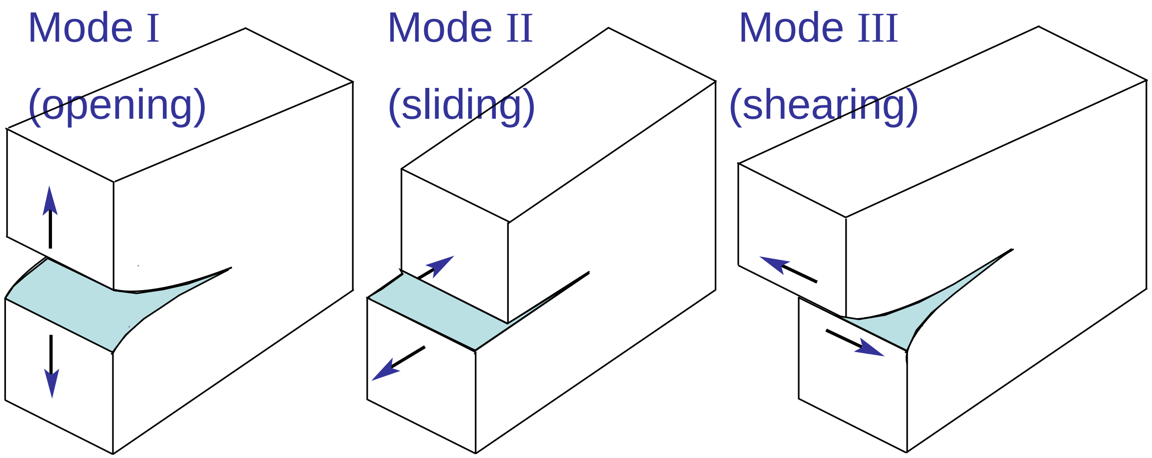
University of Liège
Aerospace & Mechanical Engineering

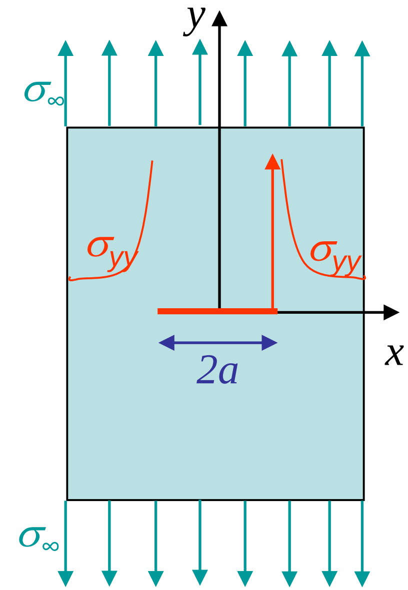
Fracture mechanics, Damage and Fatigue
Linear Elastic Fracture Mechanics – Crack Growth

Ludovic Noels

Computational & Multiscale Mechanics of Materials – CM3
http://www.ltas-cm3.ulg.ac.be/
Chemin des Chevreuils 1, B4000 Liège
L.Noels@ulg.ac.be

- Cracked body: summary
  - 3 failure modes

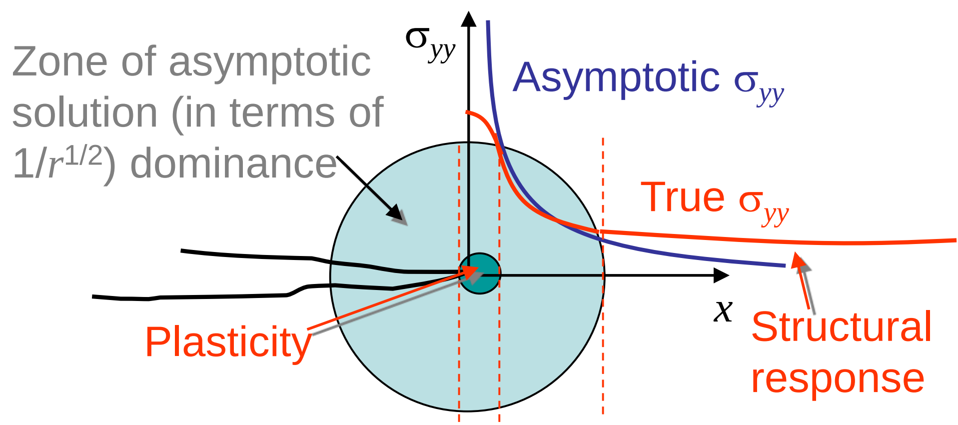




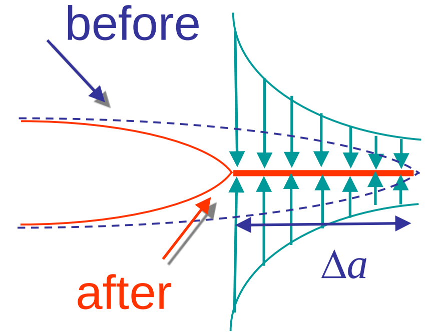
  - Asymptotic solution governed by stress intensity factors

$$\boldsymbol{\sigma}^{\text{mode } i} = \frac{K_i}{\sqrt{2\pi r}} f^{\text{mode } i}(\theta) + \mathcal{O}(r^0)$$

$$\boldsymbol{u}^{\text{mode } i} = K_i \sqrt{\frac{r}{2\pi}}\, g^{\text{mode } i}(\theta) + \mathcal{O}(r^0)$$



- Cracked body: summary
  - Potential energy $\Pi_T = E_{\text{int}} - Qu$
  - Crack closure integral
    - Energy required to close crack tip



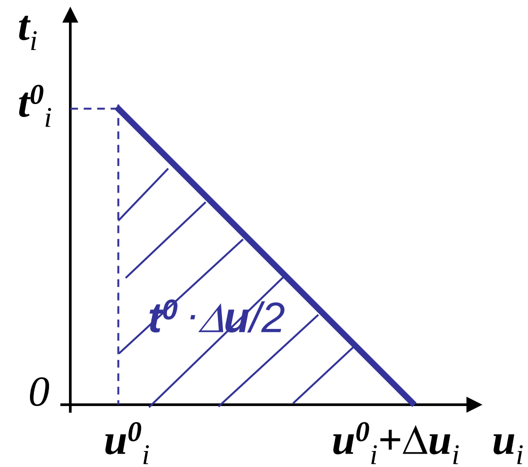
$$\Delta\Pi_T = \int_{\Delta A}\int_{\boldsymbol{u}}^{\boldsymbol{u}+\boldsymbol{\Delta u}} \boldsymbol{t}\cdot[\![\boldsymbol{u}']\!]d\boldsymbol{u}'\ dA$$

  - Energy release rate
    - Variation of potential energy in case of crack growth

$$G = -\partial_{\mathrm{A}}\left(E_{\text{int}} - W_{\text{ext}}\right) = -\partial_A \Delta\Pi_T$$

    - In linear elasticity

$$G = -\partial_A\Delta\Pi_T = -\lim_{\Delta A\to 0}\frac{1}{\Delta A}\int_{\Delta A}\frac{1}{2}\boldsymbol{t}^{\mathbf{0}}\cdot[\![\boldsymbol{\Delta u}]\!]\,dA$$

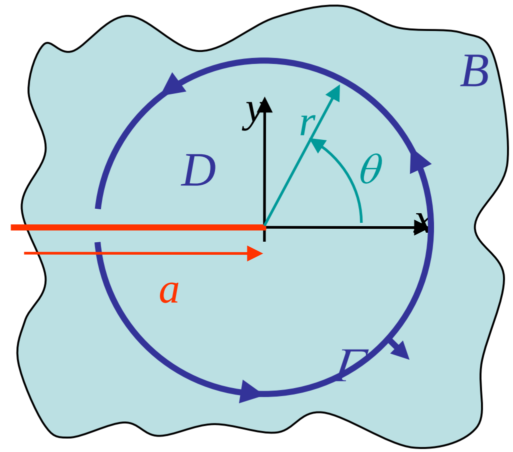


  - In linear elasticity & if crack grows straight ahead

$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \qquad E' = \frac{4E}{(1+\nu)(\kappa+1)} = \begin{cases} E & \mathrm{P}\sigma \\ \frac{E}{1-\nu^2} & \mathrm{P}\varepsilon \end{cases}$$

- Cracked body: summary
  - J-integral
    - Strain energy flow

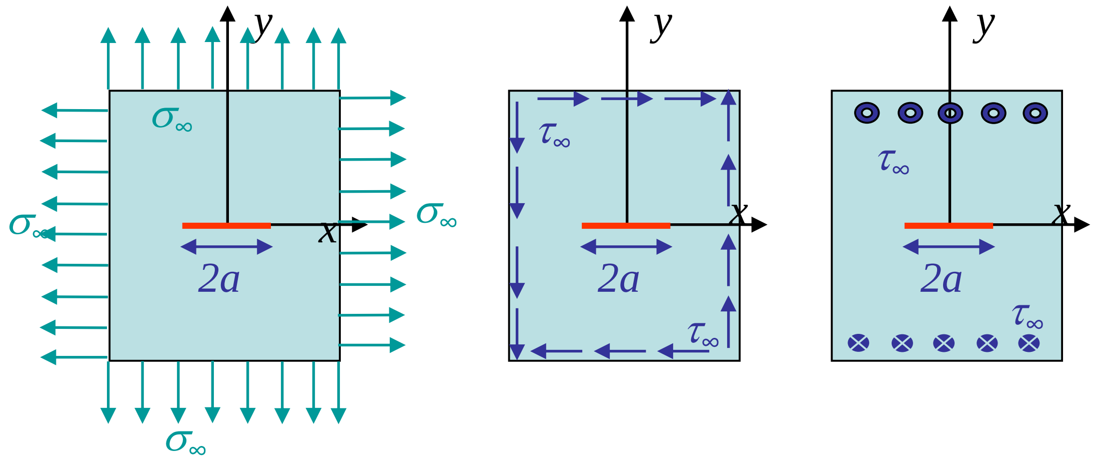
$$J = \int_{\Gamma} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T} \right] dl$$



  - Exists if an internal potential exists
    - Is path independent if the contour Γ embeds a straight crack tip
    - No assumption on subsequent growth direction
    - Can be extended to plasticity if no unloading (see later)
  - If crack grows straight ahead $\Longrightarrow$ $G=J$
  - In linear elasticity (independently of crack growth direction):

$$J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \qquad E' = \frac{4E}{(1+\nu)(\kappa+1)} = \begin{cases} E & \text{P}\sigma \\ \frac{E}{1-\nu^2} & \text{P}\varepsilon \end{cases}$$

- Analytical
  - SIF from full-field solution
    - Limited cases number



$$\Rightarrow \begin{cases} K_I = \sigma_\infty \sqrt{\pi a} \\ K_{II} = \tau_\infty \sqrt{\pi a} \\ K_{III} = \tau_\infty \sqrt{\pi a} \end{cases}$$

  - From energetic consideration
    - Growing straight ahead crack

$$G = -\partial_{\mathrm{A}} \left(E_{\mathrm{int}} - W_{\mathrm{ext}}\right) \Rightarrow G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

    - From J-integral

$$J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

- Numerical (e.g. FEM)

$$\Rightarrow \begin{cases} K_I = \beta_I \sigma_\infty \sqrt{\pi a} \\ K_{II} = \beta_{II} \tau_\infty \sqrt{\pi a} \\ K_{III} = \beta_{III} \tau_\infty \sqrt{\pi a} \end{cases}$$

  - β_i depends on geometry & crack length
- Tabulated solutions (handbooks)
  - http://ebooks.asmedigitalcollection.asme.org/book.aspx?bookid=230

- Crack growth criterion in mode I
  - For a crack to grow

Energy released by the system if the crack grows by a unit surface → $G \geq G_C$ ← Energy required at the material level to form a crack of unit surface

  - The energy release rate G depends on
    - The geometry, including the crack length
    - The loading
  - What does the fracture energy G_c depend on ?
    - For perfectly brittle materials: this is the energy to break atomic bonding

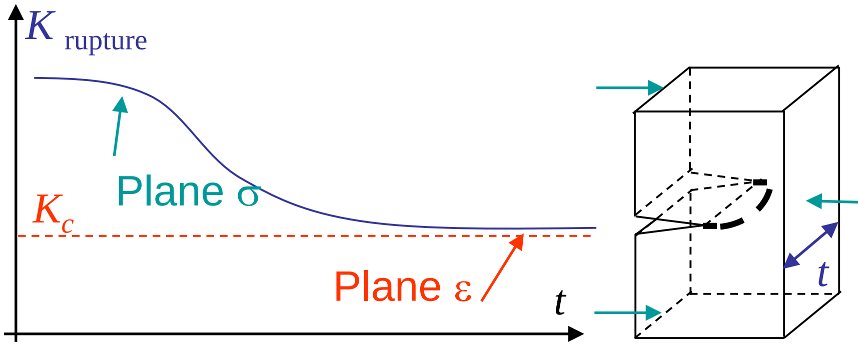
$$G_C = 2\gamma_s$$

    - Practically: should account for the inelastic deformations in the process zone

$$G_C = 2\gamma_s + W_{\text{pl}}$$

    - For (an)isotropic materials, G_c is (not) the same in all the directions
      - Examples: wood, composites

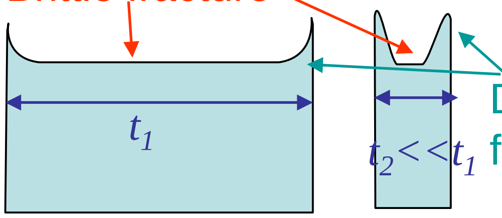
- Crack growth criterion in mode I (2)
  - For an initial straight crack under mode I: in an isotropic material
    - The crack will grow straight ahead (see next slides) ⟹ $G = \frac{K_I^2}{E'}$
    - So toughness and fracture energy are related ⟹ $K_C = \sqrt{E' G_C}$
  - Toughness is defined for plane ε state
    - To be conservative
    - Near the free surfaces
      - Deformations along thickness release the stress state
      - The SIF is lower
      - Process zone is not negligible (see lecture on NLFM)



⟹ G_c is larger near free surfaces

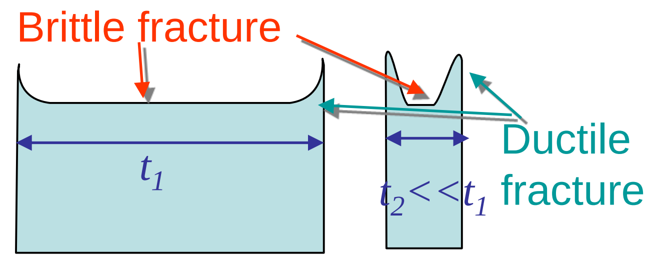
- There is a thickness effect

⟹ a thin specimen is more ductile



$$K_C(t) \simeq K_C(t \to \infty)\left[1 + \frac{1.4}{t^2}\left(\frac{K_C(t\to\infty)}{\sigma_p^0}\right)^4\right]^{\frac{1}{2}}$$

- LEFM validity
  - Plane strain & elastic fracture can be assumed if the process zone is small compared to
    - The specimen
      - Thick specimen to be under plane strain
      - See lecture on NLFM



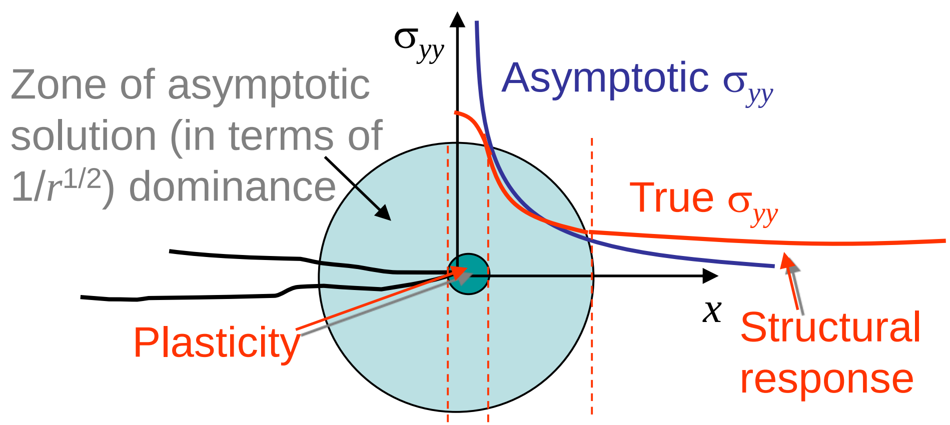
$$\Rightarrow \quad t > 2.5 \left(\frac{K_I}{\sigma_p^0}\right)^2$$

Plane ε

$$\Rightarrow \quad K_C = \sqrt{\frac{E G_C}{1-\nu^2}}$$

- The crack/ligament length
  - Crack large enough
  - See lecture on NLFM

$$\Rightarrow \quad a, W-a > 2.5 \left(\frac{K_I}{\sigma_p^0}\right)^2$$

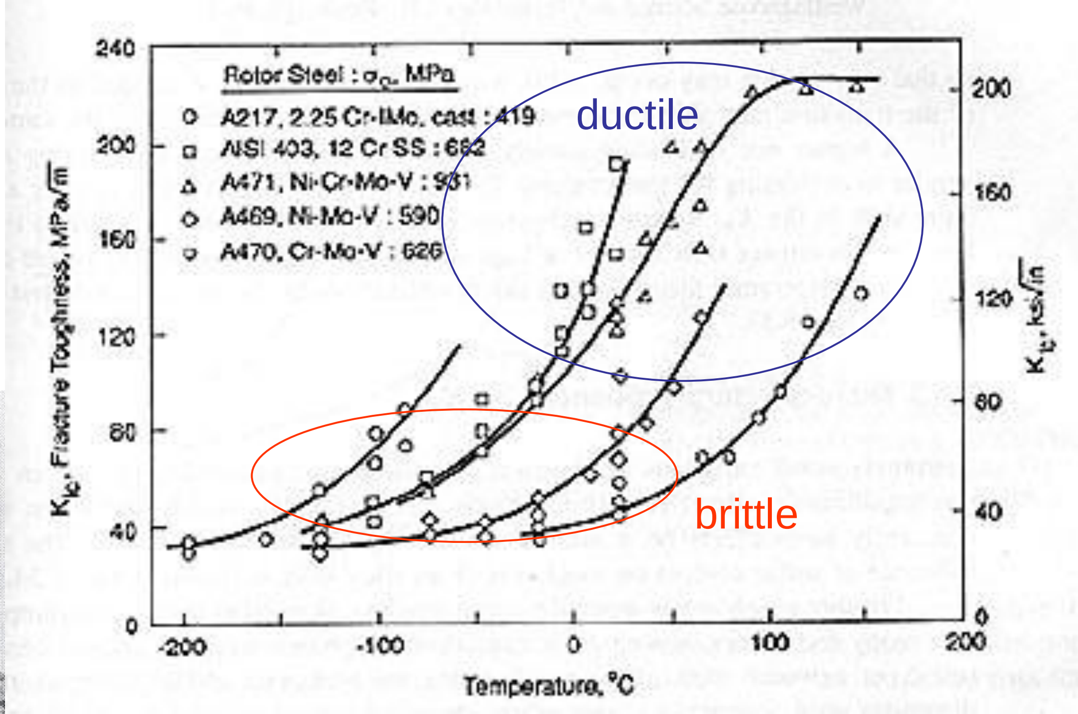


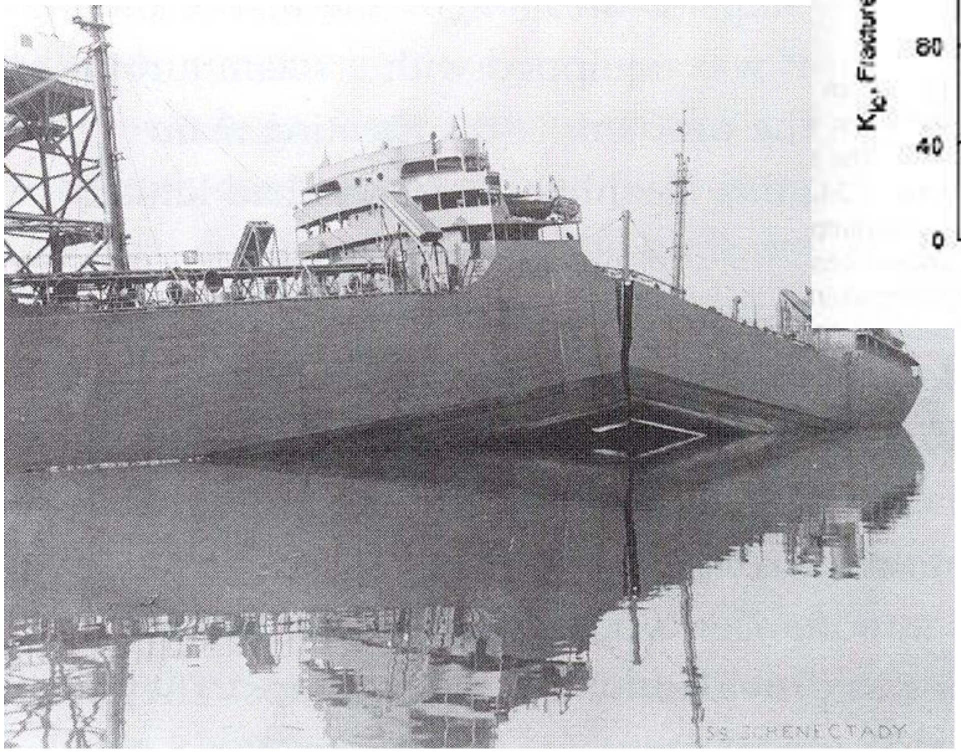
- Crack growth criterion in mode I (3)
  - Toughness & fracture energy of brittle materials

Material	K_C [MPa · m$^{1/2}$]	G_C [J · m^{-2}]
Borosilicate Glass	0.8	9.
Alumina 99% polycrystalline	4.	39.
Zirconia-Toughened Alumina	6.	90.
Yttria Partially Stabilized Zirconia	13.	730.
Aluminum 7075-T6	25.	7800.
AlSiC Metal Matrix Composite	10.	400.
Epoxy	0.4	200.

  - Toughness and fracture energy are related
    - $K_C = \sqrt{\frac{E G_C}{1-\nu^2}}$ since toughness is defined under plane strain condition

- Crack growth criterion in mode I (4)
  - Behavior depends on environmental conditions
    - Example: High strength steel alloys exhibit a DBTT





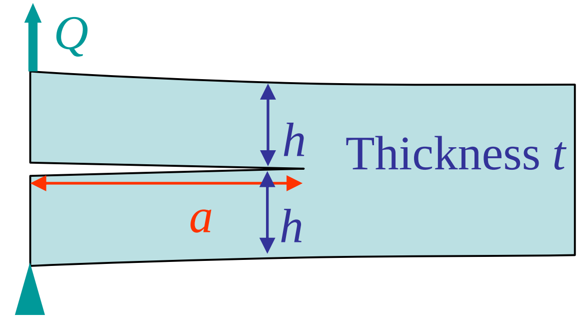
- We know whether the crack grows
- But
  - How fast ?
  - How far ?
  - In which direction ?

- Considering a cracked body with $G \geq G_C$: the crack will grow
  - If G_C is constant: two possible behaviors as the crack grows
    - G decreases ⟹ the crack stops growing (unless the loading increases)
    - G increases ⟹ instability and the specimen fractures
  - If G_C is not constant: the question becomes
    - Which one of G or G_C will grow at the highest rate
  - The stability of a crack growth depends on
    - The geometry
    - The loading
    - The material behavior

Crack grow stability

- Example: Delamination of composites (DCB specimen)
  - Compliance (Linear elasticity)

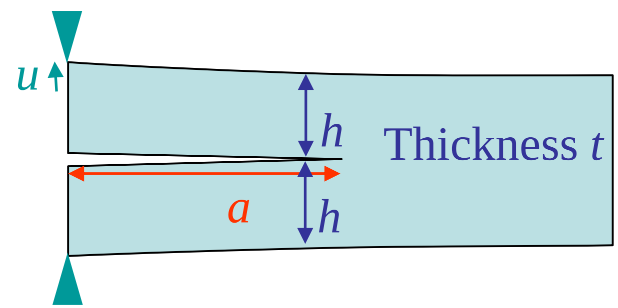
$$C = \frac{u}{Q} = \frac{8a^3}{Eth^3} \Longrightarrow \partial_A C = \frac{1}{t}\partial_a \frac{8a^3}{Eth^3} = \frac{24a^2}{Et^2h^3}$$



  - Prescribed loading $G = \frac{Q^2}{2}\partial_A C = \frac{12Q^2a^2}{Et^2h^3}$
    - As the crack is growing: G increases
    - For a perfectly brittle material G_C is constant

    $\Longrightarrow$ unstable

  - Prescribed displacement



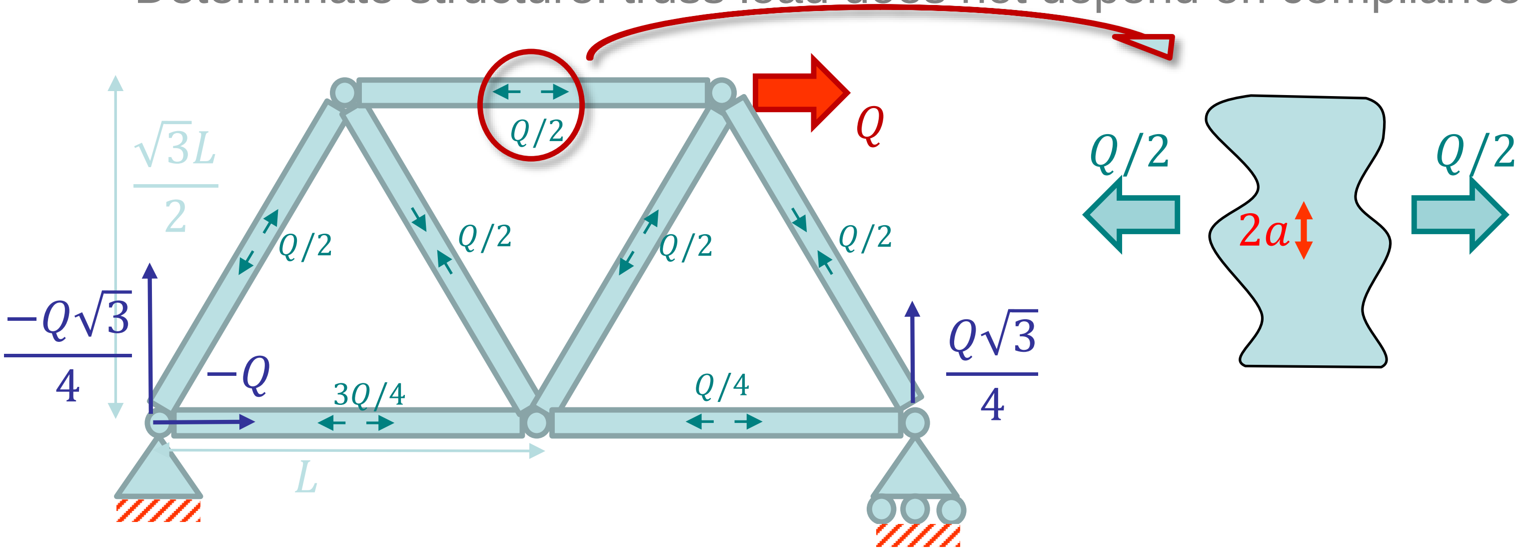
$$G = \frac{u^2}{2C^2}\partial_A C = \frac{3u^2Eh^3}{16a^4}$$

    - As the crack is growing: G decreases
    - For a perfectly brittle material G_C is constant

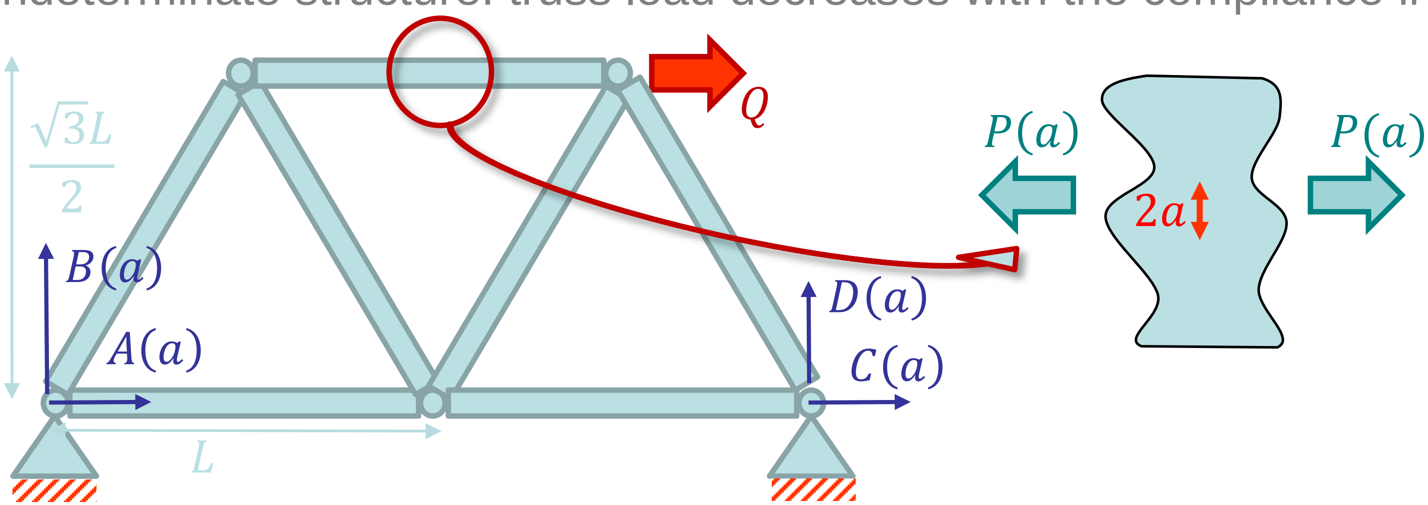
    $\Longrightarrow$ stable

  - Is it a general rule? What happens for a general loading?

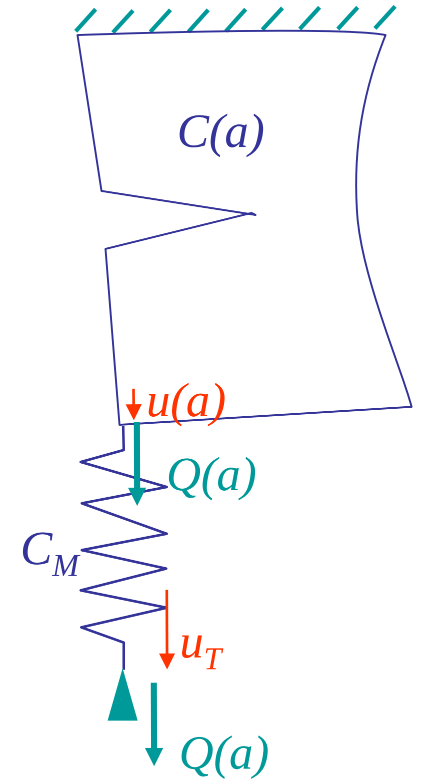
- Statically determinate structure vs. statically indeterminate structure
  - Determinate structure: truss load does not depend on compliance



  - Indeterminate structure: truss load decreases with the compliance increase



- General loading conditions
  - Practically, when a crack propagates
    - Part of the structure becomes more compliant
    - Part of the load is transferred to other parts
    - ⟹ neither fixed load nor fixed displacement
  - This corresponds to a compliant machine loading
    - Spring of compliance C_M
    - Generalized loading $Q(a)$ (spring and cracked body)
    - Generalized displacement at the cracked body $u(a)$



$$\begin{cases} u_T = \left(C\left(a\right) + C_M\right) Q\left(a\right) \\ u = C\left(a\right) Q\left(a\right) = \dfrac{C\left(a\right)}{C\left(a\right) + C_M} u_T \end{cases}$$

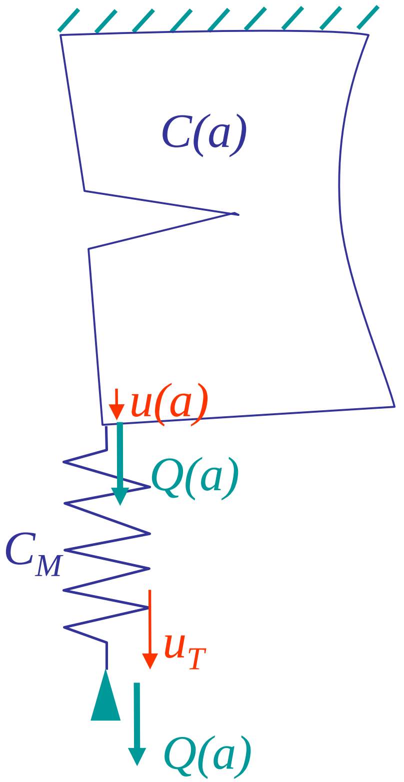
- Internal energy

$$E_{\text{int}} = \frac{QU}{2} + \frac{Q(u_T - u)}{2} = \frac{C(a)Q^2}{2} + \frac{C_M Q^2}{2}$$

$$\Longrightarrow \quad E_{\text{int}} = \frac{(C(a) + C_M)Q^2}{2} = \frac{u_T^2}{2(C(a) + C_M)}$$

- General loading conditions (2)
  - This corresponds to a compliant machine loading
    - Internal energy

$$E_{\text{int}} = \frac{(C(a) + C_M)Q^2}{2} = \frac{u_T^2}{2(C(a) + C_M)}$$



- Prescribed displacement u_T at one extremity of the spring

$$\Rightarrow \quad G = -\partial_A E_{\text{int}}\Big|_{u_T} = \frac{u_T^2}{2(C(a) + C_M)^2}\partial_A C$$

$$\Rightarrow \quad \partial_A G = \frac{u_T^2}{2(C(a) + C_M)^2}\partial_{A^2}^2 C - \frac{u_T^2}{(C(a) + C_M)^3}(\partial_A C)^2$$

- In terms of the load

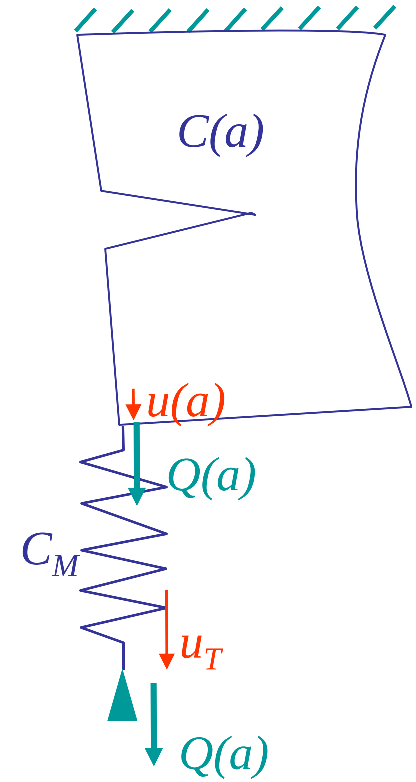
$$u_T = (C(a) + C_M)\,Q(a)$$

$$\Rightarrow \quad \partial_A G = \frac{Q^2}{2}\partial_{A^2}^2 C - \frac{Q^2}{C(a) + C_M}(\partial_A C)^2$$

- General loading conditions (3)
  - Variation of the energy release rate

$$\partial_A G = \frac{Q^2}{2}\partial^2_{A^2}C - \frac{Q^2}{C(a)+C_M}(\partial_A C)^2$$

    - Dead load
      - Spring of infinite compliance ($C_M \rightarrow \infty$)
      - Positive $\partial_A G$ ⟹ unstable
      - Since second term is <0, this is maximum of $\partial_A G$
    - As the spring stiffens (toward a fixed grip)
      - C_M decreases
      - $\partial_A G$ also decreases, which stabilizes the crack growth
      - If $\partial_A G$ becomes <0, the crack is stable

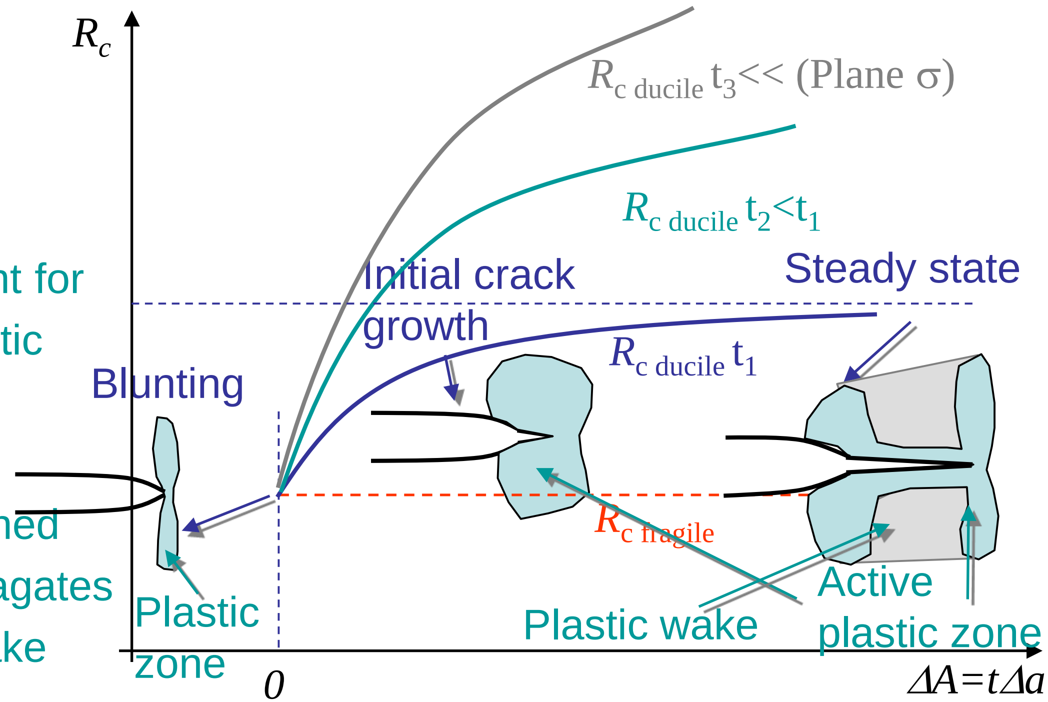


  - A fixed grip is always more stable than a dead load
    - Fixed grip $\partial_A G = -\frac{Q^2}{C}(\partial_A C)^2 + \frac{Q^2}{2}\partial^2_{A^2}C = -\frac{u^2}{C^3}(\partial_A C)^2 + \frac{u^2}{2C^2}\partial^2_{A^2}C$
    - Dead load $\partial_A G = \frac{Q^2}{2}\partial^2_{A^2}C$

Crack grow stability

- Non-perfectly brittle material: Resistance curve
  - For non-perfectly brittle materials G_C depends on the crack surface
    - Therefore G_C will be renamed the resistance $R_c(A)$
  - Elastoplastic behavior
    - Active plastic zone in front of the crack tip
    - Plastic work increases

      ⇒ Resistance increases
    - This effect is more important for thin specimens (elasto-plastic behavior more pronounced under plane σ)
    - A steady state can be reached in which case a crack propagates in a compressive plastic wake

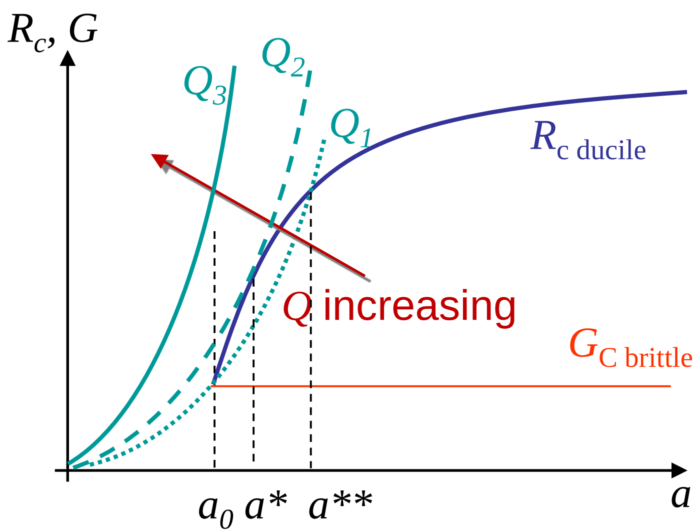


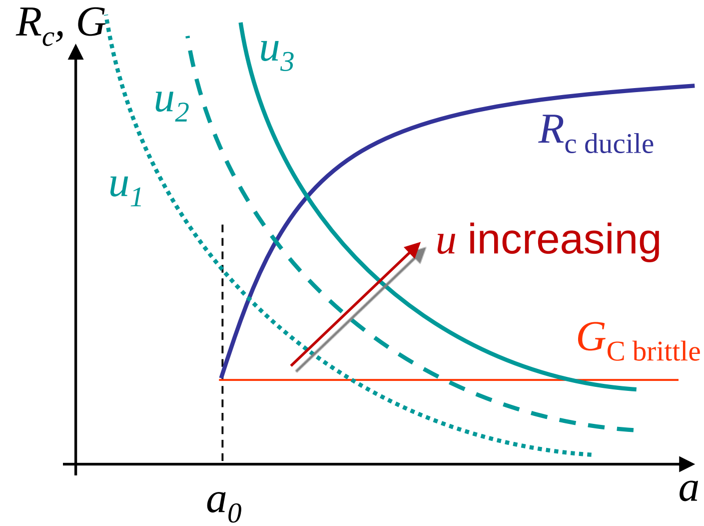
  - Composites
    - As the crack propagates, fibers in the wake tend to close crack tip

      ⇒ more and more energy is required for the crack to grow

- Example: Delamination of composites with initial crack a_0

Dead load $G = \frac{Q^2}{2}\partial_A C = \frac{12Q^2a^2}{Et^2h^3}$ vs **Fixed grip** $G = \frac{u^2}{2C^2}\partial_A C = \frac{3u^2Eh^3}{16a^4}$





- Dead load Q_1
  - Perfectly brittle materials: unstable
  - Ductile materials: stable, but if a is larger than a^{**} it turns unstable
- Dead load $Q_2 > Q_1$
  - This is the limit of stability for ductile materials (always unstable for perfectly brittle material)
- For Q_3: unstable

- Fixed grip u_1 ,u_2 or u_3
  - The crack is stable for any material

Crack grow stability

- Crack growth stability criterion
  - Crack growth criterion is $G \geq G_C$
  - Stability of the crack is reformulated (in 2D)
    - Stable crack growth if $\partial_a G \leq \partial_a R_c$
    - Unstable crack growth if $\partial_a G > \partial_a R_c$

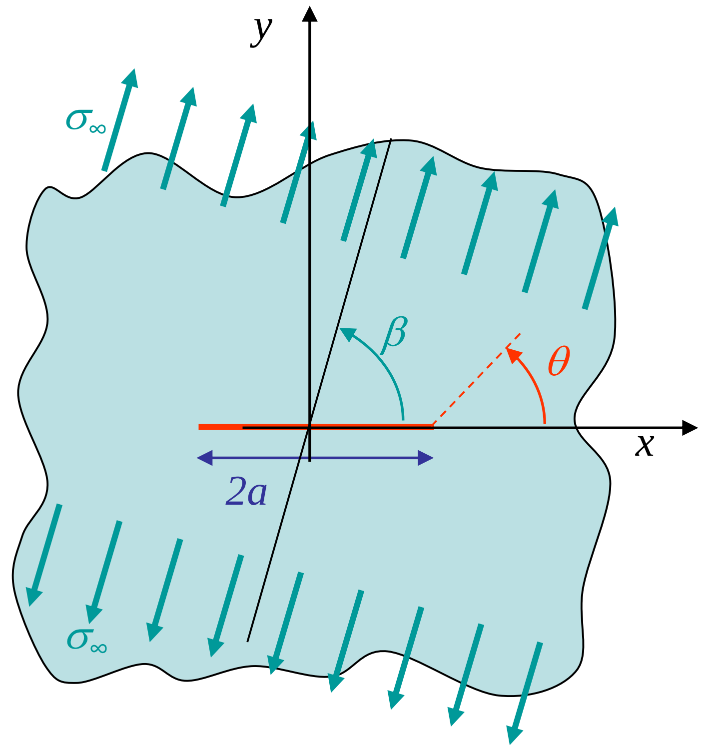
- In which direction will the crack grow?
  - For anisotropic & isotropic material
  - Under mixed mode loading
  - In composites
- Cracks follow the path of least resistance
  - Important to predict this path during the design
  - “Fail safe design”

- In the following we assume homogeneous & isotropic materials under mixed mode loading

- Combination of mode I & mode II loadings
  - Loadings in mode I and mode II
    - Rotation

$$\begin{pmatrix} \boldsymbol{\sigma}_{xx} & \boldsymbol{\sigma}_{xy} \\ \boldsymbol{\sigma}_{xy} & \boldsymbol{\sigma}_{yy} \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}^T \begin{pmatrix} \sigma_\infty & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$$

$$\Rightarrow \begin{cases} \boldsymbol{\sigma}_{yy} = \sigma_\infty \sin^2\beta \\ \boldsymbol{\sigma}_{xy} = \sigma_\infty \sin\beta\cos\beta \\ \boldsymbol{\sigma}_{xx} = \sigma_\infty \cos^2\beta \end{cases}$$



    - Mode I loading (infinite plate):

$$\boldsymbol{\sigma}_{yy} = \sigma_\infty \sin^2\beta \quad \Rightarrow K_I = \sigma_\infty \sin^2\beta\sqrt{a\pi}$$

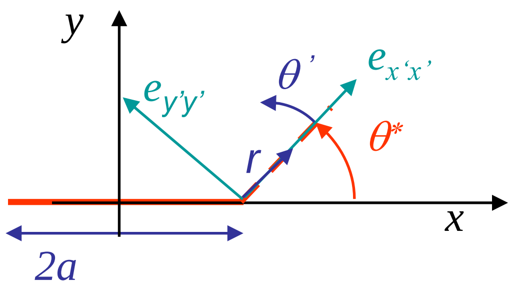
    - Mode II loading (infinite plate):

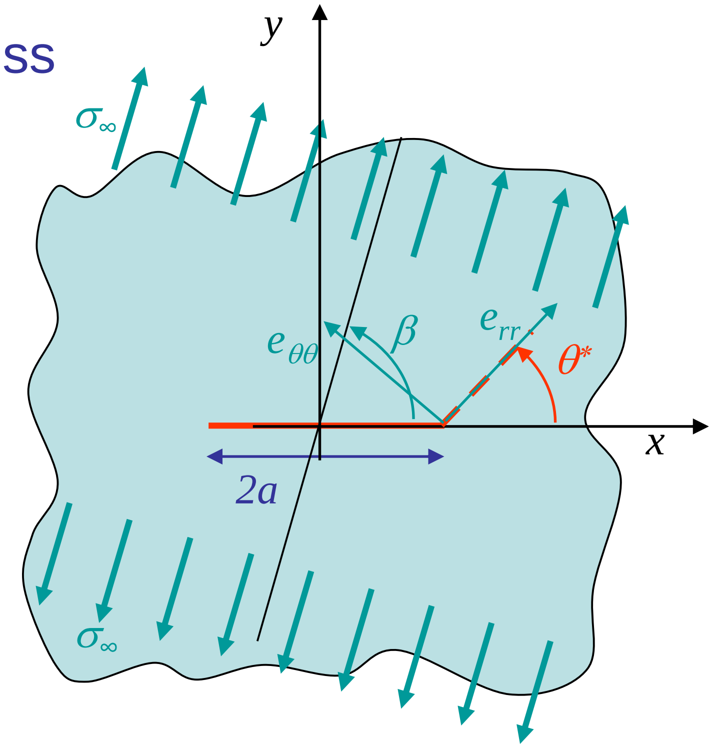
$$\boldsymbol{\sigma}_{xy} = \sigma_\infty \sin\beta\cos\beta \quad \Rightarrow K_{II} = \sigma_\infty \sin\beta\cos\beta\sqrt{a\pi}$$

  - The crack will propagate with a kink angle θ
  - $\theta=0$ only if it corresponds to a weak plane of the material (e.g. delamination)

- Method of the maximum circumferential stress
  - Erdogan & Sih, 1963
  - Assumptions:
    - Direction maximizes mode I SIF in the new frame $\mathbf{e}_{x'x'},\ \mathbf{e}_{y'y'}$

$$K_I' = \lim_{r\to 0} \sqrt{2\pi r}\, \sigma_{y'y'}(r, \theta' = 0)$$





⟹ This corresponds to maximizing hoop stress $\sigma_{\theta\theta}$ in the frame $\mathbf{e}_{rr},\ \mathbf{e}_{\theta\theta}$

⟹ θ^* such that $\left.\partial_\theta \boldsymbol{\sigma}_{\theta\theta}\right|_{\theta^*} = 0$ & $\left.\partial^2_{\theta\theta} \boldsymbol{\sigma}_{\theta\theta}\right|_{\theta^*} < 0$

- Crack grows if new mode I SIF K_I' reaches the toughness

⟹ $$K_I' = \lim_{r\to 0} \sqrt{2\pi r}\, \sigma_{\theta\theta}(r, \theta^*) \geq K_C$$

- Method of the maximum circumferential stress (2)
  - Mode I

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] \\ \boldsymbol{\sigma}_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] \\ \boldsymbol{\sigma}_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} \end{cases}$$

  - Mode II

$$\begin{cases} \boldsymbol{\sigma}_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left[2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right] \\ \boldsymbol{\sigma}_{yy} = \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \\ \boldsymbol{\sigma}_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] \end{cases}$$

  - Mixed mode I & mode II loadings

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \frac{1}{\sqrt{2\pi r}}\left[K_I\cos\frac{\theta}{2} - \frac{K_I}{2}\sin\theta\sin\frac{3\theta}{2} - 2K_{II}\sin\frac{\theta}{2} - \frac{K_{II}}{2}\sin\theta\cos\frac{3\theta}{2}\right] \\ \boldsymbol{\sigma}_{yy} = \frac{1}{\sqrt{2\pi r}}\left[K_I\cos\frac{\theta}{2} + \frac{K_I}{2}\sin\theta\sin\frac{3\theta}{2} + \frac{K_{II}}{2}\sin\theta\cos\frac{3\theta}{2}\right] \\ \boldsymbol{\sigma}_{xy} = \frac{1}{\sqrt{2\pi r}}\left[\frac{K_I}{2}\sin\theta\cos\frac{3\theta}{2} + K_{II}\cos\frac{\theta}{2} - \frac{K_{II}}{2}\sin\theta\sin\frac{3\theta}{2}\right] \end{cases}$$

- Method of the maximum circumferential stress (3)
  - Mixed mode I & mode II loadings in new frame

$$\boldsymbol{\sigma}_{\theta\theta} = \boldsymbol{\sigma}_{yy}\cos^2\theta + \boldsymbol{\sigma}_{xx}\sin^2\theta - 2\boldsymbol{\sigma}_{xy}\sin\theta\cos\theta$$

$$\boldsymbol{\sigma}_{\theta\theta} = \frac{K_I}{2\sqrt{2\pi r}}\left[2\cos\frac{\theta}{2} + \sin\theta\cos 2\theta\sin\frac{3\theta}{2} - \sin 2\theta\sin\theta\cos\frac{3\theta}{2}\right] + \frac{K_{II}}{2\sqrt{2\pi r}}\left[\sin\theta\cos 2\theta\cos\frac{3\theta}{2} - 4\sin^2\theta\sin\frac{\theta}{2} - 2\sin 2\theta\cos\frac{\theta}{2} + \sin 2\theta\sin\theta\sin\frac{3\theta}{2}\right]$$

$$\boldsymbol{\sigma}_{\theta\theta} = \frac{K_I}{2\sqrt{2\pi r}}\left[2\cos\frac{\theta}{2} - \sin\theta\sin\frac{\theta}{2}\right] + \frac{K_{II}}{2\sqrt{2\pi r}}\left[\sin\theta\cos\frac{\theta}{2} - 4\sin^2\theta\sin\frac{\theta}{2} - 2\sin 2\theta\cos\frac{\theta}{2}\right]$$

$$\boldsymbol{\sigma}_{\theta\theta} = \frac{K_I}{2\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[2 - 2\sin^2\frac{\theta}{2}\right] + \frac{K_{II}}{2\sqrt{2\pi r}}\left[\sin\theta\cos\frac{\theta}{2} - 4\sin\theta\cos\frac{\theta}{2}\right]$$

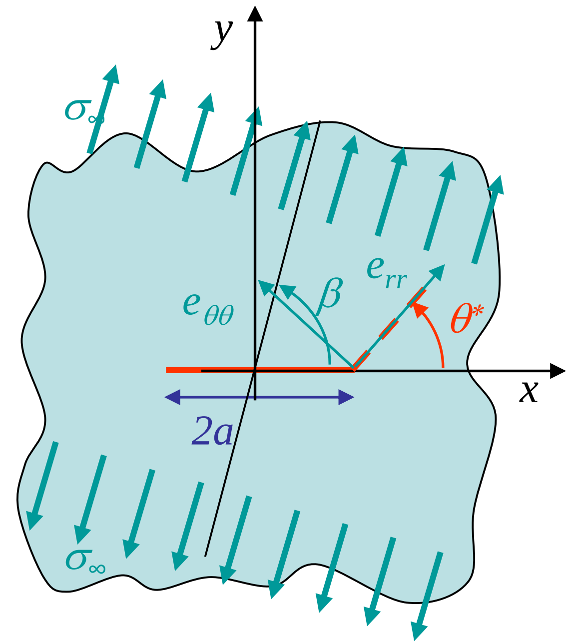
$$\boxed{\boldsymbol{\sigma}_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\cos^3\frac{\theta}{2} - \frac{3K_{II}}{2\sqrt{2\pi r}}\sin\theta\cos\frac{\theta}{2}}$$

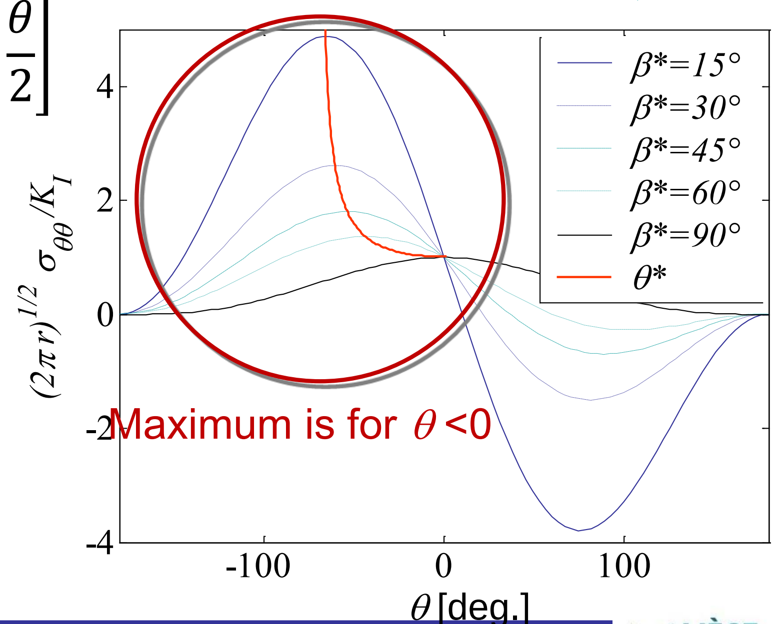
Mixed mode fracture

- Method of the maximum circumferential stress (4)
  - Maximum hoop stress
    - $$\boldsymbol{\sigma}_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\cos^3\frac{\theta}{2} - \frac{3K_{II}}{2\sqrt{2\pi r}}\sin\theta\cos\frac{\theta}{2}$$
    - Direction of loading defined by

      $$\cot\beta^* = \frac{K_{II}}{K_I}$$

      In case of the infinite plate $\beta^* = \beta$
    - $$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\cos^3\frac{\theta}{2} - \frac{3\cot\beta^*}{2}\sin\theta\cos\frac{\theta}{2}\right]$$
  - Particular cases
    - For $\beta^*=90°$: $\theta^*=0$
    - For $\beta^*>0$: $\theta^*<0$





- Method of the maximum circumferential stress (5)
  - Kink angle obtained when hoop stress is maximum
    - $\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\cos^3\frac{\theta}{2} - \frac{3\cot\beta^*}{2}\sin\theta\cos\frac{\theta}{2}\right]$

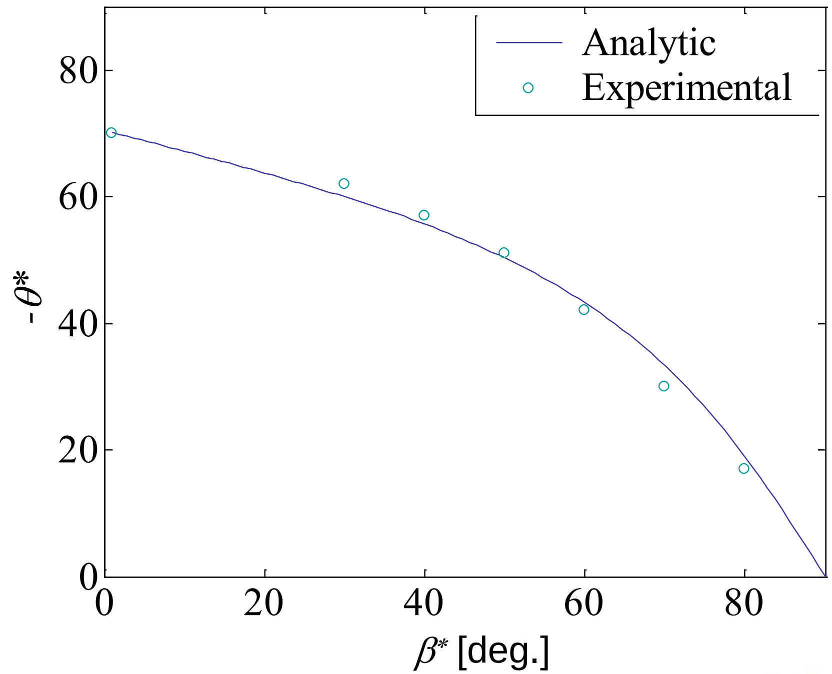
$$\Rightarrow \frac{\sqrt{2\pi r}}{K_I}\partial_\theta\sigma_{\theta\theta} = -\frac{3}{2}\cos^2\frac{\theta}{2}\sin\frac{\theta}{2} - \frac{3\cot\beta^*}{2}\cos^3\frac{\theta}{2} + 3\cot\beta^*\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\Rightarrow \frac{\sqrt{2\pi r}}{K_I}\partial_\theta\sigma_{\theta\theta} = \frac{3}{2}\cos^3\frac{\theta}{2}\left[-\tan\frac{\theta}{2} - \cot\beta^*\left(1 - 2\tan^2\frac{\theta}{2}\right)\right]$$

    - Maximum hoop stress for θ^* with

$$\boxed{2\tan\frac{\theta^*}{2} - \cot\frac{\theta^*}{2} = \tan\beta^*}$$

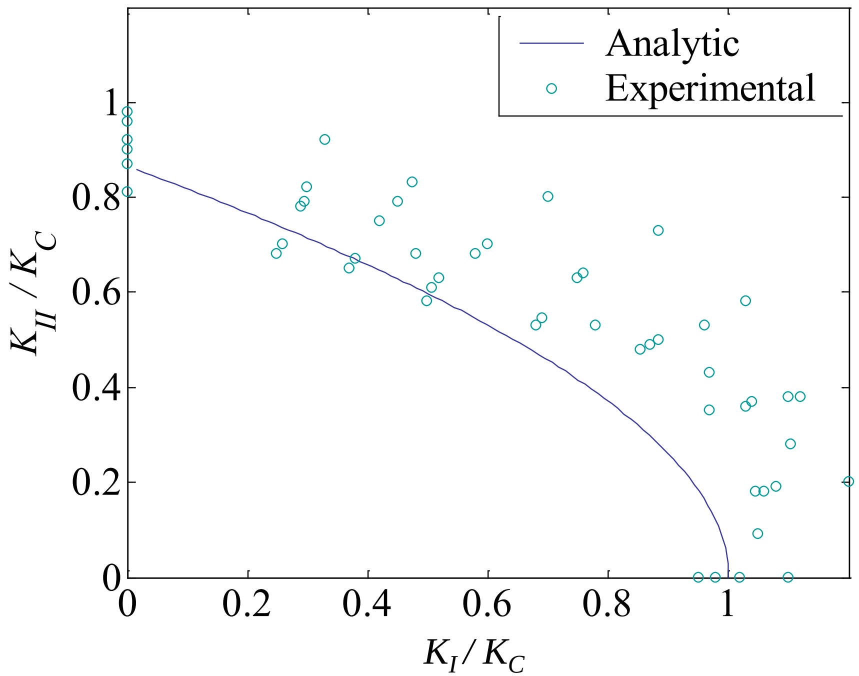
  - Prediction in good agreement with experimental results
    - (Erdogan and Sih, 1963)



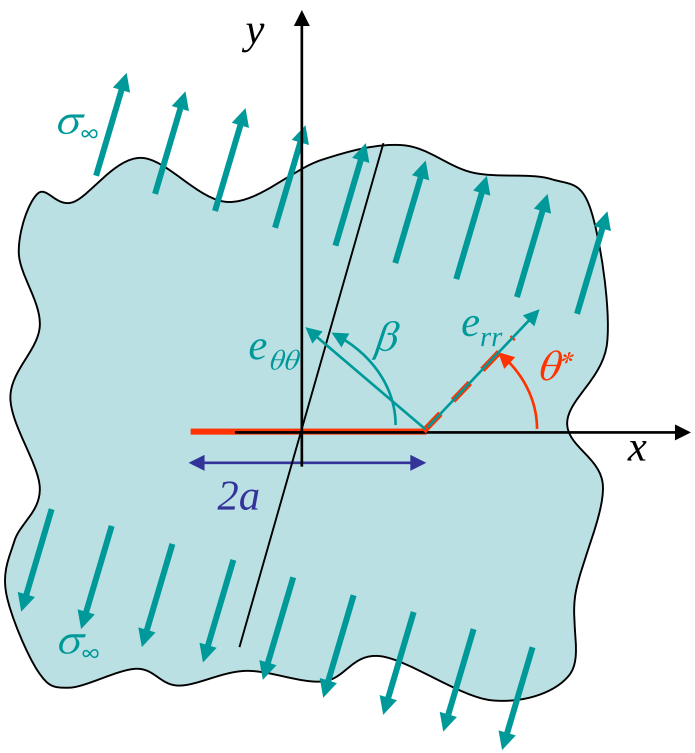
- Method of the maximum circumferential stress (6)
  - Mode I SIF associated to maximum hoop-stress
    - New mode I SIF $K_I' = \lim_{r\to 0} \sqrt{2\pi r}\, \sigma_{\theta\theta}(r,\theta^*) = K_I \cos^3\frac{\theta^*}{2} - \frac{3K_{II}}{2}\sin\theta^* \cos\frac{\theta^*}{2}$

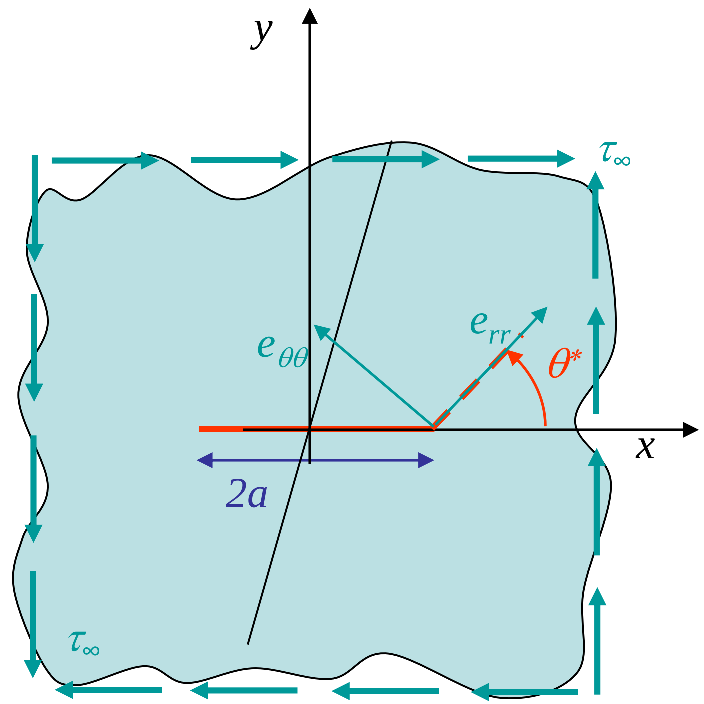
      ⇒ at failure $K_C = \lim_{r\to 0} \sqrt{2\pi r}\, \sigma_{\theta\theta}(r,\theta^*) = K_I \cos^3\frac{\theta^*}{2} - \frac{3K_{II}}{2}\sin\theta^* \cos\frac{\theta^*}{2}$

      with kink angle from $2\tan\frac{\theta^*}{2} - \cot\frac{\theta^*}{2} = \tan\beta^* = \frac{K_I}{K_{II}}$
  - Failure envelope
    - Resolution of this linear system of 2 equations with 2 unknowns
    - Toughness tests for different loading directions
    - Maximum circumferential stress theory is conservative

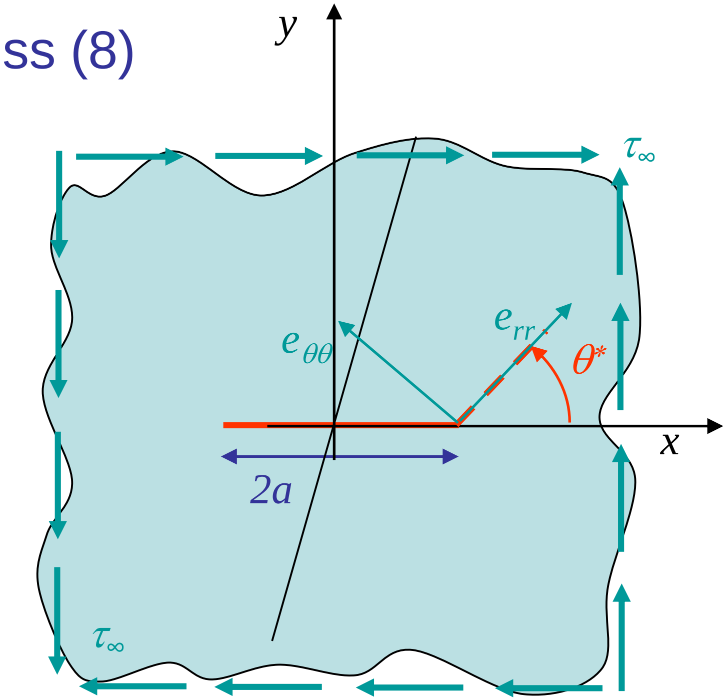


- Method of the maximum circumferential stress (7)
  - Pure mode II theory ($\beta^* = 0$, $K_I = 0$)
    - For pure mode II the case of a plate in tension is meaningless $\beta^* \neq \beta$





- Method of the maximum circumferential stress (8)
  - Pure mode II theory ($\beta^* = 0$, $K_I = 0$)



- Maximum hoop stress direction

$$2\tan\frac{\theta^*}{2} - \cot\frac{\theta^*}{2} = \tan\beta^* = 0$$

$$\Rightarrow \tan^2\frac{\theta^*}{2} = \frac{1}{2}$$

$$\Rightarrow \tan\frac{\theta^*}{2} = \pm\frac{\sqrt{2}}{2}$$

- Negative kink angle $\Rightarrow$ $\theta^*|_{\beta*=0} \sim -70.53°$

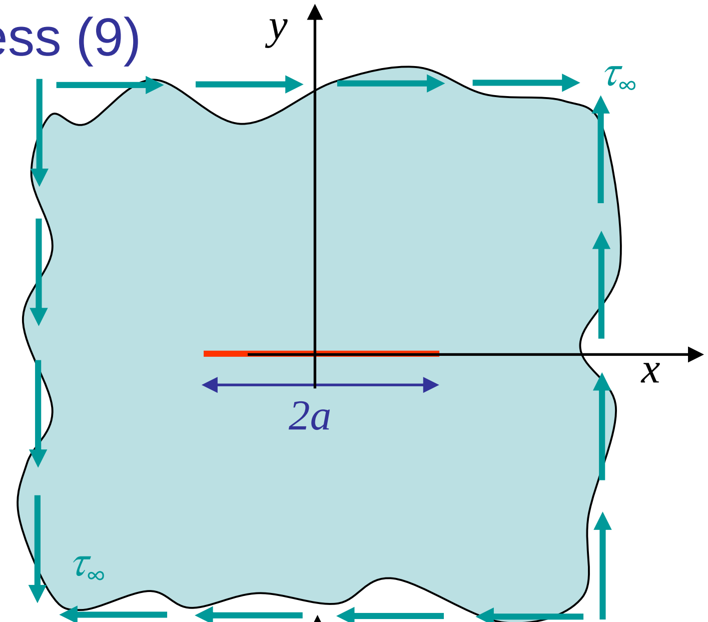
- New mode I SIF after kinking

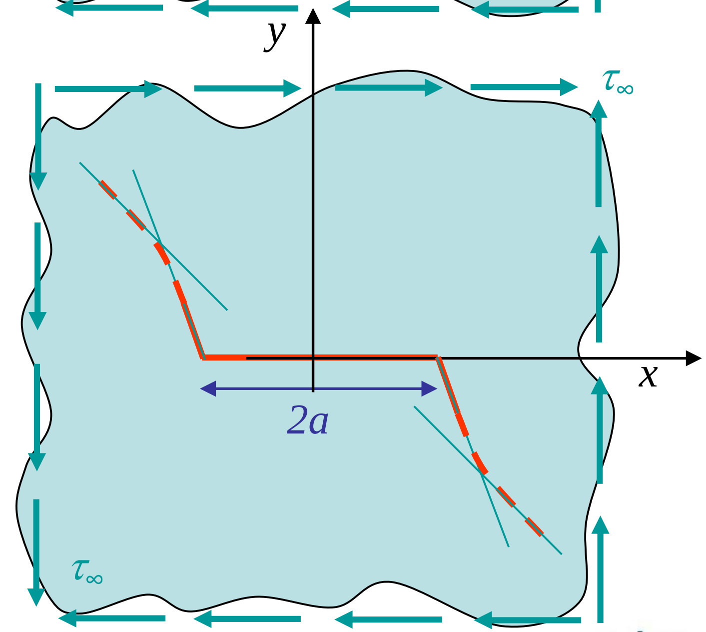
$$K_I' = \lim_{r\to 0}\sqrt{2\pi}\,\sigma_{\theta\theta}(r,\theta^*) = K_I\cos^3\frac{\theta^*}{2} - \frac{3K_{II}}{2}\sin\theta^*\ \cos\frac{\theta^*}{2}$$

$$\Rightarrow K_I' = -\frac{3K_{II}}{2}\sin\theta^*\cos\frac{\theta^*}{2} = -3K_{II}\sin\frac{\theta^*}{2}\cos^2\frac{\theta^*}{2} = -\frac{3K_{II}\tan\frac{\theta^*}{2}}{\left(1+\tan^2\frac{\theta^*}{2}\right)^{\frac{3}{2}}}$$

- At failure $K_I' = K_C = -\frac{3K_{II}\tan\frac{\theta^*}{2}}{\left(1+\tan^2\frac{\theta^*}{2}\right)^{\frac{3}{2}}}$ $\Rightarrow$ $\frac{K_{II}}{K_C} = -\frac{\left(1+\tan^2\frac{\theta^*}{2}\right)^{\frac{3}{2}}}{3\tan\frac{\theta^*}{2}} = \sqrt{\frac{3}{4}}$

- Method of the maximum circumferential stress (9)
  - Pure mode II: plate under shearing
    - Circumferential stress:
      - $\boldsymbol{\sigma}_{\theta\theta} = -\tau_{\infty} \sin 2\theta$
      - Maximum for θ=-45°
    - Theory predicts a kink angle $\theta^*|_{\beta*=0}$ ~ -70.53°
    - Why is there a difference?
      - Due to the stress field at crack tip the maximum circumferential stress is at -70.53° ⇒ crack will grow with this angle
      - As it grows, the crack will not remain under pure mode II loading
      - Crack will turn until growing with a -45-degree angle





- Method of the maximum circumferential stress (10)
  - Observation
    - For a plate in shearing the kink |angle| decreases with the propagation
    - What is the general rule?
  - Second order theory
    - Full field solution

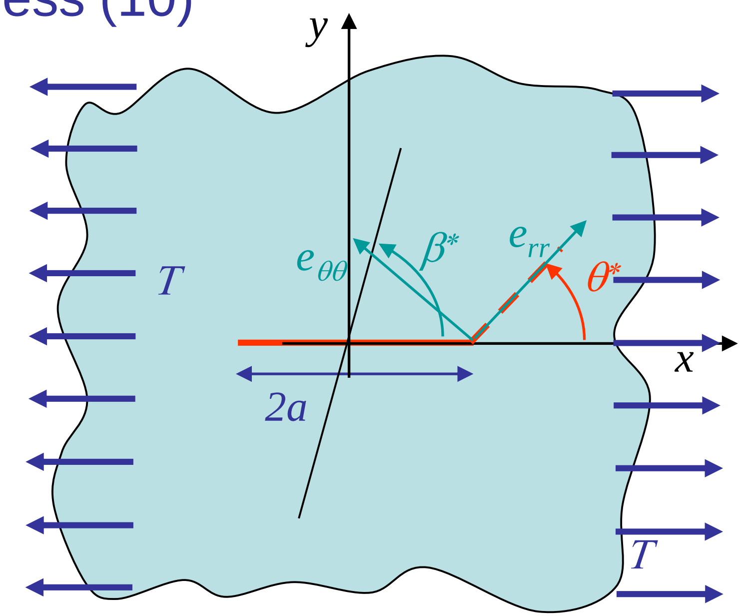
$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] +$$

$$C(\theta)r^0 + D(\theta)\sqrt{r}\ \ldots$$

    - Hoop stress at distance r_c actually reads

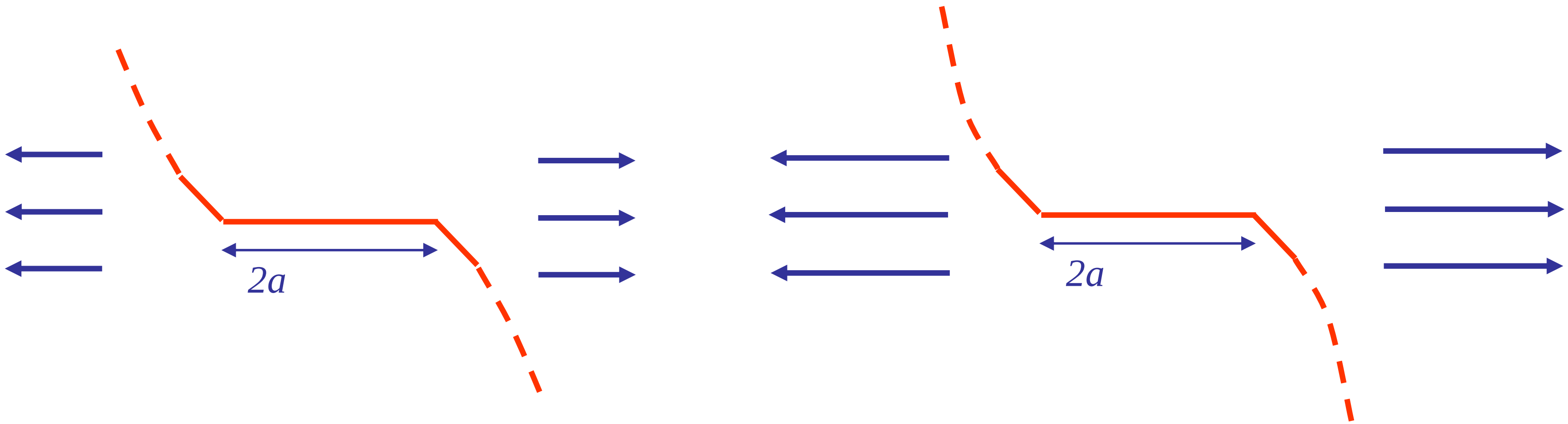
$$\sigma_{\theta\theta}(r_c) = \frac{K_I}{\sqrt{2\pi r_c}}\left[\cos^3\frac{\theta}{2} - \frac{3\cot\beta^*}{2}\sin\theta\cos\frac{\theta}{2}\right] + T\sin^2\theta + \cdots.$$

    - Kink angle θ^* which maximizes the hoop stress depends on T
      - $\theta^*(\beta^*, T, r_c, K_I)$ depends on stress field at r_c
      - r_c determined by experiment (Aluminum alloy: r_c ~1.5 mm)



- Method of the maximum circumferential stress (11)
  - Second order theory (2)
    - Results obtained numerically can be summarized as: after initial kinking
      - For compression (T < 0) or low traction: crack will tend to be aligned with T

      - For traction of the order of $K_I/r_c^{1/2}$ or larger: there is bifurcation



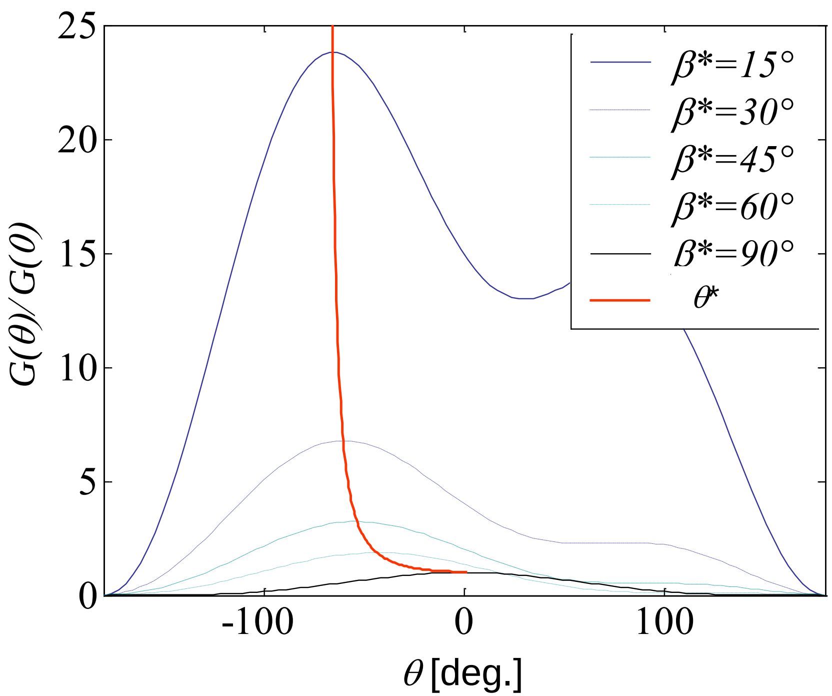
- Method of the maximum energy release rate
  - In thermodynamics: equilibrium corresponds to the lowest potential energy
    - So the crack will propagate to minimize the potential energy
    - Since the energy release rate is defined by $G = -\partial_A \left(E_{\text{int}} - W_{\text{ext}}\right) = -\partial_A \Pi_T$ the crack will grow in the direction maximizing G
  - Energy release rate
    - K_i: SIFs at the crack tip before kink propagation
    - K_I': SIFs at extremity of a kink of infinitesimal length and angle θ

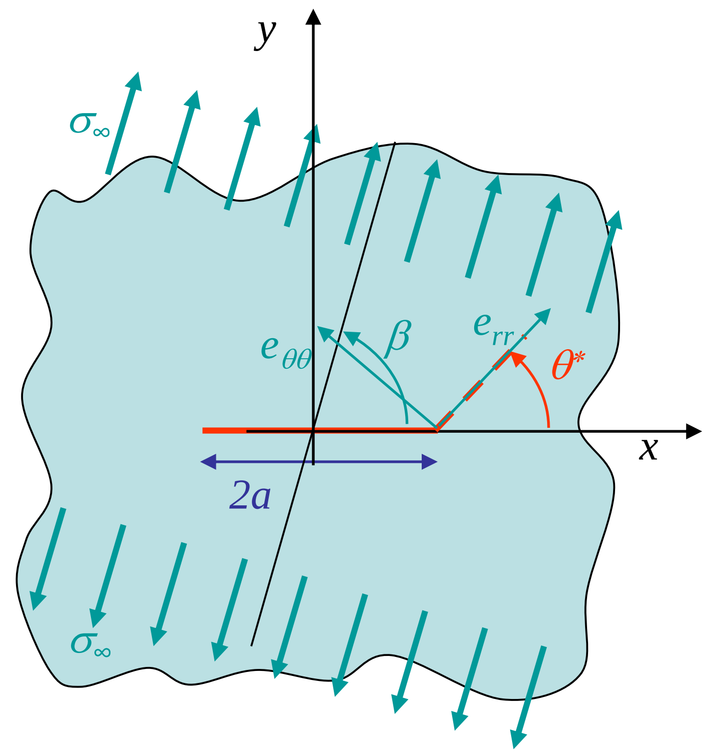
    - $K_I'(\theta)$ can be deduced from K_i: (Cotterell & Rice, 1980)

$$\begin{pmatrix} K_I'(\theta) \\ K_{II}'(\theta) \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3\cos\frac{\theta}{2} + \cos\frac{3\theta}{2} & -3\sin\frac{\theta}{2} - 3\sin\frac{3\theta}{2} \\ \sin\frac{\theta}{2} + \sin\frac{3\theta}{2} & \cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2} \end{pmatrix} \begin{pmatrix} K_I \\ K_{II} \end{pmatrix}$$

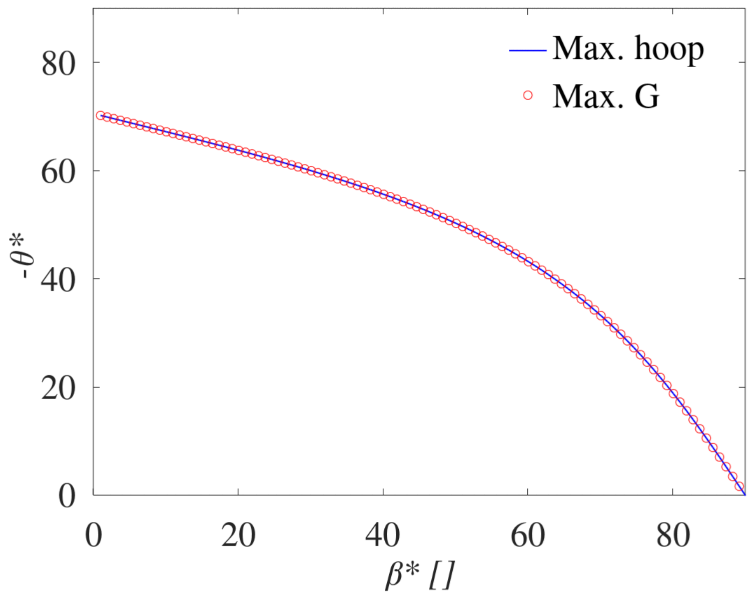
      - First order accurate in θ
    - Since the kink grows straight ahead: $G(\theta) = \dfrac{{K_I'}^2(\theta) + {K_{II}'}^2(\theta)}{E'} = g(K_I, K_{II}, \theta)$

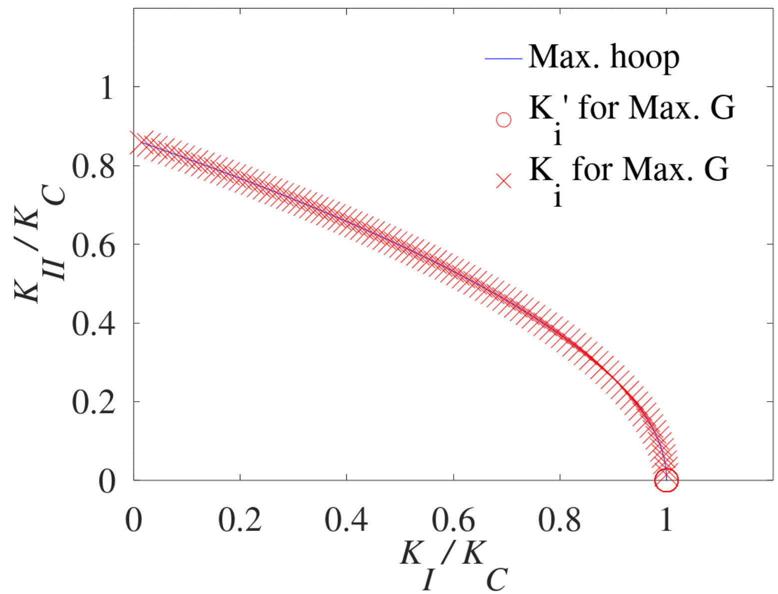
- Method of the maximum energy release rate (2)
  - Energy release rate fro mixed mode loading
    - $G(\theta,\beta^*) = \dfrac{K_I'^2(\theta) + K_{II}'^2(\theta)}{E'} = g(K_I, K_{II}, \theta)$ with $\cot\beta^* = \dfrac{K_{II}}{K_I}$
    - Kinking such that $\partial_\theta G(\theta)\big|_{\theta^*} = 0$ & $\partial^2_{\theta\theta} G(\theta)\big|_{\theta^*} < 0$
    - Propagation if $G(\theta^*) \geq G_C$





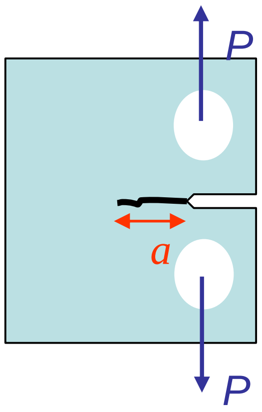
- Method of the maximum energy release rate (3)
  - Comparison against the method of maximum hoop stress





  - Remember we used a formula first order accurate in θ
    - < 5% error for θ < 40°
    - ~ 5% error on K_I for θ > 40° (Cotterell & Rice, 1980)
  - For the method of maximum energy release rate:
    - Before the crack propagates: K_I & K_{II} can be ≠ 0
    - After kinking: K'_{II}= 0 $\Longrightarrow$ the crack propagates under pure mode I

- Under cyclic loading a crack can
  - Be initiated for loadings with $\sigma < \sigma_p^0$
  - Propagate for $K_i < K_C$



- Total life approaches
  - Unable to account for inherent defects
    - Example: design of the De Havilland Comet using total life approach. Defects resulting from punched riveting of square windows caused failure of aircrafts
  - Unable to predict crack propagation

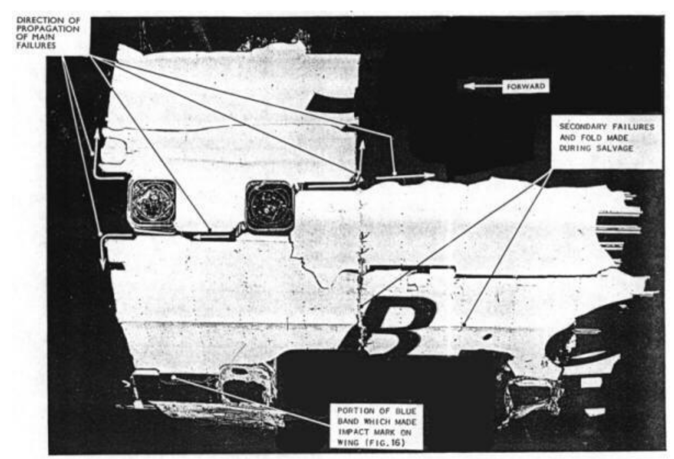
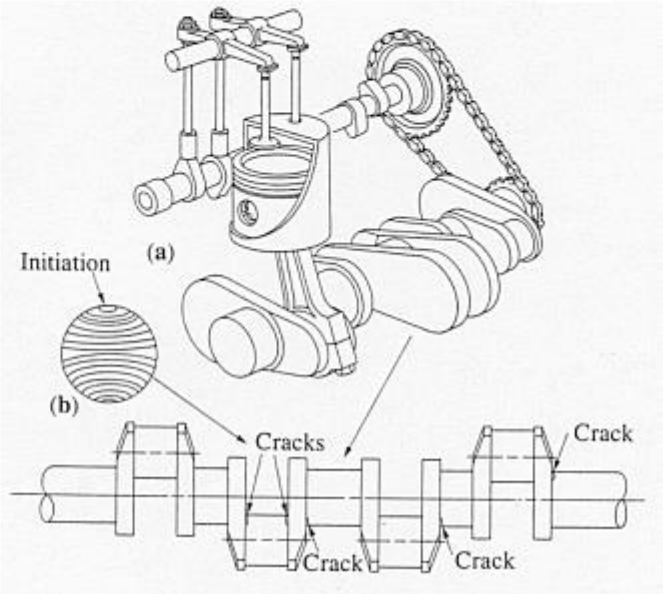
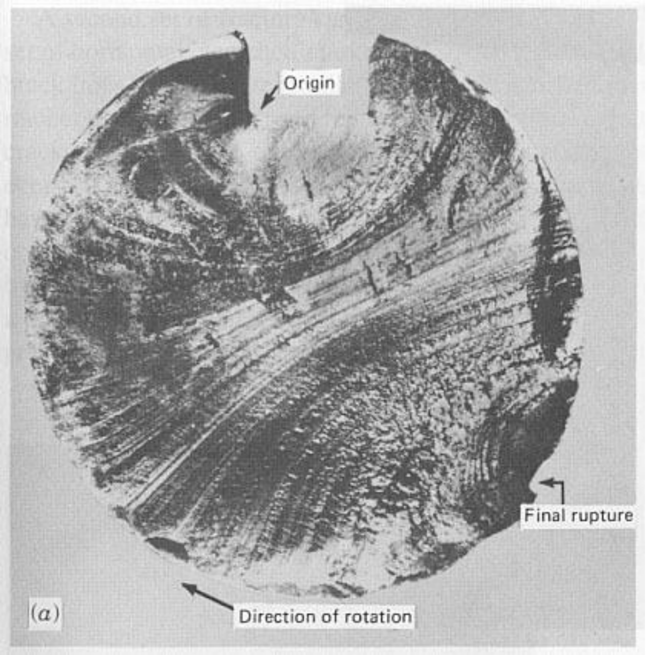


FIG. 12. PHOTOGRAPH OF WRECKAGE AROUND ADF AERIAL WINDOWS—G-ALYP.

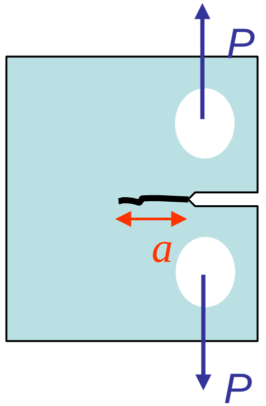
- Microscopic observations for cycling loading
  - I. Crack initiated at stress concentrations (nucleation)
  - II. Crack growth resulting into surface striations
  - III. Failure of the structure when the crack reaches a critical size
  - Example of a crank axis

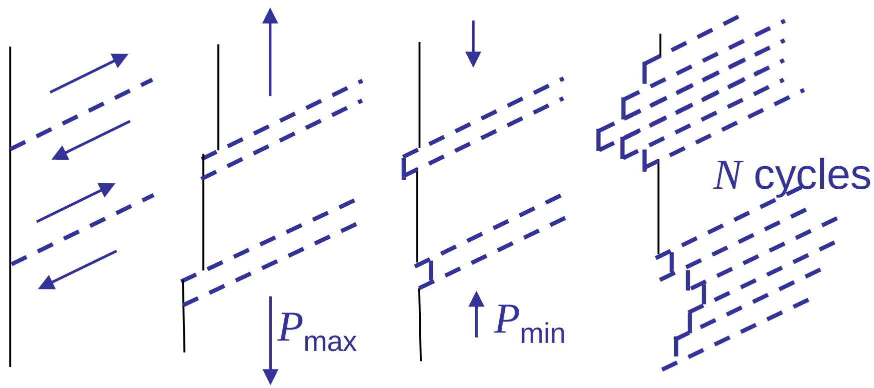


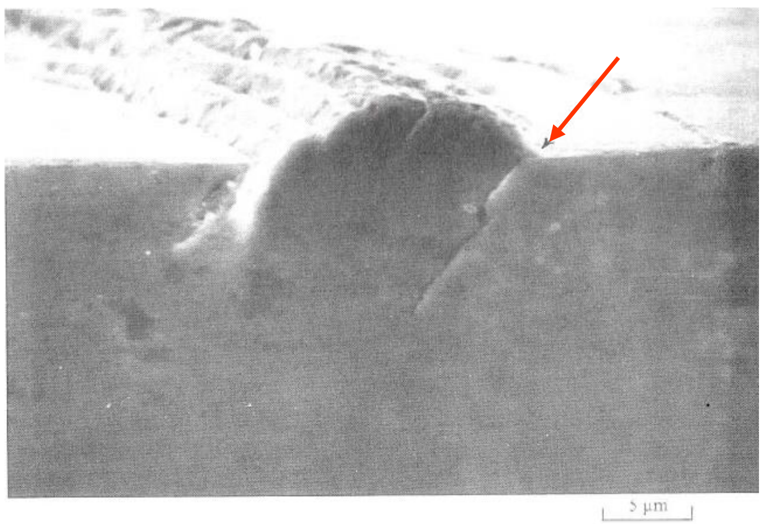


- Development of damage tolerant design
  - Assume cracks are present from the beginning of service
  - Predict crack growth and end of life

- I. Crack initiation
  - Nucleation: cracks initiated for $K << K_C$
    - Surface: deformations result from dislocations motion along slip planes

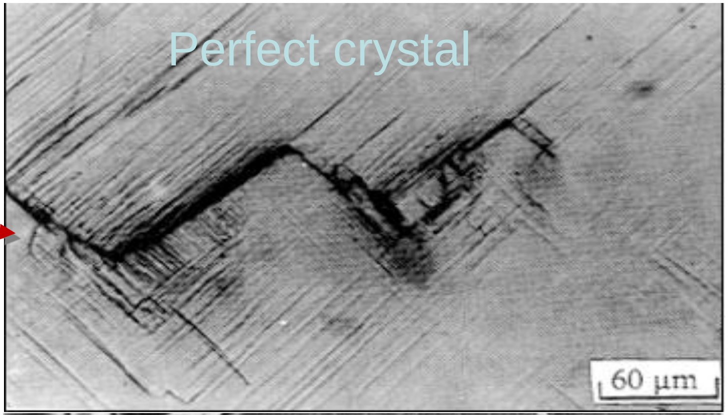


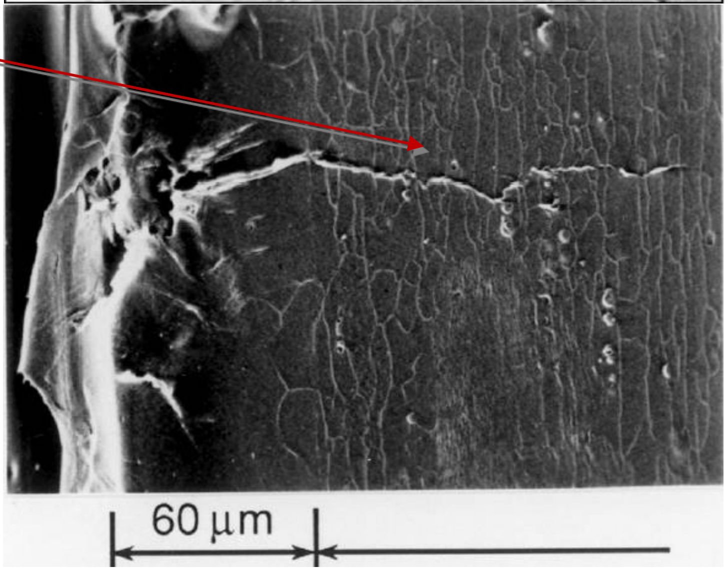


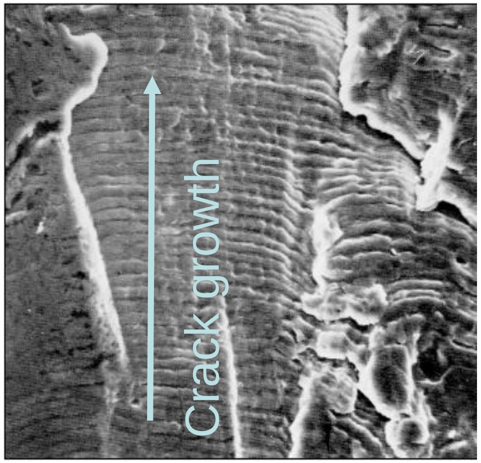


    - Can also happen around a bulk defect

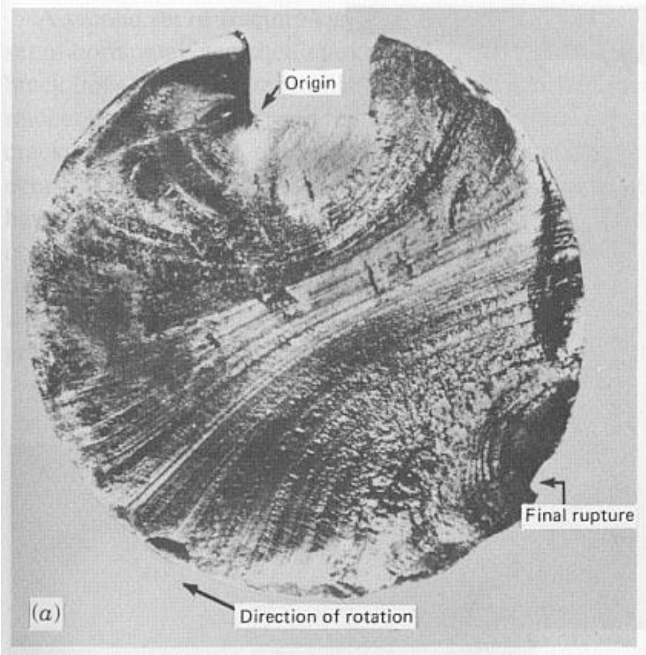
- II. Crack growth
  - Stage I fatigue crack growth:
    - Along a slip plane
  - Stage II fatigue crack growth:
    - Across several grains
      - Along a slip plane in each grain
      - Straight ahead macroscopically
    - Striation of the failure surface: corresponds to the cycles



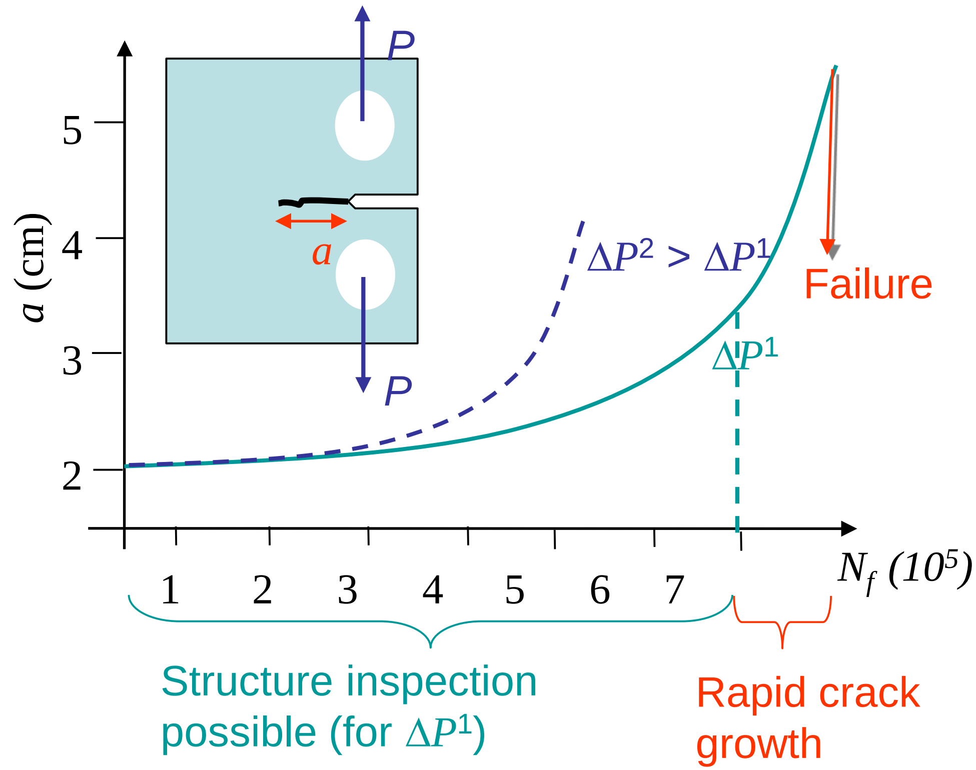




- III. Structure failure
  - As the crack growth K tends toward K_C
  - For a critical size of the crack there is a static failure



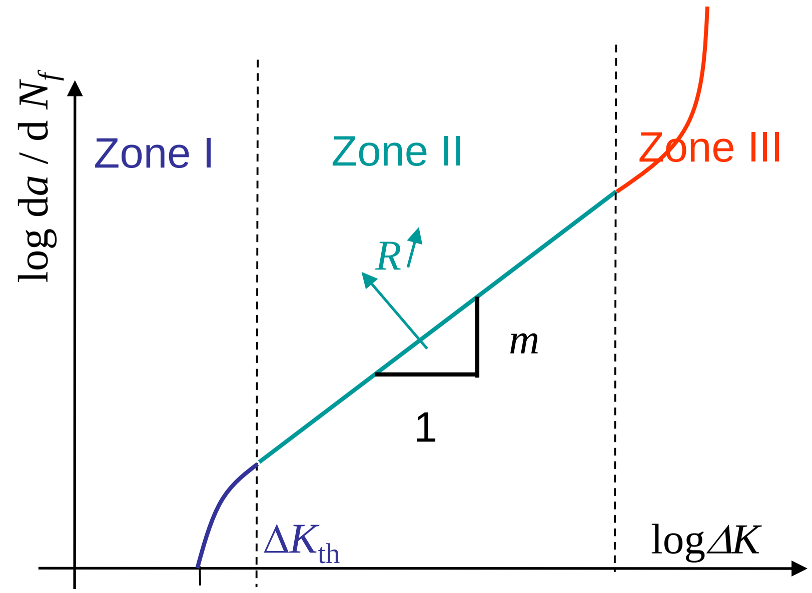
- Crack growth rate
  - Tests: conditioning parameters
    - ΔP &
    - P_{min} / P_{max}
  - SSY assumption
    - Fatigue occurs for $K_i < K_C$
    - SSY usually satisfied, at least during crack growth stage II
  - Since SSY assumption holds,
    - Fatigue failure can be described solely by
      - $\Delta K = K_{\text{max}} - K_{\text{min}}$ &
      - $R = K_{\text{min}}/K_{\text{max}}$



Interval of striations

$$\boxed{\frac{da}{dN_f} = f\left(\Delta K, R\right)}$$

- Crack growth rate (2)
  - Zone I: Stage I growth
    - Existence of a threshold
    - Crack grows only if $\Delta K > \Delta K_{th}(R)$
  - Zone II: Stage II growth
    - 1963, Paris-Erdogan
      - $$\frac{da}{dN_f} = C\Delta K^m$$
      - Depends on the geometry, the loading, the frequency
      - Be careful: K depends on a ⟹ integration needed to get $a(N_f)$
      - Infinite plate, mode I : $K_I = \sigma_\infty \sqrt{\pi a}$ ⟹ $\Delta K = (\sigma_{\infty,\max} - \sigma_{\infty,\min})\sqrt{\pi a}$
  - Zone III: Rapid crack growth
    - Until failure
    - Static behavior (cleavage) due to the effect of $K_{max}(a)$
    - There is failure once a_f is reached, with a_f such that $K_{max}(a_f) = K_c$

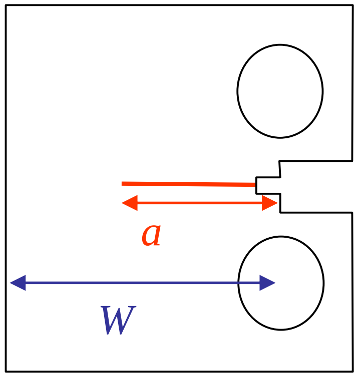


Fatigue failure

- Experimental results
  - Norm: ASTM E647
  - Parameters in Paris law

Material	ΔK_{th} [MPa · m^{½}]	m [-]	C [m (MPa · m$^{1/2}$)$^{-m}$]
Mild steel	3.2-6.6	3.3	$0.24 \cdot 10^{-11}$
Structural steel	2.0-5.0	3.85-4.2	$0.07\text{-}0.11 \cdot 10^{-11}$
Structural steel is sea water	1.0-1.5	3.3	$1.6 \cdot 10^{-11}$
Aluminum	1.0-2.0	2.9	$4.56 \cdot 10^{-11}$
Aluminum alloy	1.0-2.0	2.6-2.9	$3\text{-}19 \cdot 10^{-11}$
Copper	1.8-2.8	3.9	$0.34 \cdot 10^{-11}$
Titanium alloy (6Al-4V, R=0.1)	2.0-3.0	3.22	$1 \cdot 10^{-11}$

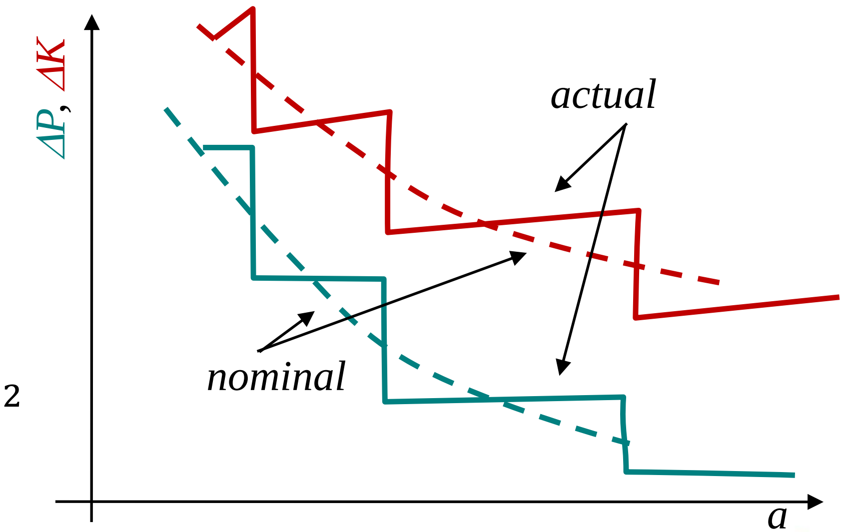
- Experimental fatigue curve
  - Specimen
    - Samples: CTS
    - Fatigue pre-cracked



  - Fatigue curves
    - SIF evaluated following the norm

$$\Delta K_I = \frac{\Delta P}{tW^{\frac{1}{2}}} \frac{\left(2+\frac{a}{W}\right)\left(0.886+4.64\frac{a}{W}-13.32\left(\frac{a}{W}\right)^2+14.72\left(\frac{a}{W}\right)^3-5.6\left(\frac{a}{W}\right)^4\right)}{\left(1-\frac{a}{W}\right)^{\frac{3}{2}}}$$

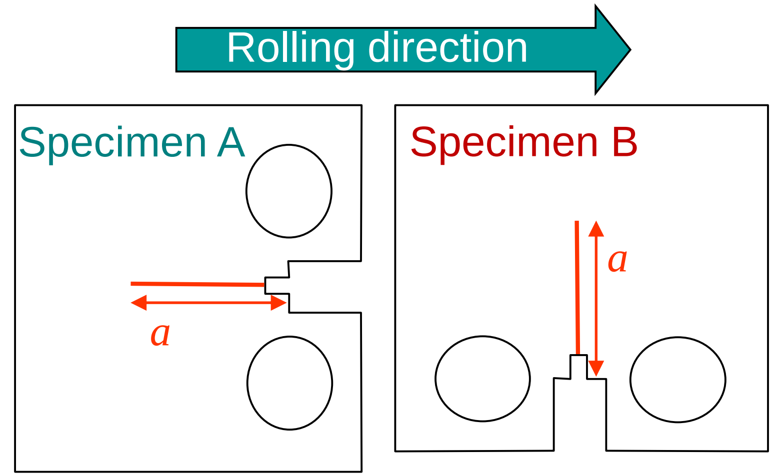
    - Crack length a evolution by measuring on both sides



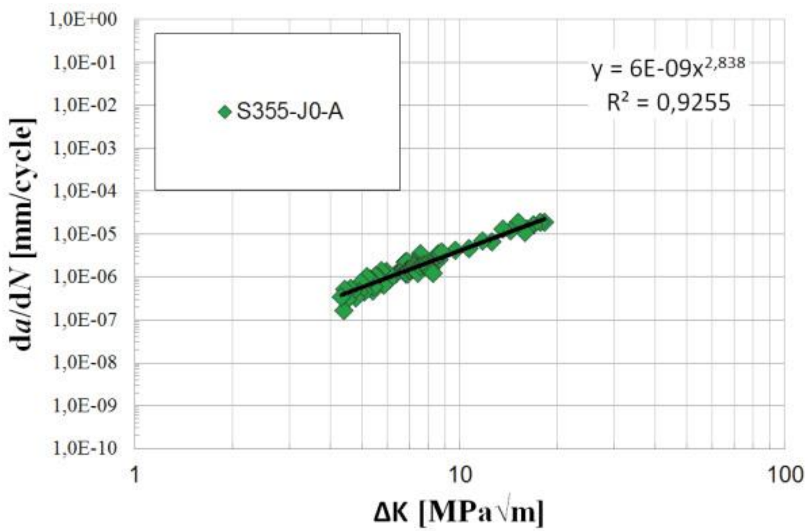
  - Control of ΔP: 2 methods
    - K-decreasing ⟹ ΔP decreases
    - K-increasing ⟹ ΔP constant
  - Check SSY valid: $W - a \geq \frac{4}{\pi}\left(\frac{K_{\max}}{\sigma_p^0}\right)^2$

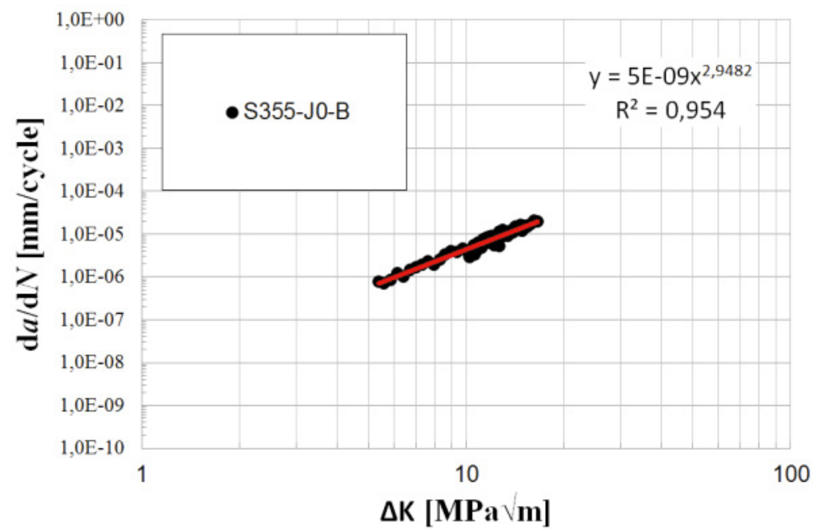
- Ex: S355 J0 steel
  - Known properties

Toughness ΔK_C [MPa · m$^{1/2}$]	40
Yield σ_p^0 [MPa]	355



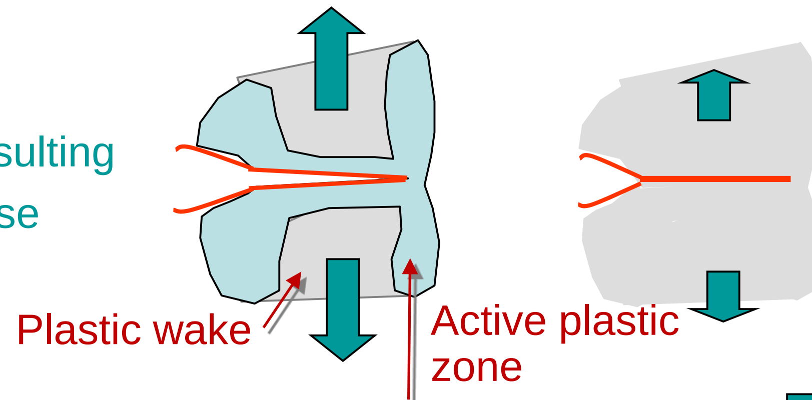
  - Extraction of fatigue curves
    - Loading ration R=0.1



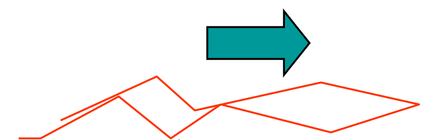


Stanislav Seitl, Petr Miarka, Jan Klusák, Zdeněk Kala, Martin Krejsa, Sergio Blasón, Alfonso F. Canteli, Evaluation of fatigue properties of S355 J0 steel using ProFatigue and ProPagation software, Procedia Structural Integrity,13, 2018, 1494-1501,

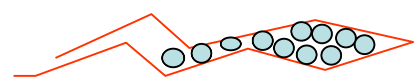
- Effect of $R=K_{min}/K_{max}$: crack closure
  - During loading phases at maximum stress
    - There is an active plastic zone (phase transformation can happen)
    - Crack opening allows fluid or products to enter
  - If R <0.7 or negative, at low stress crack lips can enter into contact due to



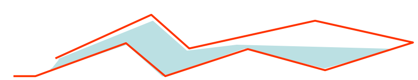
    - Plasticity
      - Residual plastic strain resulting from plastic wake will close the crack
    - Roughness
      - Non flat lips prevent sliding (mode II)



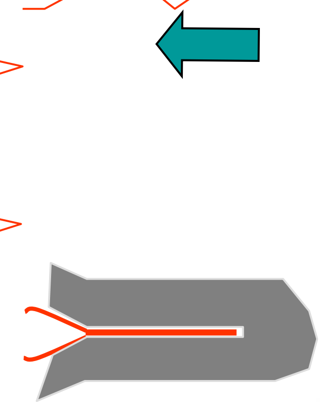
    - Corrosion
      - Corrosion products fill the opening



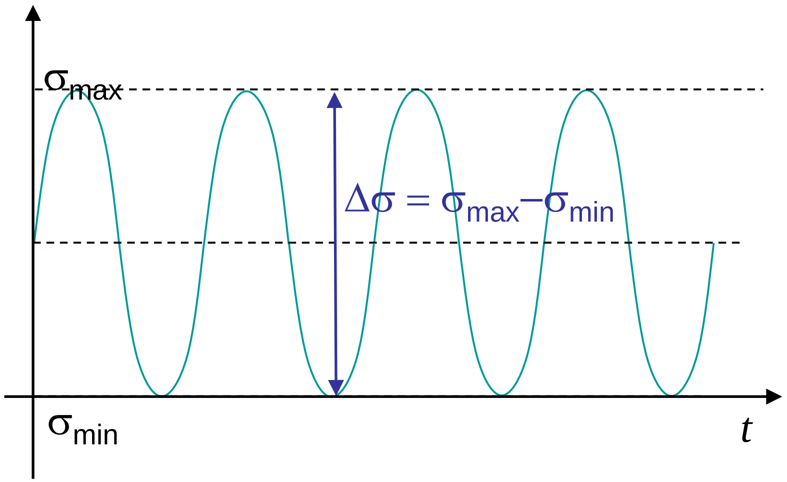
    - Viscous fluid
      - Lubricant fluids fill the opening

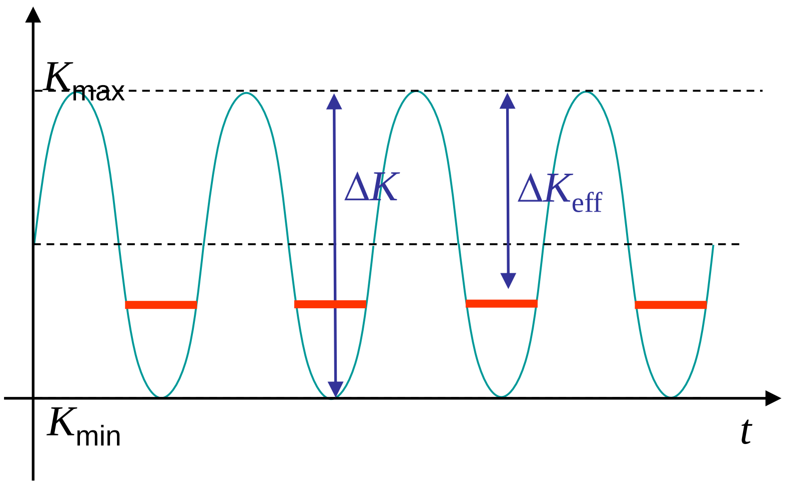


    - Phase transformation
      - Phase transformation put crack tip under compression



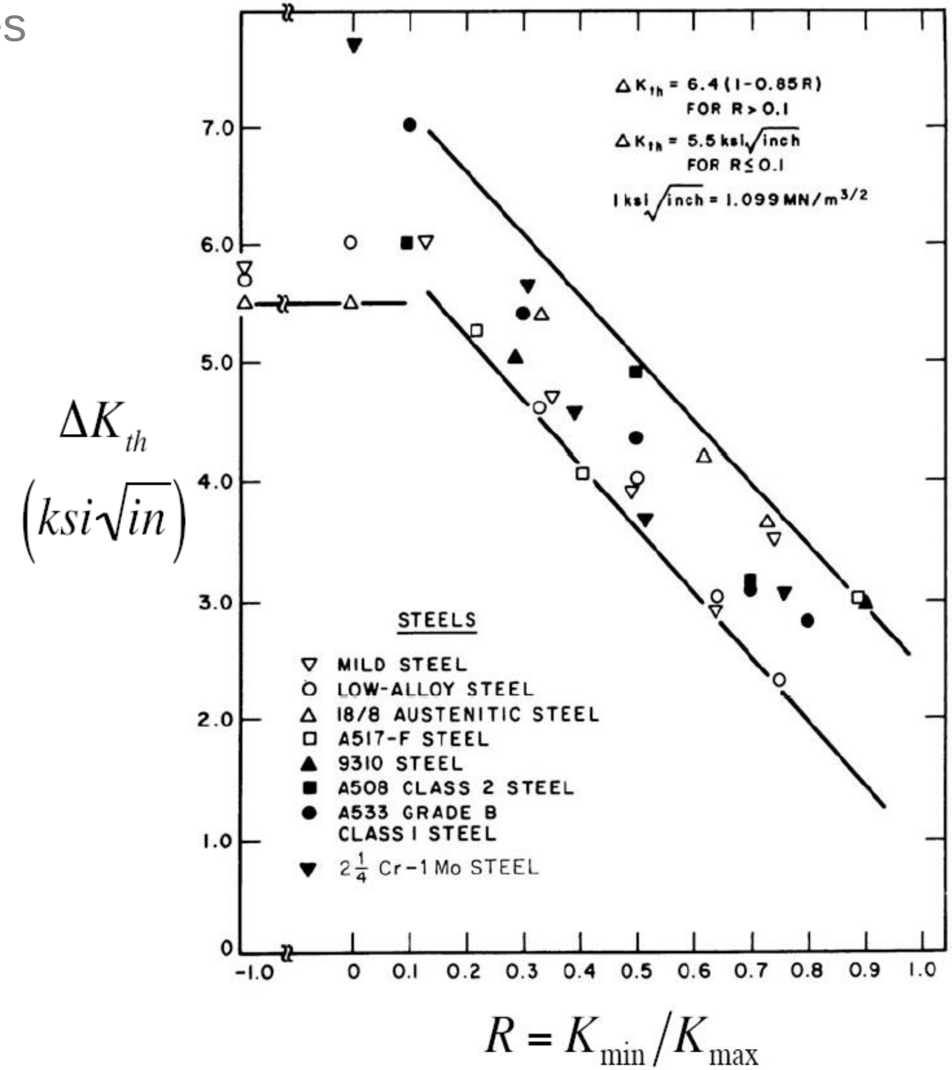
- Effect of crack closure on fatigue
  - When the loading decreases ⇒ local compressive effects ⇒ parts of the cracks are kept opened
    - The stress intensity factor at the minimum of the cycle is more important than predicted ⇒ the effective ΔK is actually reduced



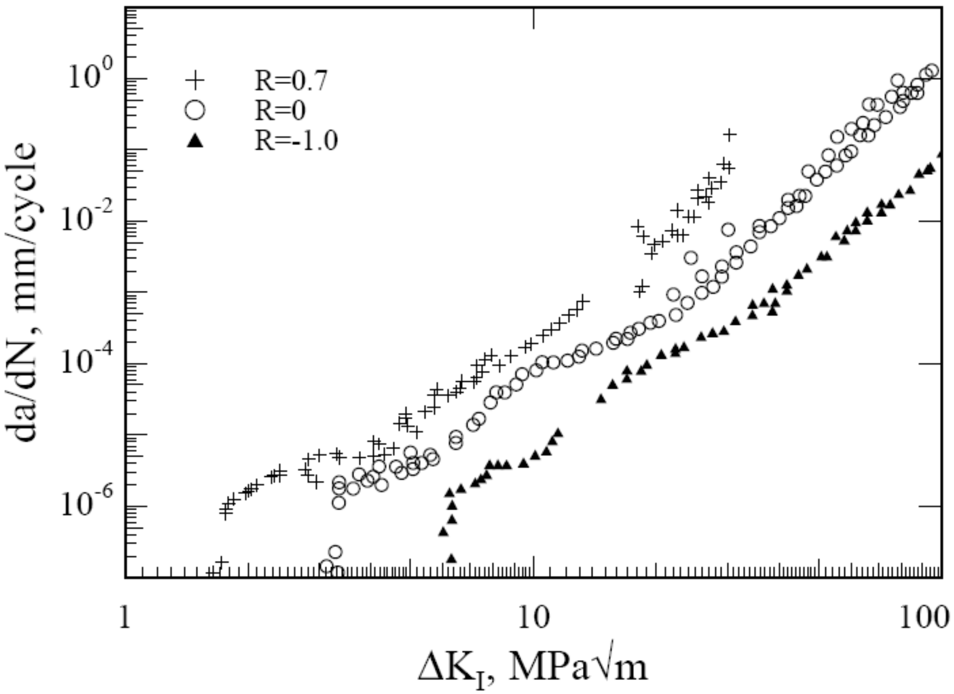


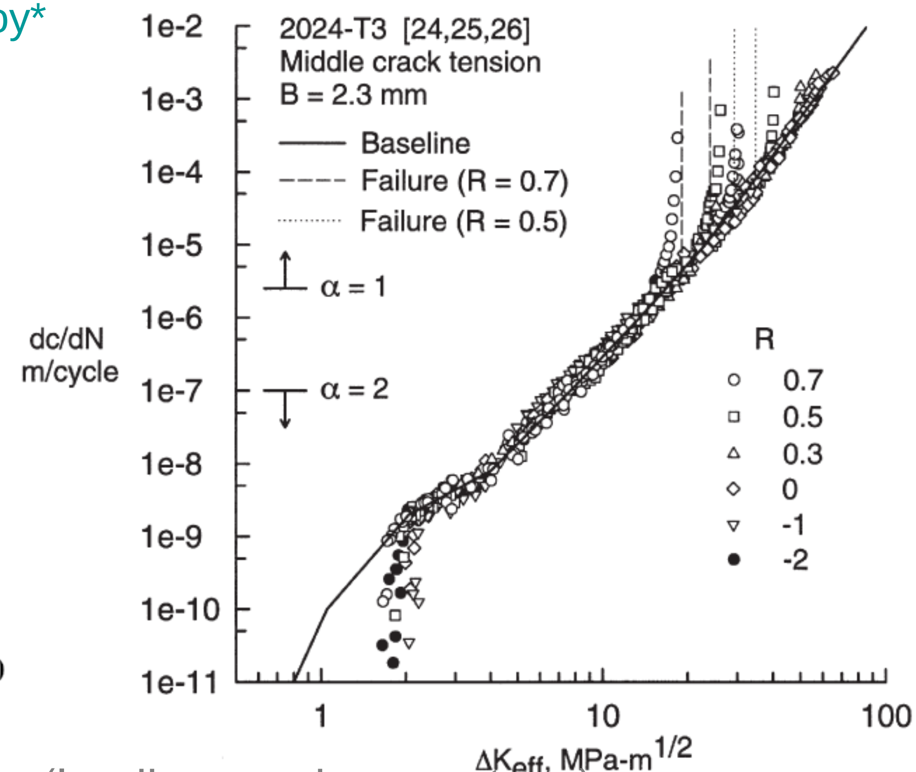
    - The crack closure effect is therefore beneficial to structure life

- Effect of $R=K_{min}/K_{max}$ on threshold (Zone I)
  - Due to Plasticity Induced Crack Closure (PICC)
  - ΔK_{th} decreases when R increases



- Effect of $R=K_{min}/K_{max}$ on crack growth rate (Zone II)
  - Due to crack closure life of structure is improved for low R
    - Example of 2024-T3 aluminum alloy*

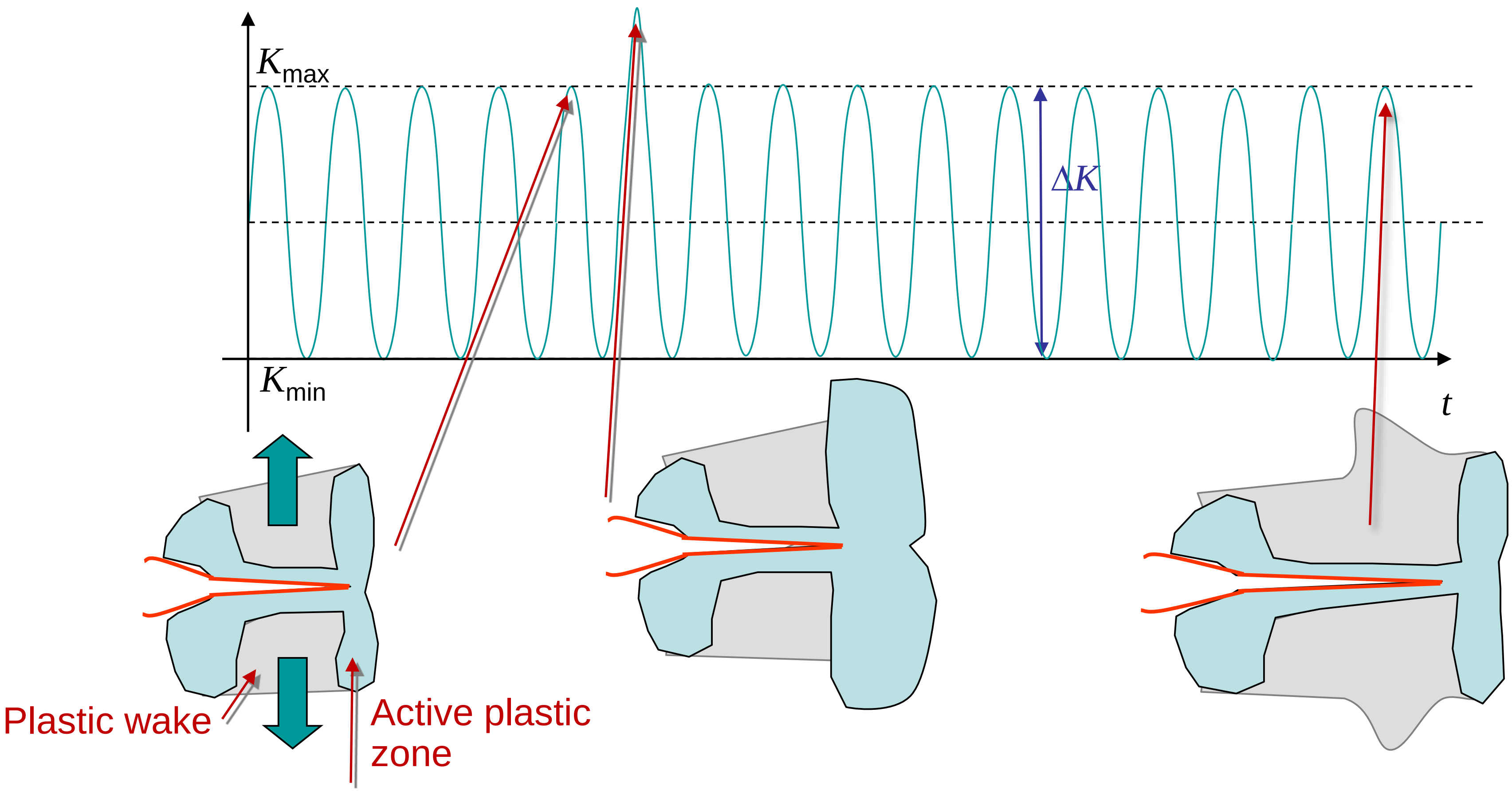




  - ΔK_{eff} depends on many parameters (loading, environment, …)
    - Example: model of Elber & Schijve for Al. 2024-T3
      - $\Delta K_{eff} = (0.55 + 0.33\,R + 0.12\,R^2)\ \Delta K$ for $-1<R<0.54$
    - Models can be inaccurate in non-adequate circumstances

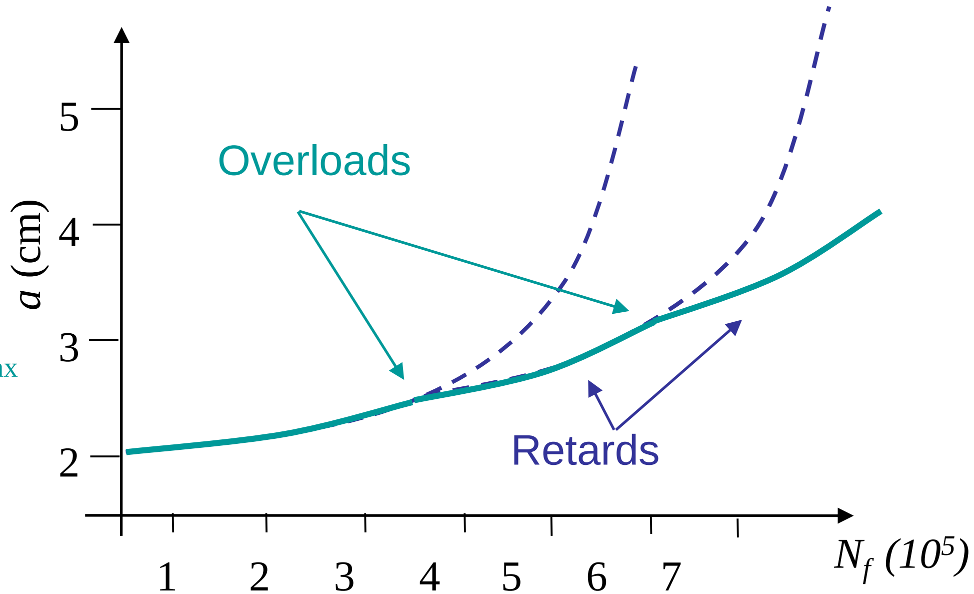
*J.C. Newman Jr, E.P. Phillips, M.H. Swain, Fatigue-life prediction methodology using small-crack theory, International Journal of Fatigue 21 (1999)

- Overload effect
  - What happens if there is a few (or a moderate) number of overloads ?



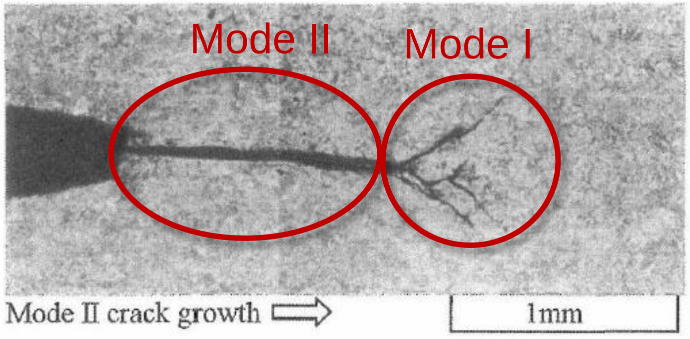
  - Plastic wake is temporarily increased
  - Until active plastic zone at crack tip passes the extended zone due to the overload
  - So which effect ?

- Overload effect (2)
  - As the plastic wake is temporarily increased, ΔK_{eff} is reduced due to PICC
    ⇒ there is a retard effect in the crack propagation
    - Infrequent overloads help
    - Frequent overloads may help
    - Too frequent overloads are damaging, as it actually corresponds to increasing $K_{\max}$

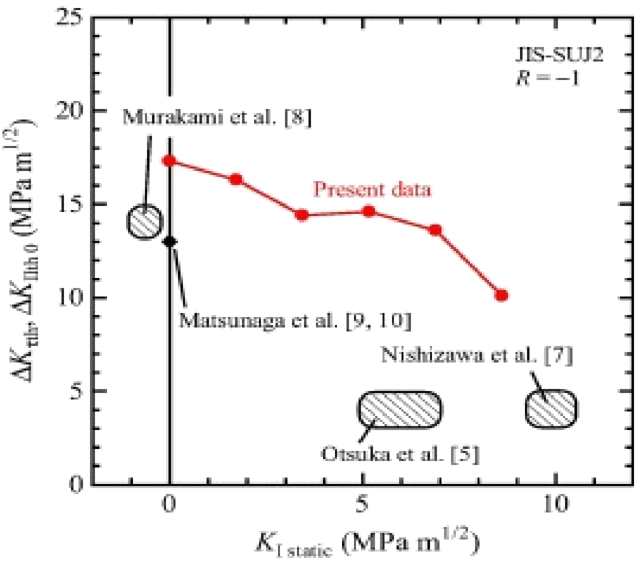


  - There exist overload models
    - Wheeler e.g.
  - N.B. 1952, a fuselage of the Comet was tested against fatigue
    - Static loading at 1.12 atm, followed by
    - 10 000 cycles at 0.7 atm (> cabin pressurization at 0.58 atm)
    - Production fuselage without the static loading failed after a few 1000 cycles (pressurization at 0.58 atm)

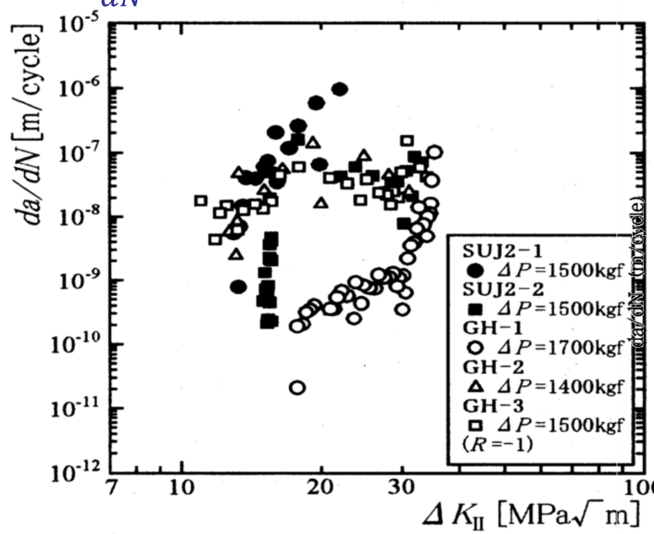
- Cyclic loading under mode II
  - We consider
    - Cyclic loading under mode II: ΔK_{II} &
    - Static loading under mode I: K_{I}
  - It is possible to have mode II crack growth, e.g.
    - SUJ2 steel (Japanese norm of 52100-steel)
    - Image**: Fully reverted mode II loading (R=-1) under mode I compression



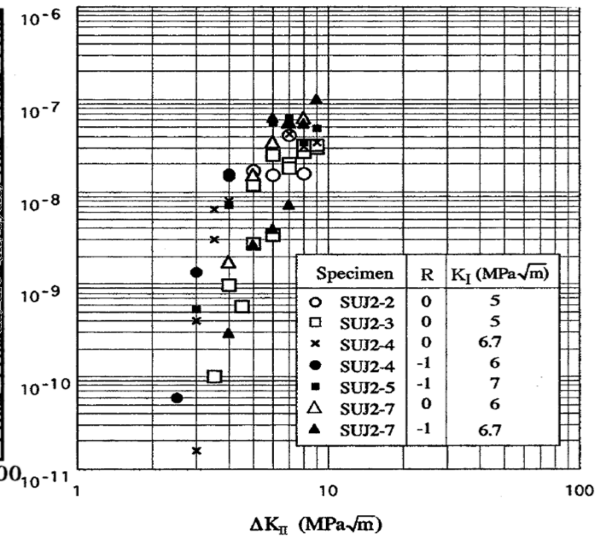
$\Delta K_{II\ th}$ as a function of K_I*



$\frac{da}{dN}$ for function of $K_I < 0$**



$\frac{da}{dN}$ for function of $K_I > 0$***

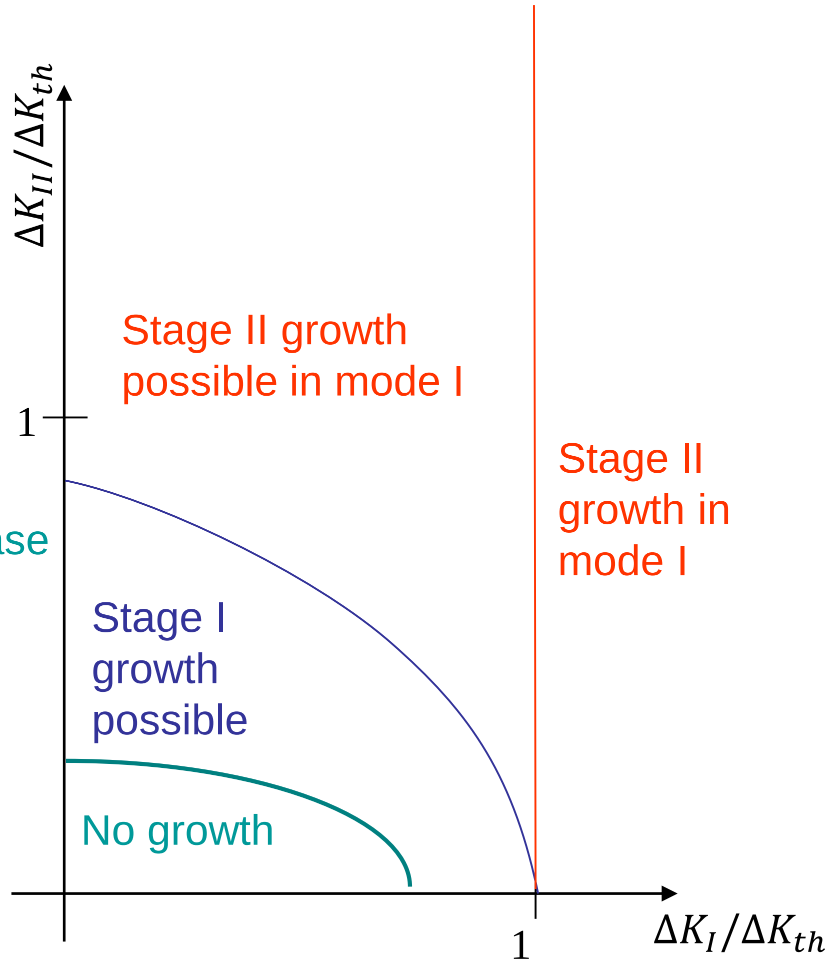


*K. Okazaki, K. Wada, H. Matsunaga, M. Endo, , Engineering Fracture Mechanics, 174 (2017), 127-138
**Y. Murakami, T. Fukuhara, S. Hamada Journal Society Material Sciences, 51 (8) (2002), 918-925
***A. Otsuka, Y Fuji, K. Maeda. Fatigue Fracture Engineering Material Structures, 27 (2004), 203-212

- Mixed mode cyclic loading
  - We consider
    - Cyclic loading under mode I: ΔK_{I}
    - Static loading under mode II: ΔK_{II} &
    - **Proportional** loading
  - Threshold
    - Depends on ΔK_{I} & ΔK_{II}
  - Direction
    - Predicted from K_{I} and K_{II} as for static case
    - E.g. maximum hoop stress

$$2\tan\frac{\theta^*}{2} - \cot\frac{\theta^*}{2} = \tan\beta^*$$

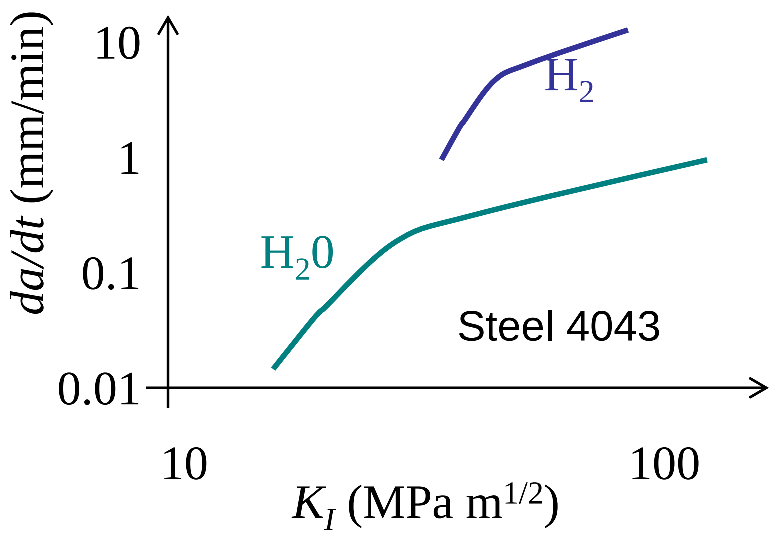
Not always validated*

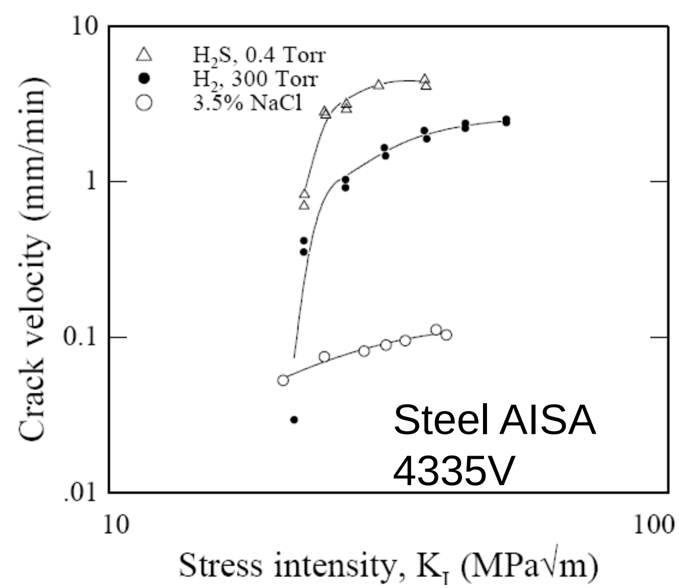


  - Crack growth rate
    - Adapt Paris' law: $\frac{da}{dN_f} = C\Delta K_{eff}^m$
    - Different approaches** to evaluate ΔK_{eff}
      - E.g. Tanaka: $\Delta K_{eff} = [\Delta K_I^4 + \Delta K_{II}^4]^{1/4}$

*Pook, L. P., A failure mechanism map for mixed mode I and II fatigue crack growth thresholds. Int. J. Fracture, 1985, 28, R21-23.
**J. Qian, A. Fatemi, Mixed mode fatigue crack growth: A literature survey, Engineering Fracture Mechanics 55(6), 1996, 969-990

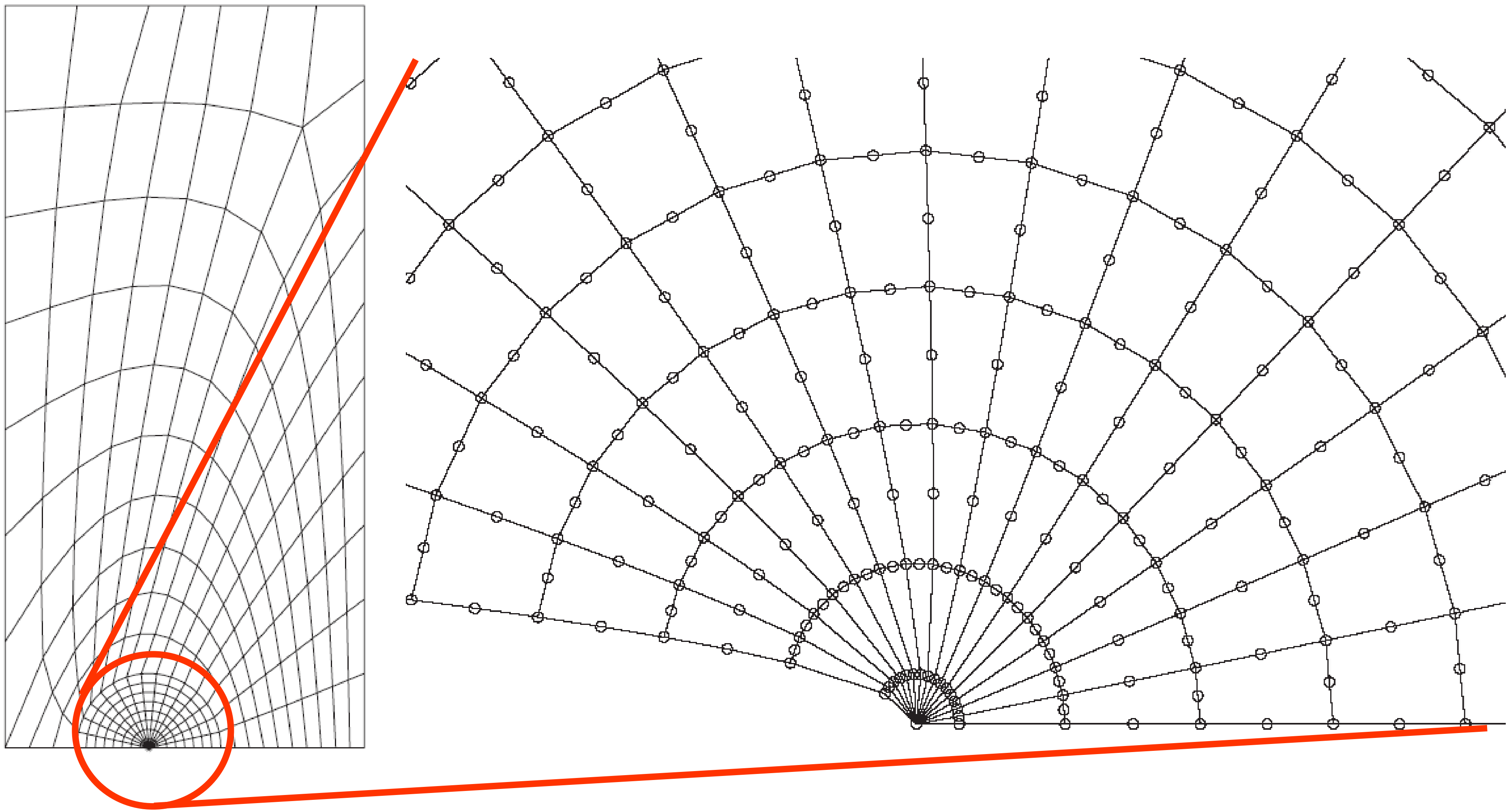
- A crack can grow due to the combination of stress and chemical attack
  - This is not only for fatigue but also for static stress with $K<K_C$
  - It happens in particular environments
    - Salt water
    - Hydrogen
    - Chlorides
    - …
  - Mainly for metals
    - Steel in salt water, chloride, hydrogen
    - Aluminum alloys in salt water





Simulation of crack propagation

- A simple method is a FE simulation where the crack is used as BCs
  - The mesh is conforming with the crack lips



- A simple method is a FE simulation where the crack is used as BCs (2)
  - Mesh the structure in a conforming way with the crack
  - Extract SIFs K_i (see lecture on SIF)
  - Use criterion on crack propagation: e.g. the maximal hoop stress criterion
    - Crack growth criterion: new SIF with crack in mode I after kinking

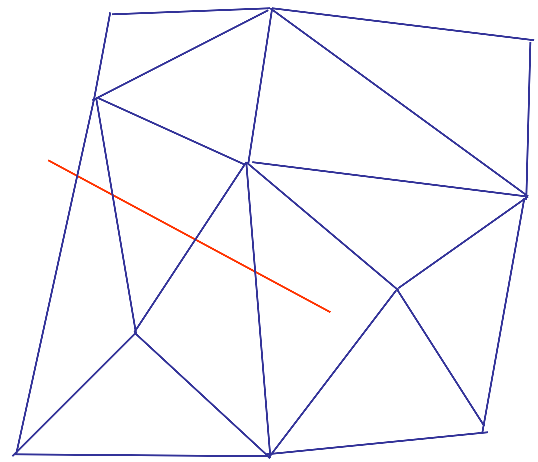
$$K_I' = \lim_{r\to 0} \sqrt{2\pi r}\, \sigma_{\theta\theta}(r,\theta^*) \geq K_C$$

    - With crack propagation direction (kinking) obtained by

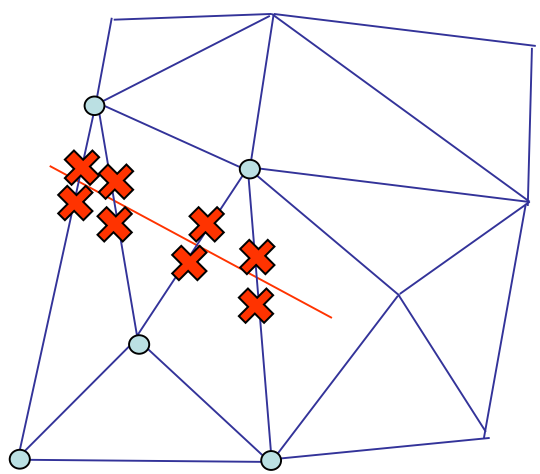
$$\partial_\theta \boldsymbol{\sigma}_{\theta\theta}\big|_{\theta^*} = 0 \ \ \& \ \ \partial^2_{\theta\theta} \boldsymbol{\sigma}_{\theta\theta}\big|_{\theta^*} < 0$$

  - If the crack propagates
    - Move crack tip by Δa in the θ^*-direction
    - A new mesh is required as the crack has changed (since the mesh has to be conforming)
      - Involves a large number of remeshing operations (time consuming)
      - Is not always fully automatic
      - Requires fine meshes and Barsoum elements
  - Not used

- How to get rid of conformity requirements?
- Key principles
  - For a FE discretization, the displacement field is approximated by $\boldsymbol{u}_h\left(\xi^i\right)=\sum_{a\in I} N^a\left(\xi^i\right)\boldsymbol{u}^a$
    - Sum on nodes a in the set I (11 nodes here)
    - u^a are the nodal displacements
    - N^a are the shape functions
    - ξ^i are the reduced coordinates

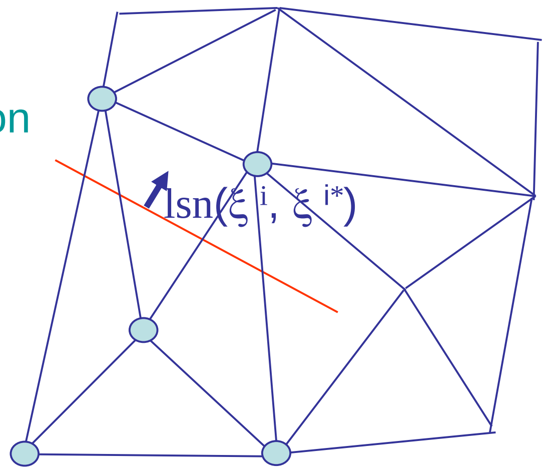


  - XFEM
    - New degrees of freedom are introduced to account for the discontinuity
    - It could be done by inserting new nodes (✖) near the crack tip, but this would be inefficient (remeshing)
    - Instead, shape functions are modified
      - Only shape functions that intersect the crack
      - This implies adding new degrees of freedom to the related nodes (○)



- Key principles (2)
  - New degrees of freedom are introduced to account for the discontinuity

$$\boldsymbol{u}_h\left(\xi^i\right)=\sum_{a\in I}N^a\left(\xi^i\right)\boldsymbol{u}^a+\sum_{a\in J}N^a\left(\xi^i\right)F^a\left(\xi^i\right)\boldsymbol{u}^{*a}$$

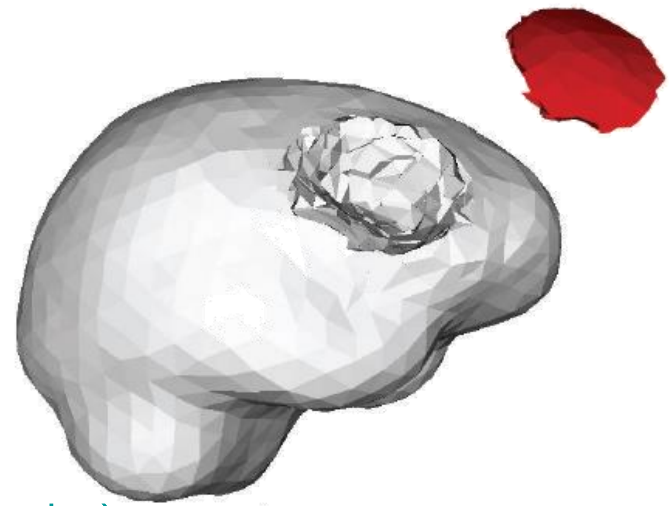


    - J, subset of I, is the set of nodes whose shape-function support is entirely separated by the crack (5 here)
    - u^{*a} are the new degrees of freedom at node a
  - Form of F^a the shape functions related to u^{*a}?
    - Use of Heaviside's function, and we want +1 above and -1 below the crack
    - In order to know if we are above or below the crack, signed-distance has to be computed
    - Normal level set $\mathrm{lsn}(\xi^i, \xi^{i*})$ is the signed distance between a point ξ^i of the solid and its projection ξ^{i*} on the crack

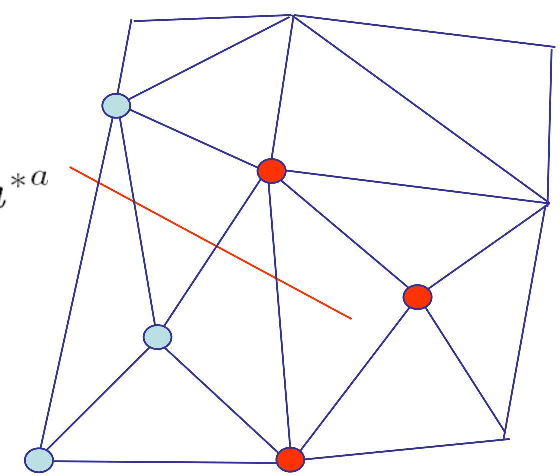
$$\Longrightarrow \boldsymbol{u}_h\left(\xi^i\right)=\sum_{a\in I}N^a\left(\xi^i\right)\boldsymbol{u}^a+\sum_{a\in J}N^a\left(\xi^i\right)H\left(\mathrm{lsn}\left(\xi^i,\,\xi^{i*}\right)\right)\boldsymbol{u}^{*a}$$

with $H(x) = \pm 1$ if $x >< 0$

- Key principles (3)
  - Example: removing of a brain tumor (L. Vigneron et al.)
  - At this point
    - A discontinuity can be introduced in the mesh
    - Fracture mechanics is not introduced yet
  - New enrichment with LEFM solution
    - Zone J of Heaviside enrichment is reduced (3 nodes)
    - A zone K of LEFM solution is added to the nodes (●) of elements containing the crack tip



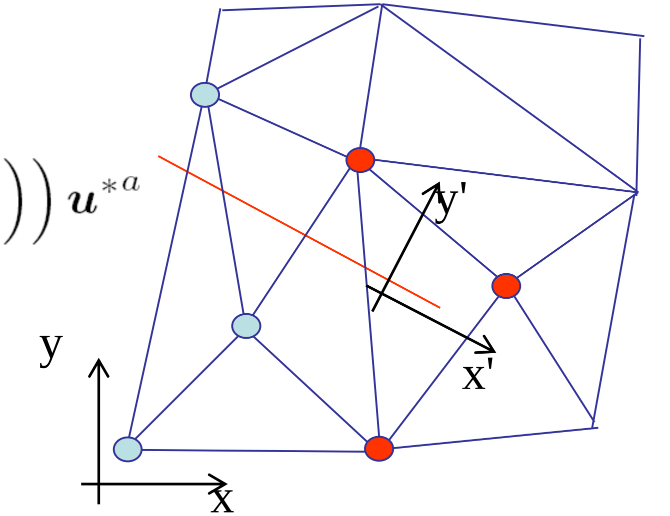
$$\boldsymbol{u}_h\left(\xi^i\right)=\sum_{a\in I}N^a\left(\xi^i\right)\boldsymbol{u}^a+\sum_{a\in J}N^a\left(\xi^i\right)H\left(\mathrm{lsn}\left(\xi^i,\,{\xi^i}^*\right)\right)\boldsymbol{u}^{*a}$$
$$+\sum_{a\in K}N^a\left(\xi^i\right)\sum_b\Psi_b\left(\xi^i\right)\boldsymbol{\psi}_b^a$$



- LEFM solution is asymptotic $\Longrightarrow$ only nodes close to crack tip can be enriched
- $\psi_b{}^a$ is the new degree b at node a (more than one see next slide)
- Ψ_b is the new shape function b (more than one see next slide)

- Key principles (4)
  - New enrichment with LEFM solution (2)

$$\boldsymbol{u}_h\left(\xi^i\right)=\sum_{a\in I}N^a\left(\xi^i\right)\boldsymbol{u}^a+\sum_{a\in J}N^a\left(\xi^i\right)H\left(\mathrm{lsn}\left(\xi^i,\,\xi^{i*}\right)\right)\boldsymbol{u}^{*a}$$
$$+\sum_{a\in K}N^a\left(\xi^i\right)\sum_b\Psi_b\left(\xi^i\right)\boldsymbol{\psi}_b^a$$



  - Ψ_b and $\psi_b{}^a$ from LEFM solutions

$$\boldsymbol{u}_{x'}=\frac{1+\nu}{E}\sqrt{\frac{r}{2\pi}}\left\{K_I\cos\frac{\theta}{2}\left[\kappa-1+2\sin^2\frac{\theta}{2}\right]+K_{II}\sin\frac{\theta}{2}\left[\kappa+1+2\cos^2\frac{\theta}{2}\right]\right\}$$

$$\boldsymbol{u}_{y'}=\frac{1+\nu}{E}\sqrt{\frac{r}{2\pi}}\left\{K_I\sin\frac{\theta}{2}\left[\kappa+1-2\cos^2\frac{\theta}{2}\right]+K_{II}\cos\frac{\theta}{2}\left[1-\kappa+2\sin^2\frac{\theta}{2}\right]\right\}$$

$$\boldsymbol{u}_{z'}=\frac{2K_{III}\left(1+\nu\right)}{E}\sqrt{\frac{2r}{\pi}}\sin\frac{\theta}{2}$$

- Key principles (5)
  - New enrichment with LEFM solution (3)
    - But

$$\boldsymbol{u}_{x'} = \frac{1+\nu}{E}\sqrt{\frac{r}{2\pi}}\left\{K_I \cos\frac{\theta}{2}\left[\kappa - 1 + 2\sin^2\frac{\theta}{2}\right] + K_{II}\sin\frac{\theta}{2}\left[\kappa + 1 + 2\cos^2\frac{\theta}{2}\right]\right\}$$

$$\boldsymbol{u}_{x'} = \frac{1+\nu}{E}\sqrt{\frac{r}{2\pi}}\left\{K_I \cos\frac{\theta}{2}[\kappa - 1] + K_I \sin\frac{\theta}{2}\sin\theta + K_{II}\boxed{\sin\frac{\theta}{2}}[\kappa + 1] + K_{II}\boxed{\cos\frac{\theta}{2}\sin\theta}\right\}$$

$$\boldsymbol{u}_{y'} = \frac{1+\nu}{E}\sqrt{\frac{r}{2\pi}}\left\{K_I \sin\frac{\theta}{2}\left[\kappa + 1 - 2\cos^2\frac{\theta}{2}\right] + K_{II}\cos\frac{\theta}{2}\left[1 - \kappa + 2\sin^2\frac{\theta}{2}\right]\right\}$$

$$\boldsymbol{u}_{y'} = \frac{1+\nu}{E}\sqrt{\frac{r}{2\pi}}\left\{K_I \boxed{\sin\frac{\theta}{2}}[\kappa + 1] - K_I \boxed{\cos\frac{\theta}{2}\sin\theta} + K_{II}\cos\frac{\theta}{2}[1 - \kappa] + K_{II}\sin\frac{\theta}{2}\sin\theta\right\}$$

    - We still have $\boldsymbol{u}_{z'} = \frac{2K_{III}(1+\nu)}{E}\sqrt{\frac{2r}{\pi}}\boxed{\sin\frac{\theta}{2}}$
  - We have determined 4 independent shape functions Ψ_b

- Key principles (6)
  - New enrichment with LEFM solution (4)
    - Vectors of unknowns $\boldsymbol{\psi}_b$ and shape functions Ψ_b are now defined

$$\boldsymbol{u}_x = (\boldsymbol{\psi}_x)_1 \underbrace{\sqrt{r}\sin\frac{\theta}{2}}_{\Psi_1} + (\boldsymbol{\psi}_x)_2 \underbrace{\sqrt{r}\cos\frac{\theta}{2}}_{\Psi_2} + (\boldsymbol{\psi}_x)_3 \underbrace{\sqrt{r}\sin\frac{\theta}{2}\sin\theta}_{\Psi_3} + (\boldsymbol{\psi}_x)_4 \underbrace{\sqrt{r}\cos\frac{\theta}{2}\sin\theta}_{\Psi_4}$$

$$\boldsymbol{u}_y = (\boldsymbol{\psi}_y)_1 \underbrace{\sqrt{r}\sin\frac{\theta}{2}}_{\Psi_1} + (\boldsymbol{\psi}_y)_2 \underbrace{\sqrt{r}\cos\frac{\theta}{2}}_{\Psi_2} + (\boldsymbol{\psi}_y)_3 \underbrace{\sqrt{r}\sin\frac{\theta}{2}\sin\theta}_{\Psi_3} + (\boldsymbol{\psi}_y)_4 \underbrace{\sqrt{r}\cos\frac{\theta}{2}\sin\theta}_{\Psi_4}$$

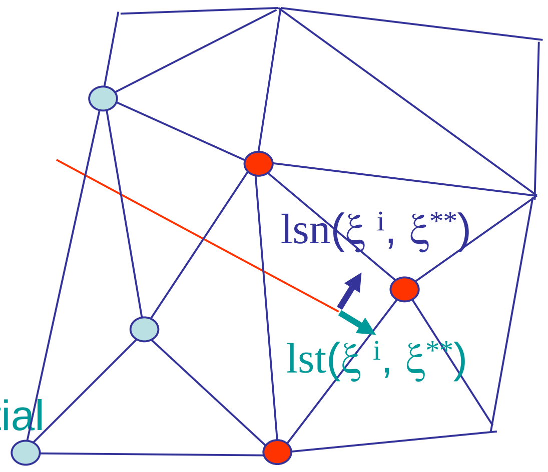
$$\boldsymbol{u}_z = (\boldsymbol{\psi}_z)_1 \underbrace{\sqrt{r}\sin\frac{\theta}{2}}_{\Psi_1} + (\boldsymbol{\psi}_z)_2 \underbrace{\sqrt{r}\cos\frac{\theta}{2}}_{\Psi_2} + (\boldsymbol{\psi}_z)_3 \underbrace{\sqrt{r}\sin\frac{\theta}{2}\sin\theta}_{\Psi_3} + (\boldsymbol{\psi}_z)_4 \underbrace{\sqrt{r}\cos\frac{\theta}{2}\sin\theta}_{\Psi_4}$$

    - We have 12 new degrees of freedom on the LEFM-enriched nodes

$$\boldsymbol{u}_h\left(\xi^i\right) = \sum_{a\in I} N^a\left(\xi^i\right)\boldsymbol{u}^a + \sum_{a\in J} N^a\left(\xi^i\right) H\left(\operatorname{lsn}\left(\xi^i, \xi^{i*}\right)\right)\boldsymbol{u}^{*a} +$$
$$\sum_{a\in K} N^a\left(\xi^i\right)\sum_{b=1}^{4}\Psi_b\left(r\left(\xi^i\right), \theta\left(\xi^i\right)\right)\boldsymbol{\psi}_b^a$$

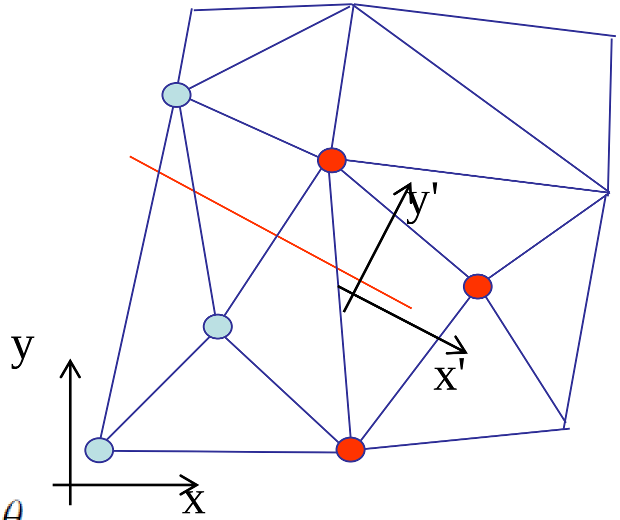
    - Remark: as Ψ_1 is discontinuous we do not need Heaviside functions for K-nodes

- Key principles (7)
  - How are $\Psi_b\left(r\left(\xi^i\right), \theta\left(\xi^i\right)\right)$ evaluated?
    - New level sets
      - Normal level set $\mathrm{lsn}(\xi^i, \xi^{**})$ is the normal signed distance between a point ξ^i of the solid and the crack tip ξ^{i**}
      - Tangent level set $\mathrm{lst}(\xi^i, \xi^{**})$ is the tangential signed distance between a point ξ^i of the solid and the crack tip ξ^{i**}



- $$\Rightarrow \begin{cases} r = \sqrt{\mathrm{lsn}^2\left(\xi^i, \xi^{**}\right) + \mathrm{lst}^2\left(\xi^i, \xi^{**}\right)} \\ \theta = \pm \arctan \dfrac{\mathrm{lsn}\left(\xi^i, \xi^{**}\right)}{\mathrm{lst}\left(\xi^i, \xi^{**}\right)} \end{cases}$$

- Crack propagation criterion
  - Requires the values of the SIFs
    - Using ψ_b^a as



$$\boldsymbol{u}_{x'} = \frac{1+\nu}{E}\sqrt{\frac{r}{2\pi}}\left\{K_I \cos\frac{\theta}{2}\left[\kappa-1\right] + K_I \sin\frac{\theta}{2}\sin\theta + K_{II}\sin\frac{\theta}{2}\left[\kappa+1\right] + K_{II}\cos\frac{\theta}{2}\sin\theta\right\}$$

was substituted by

$$\boldsymbol{u}_x = (\boldsymbol{\psi}_x)_1 \underbrace{\sqrt{r}\sin\frac{\theta}{2}}_{\Psi_1} + (\boldsymbol{\psi}_x)_2 \underbrace{\sqrt{r}\cos\frac{\theta}{2}}_{\Psi_2} + (\boldsymbol{\psi}_x)_3 \underbrace{\sqrt{r}\sin\frac{\theta}{2}\sin\theta}_{\Psi_3} + (\boldsymbol{\psi}_x)_4 \underbrace{\sqrt{r}\cos\frac{\theta}{2}\sin\theta}_{\Psi_4}$$

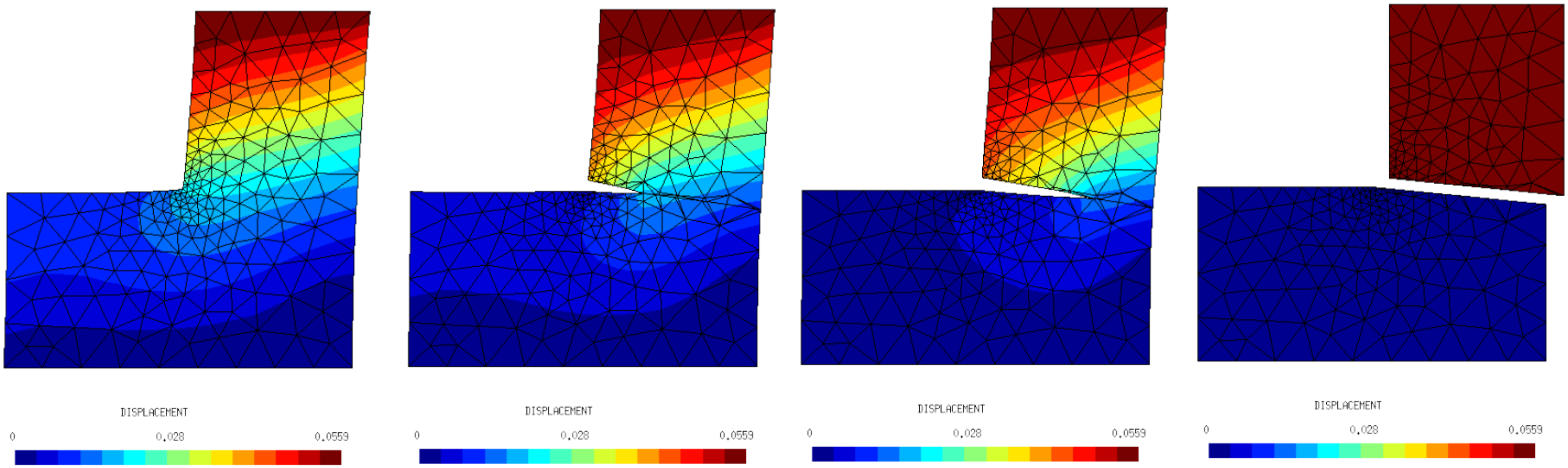
- Crack propagation criterion
  - Requires the values of the SIFs (2)
    - A more accurate solution is to compute J
      - But K_I, K_{II} & K_{III} have to be extracted from $J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$
        - » Define an adequate auxiliary field u^{aux}
        - » Compute $J^{aux}(u^{aux})$ and $J^{s}(u+u^{aux})$
        - » On can show that the interaction integral (see lecture on SIFs)

$$I^s = J^s - J - J^{aux} = \frac{2}{E'}\left(K_I K_I^{aux} + K_{II} K_{II}^{aux}\right) + \frac{1}{\mu} K_{III} K_{III}^{aux}$$

        - » If u^{aux} is chosen such that only $K_i^{aux} \neq 0$, K_i is obtained directly
  - Then the maximum hoop stress criterion, e.g., can be used
    - $K_I' = \lim_{r\to 0} \sqrt{2\pi r}\, \sigma_{\theta\theta}(r, \theta^*) \geq K_C$

with $\partial_\theta \boldsymbol{\sigma}_{\theta\theta}\big|_{\theta^*} = 0$ & $\partial^2_{\theta\theta} \boldsymbol{\sigma}_{\theta\theta}\big|_{\theta^*} < 0$

- Numerical example
  - Crack propagation (E. Béchet)

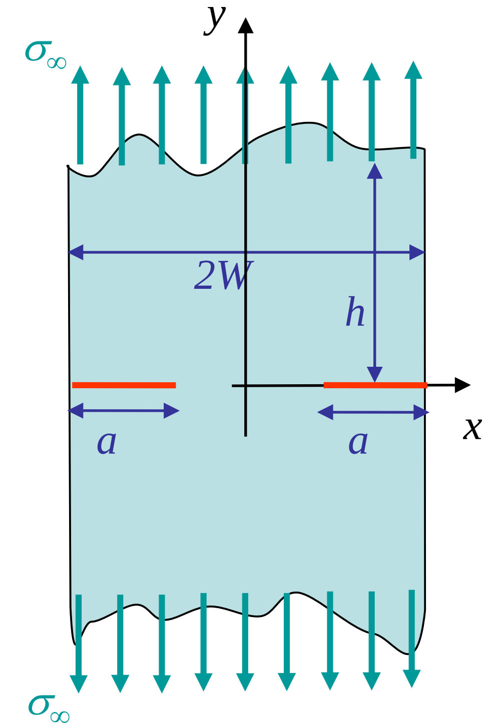


  - Advantages:
    - No need for a conforming mesh (but mesh has still to be fine near crack tip)
    - Mesh independency
    - Computationally efficient
  - Drawbacks:
    - Require radical changes to the FE code
      - New degrees of freedom
      - Gauss integration
      - Time integration algorithm

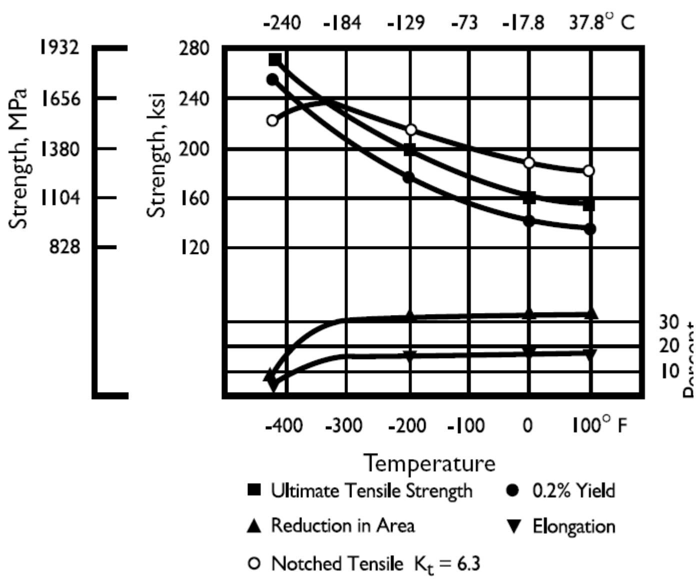
Exercise 1

- Edge notch specimen under cyclic loading
  - Assume titanium alloy 6%Al - 4%V
    - See figures below
  - Cyclic loading between
    - Minimum value: 12 MPA
    - Maximum value: 120 MPa
  - What is the structure life ?

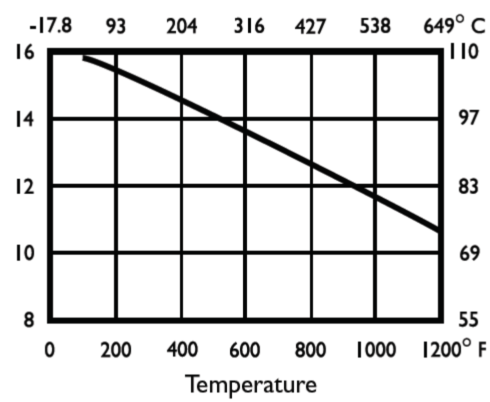
Half width W [m]	0.03
Half width h [m]	0.1
Thickness t [m]	0.0125
Initial crack a_0 [m]	0.003



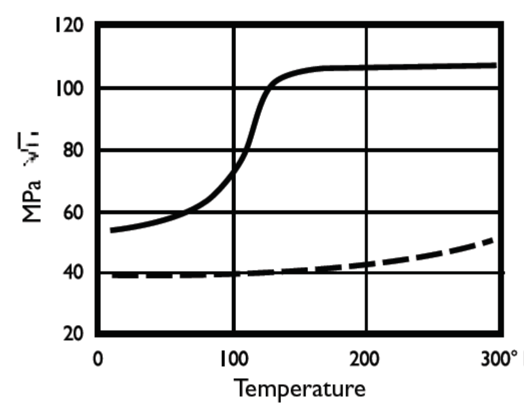
Tensile Properties vs. Temperature



Modulus of Elasticity



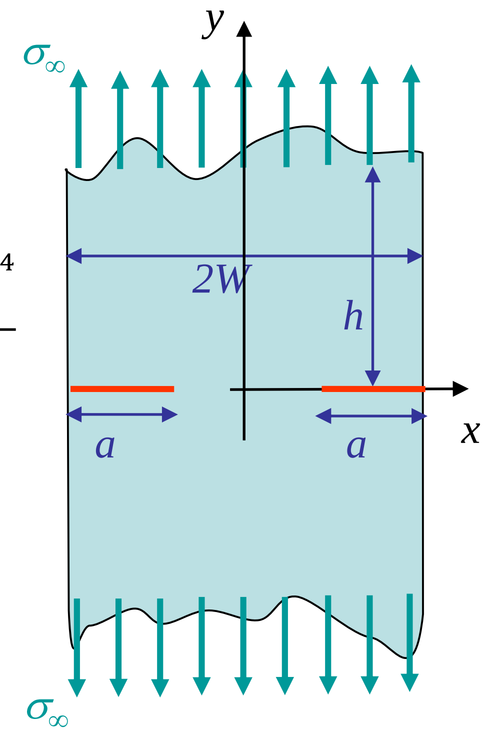
Fracture Toughness vs Temperature



- Initial SIF
  - SIF from handbook
    - $K_I = \sigma\sqrt{\pi a}F\left(\frac{a}{W}\right)$ with

$$F\left(\frac{a}{W}\right) = \frac{1.122 - 0.561\frac{a}{W} - 0.205\left(\frac{a}{W}\right)^2 + 0.471\left(\frac{a}{W}\right)^3 - 0.190\left(\frac{a}{W}\right)^4}{\sqrt{1-\frac{a}{W}}}$$

  - For initial crack length
    - $K_{I_{\min}}(a = 0.003\text{ m}) = 12\,10^6\,\sqrt{\pi 0.003}F(0.1)$
      $= 1.31\text{ MPa}\cdot\text{m}^{\frac{1}{2}}.$
    - $K_{I_{\max}}(a = 0.003\text{ m}) = 120\,10^6\,\sqrt{\pi 0.003}F(0.1)$
      $13.1\text{ MPa}\cdot\text{m}^{\frac{1}{2}}.$
  - $K_{I_{\max}}$<< toughness so no static failure



Half width W [m]	0.03
Half width h [m]	0.1
Thickness t [m]	0.0125
Initial crack a_0 [m]	0.003

Exercise 1: Solution

- Assumptions
  - Material properties at room temperature

Toughness K_C [MPa · m^{½}]	55
Yield σ_p^0 [MPa]	830

  - Plane strain & linear elasticity
    - Initially plain strain: OK

$$t > 2.5 \left(\frac{K_{I\text{max}}}{\sigma_p^0}\right)^2 = 0.62\ \text{mm}$$

    - Initially in K-dominance zone: OK

$$a > 2.5 \left(\frac{K_{I\text{max}}}{\sigma_p^0}\right)^2 = 0.62\ \text{mm}$$

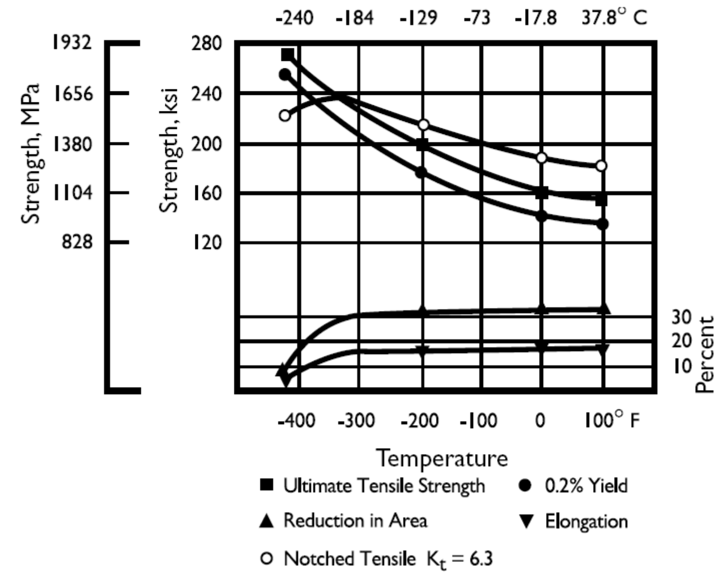
    - Plane strain at failure: OK

$$t > 2.5 \left(\frac{K_C}{\sigma_p^0}\right)^2 = 11\ \text{mm}$$

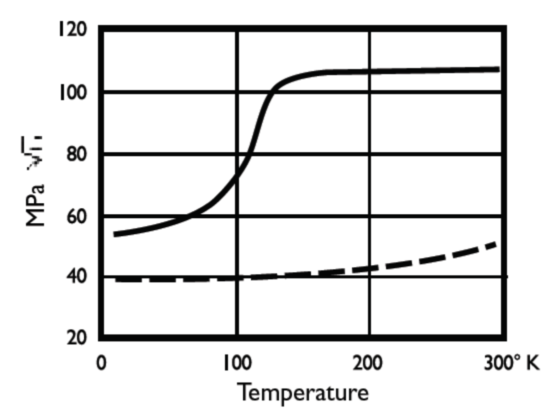
    - K-dominance zone at failure: check later

$$a_{cr}, W - a_{cr} >^{?} 2.5 \left(\frac{K_C}{\sigma_p^0}\right)^2 = 11\ \text{mm}$$

Tensile Properties vs. Temperature



Fracture Toughness vs Temperature



Exercise 1: Solution

- Fatigue law
  - Parameters in Paris' law
    - $\frac{da}{dN_f} = C\ \Delta K^m$
    - For $\Delta K < 16$ MPa · m½
      - $m = 7.38$

$\Rightarrow$ $8.7\ 10^{-8} \cdot m^{-1} = C\ 16^{7.38}$

$\Rightarrow$ $C = 1.13\ 10^{-16}$ $\mathrm{m} \cdot (\mathrm{MPA} \cdot \sqrt{\mathrm{m}})^{-7.38}$

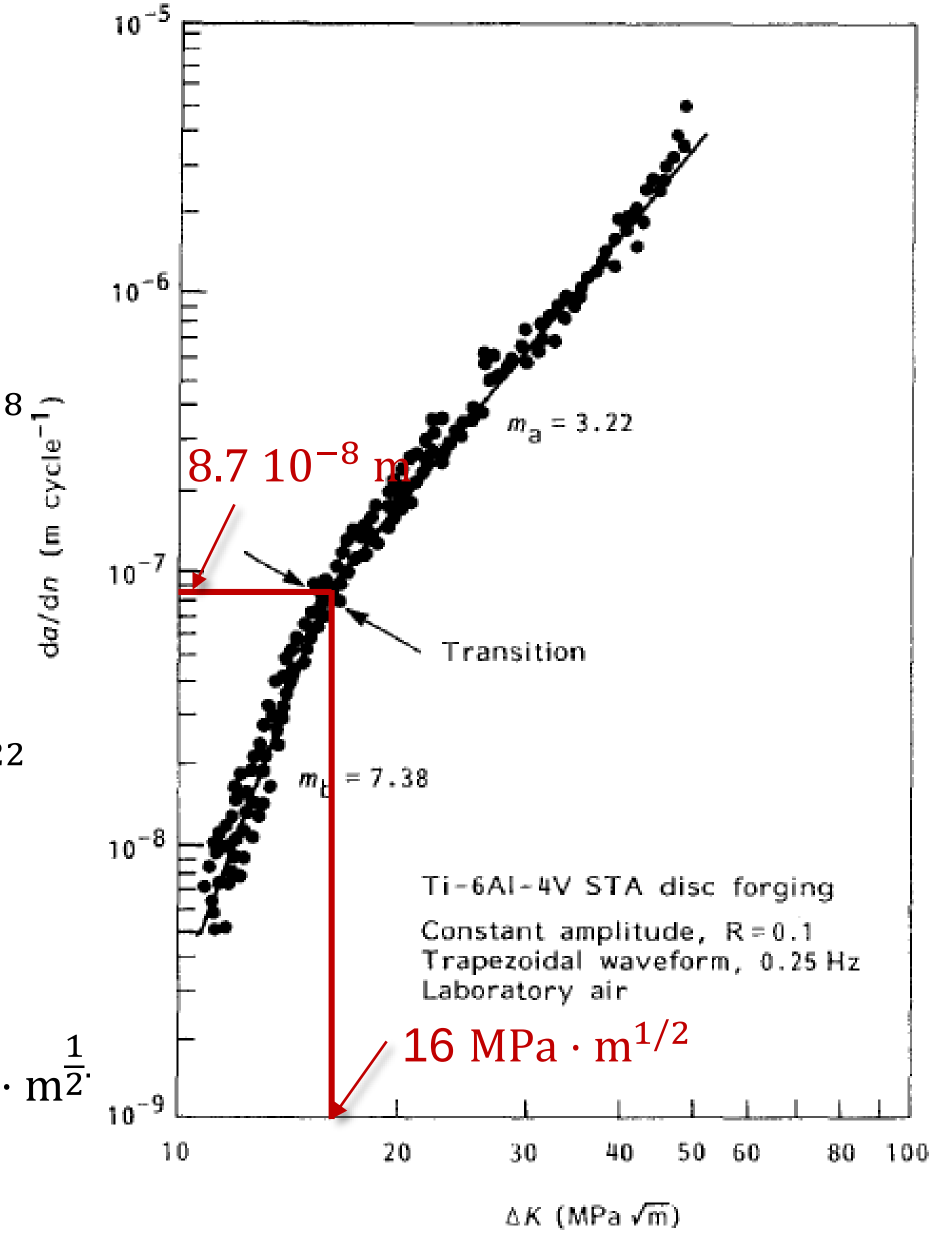
    - For $\Delta K > 16$ MPa · m½
      - $m = 3.22$

$\Rightarrow$ $8.7\ 10^{-8} \cdot m^{-1} = C\ 16^{3.22}$

$\Rightarrow$ $C = 1.15\ 10^{-11}$ $\mathrm{m} \cdot (\mathrm{MPA} \cdot \sqrt{\mathrm{m}})^{-3.22}$

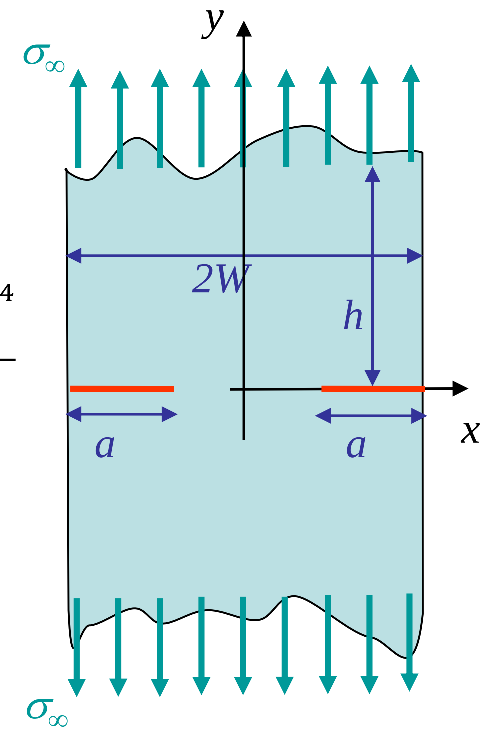
  - Threshold $\Delta K_{th} < 11$ MPa · m½
    - $\Delta K_I(a = 0.003\ \mathrm{m}) = 11.8\ \mathrm{MPa} \cdot \mathrm{m}^{\frac{1}{2}}$

$\Rightarrow$ crack propagation by fatigue

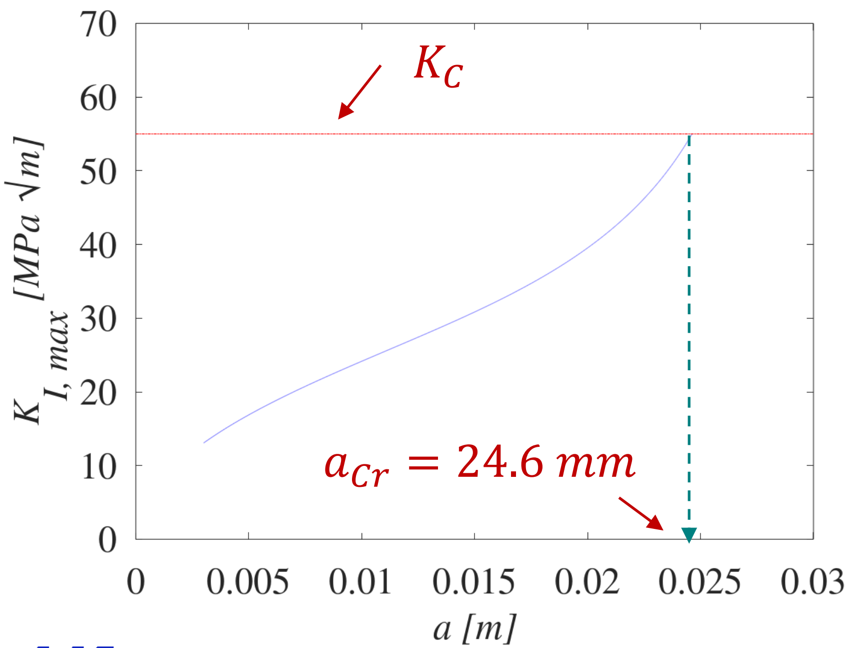


- Critical crack length
  - When is toughness reached
    - $$K_{I_{\max}} = \sigma_{\max}\sqrt{\pi a_{cr}}F\left(\frac{a_{cr}}{W}\right) = K_C$$ with

$$F\left(\frac{a}{W}\right) = \frac{1.122 - 0.561\frac{a}{W} - 0.205\left(\frac{a}{W}\right)^2 + 0.471\left(\frac{a}{W}\right)^3 - 0.190\left(\frac{a}{W}\right)^4}{\sqrt{1-\frac{a}{W}}}$$



    - Resolution by iterations or graphical

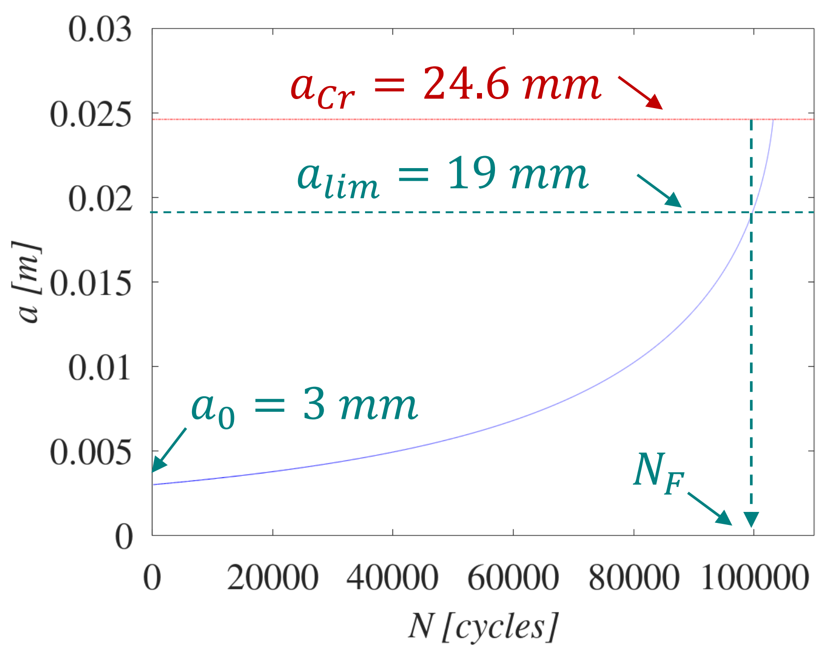


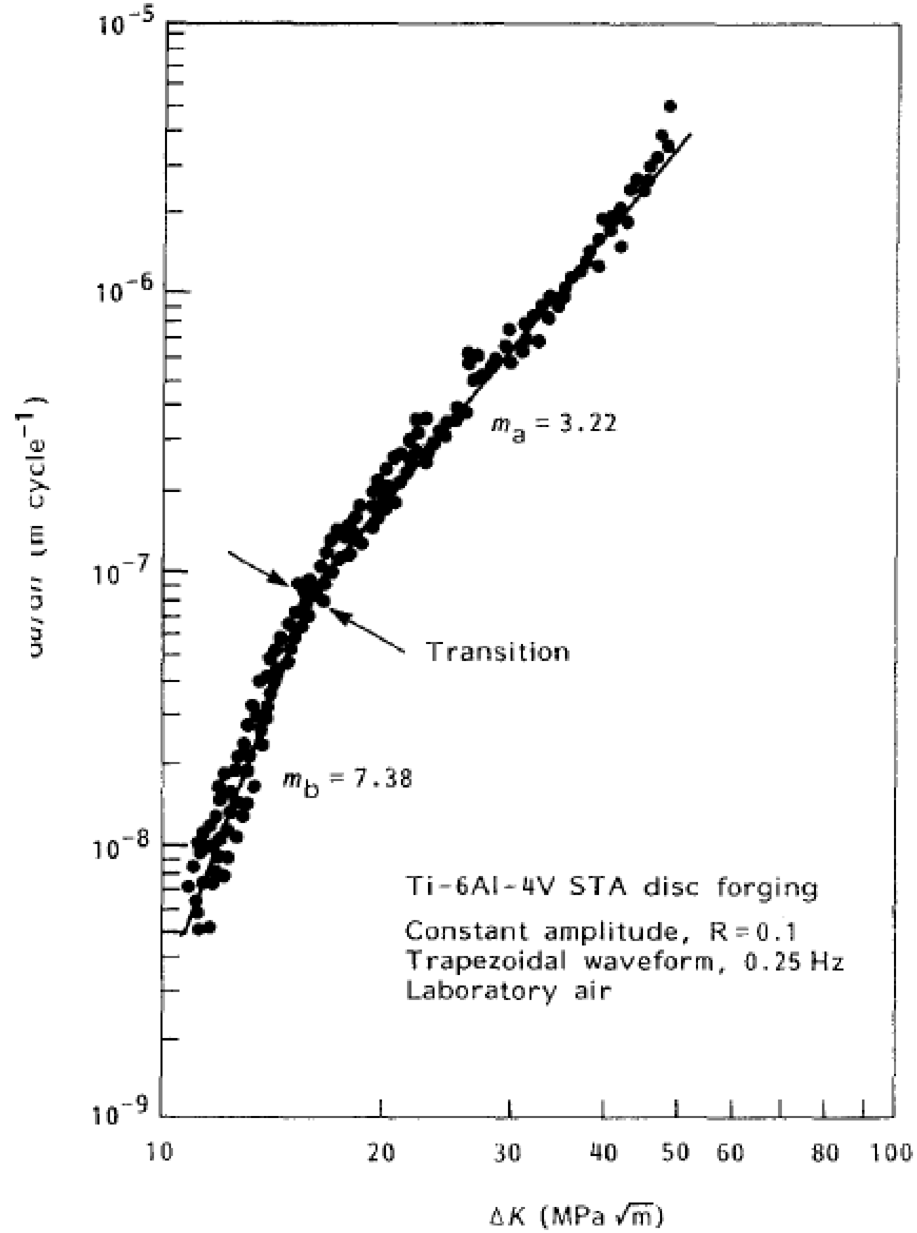
  - K-dominance zone at failure?
    - $a_{cr}, W - a_{cr} >^{?} 2.5\left(\frac{K_C}{\sigma_p^0}\right)^2 = 11\text{ mm}$
    - Ligament too short so analysis valid up to $a_{lim} = 19\text{ mm}$

- Structure life
  - Paris' law
    - $$\frac{da}{dN_f} = C\,\Delta K^m$$

$$\Rightarrow N_f = \int_{a_0}^{\Delta K=16} \frac{da'}{C\,\Delta K^m(a')} + \int_{\Delta K=16}^{a} \frac{da'}{C\,\Delta K^m(a')}$$

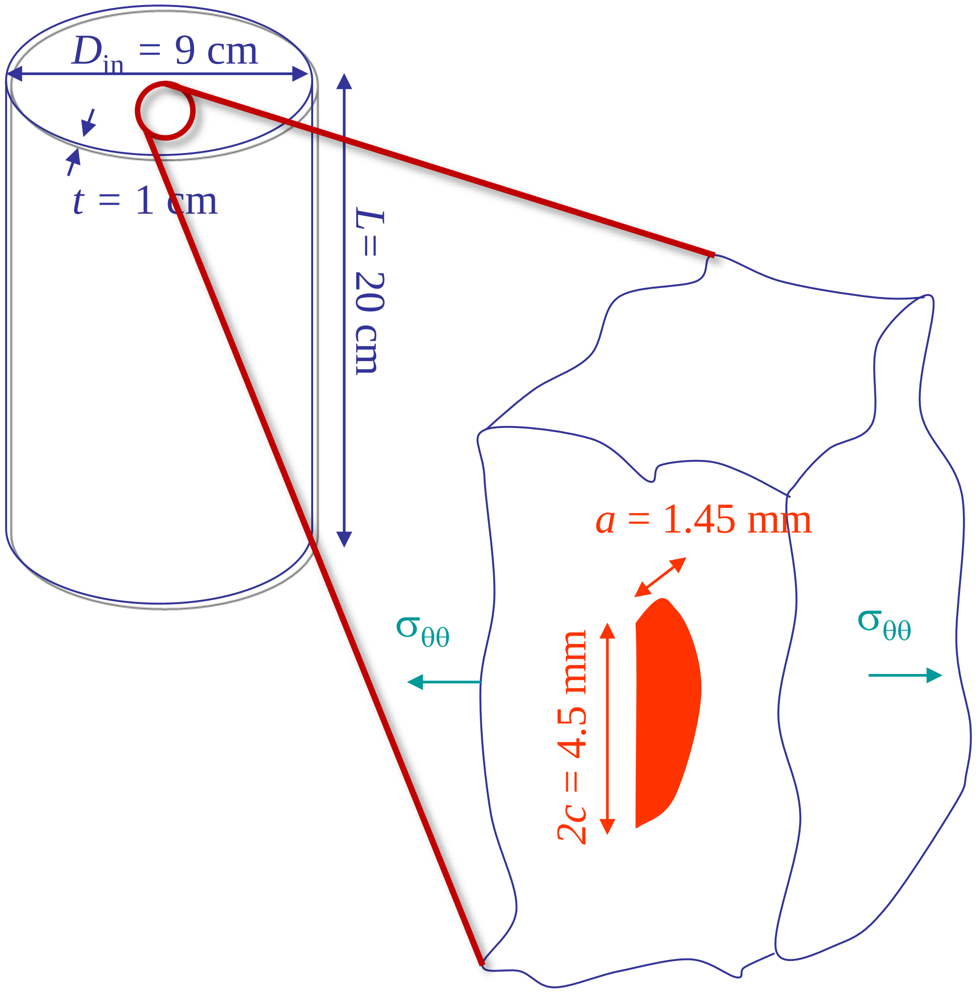
- Graphical resolution





Exercise 2

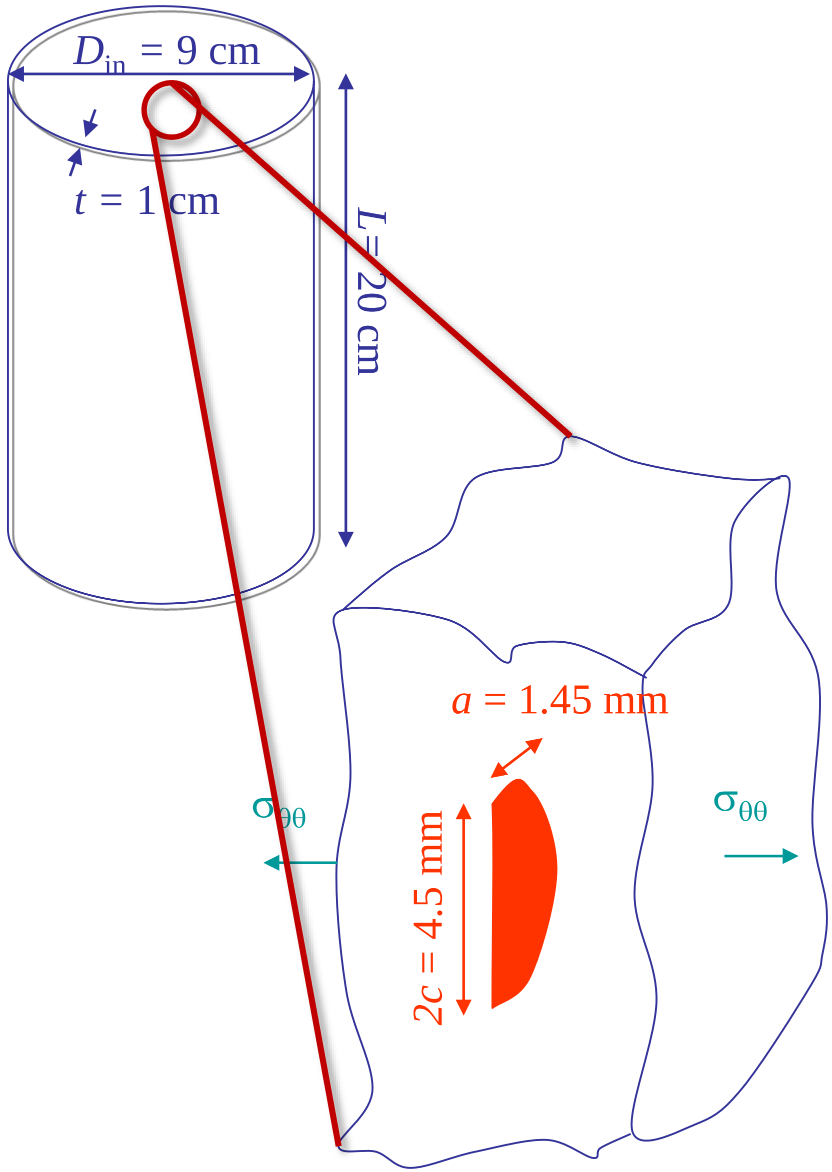
- Flawed cylinder
  - A piston is used to increase inner pressure
    - From 0 to 55 MPa
  - Cylinder made of
    - Peaked-aged aluminum alloy
      - 7075-T651
    - Yield σ_p^0 = 550 Mpa
    - Toughness K_{IC} = 30 MPa m$^{1/2}$
  - Malfunction
    - Cylinder burst
    - Post failure analysis reveals an initial elliptical flaw at inner wall
      - 4.5 mm long
      - 1.45 mm deep
      - Normal to hoop stress
  - Origin of burst?



- Stress field
  - Consider thick cylinder with
    - r_{in} = 0.045 m & r_{out} = 0.055 m
    - Inner pressure p

$$\begin{cases} \sigma_{rr}(r) = \dfrac{r_{\text{in}}^2 p}{r_{\text{out}}^2 - r_{\text{in}}^2}\left(1 - \dfrac{r_{\text{out}}^2}{r^2}\right) \\ \sigma_{\theta\theta}(r) = \dfrac{r_{\text{in}}^2 p}{r_{\text{out}}^2 - r_{\text{in}}^2}\left(1 + \dfrac{r_{\text{out}}^2}{r^2}\right) \end{cases}$$

$$\Rightarrow \begin{cases} \sigma_{rr}(r_{\text{in}} + a) = -0.808\, p \\ \sigma_{\theta\theta}(r_{\text{in}} + a) = 4.86\, p \end{cases}$$



Exercise 2: Solution

- SIF
  - Use SIF for semi-elliptical crack in large plate
    - See SIF handbook

$$K_I = \frac{1.12\sigma_{\theta\theta}\sqrt{\pi a}}{\Psi}$$

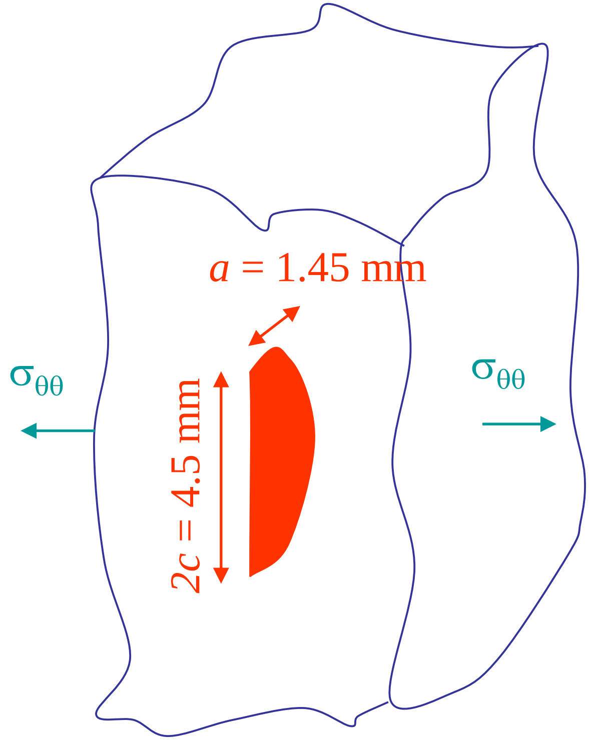
    - Geometrical effect

$$\Psi \simeq \frac{3\pi}{8} + \frac{\pi}{8}\left(\frac{a}{c}\right)^2 = 1.3412$$

    - Using $\sigma_{\theta\theta}\left(r_{\text{in}} + a\right) = 4.86\, p$

$$K_I = \frac{1.12\sigma_{\theta\theta}\sqrt{\pi a}}{\Psi} = 0.27392\, p\sqrt{\text{m}}$$

$$\Longrightarrow K_I(p = 55\text{ MPa}) = 15\text{ MPa}\cdot\sqrt{\text{m}}$$



  - Check LEFM validity
    - Initial crack $t, a_0 > 2.5\left(\frac{K_I}{\sigma_p^0}\right)^2 = 1.9\text{ mm}$?
    - At failure $t - a_{cr} > 2.5\left(\frac{K_C}{\sigma_p^0}\right)^2 = 7.4\text{ mm}$?
    - We need a plastic correction

- SIF (2)
  - Use SIF for semi-elliptical crack with plastic correction
    - See SIF handbook

$$K_I = \frac{1.12\sigma_{\theta\theta}\sqrt{\pi a}}{\sqrt{Q}}$$

    - Geometrical effect

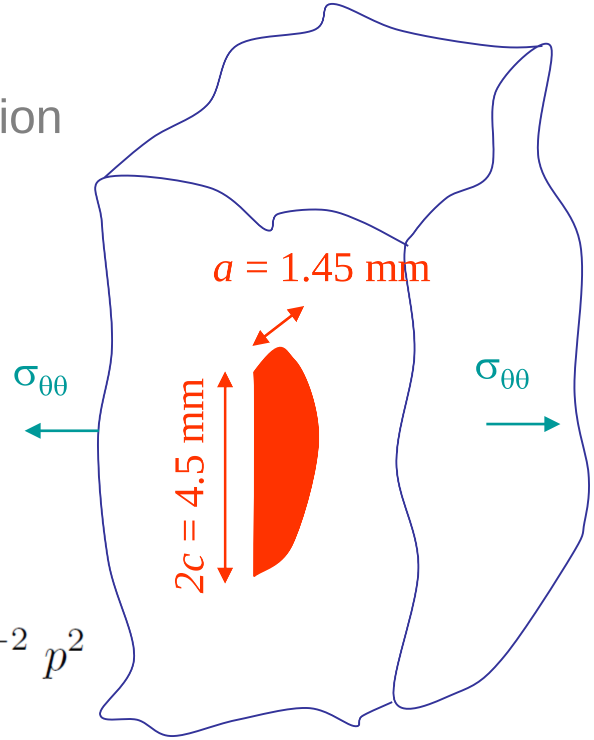
$$\Psi \simeq \frac{3\pi}{8} + \frac{\pi}{8}\left(\frac{a}{c}\right)^2 = 1.3412$$

    - Plastic correction

$$Q = \Psi^2 - 0.212\frac{\sigma_{\theta\theta}^2}{\sigma_p^{0\,2}} = 1.7988 - 1.654\,10^{-5}\ \text{MPa}^{-2}\ p^2$$

    - Using $\sigma_{\theta\theta}\left(r_{\text{in}} + a\right) = 4.86\ p$

$$K_I = 0.3672\sqrt{\text{m}}\ p\ \frac{1}{\sqrt{1.7988 - 1.654\,10^{-5}\ \text{MPa}^{-2}\ p^2}}$$



  - Limit load
    - $K_I\left(p = 104\ \text{MPa}\right) = 30\ \text{MPa}\sqrt{\text{m}} = K_{IC}$
    - Maximum pressure in the cylinder is 55MPa ⟹ failure by fatigue

- Cyclic loading
  - p from 0 to 55 Mpa
    - Hoop stress from 0 to $\sigma_{\theta\theta}(r) = \frac{r_{\text{in}}^2 p}{r_{\text{out}}^2 - r_{\text{in}}^2}\left(1 + \frac{r_{\text{out}}^2}{r^2}\right)$

$$\Longrightarrow \sigma_{\theta\theta}(r = r_{\text{in}},\, p = 55\,\text{MPa}) = 277.75\ \text{MPa}$$

  - SIF evolution
    - Assuming a/c remains constant: $\Psi \simeq \frac{3\pi}{8} + \frac{\pi}{8}\left(\frac{a}{c}\right)^2 = 1.3412$

$$\Longrightarrow Q = \Psi^2 - 0.212\frac{\sigma_{\theta\theta}^2}{\sigma_p^{0\,2}} = 1.7988 - 0.054 = 1.745$$

$$\Longrightarrow K_{\max} = 1.12\frac{\boldsymbol{\sigma}_{\theta\theta}\sqrt{\pi a}}{\sqrt{Q}} = 417.42\ \text{MPa}\sqrt{a}$$

    - At each cycle the pressure vanishes

$$\Longrightarrow K_{\min} = 0$$

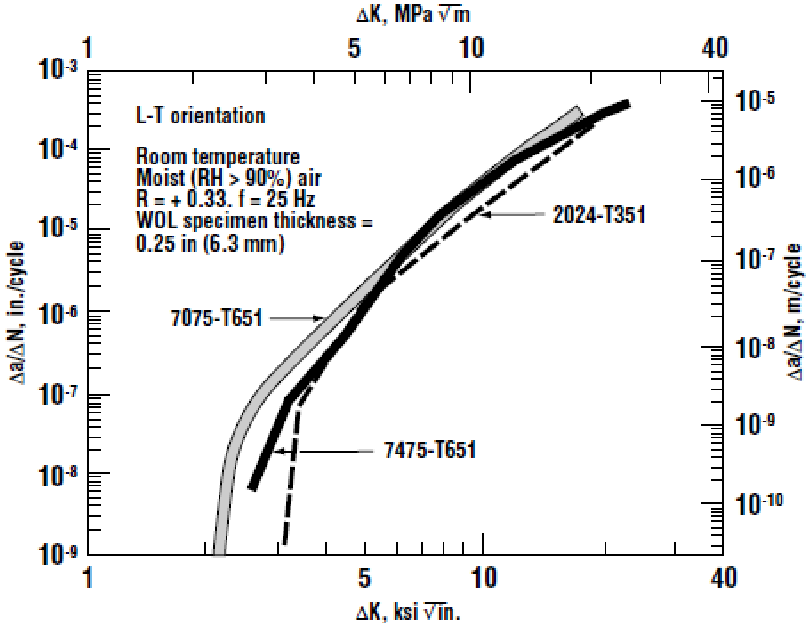
$$\Longrightarrow \Delta K(a) = 417.42\ \text{MPa} \cdot \sqrt{a}$$

- Cyclic loading
  - p from 0 to 55 Mpa

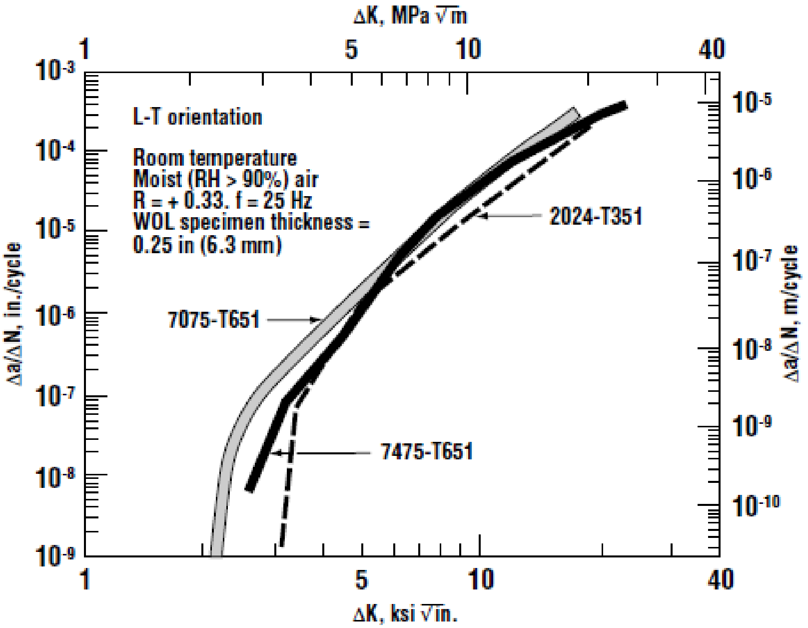
$$\begin{cases} K_{\max}(a) = 417.42 \text{ MPa} \cdot \sqrt{a} \\ \Delta K(a) = 417.42 \text{ MPa} \cdot \sqrt{a} \end{cases}$$

  - Due to initial flaw
    - $\Delta K(a_0) = 15.89 \text{ MPa} \cdot \sqrt{\text{m}}$
    - Assuming curves are valid for R=0
    - We are in Paris regime

      ⇒ crack propagation
  - Critical crack:
    - $a_{cr} = \left(\frac{K_{I_C}}{417.42 \text{ MPa}}\right)^2$=5.17 mm
  - Life of cylinder
    - Number of cycles?



- Cyclic loading (2)
  - Cylinder life
    - $\Delta K(a) = 417.42 \text{ MPa} \cdot \sqrt{a}$
    - Assuming curves are valid for R=0
    - We are in Paris regime



$$\Rightarrow \frac{da}{dN} = 3\ 10^{-11}\ \Delta K^{4.3}$$

$$\Rightarrow \frac{da}{dN} = 3\ 10^{-11}\ 417.42^{4.3}\sqrt{a}^{4.3} = 5.567a^{2.15}$$

$$\Rightarrow \frac{da}{a^{2.15}} = 5.567dN$$

$$\Rightarrow -\frac{1}{1.15}\left(\frac{1}{a_{cr}^{1.15}} - \frac{1}{a_0^{1.15}}\right) = 5.567N_f$$

$$\Rightarrow N_f = \frac{1}{1.15\ 5.567}\left(\frac{1}{0.00145^{1.15}} - \frac{1}{0.00517^{1.15}}\right) = 220 \text{ cycles}$$

- !!!Life strongly depends on the maximum pressure reached during accidents

References

- Lecture notes
  - Lecture Notes on Fracture Mechanics, Alan T. Zehnder, Cornell University, Ithaca, http://hdl.handle.net/1813/3075
  - Fracture Mechanics Online Class, L. Noels, ULg, http://www.ltas-cm3.ulg.ac.be/FractureMechanics
- Book
  - Fracture Mechanics: Fundamentals and applications, D. T. Anderson. CRC press, 1991.
  - S. Suresh, Fatigue of Materials, Cambridge University Press, 2001