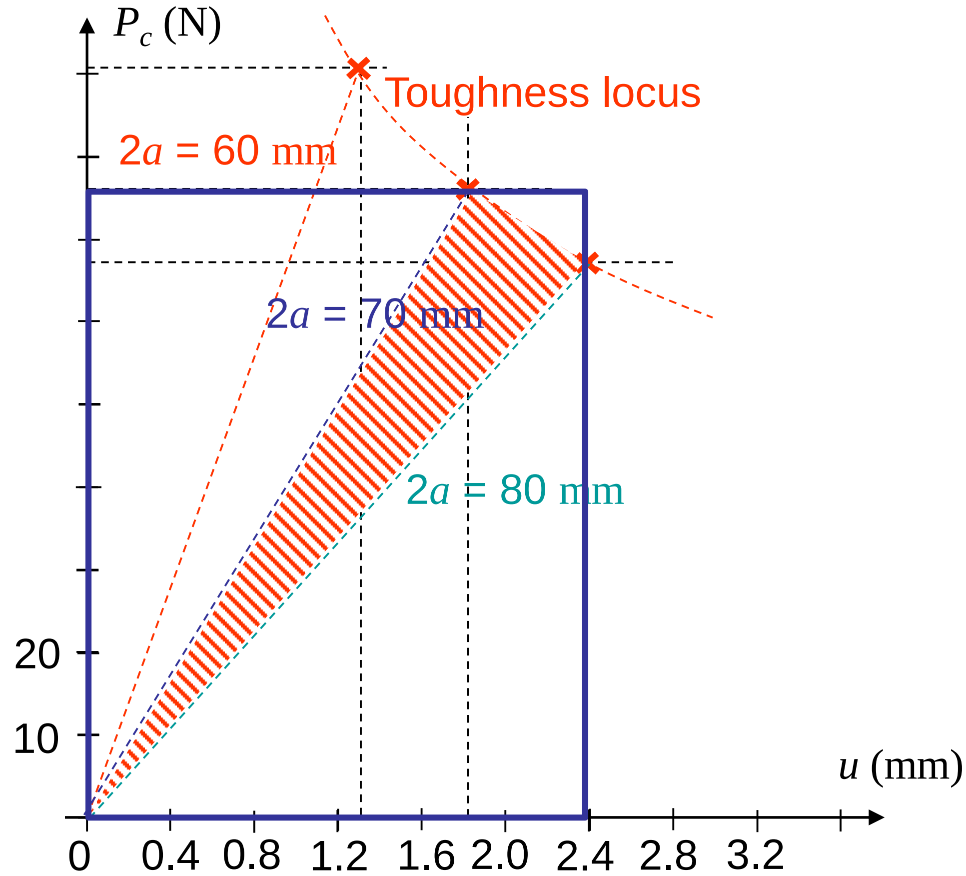
- Failure locus
  - Using the same method

$$\begin{cases} P_c(2a = 70\,\mathrm{mm}) = 77.98\,\mathrm{N} \\ P_c(2a = 80\,\mathrm{mm}) = 68.23\,\mathrm{N} \end{cases}$$

$$\begin{cases} u_c(2a = 70\,\mathrm{mm}) = \dfrac{P_c a^3}{Eth^3} = 0.00181\ \mathrm{m} \\ u_c(2a = 80\,\mathrm{mm}) = \dfrac{P_c a^3}{Eth^3} = 0.00237\ \mathrm{m} \end{cases}$$

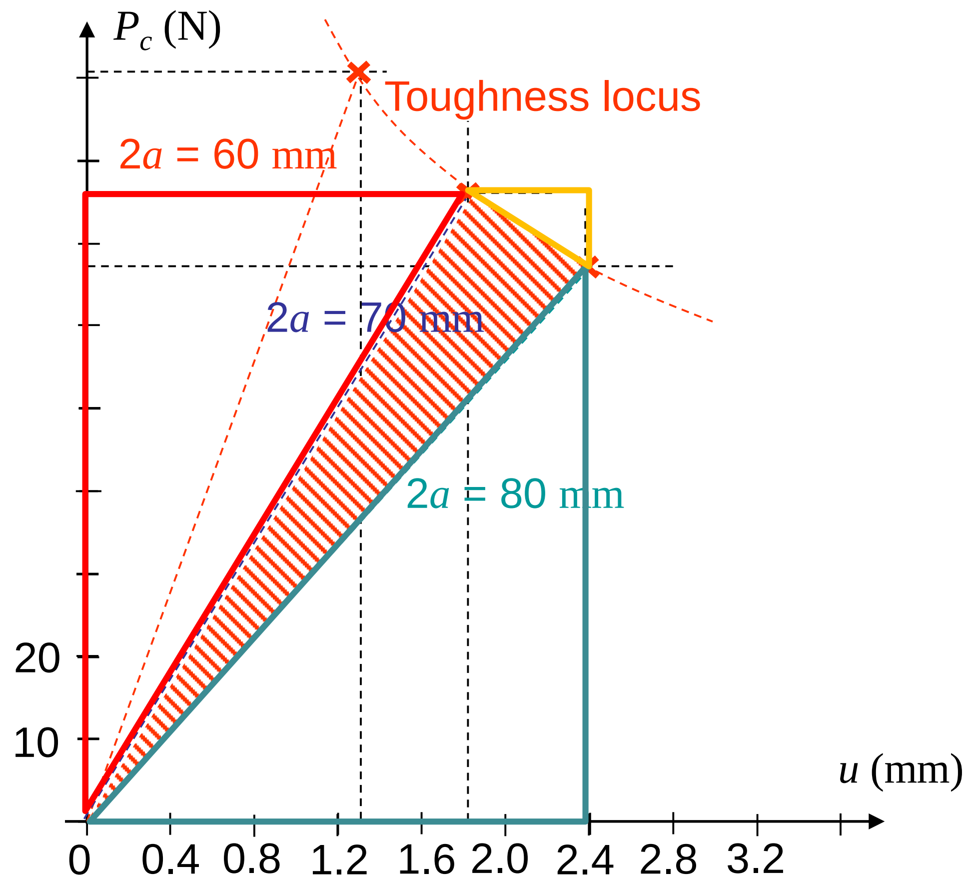


- Failure locus
  - Assuming the graph is deduced from experiments
    - Critical energy release rate



$$\text{Area} = \boxed{0.00237\ 77.98 -} \frac{0.00181\ 77.98 + 0.00237\ 68.23 + (0.00237 - 0.00181)\,(77.98 - 68.23)}{2} = 0.03066\ \text{Nm}$$

- Failure locus
  - Assuming the graph is deduced from experiments
    - Critical energy release rate



$$\text{Area} = \frac{\boxed{0.00237\ 77.98-} \boxed{0.00181\ 77.98} + \boxed{0.00237\ 68.23} + \boxed{(0.00237 - 0.00181)\ (77.98 - 68.23)}}{2} = 0.03066\ \text{Nm}$$

$$\Longrightarrow G_c \simeq \frac{\text{Area}}{2t\Delta a} = \frac{0.03066}{2\ 0.0101\ 0.005} = 303.5\ \text{N} \cdot \text{m}^{-1}$$

References

- Lecture notes
  – Lecture Notes on Fracture Mechanics, Alan T. Zehnder, Cornell University, Ithaca, http://hdl.handle.net/1813/3075
  – Fracture Mechanics Online Class, L. Noels, ULg, http://www.ltas-cm3.ulg.ac.be/FractureMechanics
- Book
  – Fracture Mechanics: Fundamentals and applications, D. T. Anderson. CRC press, 1991.
  – S. Suresh, Fatigue of Materials, Cambridge University Press, 2001

- In elasticity (linear or not) and if ***b*** assumed equal to 0
  - Potential energy variation

$$\begin{cases} \Pi_T = \int_B U(\boldsymbol{\nabla u})\, dB - \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \boldsymbol{u} d\partial B \\ \Pi_T + \Delta\Pi_T = \int_{B-\Delta B} U(\boldsymbol{\nabla u} + \boldsymbol{\nabla}\Delta\boldsymbol{u})\, dB - \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot (\boldsymbol{u} + \Delta\boldsymbol{u})\, d\partial B \end{cases}$$

$$\Delta\Pi_T = \int_{B-\Delta B} U(\boldsymbol{\nabla u} + \boldsymbol{\nabla}\Delta\boldsymbol{u}) - U(\boldsymbol{\nabla u})\, dB - \int_{\Delta B} U(\boldsymbol{\nabla u})\, dB - \int_{\partial_N B} \bar{\boldsymbol{T}} \cdot \Delta\boldsymbol{u} d\partial B$$

  - Stress derives from a potential

$$\int_{B-\Delta B} U(\boldsymbol{\varepsilon} + \Delta\boldsymbol{\varepsilon}) - U(\boldsymbol{\varepsilon})\, dB = \int_{B-\Delta B} \left\{ \int_0^{\boldsymbol{\varepsilon}+\Delta\boldsymbol{\varepsilon}} \boldsymbol{\sigma}(\boldsymbol{\varepsilon}') : d\boldsymbol{\varepsilon}' - \int_0^{\boldsymbol{\varepsilon}} \boldsymbol{\sigma}(\boldsymbol{\varepsilon}') : d\boldsymbol{\varepsilon}' \right\} dB$$

    - Since $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = 0$ (as ***b***=0)

$$\int_{B-\Delta B} U(\boldsymbol{\varepsilon} + \Delta\boldsymbol{\varepsilon}) - U(\boldsymbol{\varepsilon})\, dB = \int_{B-\Delta B} \int_{\boldsymbol{u}}^{\boldsymbol{u}+\Delta\boldsymbol{u}} \boldsymbol{\nabla} \cdot \left(\boldsymbol{\sigma}^T(\boldsymbol{u}') \cdot d\boldsymbol{u}'\right) dB$$

    - Applying Gauss theorem

$$\int_{B-\Delta B} U(\boldsymbol{\varepsilon} + \Delta\boldsymbol{\varepsilon}) - U(\boldsymbol{\varepsilon})\, dB = \int_{S+\Delta S+\partial_N B+\partial_D B} \int_{\boldsymbol{u}}^{\boldsymbol{u}+\Delta\boldsymbol{u}} [\boldsymbol{\sigma}(\boldsymbol{u}') \cdot \boldsymbol{n}] \cdot d\boldsymbol{u}' d\partial B$$

- In elasticity (linear or not) and if $\boldsymbol{b}$ assumed equal to 0 (2)
  - Study of term

$$\int_{B-\Delta B} U(\boldsymbol{\varepsilon}+\Delta\boldsymbol{\varepsilon}) - U(\boldsymbol{\varepsilon})\, dB = \int_{S+\Delta S+\partial_N B+\partial_D B} \int_{\boldsymbol{u}}^{\boldsymbol{u}+\Delta\boldsymbol{u}} \left[\boldsymbol{\sigma}(\boldsymbol{u}')\cdot\boldsymbol{n}\right]\cdot d\boldsymbol{u}'\, d\partial B$$

- Traction is constant on $\partial_N B$ & displacement is constant on $\partial_D B$
- On the cavity surface S: $\boldsymbol{t}$ is defined as $\boldsymbol{\sigma}\cdot\boldsymbol{n}$

$$\int_{B-\Delta B} U(\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\nabla}\Delta\boldsymbol{u}) - U(\boldsymbol{\nabla}\boldsymbol{u})\, dB = \int_{S+\Delta S} \int_{\boldsymbol{u}}^{\boldsymbol{u}+\Delta\boldsymbol{u}} \boldsymbol{t}(\boldsymbol{u}')\cdot d\boldsymbol{u}'\, d\partial B + \int_{\partial_N B} \bar{\boldsymbol{T}}\cdot\Delta\boldsymbol{u}\, d\partial B$$

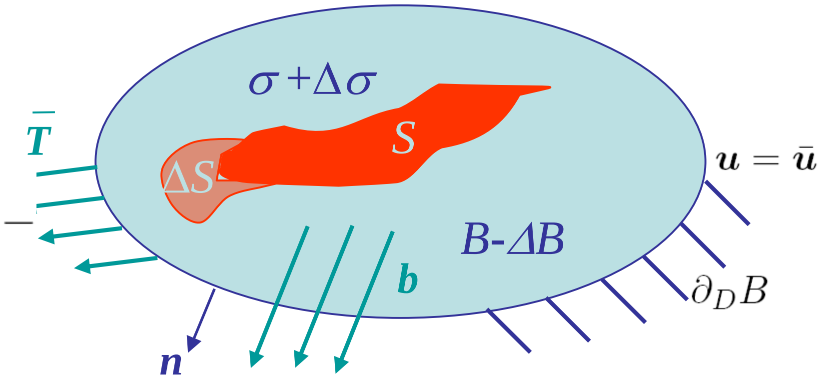
- Be careful: $S+\Delta S$ is stress free, but only in the final configuration
  - $(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})\cdot\boldsymbol{n} = 0$ on $S+\Delta S$ but,
  - $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}')\cdot\boldsymbol{n} \neq 0$ on $S+\Delta S$ $\Longrightarrow$ so the integral does not vanish
  - However, S remains stress free during the whole process

$$\Longrightarrow \int_{B-\Delta B} U(\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\nabla}\Delta\boldsymbol{u}) - U(\boldsymbol{\nabla}\boldsymbol{u})\, dB = \int_{\Delta S} \int_{\boldsymbol{u}}^{\boldsymbol{u}+\Delta\boldsymbol{u}} \boldsymbol{t}(\boldsymbol{u}')\cdot d\boldsymbol{u}'\, d\partial B + \int_{\partial_N B} \bar{\boldsymbol{T}}\cdot\Delta\boldsymbol{u}\, d\partial B$$

- Eventually $\Delta\Pi_T = \int_{\Delta S} \int_{\boldsymbol{u}}^{\boldsymbol{u}+\Delta\boldsymbol{u}} \boldsymbol{t}(\boldsymbol{u}')\cdot d\boldsymbol{u}'\, d\partial B - \int_{\Delta B} U(\boldsymbol{\nabla}\boldsymbol{u})\, dB$
  - If instead of a cavity we have a crack, the change of volume is zero and the last term disappears

Annex 2: J integral for crack growing straight ahead

- Back to crack closure integral
  - Material not necessarily linear



$$\Delta\Pi_T = \int_{B-\Delta B} U\left(\boldsymbol{\nabla u}+\boldsymbol{\nabla}\Delta\boldsymbol{u}\right) - U\left(\boldsymbol{\nabla u}\right) dB - \int_{\Delta B} U\left(\boldsymbol{\nabla u}\right) dB - \int_{\partial_N B} \bar{\boldsymbol{T}}\cdot\Delta\boldsymbol{u}\, d\partial B$$

  - For a crack (ΔB = 0), and in 2D, the energy release rate becomes
    - $$G = \lim_{\Delta a\to 0} -\frac{\Delta\Pi_T}{t\Delta a}$$
      $$= \lim_{\Delta a\to 0}\frac{1}{\Delta a}\left\{\int_B U\left(\boldsymbol{\nabla u}\right) - U\left(\boldsymbol{\nabla u}+\boldsymbol{\nabla}\Delta\boldsymbol{u}\right) dB + \int_{\partial_N B}\bar{\boldsymbol{T}}\cdot\Delta\boldsymbol{u}\, d\partial B\right\}$$
    - As the crack is stress free, as $\Delta\boldsymbol{u}$ = 0 on $\partial_D B$, and as $\Delta\boldsymbol{\sigma}$ = 0 on $\partial_N B$

$$\int_{\partial_N B}\bar{\boldsymbol{T}}\cdot\Delta\boldsymbol{u}\, d\partial B = \int_{\partial_N B+\partial_D B+S+\Delta S}\Delta\boldsymbol{u}\cdot\left(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma}\right)\cdot\boldsymbol{n}\, d\partial B$$
$$= \int_B \boldsymbol{\nabla}\cdot\left(\Delta\boldsymbol{u}\cdot\left(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma}\right)\right) dB$$

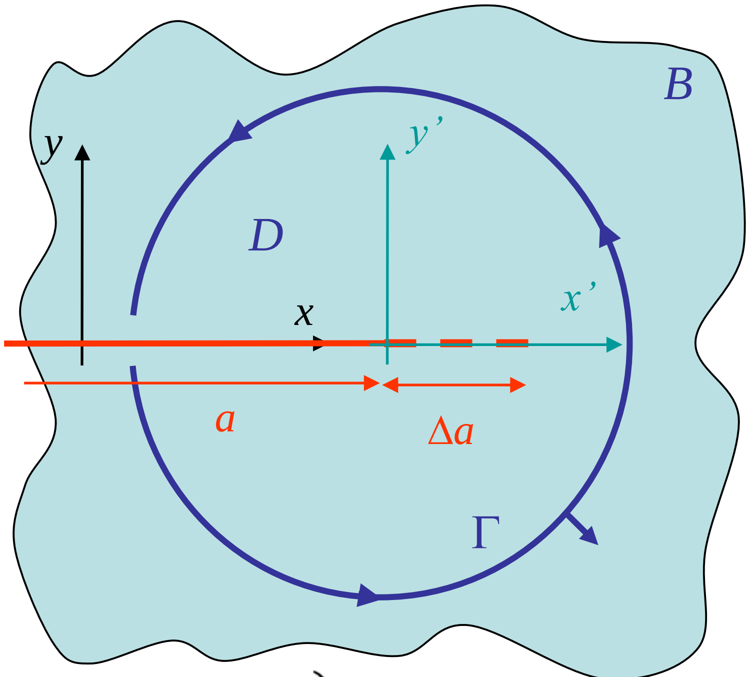
    - After using the equilibrium equation $\nabla\cdot(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma}) = 0$

$$G = \lim_{\Delta a\to 0}\frac{1}{\Delta a}\left\{\int_B U\left(\boldsymbol{\varepsilon}\right) - U\left(\boldsymbol{\varepsilon}+\Delta\boldsymbol{\varepsilon}\right) + \left(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma}\right):\Delta\boldsymbol{\varepsilon}\, dB\right\}$$

- Referential *O'x'y'* moving with the crack tip
  - Only if the crack grows straight ahead

$$x' = x - a \;\; \& \;\; y' = y$$

$$\Longrightarrow \begin{cases} f(x', y', a) = f(x - a, y, a) \\ \dfrac{df}{da} = -\partial_{x'} f + \partial_a f \end{cases}$$



- Energy release rate

$$G = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \left\{ \int_B U(\boldsymbol{\varepsilon}) - U(\boldsymbol{\varepsilon} + \Delta\boldsymbol{\varepsilon}) + (\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}) : \Delta\boldsymbol{\varepsilon} dB \right\}$$

  - Involves the whole body B
  - However, as $\Delta a \to 0$, the non vanishing contributions are around the crack tip
    - The equation is then limited to the FIXED region D of boundary Γ

$$G = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \left\{ \int_D U(\boldsymbol{\varepsilon}) - U(\boldsymbol{\varepsilon} + \Delta\boldsymbol{\varepsilon}) + (\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}) : \Delta\boldsymbol{\varepsilon} dD \right\}$$

- Energy release rate (2)
  - In static $\nabla\cdot(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})=0$ in D

$$\Longrightarrow \int_D \nabla\cdot(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})\cdot\Delta\boldsymbol{u}dD=0$$

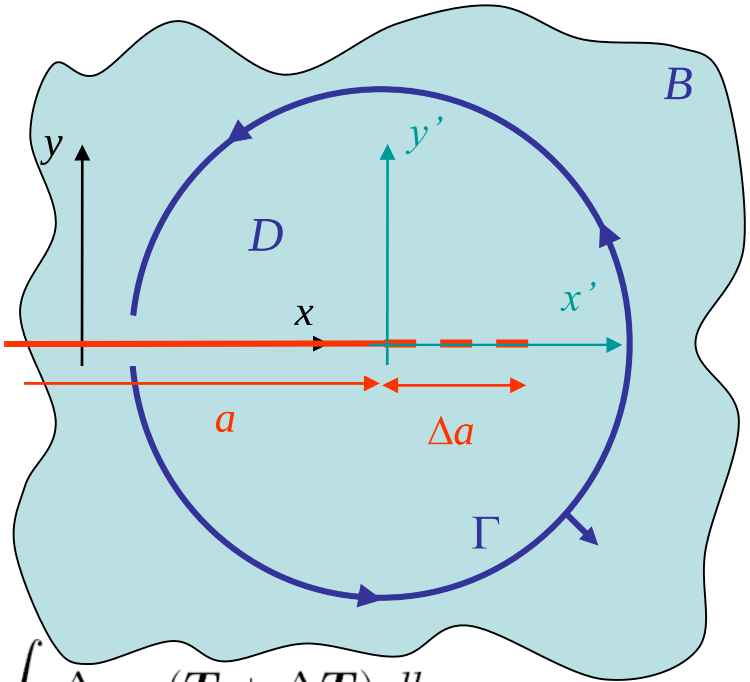
  - Since the crack is stress free

    Gauss theorem leads to

$$\int_D (\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma}):\Delta\boldsymbol{\varepsilon}dD=\int_\Gamma \Delta\boldsymbol{u}\cdot(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})\cdot\boldsymbol{n}dl$$

  - Last term of energy release rate becomes

$$\lim_{\Delta a\to 0}\frac{1}{\Delta a}\int_D(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma}):\Delta\boldsymbol{\varepsilon}dD=\lim_{\Delta a\to 0}\frac{1}{\Delta a}\int_\Gamma\Delta\boldsymbol{u}\cdot(\boldsymbol{T}+\Delta\boldsymbol{T})\,dl$$



    - Using limit definition and after changing frame

$$\Longrightarrow \lim_{\Delta a\to 0}\frac{1}{\Delta a}\int_D(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma}):\Delta\boldsymbol{\varepsilon}dD=\int_\Gamma\frac{d\boldsymbol{u}}{da}\cdot\boldsymbol{T}dl=\int_\Gamma\left(-\partial_{x'}\boldsymbol{u}+\partial_a\boldsymbol{u}\right)\cdot\boldsymbol{T}dl$$

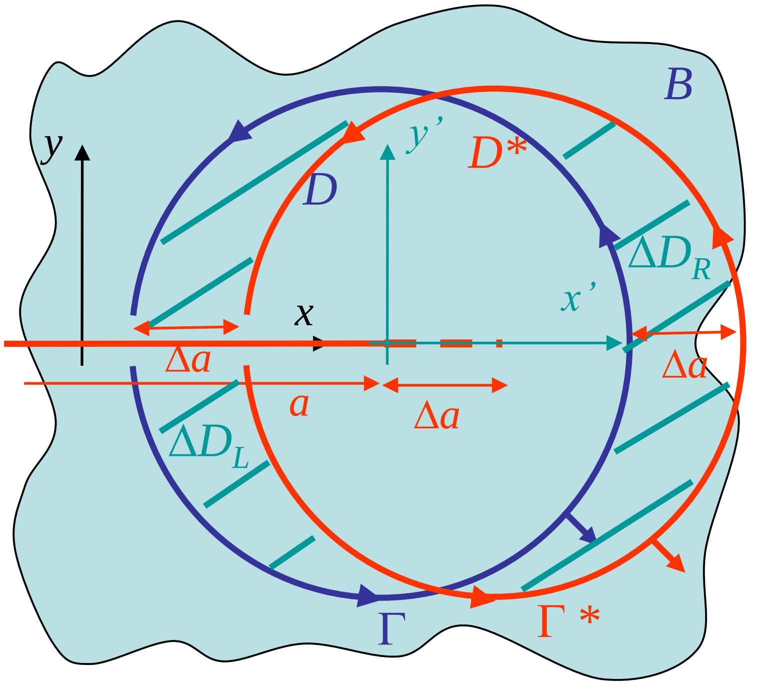
    - Using momentum equilibrium equation and internal potential definition

$$\int_\Gamma\partial_a\boldsymbol{u}\cdot\boldsymbol{T}dl=\int_\Gamma\partial_a\boldsymbol{u}\cdot\partial_{\boldsymbol{\varepsilon}}U\cdot\boldsymbol{n}dl=\int_D\partial_a\boldsymbol{\varepsilon}:\partial_{\boldsymbol{\varepsilon}}UdD=\int_D\partial_aUdD$$

$$\Longrightarrow \boxed{\lim_{\Delta a\to 0}\frac{1}{\Delta a}\int_D(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma}):\Delta\boldsymbol{\varepsilon}dD=-\int_\Gamma\partial_{x'}\boldsymbol{u}\cdot\boldsymbol{T}dl+\int_D\partial_aUdD}$$

- Energy release rate (3)
  - As D is fixed, let us define D^* moving with the crack tip: $D = D^* + \Delta D_L - \Delta D_R$
  - First part of energy release rate becomes

$$\lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_D U(\varepsilon) - U(\varepsilon + \Delta\varepsilon)\, dD \quad = \quad \lim_{\Delta a \to 0} \frac{1}{\Delta a} \left\{ \int_D U(\varepsilon(a))\, dD - \int_{D* + \Delta D_L - \Delta D_R} U(\varepsilon(a + \Delta a))\, dD \right\}$$



- Since $D^* \rightarrow D$, this relation tends toward

$$\lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_D U(\varepsilon) - U(\varepsilon + \Delta\varepsilon)\, dD \quad = \quad \lim_{\Delta a \to 0} \left\{ \int_D \frac{U(\varepsilon(a)) - U(\varepsilon(a + \Delta a))}{\Delta a} dD - \frac{1}{\Delta a} \int_{\Delta D_L - \Delta D_R} U(\varepsilon(a + \Delta a))\, dD \right\}$$

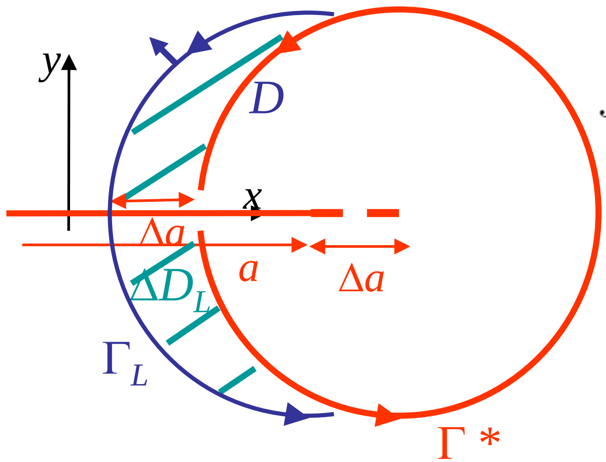
- N.B.: Formally, one should use derivatives & limits of integrals with non-constant intervals

Annex 2: *J* integral for crack growing straight ahead

- Energy release rate (4)
  - First part of energy release rate becomes (2)
    - Homogeneous materials ($\partial_{x'} U = 0$)

$$\lim_{\Delta a \to 0} \int_D \frac{U(\boldsymbol{\varepsilon}(a)) - U(\boldsymbol{\varepsilon}(a+\Delta a))}{\Delta a} dD = -\int_D \partial_a U dD$$

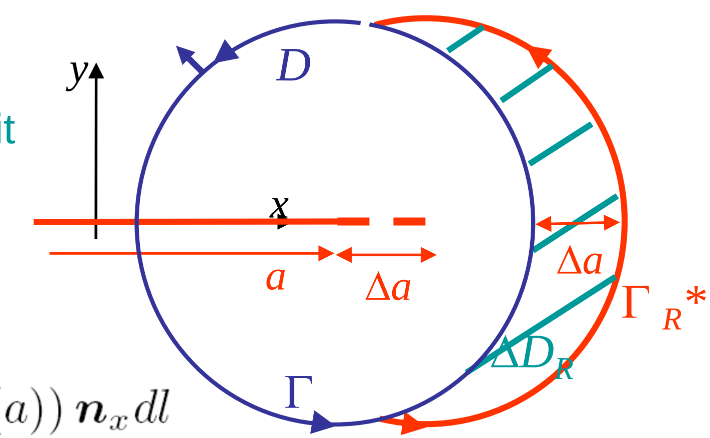
- Considering the opened curve Γ_L , at the limit



$$\int_{\Delta D_L} U dD = \int_{-\Gamma_L} U \boldsymbol{n}_x \Delta a dl \Rightarrow$$

$$\lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_{\Delta D_L} U(\boldsymbol{\varepsilon}(a+\Delta a)) dD = -\int_{\Gamma_L} U(\boldsymbol{\varepsilon}(a)) \boldsymbol{n}_x dl$$

- Considering the opened curve Γ_R* , at the limit



$$\int_{\Delta D_R} U dD = \int_{\Gamma_R^*} U \boldsymbol{n}_x \Delta a dl \Rightarrow$$

$$\lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_{\Delta D_R} U(\boldsymbol{\varepsilon}(a+\Delta a)) dD = \int_{\Gamma_r^*} U(\boldsymbol{\varepsilon}(a)) \boldsymbol{n}_x dl$$

- Energy release rate (5)
  - First part of energy release rate becomes (3)
    - As $\Gamma_L + \Gamma_R{}^* \rightarrow \Gamma$, it yields

$$\lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_D U(\boldsymbol{\varepsilon}) - U(\boldsymbol{\varepsilon} + \Delta\boldsymbol{\varepsilon})\, dD \quad = \quad -\int_D \partial_a U dD + \int_\Gamma U(\boldsymbol{\varepsilon}(a))\, \boldsymbol{n}_x dl$$

  - And as

$$G = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \left\{ \int_D U(\boldsymbol{\varepsilon}) - U(\boldsymbol{\varepsilon} + \Delta\boldsymbol{\varepsilon}) + (\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}) : \Delta\boldsymbol{\varepsilon} dD \right\}$$

    - with $\lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_D (\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}) : \Delta\boldsymbol{\varepsilon} dD = -\int_\Gamma \partial_{x'} \boldsymbol{u} \cdot \boldsymbol{T} dl + \int_D \partial_a U dD$
    - The energy rate is rewritten $G = \int_\Gamma U(\boldsymbol{\varepsilon}(a))\, \boldsymbol{n}_x dl - \int_\Gamma \partial_x \boldsymbol{u} \cdot \boldsymbol{T} dl = J$

- So $G=J$
  - For materials defined by an internal potential (linear response or not)
  - AND if the crack grows straight ahead