University of Liège
Aerospace & Mechanical Engineering

Fracture Mechanics, Damage and Fatigue
Linear Elastic Fracture Mechanics - Asymptotic Solution

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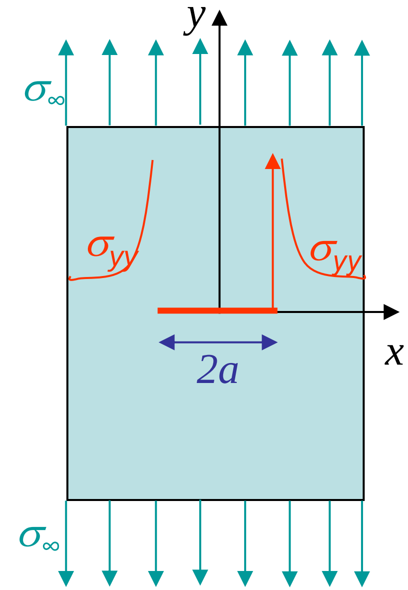
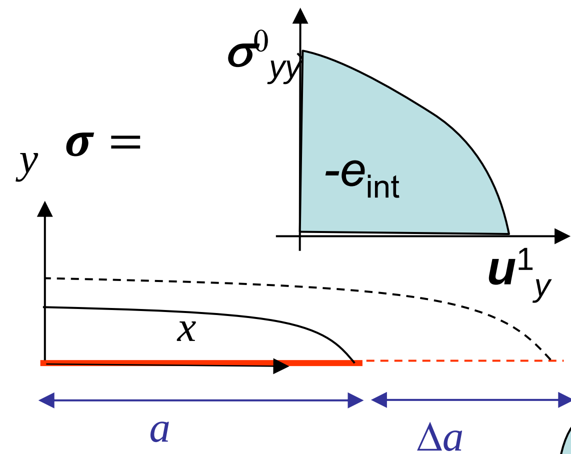
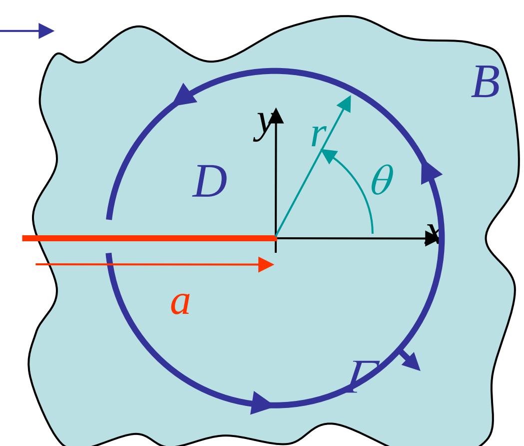


- Cracked body: different concepts
  - Stress intensity factor
    - Governing the asymptotic solution
    - $\boldsymbol{\sigma}_{yy} = \frac{K_I}{\sqrt{2\pi \mathrm{r}}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + o\left(r^0\right)$
  - Energy release rate
    - Variation of strain energy if crack grows
    - $G = -\frac{\partial \mathrm{Energy}}{\partial A}$



  - J-integral
    - Strain energy flow
    - $J = \int_{\Gamma} \left[U\left(\boldsymbol{\varepsilon}\right) \boldsymbol{n}_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T}\right] dl$



- More in depth
  - How are they defined?
  - How are they related?

- Cracked body
  - Where do the asymptotic solutions come form?

Mode I

$$
\begin{cases}
\sigma_{xx} = \frac{C}{\sqrt{r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{3\theta}{2}\sin\frac{\theta}{2}\right] + \cancel{\mathcal{O}\left(r^0\right)} \\
\boxed{\sigma_{yy} = \frac{C}{\sqrt{r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{3\theta}{2}\sin\frac{\theta}{2}\right] + \cancel{\mathcal{O}\left(r^0\right)}} \\
\sigma_{xy} = \frac{C}{\sqrt{r}} \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\sin\frac{\theta}{2} + \cancel{\mathcal{O}\left(r^0\right)} \\
u_x = \frac{C\left(1+\nu\right)}{E}\sqrt{r}\cos\left(\frac{\theta}{2}\right)\left[\kappa - 1 + 2\sin^2\left(\frac{\theta}{2}\right)\right] \\
u_y = \frac{C\left(1+\nu\right)}{E}\sqrt{r}\sin\left(\frac{\theta}{2}\right)\left[\kappa + 1 - 2\cos^2\left(\frac{\theta}{2}\right)\right]
\end{cases}
$$

Mode II

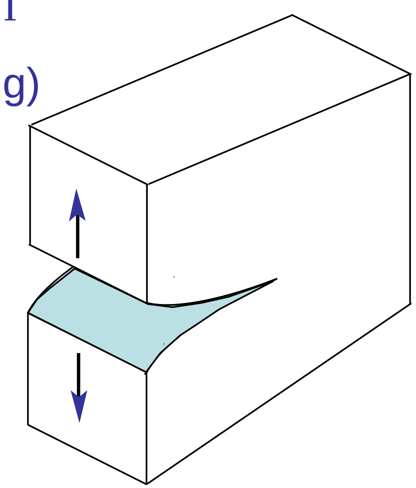
$$
\begin{cases}
\sigma_{xx} = -\frac{C}{\sqrt{r}} \sin\frac{\theta}{2}\left[2 + \cos\frac{3\theta}{2}\cos\frac{\theta}{2}\right] + \cancel{\mathcal{O}\left(r^0\right)} \\
\sigma_{yy} = \frac{C}{\sqrt{r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + \cancel{\mathcal{O}\left(r^0\right)} \\
\boxed{\sigma_{xy} = \frac{C}{\sqrt{r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{3\theta}{2}\sin\frac{\theta}{2}\right] + \cancel{\mathcal{O}\left(r^0\right)}} \\
u_x = \frac{C\left(1+\nu\right)}{E}\sqrt{r}\sin\left(\frac{\theta}{2}\right)\left[\kappa + 1 + 2\cos^2\left(\frac{\theta}{2}\right)\right] \\
u_y = -\frac{C\left(1+\nu\right)}{E}\sqrt{r}\cos\left(\frac{\theta}{2}\right)\left[\kappa - 1 - 2\sin^2\left(\frac{\theta}{2}\right)\right]
\end{cases}
$$

Mode III

$$
\begin{cases}
\sigma_{xz} = -\frac{C}{\sqrt{r}} \sin\frac{\theta}{2} + \cancel{\mathcal{O}\left(r^0\right)} \\
\boxed{\sigma_{yz} = \frac{C}{\sqrt{r}} \cos\frac{\theta}{2} + \cancel{\mathcal{O}\left(r^0\right)}} \\
u_z = \frac{4C\left(1+\nu\right)}{E}\sqrt{r}\sin\left(\frac{\theta}{2}\right)
\end{cases}
$$

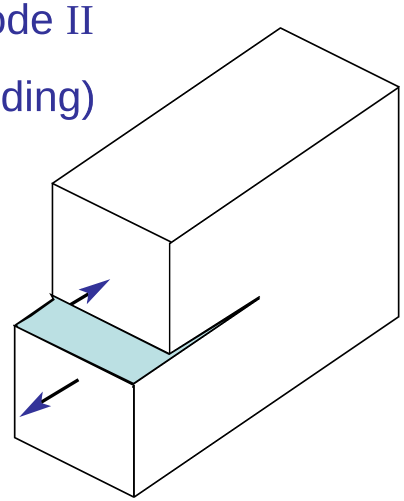
Mode I

(opening)



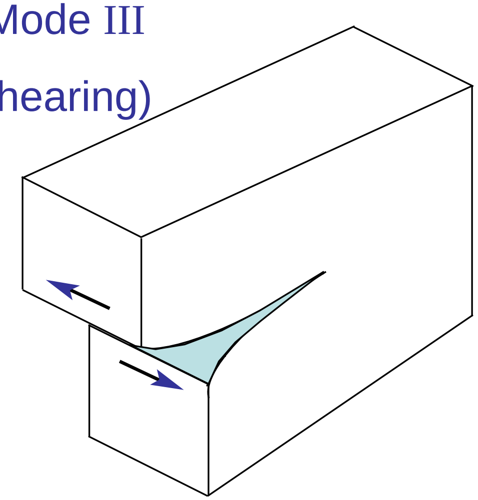
Mode II

(sliding)



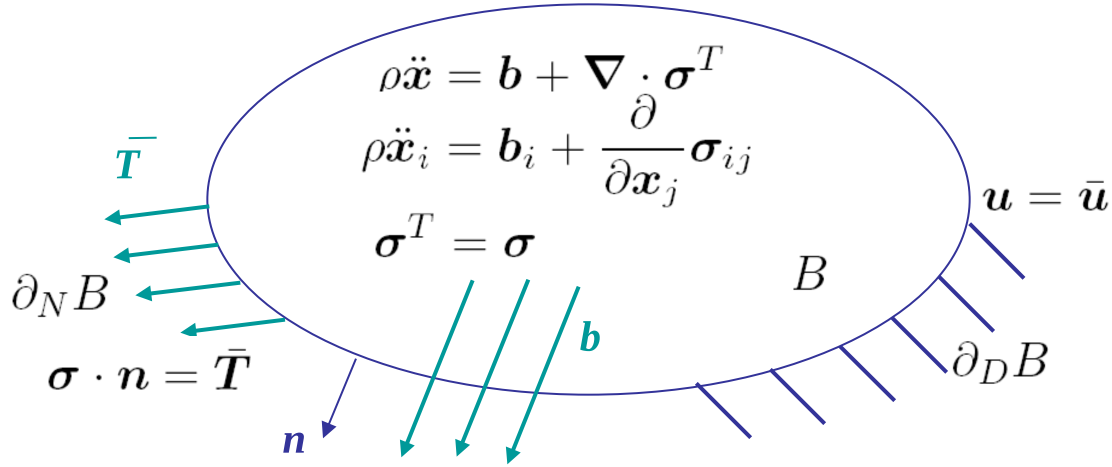
Mode III

(shearing)



LEFM assumptions

- Balance of body B
  - Momenta balance
    - Linear
    - Angular
  - Boundary conditions
    - Neumann
    - Dirichlet

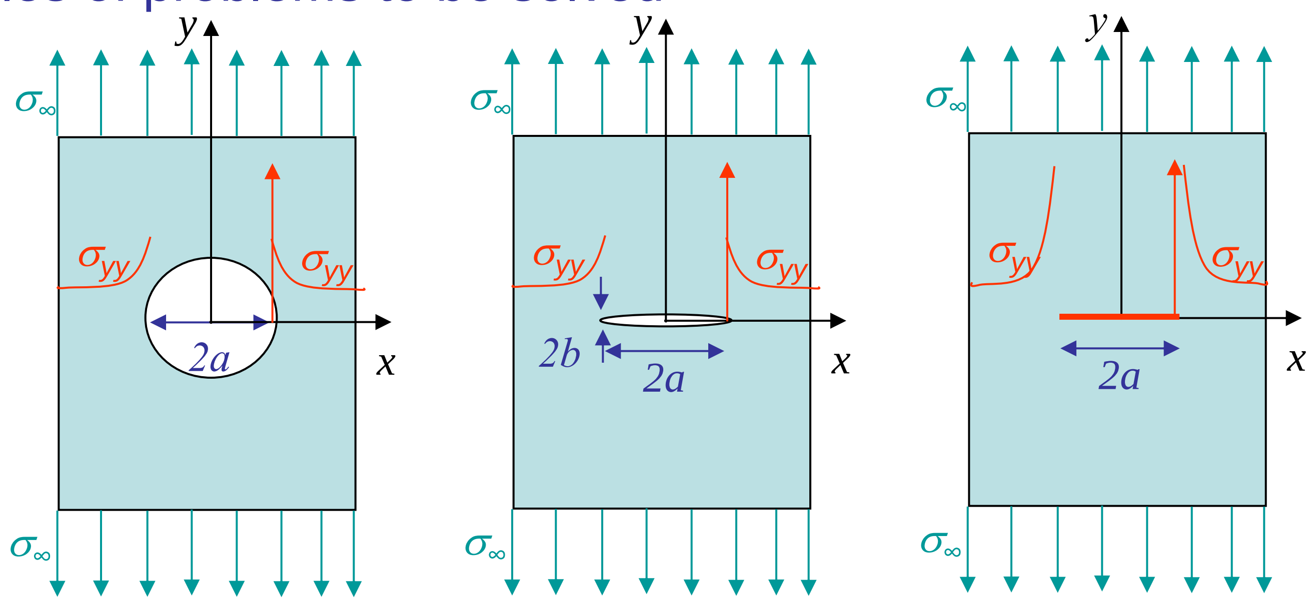


- Small deformations with linear elastic, homogeneous & isotropic material
  - (Small) Strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla}\right)$, or $\begin{cases} \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{x}_i}\boldsymbol{u}_j + \frac{\partial}{\partial \boldsymbol{x}_j}\boldsymbol{u}_i\right) \\ \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\boldsymbol{u}_{j,i} + \boldsymbol{u}_{i,j}\right) \end{cases}$
  - Hooke's law $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon}$, or $\boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl}\boldsymbol{\varepsilon}_{kl}$

    with $$\mathcal{H}_{ijkl} = \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda = K - 2\mu/3}\delta_{ij}\delta_{kl} + \underbrace{\frac{E}{1+\nu}}_{2\mu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$$
  - Inverse law $\boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma}$

    with $$\mathcal{G}_{ijkl} = \frac{1+\nu}{E}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right) - \frac{\nu}{E}\delta_{ij}\delta_{kl}$$

- Examples of problems to be solved



  – Static
  – 2D &
    - Plane σ ⟹ $\boldsymbol{\sigma}_{xz} = \boldsymbol{\sigma}_{yz} = \boldsymbol{\sigma}_{zz} = 0$
    - Or plane ε ⟹ $\left\{ \begin{array}{l} \boldsymbol{\varepsilon}_{xz} = \boldsymbol{\varepsilon}_{yz} = \boldsymbol{\varepsilon}_{zz} = 0 \\ \dfrac{\partial}{\partial z} = 0 \end{array} \right.$

- Static & 2D assumptions: Plane-ε case
  - Plane ε :
    - $\boldsymbol{\varepsilon}_{xz} = \boldsymbol{\varepsilon}_{yz} = \boldsymbol{\varepsilon}_{zz} = 0$ & $\dfrac{\partial}{\partial z} = 0$
  - Hooke's law
    - General $\begin{cases} \boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl}\boldsymbol{\varepsilon}_{kl} \\ \mathcal{H}_{ijkl} = \dfrac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \dfrac{E}{1+\nu}\left(\dfrac{1}{2}\delta_{ik}\delta_{jl} + \dfrac{1}{2}\delta_{il}\delta_{jk}\right) \end{cases}$
    - Plane ε

$$\boldsymbol{\varepsilon}_{kk} = \boldsymbol{\varepsilon}_{xx} + \boldsymbol{\varepsilon}_{yy} + \boldsymbol{\varepsilon}_{zz} = \boldsymbol{\varepsilon}_{xx} + \boldsymbol{\varepsilon}_{yy} = \boldsymbol{\varepsilon}_{\alpha\alpha}$$

$$- \begin{cases} \boldsymbol{\sigma}_{xx} = \dfrac{E\nu}{(1+\nu)(1-2\nu)}\cancelto{1}{\delta_{xx}}\boldsymbol{\varepsilon}_{kk} + \dfrac{E}{1+\nu}\boldsymbol{\varepsilon}_{xx} \\ \boldsymbol{\sigma}_{xy} = \dfrac{E\nu}{(1+\nu)(1-2\nu)}\cancelto{0}{\delta_{xy}}\boldsymbol{\varepsilon}_{kk} + \dfrac{E}{1+\nu}\boldsymbol{\varepsilon}_{xy} \end{cases}$$

$$\Rightarrow \boldsymbol{\sigma}_{\alpha\beta} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{\alpha\beta}\boldsymbol{\varepsilon}_{\gamma\gamma} + \frac{E}{1+\nu}\boldsymbol{\varepsilon}_{\alpha\beta}$$

Greek subscripts for x or y

Roman subscripts for x, y or z

  - N.B.: $\boldsymbol{\sigma}_{\gamma\gamma} = \dfrac{E\nu}{(1+\nu)(1-2\nu)}\delta_{\gamma\gamma}\boldsymbol{\varepsilon}_{\delta\delta} + \dfrac{E}{1+\nu}\boldsymbol{\varepsilon}_{\gamma\gamma} = \dfrac{E}{(1+\nu)(1-2\nu)}\boldsymbol{\varepsilon}_{\gamma\gamma}$

$$2\boldsymbol{\varepsilon}_{\delta\delta} = 2\boldsymbol{\varepsilon}_{xx} + 2\boldsymbol{\varepsilon}_{yy} = 2\boldsymbol{\varepsilon}_{\gamma\gamma}$$

- Static & 2D assumptions: Plane-ε case (2)
  - Plane ε:
    - $\boldsymbol{\varepsilon}_{xz} = \boldsymbol{\varepsilon}_{yz} = \boldsymbol{\varepsilon}_{zz} = 0$ & $\dfrac{\partial}{\partial z} = 0$
  - Hooke's law
    - General $\begin{cases} \boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl}\boldsymbol{\varepsilon}_{kl} \\ \mathcal{H}_{ijkl} = \dfrac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \dfrac{E}{1+\nu}\left(\dfrac{1}{2}\delta_{ik}\delta_{jl} + \dfrac{1}{2}\delta_{il}\delta_{jk}\right) \end{cases}$
    - Plane ε
      - $\begin{cases} \boldsymbol{\sigma}_{\alpha\beta} = \dfrac{E\nu}{(1+\nu)(1-2\nu)}\delta_{\alpha\beta}\boldsymbol{\varepsilon}_{\gamma\gamma} + \dfrac{E}{1+\nu}\boldsymbol{\varepsilon}_{\alpha\beta} \\ \boldsymbol{\sigma}_{\gamma\gamma} = \dfrac{E}{(1+\nu)(1-2\nu)}\boldsymbol{\varepsilon}_{\gamma\gamma} \end{cases}$
      - $\boldsymbol{\sigma}_{zz} = \dfrac{E\nu}{(1+\nu)(1-2\nu)}\delta_{zz}\boldsymbol{\varepsilon}_{kk} + \dfrac{E}{1+\nu}\cancelto{0}{\boldsymbol{\varepsilon}_{zz}}$

$\boldsymbol{\varepsilon}_{\delta\delta} = \boldsymbol{\varepsilon}_{xx} + \boldsymbol{\varepsilon}_{yy} = \boldsymbol{\varepsilon}_{\gamma\gamma}$

Greek subscripts for x or y

Roman subscripts for x, y or z

$$\Rightarrow \boldsymbol{\sigma}_{zz} = \frac{E\nu}{(1+\nu)(1-2\nu)}\boldsymbol{\varepsilon}_{\gamma\gamma} = \nu\boldsymbol{\sigma}_{\gamma\gamma}$$

- Static & 2D assumptions: Plane-ε case (3)
  - Plane ε:
    - $\boldsymbol{\varepsilon}_{xz} = \boldsymbol{\varepsilon}_{yz} = \boldsymbol{\varepsilon}_{zz} = 0$ & $\dfrac{\partial}{\partial z} = 0$
    - $\boldsymbol{\sigma}_{zz} = \dfrac{E\nu}{(1+\nu)(1-2\nu)}\boldsymbol{\varepsilon}_{\gamma\gamma} = \nu\boldsymbol{\sigma}_{\gamma\gamma}$
  - Hooke's law
    - General $\begin{cases} \boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma} \\ \mathcal{G}_{ijkl} = \dfrac{1+\nu}{E}\left(\dfrac{1}{2}\delta_{ik}\delta_{jl} + \dfrac{1}{2}\delta_{il}\delta_{jk}\right) - \dfrac{\nu}{E}\delta_{ij}\delta_{kl} \end{cases}$
    - Plane ε

$$
- \begin{cases} \boldsymbol{\varepsilon}_{xx} = \dfrac{1+\nu}{E}\boldsymbol{\sigma}_{xx} - \dfrac{\nu}{E}\delta_{xx}\boldsymbol{\sigma}_{kk} = \dfrac{1+\nu}{E}\left(\boldsymbol{\sigma}_{xx} - \nu\boldsymbol{\sigma}_{\gamma\gamma}\right) \\ \boldsymbol{\varepsilon}_{xy} = \dfrac{1+\nu}{E}\boldsymbol{\sigma}_{xy} - \dfrac{\nu}{E}\delta_{xy}\boldsymbol{\sigma}_{kk} = \dfrac{1+\nu}{E}\boldsymbol{\sigma}_{xy} \end{cases}
$$

$$\boldsymbol{\sigma}_{kk} = \boldsymbol{\sigma}_{xx} + \boldsymbol{\sigma}_{yy} + \boldsymbol{\sigma}_{zz} = \boldsymbol{\sigma}_{\gamma\gamma} + \nu\boldsymbol{\sigma}_{\gamma\gamma}$$

Greek subscripts for x or y

Roman subscripts for x, y or z

$$\Longrightarrow \quad \boldsymbol{\varepsilon}_{\alpha\beta} = \frac{1+\nu}{E}\left(\boldsymbol{\sigma}_{\alpha\beta} - \nu\boldsymbol{\sigma}_{\gamma\gamma}\delta_{\alpha\beta}\right)$$

- Static & 2D assumptions: Plane-σ case
  - Plane σ:
    - $\boldsymbol{\sigma}_{xz} = \boldsymbol{\sigma}_{yz} = \boldsymbol{\sigma}_{zz} = 0$
  - Hooke's law
    - General $\begin{cases} \boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma} \\ \mathcal{G}_{ijkl} = \dfrac{1+\nu}{E}\left(\dfrac{1}{2}\delta_{ik}\delta_{jl} + \dfrac{1}{2}\delta_{il}\delta_{jk}\right) - \dfrac{\nu}{E}\delta_{ij}\delta_{kl} \end{cases}$
    - Plane σ

$\boldsymbol{\sigma}_{kk} = \boldsymbol{\sigma}_{xx} + \boldsymbol{\sigma}_{yy} + \boldsymbol{\sigma}_{zz} = \boldsymbol{\sigma}_{xx} + \boldsymbol{\sigma}_{yy} = \boldsymbol{\sigma}_{\alpha\alpha}$

$$- \begin{cases} \boldsymbol{\varepsilon}_{xx} = \dfrac{1+\nu}{E}\boldsymbol{\sigma}_{xx} - \dfrac{\nu}{E}\cancelto{1}{\delta_{xx}}\boldsymbol{\sigma}_{kk} = \dfrac{1+\nu}{E}\boldsymbol{\sigma}_{xx} - \dfrac{\nu}{E}\boldsymbol{\sigma}_{\gamma\gamma} \\ \boldsymbol{\varepsilon}_{xy} = \dfrac{1+\nu}{E}\boldsymbol{\sigma}_{xy} - \dfrac{\nu}{E}\cancelto{0}{\delta_{xy}}\boldsymbol{\sigma}_{kk} = \dfrac{1+\nu}{E}\boldsymbol{\sigma}_{xy} \end{cases}$$

$$\Rightarrow \boldsymbol{\varepsilon}_{\alpha\beta} = \frac{1+\nu}{E}\boldsymbol{\sigma}_{\alpha\beta} - \frac{\nu}{E}\delta_{\alpha\beta}\boldsymbol{\sigma}_{\gamma\gamma}$$

Greek subscripts for x or y

Roman subscripts for x, y or z

- N.B.: $\boldsymbol{\varepsilon}_{\gamma\gamma} = \dfrac{1+\nu}{E}\boldsymbol{\sigma}_{\gamma\gamma} - \dfrac{\nu}{E}\delta_{\gamma\gamma}\boldsymbol{\sigma}_{\delta\delta} = \dfrac{1-\nu}{E}\boldsymbol{\sigma}_{\gamma\gamma}$

$2\boldsymbol{\sigma}_{\delta\delta} = 2\boldsymbol{\sigma}_{xx} + 2\boldsymbol{\sigma}_{yy} = 2\boldsymbol{\sigma}_{\gamma\gamma}$

- Static & 2D assumptions: Plane-σ case (2)
  - Plane σ:
    - $\boldsymbol{\sigma}_{xz} = \boldsymbol{\sigma}_{yz} = \boldsymbol{\sigma}_{zz} = 0$
  - Hooke's law
    - General $\begin{cases} \boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma} \\ \mathcal{G}_{ijkl} = \dfrac{1+\nu}{E}\left(\dfrac{1}{2}\delta_{ik}\delta_{jl} + \dfrac{1}{2}\delta_{il}\delta_{jk}\right) - \dfrac{\nu}{E}\delta_{ij}\delta_{kl} \end{cases}$
    - Plane σ
      - $\begin{cases} \boldsymbol{\varepsilon}_{\alpha\beta} = \dfrac{1+\nu}{E}\boldsymbol{\sigma}_{\alpha\beta} - \dfrac{\nu}{E}\delta_{\alpha\beta}\boldsymbol{\sigma}_{\gamma\gamma} \\ \boldsymbol{\varepsilon}_{\gamma\gamma} = \dfrac{1+\nu}{E}\boldsymbol{\sigma}_{\gamma\gamma} - \dfrac{\nu}{E}\delta_{\gamma\gamma}\boldsymbol{\sigma}_{\delta\delta} = \dfrac{1-\nu}{E}\boldsymbol{\sigma}_{\gamma\gamma} \end{cases}$
      - $\boldsymbol{\varepsilon}_{zz} = \dfrac{1+\nu}{E}\cancelto{0}{\boldsymbol{\sigma}_{zz}} - \dfrac{\nu}{E}\boldsymbol{\sigma}_{\gamma\gamma}$

$$\Rightarrow \quad \boldsymbol{\varepsilon}_{zz} = -\frac{\nu}{E}\boldsymbol{\sigma}_{\gamma\gamma} = -\frac{\nu}{1-\nu}\boldsymbol{\varepsilon}_{\gamma\gamma}$$

Greek subscripts for x or y

Roman subscripts for x, y or z

- Static & 2D assumptions: Plane-σ case (3)
  - Plane σ:
    - $\boldsymbol{\sigma}_{xz} = \boldsymbol{\sigma}_{yz} = \boldsymbol{\sigma}_{zz} = 0$
    - $\boldsymbol{\varepsilon}_{zz} = -\frac{\nu}{E}\boldsymbol{\sigma}_{\gamma\gamma} = -\frac{\nu}{1-\nu}\boldsymbol{\varepsilon}_{\gamma\gamma}$
  - Hooke's law
    - General $\begin{cases} \boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl}\boldsymbol{\varepsilon}_{kl} \\ \mathcal{H}_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{1+\nu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right) \end{cases}$
    - Plane σ

$$
-\begin{cases} \boldsymbol{\sigma}_{xx} = \frac{E\nu}{(1+\nu)(1-2\nu)}\cancelto{1}{\delta_{xx}}\boldsymbol{\varepsilon}_{kk} + \frac{E}{1+\nu}\boldsymbol{\varepsilon}_{xx} \\ \boldsymbol{\sigma}_{xy} = \frac{E\nu}{(1+\nu)(1-2\nu)}\cancelto{0}{\delta_{xy}}\boldsymbol{\varepsilon}_{kk} + \frac{E}{1+\nu}\boldsymbol{\varepsilon}_{xy} \end{cases}
$$

$$
\boldsymbol{\varepsilon}_{kk} = \boldsymbol{\varepsilon}_{\gamma\gamma} + \boldsymbol{\varepsilon}_{zz} = \frac{(1-2\nu)}{1-\nu}\boldsymbol{\varepsilon}_{\gamma\gamma}
$$

Greek subscripts for x or y

Roman subscripts for x, y or z

$$
\Longrightarrow \boldsymbol{\sigma}_{\alpha\beta} = \frac{E\nu}{1-\nu^2}\delta_{\alpha\beta}\boldsymbol{\varepsilon}_{\gamma\gamma} + \frac{E}{1+\nu}\boldsymbol{\varepsilon}_{\alpha\beta}
$$

- Static & 2D assumptions: Hooke's law
  - Plane σ ($\boldsymbol{\sigma}_{xz} = \boldsymbol{\sigma}_{yz} = \boldsymbol{\sigma}_{zz} = 0$) or plane ε ($\boldsymbol{\varepsilon}_{xz} = \boldsymbol{\varepsilon}_{yz} = \boldsymbol{\varepsilon}_{zz} = 0$ & $\frac{\partial}{\partial z} = 0$)
  - Hooke's law
    - Plane ε:

$$\begin{cases} \boldsymbol{\sigma}_{\alpha\beta} = \dfrac{E\nu}{(1+\nu)(1-2\nu)}\delta_{\alpha\beta}\boldsymbol{\varepsilon}_{\gamma\gamma} + \dfrac{E}{1+\nu}\boldsymbol{\varepsilon}_{\alpha\beta} \\ \boldsymbol{\sigma}_{zz} = \dfrac{E\nu}{(1+\nu)(1-2\nu)}\boldsymbol{\varepsilon}_{\gamma\gamma} = \nu\boldsymbol{\sigma}_{\gamma\gamma} \end{cases}$$

&

$$\boldsymbol{\varepsilon}_{\alpha\beta} = \frac{1+\nu}{E}\left(\boldsymbol{\sigma}_{\alpha\beta} - \nu\boldsymbol{\sigma}_{\gamma\gamma}\delta_{\alpha\beta}\right)$$

    - Plane σ:

$$\begin{cases} \boldsymbol{\varepsilon}_{\alpha\beta} = \dfrac{1+\nu}{E}\boldsymbol{\sigma}_{\alpha\beta} - \dfrac{\nu}{E}\delta_{\alpha\beta}\boldsymbol{\sigma}_{\gamma\gamma} \\ \boldsymbol{\varepsilon}_{zz} = -\dfrac{\nu}{E}\boldsymbol{\sigma}_{\gamma\gamma} = -\dfrac{\nu}{1-\nu}\boldsymbol{\varepsilon}_{\gamma\gamma} \end{cases}$$

&

$$\boldsymbol{\sigma}_{\alpha\beta} = \frac{E\nu}{1-\nu^2}\delta_{\alpha\beta}\boldsymbol{\varepsilon}_{\gamma\gamma} + \frac{E}{1+\nu}\boldsymbol{\varepsilon}_{\alpha\beta}$$

  - Greek subscripts substitute for x or y (Roman for x, y or z)

- Static & 2D assumptions: Linear momentum
  - Plane σ ($\boldsymbol{\sigma}_{xz} = \boldsymbol{\sigma}_{yz} = \boldsymbol{\sigma}_{zz} = 0$) or plane ε ($\boldsymbol{\varepsilon}_{xz} = \boldsymbol{\varepsilon}_{yz} = \boldsymbol{\varepsilon}_{zz} = 0$ & $\frac{\partial}{\partial z} = 0$)

    - Linear momentum $\underbrace{\rho\ddot{\boldsymbol{x}}}_{0} = \boldsymbol{b} + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T$

$$0 = \boldsymbol{b}_\alpha + \sigma_{\alpha x,x} + \sigma_{\alpha y,y} + \underbrace{\sigma_{\alpha z,z}}_{0} \qquad \boldsymbol{\sigma}_{\alpha z} = 0 \text{ or } \partial_z = 0$$

$$\Longrightarrow \boldsymbol{\sigma}_{\alpha\beta,\beta} + \boldsymbol{b}_\alpha = 0$$

    - If $\boldsymbol{b} = 0$, there exists an Airy function Φ: $\boldsymbol{\sigma}_{\alpha\beta} = -\Phi_{,\alpha\beta} + \delta_{\alpha\beta}\Phi_{,\gamma\gamma}$
      - Indeed $\boldsymbol{\sigma}_{\alpha\beta,\beta} = -\Phi_{,\alpha\beta\beta} + \delta_{\alpha\beta}\Phi_{,\gamma\gamma\beta}$

$$\boldsymbol{\Phi}_{,\gamma\gamma\alpha} = \boldsymbol{\Phi}_{,\beta\beta\alpha} = \boldsymbol{\Phi}_{,\alpha\beta\beta}$$

$$\Longrightarrow \boldsymbol{\sigma}_{\alpha\beta,\beta} = -\Phi_{,\alpha\beta\beta} + \Phi_{,\alpha\beta\beta} = 0$$

    - Airy function:
      - Linear momentum equation is replaced by finding Φ and defining
      - $\boldsymbol{\sigma}_{\alpha\beta} = -\Phi_{,\alpha\beta} + \delta_{\alpha\beta}\Phi_{,\gamma\gamma}$

- Static & 2D assumptions: Compatibility
  - Plane σ ($\boldsymbol{\sigma}_{xz} = \boldsymbol{\sigma}_{yz} = \boldsymbol{\sigma}_{zz} = 0$) or plane ε ($\boldsymbol{\varepsilon}_{xz} = \boldsymbol{\varepsilon}_{yz} = \boldsymbol{\varepsilon}_{zz} = 0$ & $\frac{\partial}{\partial z} = 0$)
    - Strain definition

$$\boldsymbol{\varepsilon}_{\alpha\beta} = \frac{1}{2}\left(\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}\right)$$

$$\Longrightarrow \begin{cases} \boldsymbol{\varepsilon}_{xx} = \boldsymbol{u}_{x,x} \\ \boldsymbol{\varepsilon}_{yy} = \boldsymbol{u}_{y,y} \\ 2\boldsymbol{\varepsilon}_{xy} = \boldsymbol{u}_{x,y} + \boldsymbol{u}_{y,x} \end{cases} \Longrightarrow \begin{cases} \boldsymbol{\varepsilon}_{xx,yy} = \boldsymbol{u}_{x,xyy} \\ \boldsymbol{\varepsilon}_{yy,xx} = \boldsymbol{u}_{y,yxx} \\ 2\boldsymbol{\varepsilon}_{xy,xy} = \boldsymbol{u}_{x,yxy} + \boldsymbol{u}_{y,xxy} \end{cases}$$

$$\Longrightarrow \boldsymbol{\varepsilon}_{xx,yy} + \boldsymbol{\varepsilon}_{yy,xx} = 2\boldsymbol{\varepsilon}_{xy,xy}$$

- Static & 2D assumptions: Bi-harmonic equation in plane-ε
  - Airy
    - $\boldsymbol{\sigma}_{\alpha\beta} = -\Phi_{,\alpha\beta} + \delta_{\alpha\beta}\Phi_{,\gamma\gamma} \Longrightarrow \sigma_{\delta\delta} = -\Phi_{,\delta\delta} + \delta_{\delta\delta}\Phi_{,\gamma\gamma} = \Phi_{,\delta\delta}$
  - Plane ε
    - $\boldsymbol{\varepsilon}_{\alpha\beta} = \frac{1+\nu}{E}\left(\boldsymbol{\sigma}_{\alpha\beta} - \nu\boldsymbol{\sigma}_{\gamma\gamma}\delta_{\alpha\beta}\right)$

      $\Longrightarrow \boldsymbol{\varepsilon}_{\alpha\beta} = \frac{1+\nu}{E}\left[-\Phi_{,\alpha\beta} + (1-\nu)\,\delta_{\alpha\beta}\Phi_{,\gamma\gamma}\right]$
  - Compatibility
    - $$\begin{cases} \boldsymbol{\varepsilon}_{xx,yy} = \frac{1+\nu}{E}\left(\boxed{\cancel{-\Phi_{,xxyy}}} + (1-\nu)\left(\Phi_{,xxyy} + \Phi_{,yyyy}\right)\right) \\ \boldsymbol{\varepsilon}_{yy,xx} = \frac{1+\nu}{E}\left(\boxed{\cancel{-\Phi_{,yyxx}}} + (1-\nu)\left(\Phi_{,xxxx} + \Phi_{,yyxx}\right)\right) \\ 2\boldsymbol{\varepsilon}_{xy,xy} = \frac{1+\nu}{E}\left(\boxed{\cancel{-2\,\Phi_{,xyxy}}}\right) \end{cases}$$
    - $\boldsymbol{\varepsilon}_{xx,yy} + \boldsymbol{\varepsilon}_{yy,xx} = 2\boldsymbol{\varepsilon}_{xy,xy}$

      $\Longrightarrow 0 = \Phi_{,xxyy} + \Phi_{,yyyy} + \Phi_{,xxxx} + \Phi_{,yyxx}$

      $0 = \left(\partial_{xx} + \partial_{yy}\right)\left(\Phi_{,xx} + \Phi_{,yy}\right) = \left(\partial_{xx} + \partial_{yy}\right)\left(\partial_{xx} + \partial_{yy}\right)\Phi$

      $\Longrightarrow \boxed{\nabla^2\nabla^2\Phi = 0}$

- Static & 2D assumptions: Bi-harmonic equation in plane-σ
  - Airy
    - $\boldsymbol{\sigma}_{\alpha\beta} = -\Phi_{,\alpha\beta} + \delta_{\alpha\beta}\Phi_{,\gamma\gamma}$ $\Longrightarrow$ $\sigma_{\delta\delta} = -\Phi_{,\delta\delta} + \delta_{\delta\delta}\Phi_{,\gamma\gamma} = \Phi_{,\delta\delta}$
  - Plane σ
    - $\boldsymbol{\varepsilon}_{\alpha\beta} = \frac{1+\nu}{E}\boldsymbol{\sigma}_{\alpha\beta} - \frac{\nu}{E}\delta_{\alpha\beta}\boldsymbol{\sigma}_{\gamma\gamma}$

    $\Longrightarrow$ $\boldsymbol{\varepsilon}_{\alpha\beta} = -\frac{1+\nu}{E}\Phi_{,\alpha\beta} + \frac{1}{E}\delta_{\alpha\beta}\Phi_{,\gamma\gamma}$
  - Compatibility
    - $$\begin{cases} \boldsymbol{\varepsilon}_{xx,yy} = \frac{1+\nu}{E}\left(\boxed{\cancel{-\Phi_{,xxyy}}}\right) + \frac{1}{E}\left(\Phi_{,xxyy} + \Phi_{,yyyy}\right) \\ \boldsymbol{\varepsilon}_{yy,xx} = \frac{1+\nu}{E}\left(\boxed{\cancel{-\Phi_{,yyxx}}}\right) + \frac{1}{E}\left(\Phi_{,xxxx} + \Phi_{,yyxx}\right) \\ 2\boldsymbol{\varepsilon}_{xy,xy} = \frac{1+\nu}{E}\left(\boxed{\cancel{-2\,\Phi_{,xyxy}}}\right) \end{cases}$$
    - $\boldsymbol{\varepsilon}_{xx,yy} + \boldsymbol{\varepsilon}_{yy,xx} = 2\boldsymbol{\varepsilon}_{xy,xy}$

    $\Longrightarrow$ $0 = \Phi_{,xxyy} + \Phi_{,yyyy} + \Phi_{,xxxx} + \Phi_{,yyxx}$

    $0 = \left(\partial_{xx} + \partial_{yy}\right)\left(\Phi_{,xx} + \Phi_{,yy}\right) = \left(\partial_{xx} + \partial_{yy}\right)\left(\partial_{xx} + \partial_{yy}\right)\Phi$

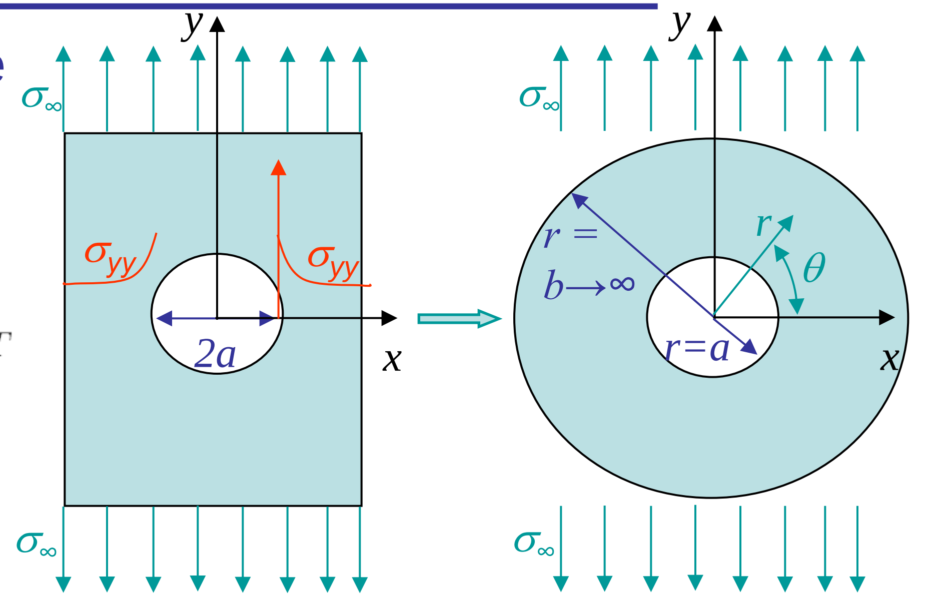
    $\Longrightarrow$ $\boxed{\nabla^2\nabla^2\Phi = 0}$

- Static & 2D assumptions: Summary
  - Plane σ ($\boldsymbol{\sigma}_{xz} = \boldsymbol{\sigma}_{yz} = \boldsymbol{\sigma}_{zz} = 0$) or plane ε ($\boldsymbol{\varepsilon}_{xz} = \boldsymbol{\varepsilon}_{yz} = \boldsymbol{\varepsilon}_{zz} = 0$ & $\frac{\partial}{\partial z} = 0$)
  - Hooke's law
    - Plane ε : $\begin{cases} \boldsymbol{\sigma}_{\alpha\beta} = \dfrac{E\nu}{(1+\nu)(1-2\nu)}\delta_{\alpha\beta}\boldsymbol{\varepsilon}_{\gamma\gamma} + \dfrac{E}{1+\nu}\boldsymbol{\varepsilon}_{\alpha\beta} \\ \boldsymbol{\sigma}_{zz} = \dfrac{E\nu}{(1+\nu)(1-2\nu)}\boldsymbol{\varepsilon}_{\gamma\gamma} = \nu\boldsymbol{\sigma}_{\gamma\gamma} \end{cases}$

      & $\boldsymbol{\varepsilon}_{\alpha\beta} = \dfrac{1+\nu}{E}\left(\boldsymbol{\sigma}_{\alpha\beta} - \nu\boldsymbol{\sigma}_{\gamma\gamma}\delta_{\alpha\beta}\right)$
    - Plane σ : $\begin{cases} \boldsymbol{\varepsilon}_{\alpha\beta} = \dfrac{1+\nu}{E}\boldsymbol{\sigma}_{\alpha\beta} - \dfrac{\nu}{E}\delta_{\alpha\beta}\boldsymbol{\sigma}_{\gamma\gamma} \\ \boldsymbol{\varepsilon}_{zz} = -\dfrac{\nu}{E}\boldsymbol{\sigma}_{\gamma\gamma} = -\dfrac{\nu}{1-\nu}\boldsymbol{\varepsilon}_{\gamma\gamma} \end{cases}$

      & $\boldsymbol{\sigma}_{\alpha\beta} = \dfrac{E\nu}{1-\nu^2}\delta_{\alpha\beta}\boldsymbol{\varepsilon}_{\gamma\gamma} + \dfrac{E}{1+\nu}\boldsymbol{\varepsilon}_{\alpha\beta}$
  - Linear momentum equation
    - Is replaced by finding Φ and defining $\boxed{\boldsymbol{\sigma}_{\alpha\beta} = -\Phi_{,\alpha\beta} + \delta_{\alpha\beta}\Phi_{,\gamma\gamma}}$
  - Hooke's law
    - Is replaced by bi-harmonic equations $\boxed{\nabla^2\nabla^2\Phi = 0}$

- Application: infinite plate with a hole
  - Cylindrical coordinates



$$\begin{pmatrix} \boldsymbol{\sigma}_{rr} & \boldsymbol{\sigma}_{r\theta} \\ \boldsymbol{\sigma}_{r\theta} & \boldsymbol{\sigma}_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_{xx} & \boldsymbol{\sigma}_{xy} \\ \boldsymbol{\sigma}_{xy} & \boldsymbol{\sigma}_{yy} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^T$$

- Boundary conditions
  - At $r = a$ $\quad \boldsymbol{\sigma}_{rr}(r=a,\theta) = \boldsymbol{\sigma}_{r\theta}(r=a,\theta) = 0$
  - At $r = b \rightarrow \infty$: only $\sigma_{yy} = \sigma_\infty$ is non zero

$$\begin{cases} \boldsymbol{\sigma}_{rr}(r=b,\theta) = \sigma_\infty \sin^2\theta = \frac{1}{2}\sigma_\infty - \frac{1}{2}\sigma_\infty \cos 2\theta \\ \boldsymbol{\sigma}_{r\theta}(r=b,\theta) = \sigma_\infty \cos\theta \sin\theta = \frac{1}{2}\sigma_\infty \sin 2\theta \end{cases}$$

Load case 1 independent from θ

Load case 2 dependent on $\cos 2\theta$ (symmetry with respect to x)

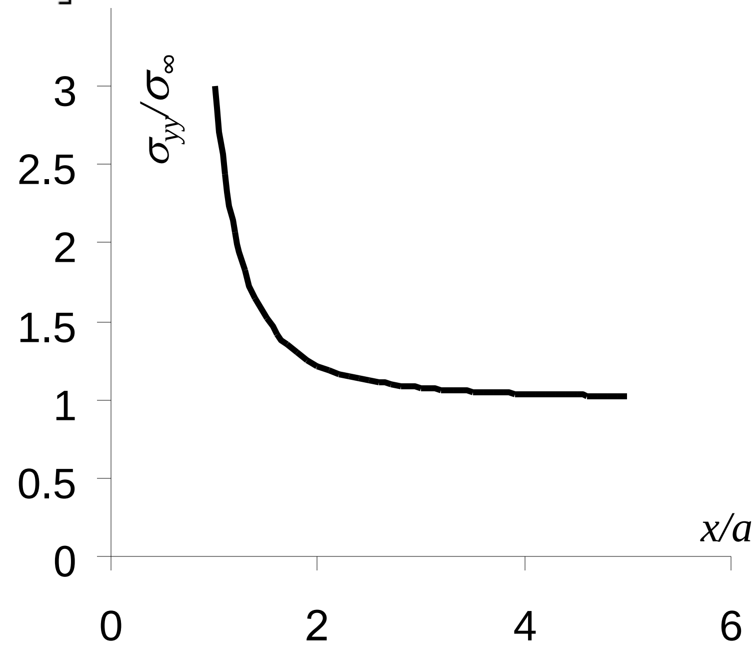
- Application: infinite plate with a hole (2)
  - For both loading cases: find potential Φ such that
    - Bi-harmonic equation is verified $\nabla^2\nabla^2\Phi = 0$
    - Stress field $\boldsymbol{\sigma}_{\alpha\beta} = -\Phi_{,\alpha\beta} + \delta_{\alpha\beta}\Phi_{,\gamma\gamma}$ satisfies the boundary conditions
  - Superposition of cases 1 and 2 (see appendix 1)
    - Stresses:

$$\begin{cases} \boldsymbol{\sigma}_{rr} = \dfrac{\sigma_\infty}{2}\left[\left(\dfrac{4a^2}{r^2} - 1 - \dfrac{3a^4}{r^4}\right)\cos 2\theta + 1 - \dfrac{a^2}{r^2}\right] \\ \boldsymbol{\sigma}_{\theta\theta} = \dfrac{\sigma_\infty}{2}\left[\left(1 + \dfrac{3a^4}{r^4}\right)\cos 2\theta + \dfrac{a^2}{r^2} + 1\right] \\ \boldsymbol{\sigma}_{r\theta} = \dfrac{\sigma_\infty}{2}\left(1 - \dfrac{3a^4}{r^4} + \dfrac{2a^2}{r^2}\right)\sin 2\theta \end{cases}$$

    - For $\theta = 0$: $\sigma_{yy} = \sigma_{\theta\theta}$

$$\Rightarrow \boldsymbol{\sigma}_{yy}(x, y = 0) = \frac{\sigma_\infty}{2}\left(2 + \frac{3a^4}{x^4} + \frac{a^2}{x^2}\right)$$

    - Stress concentration factor: $K_{\text{circle}} = 3$

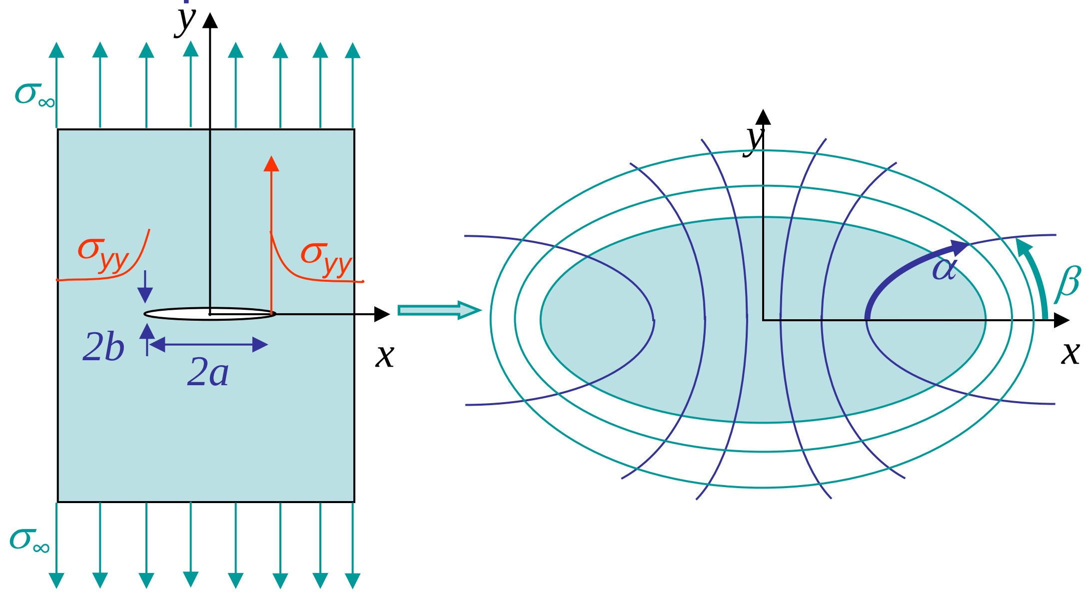


- Application: infinite plate with an elliptical hole
  - Elliptical coordinates

$$\begin{cases} x = c\cosh\alpha\cos\beta \\ y = c\sinh\alpha\sin\beta \end{cases}$$

with, on the boundary,

$$\begin{cases} c\cosh\alpha_0 = a \\ c\sinh\alpha_0 = b \end{cases}$$



  - Resolution using complex functions of z (see next slides)

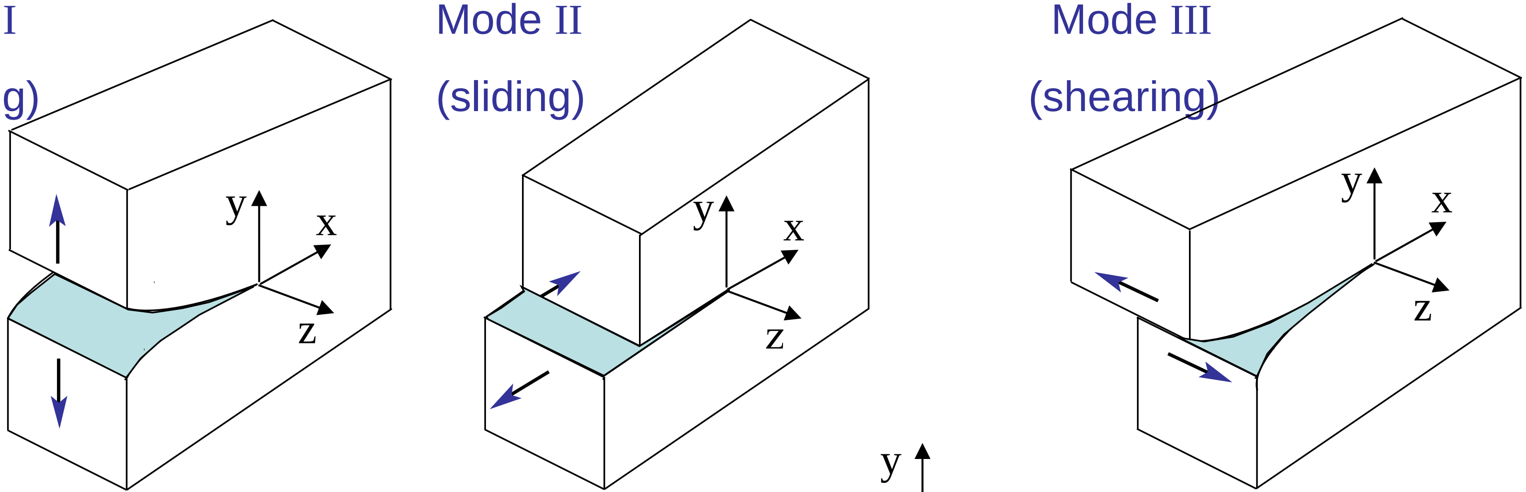
$z = c\cosh\zeta$ with the complex variable $\zeta = \alpha + i\beta$

- 1913, Inglis $\sigma_{\max} = \sigma_{yy}(a,0) = \sigma_\infty\left(1 + \frac{2a}{b}\right)$

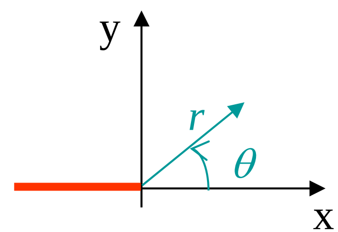
  - Stress concentration factor
    - $K_{\text{ell}} = 1 + \frac{2a}{b} = 1 + 2\sqrt{\frac{a}{\rho}}$ with ρ the curvature radius at the tip
    - Singularity when $\rho \to 0$

- Fracture modes
  - 1957, Irwin, 3 fracture modes

Mode I (opening) — Mode II (sliding) — Mode III (shearing)



- Boundary conditions



Mode I

$$\begin{cases} \boldsymbol{\sigma}_{zz} = 0 \quad \text{or} \quad \boldsymbol{\varepsilon}_{zz} = 0 \\ \boldsymbol{\sigma}_{yy}(\theta = \pm\pi) = 0 \\ \boldsymbol{\sigma}_{xy}(\theta = \pm\pi) = 0 \\ \boldsymbol{u}_x(\theta > 0) = \boldsymbol{u}_x(\theta < 0) \\ \boldsymbol{u}_y(\theta > 0) = -\boldsymbol{u}_y(\theta < 0) \end{cases}$$

Mode II

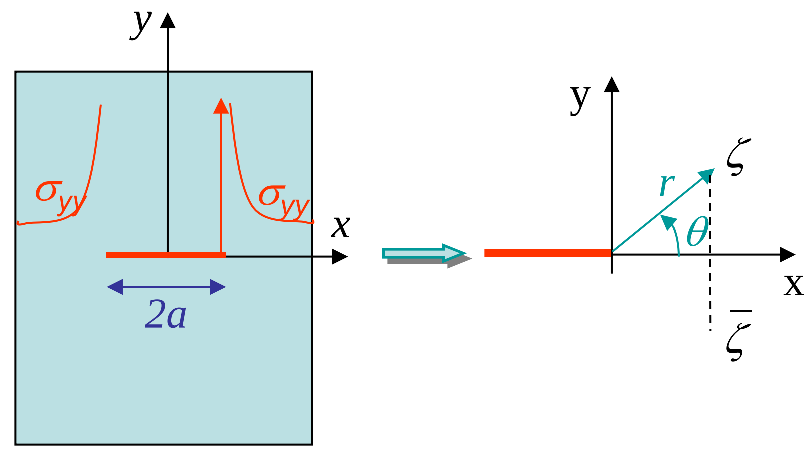
$$\begin{cases} \boldsymbol{\sigma}_{zz} = 0 \quad \text{or} \quad \boldsymbol{\varepsilon}_{zz} = 0 \\ \boldsymbol{\sigma}_{yy}(\theta = \pm\pi) = 0 \\ \boldsymbol{\sigma}_{xy}(\theta = \pm\pi) = 0 \\ \boldsymbol{u}_x(\theta > 0) = -\boldsymbol{u}_x(\theta < 0) \\ \boldsymbol{u}_y(\theta > 0) = \boldsymbol{u}_y(\theta < 0) \end{cases}$$

Mode III

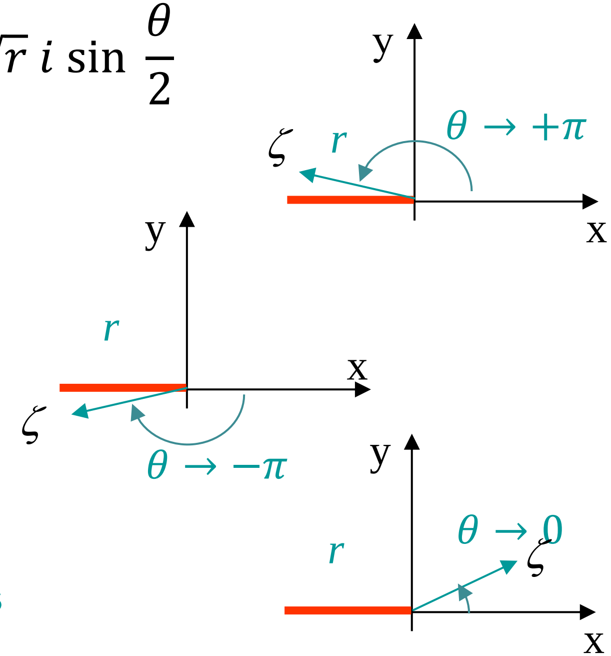
$$\begin{cases} \boldsymbol{\sigma}_{xx} = \boldsymbol{\sigma}_{xy} = \boldsymbol{\sigma}_{yy} = \boldsymbol{\sigma}_{zz} = 0 \\ \boldsymbol{u}_y = \boldsymbol{u}_x = 0 \\ \boldsymbol{u}_z(\theta > 0) = -\boldsymbol{u}_z(\theta < 0) \end{cases}$$

Resolution method

- Application: cracked body
  - Change of variables
    - Complex variables
      $$\zeta = x + iy = r \exp i\theta$$
    - Complex functions
      $$z(\zeta) = \alpha(x,\, y) + i\beta(x,\, y)$$
      with α & β real functions



  - Interest
    - $\zeta^{\frac{1}{2}} = \sqrt{r} \exp \dfrac{i\theta}{2} = \sqrt{r} \cos \dfrac{\theta}{2} + \sqrt{r}\, i \sin \dfrac{\theta}{2}$
    - Upper lip: $\lim\limits_{\theta \to +\pi} \zeta^{\frac{1}{2}} = i\sqrt{r}$
    - Lower lip: $\lim\limits_{\theta \to -\pi} \zeta^{\frac{1}{2}} = -i\sqrt{r}$
    - Ahead: $\lim\limits_{\theta \to 0} \zeta^{\frac{1}{2}} = \sqrt{r}$



⇒ A function of $\zeta^{\frac{1}{2}}$ is discontinuous

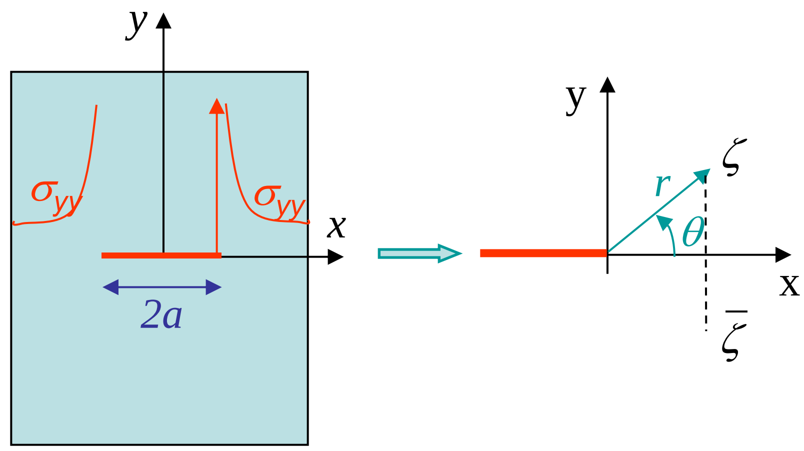
- Method for linear and elastic materials
  - Use of
    - Complex variables

$$\zeta = x + iy = r \exp i\theta$$

    - Complex functions

$$z(\zeta) = \alpha(x,\, y) + i\beta(x,\, y)$$

with α & β real functions $\begin{cases} \alpha = \mathcal{R}(z(\zeta)) \\ \beta = \mathcal{I}(z(\zeta)) \end{cases}$



  - If z is differentiable (analytic) $\Longrightarrow$ $z'(\zeta) = \partial_x z = -i\partial_y z$
    - This yields the Cauchy-Riemann relations

$$\partial_x z = \alpha_{,x} + i\beta_{,x} = -i\partial_y z = -i\alpha_{,y} + \beta_{,y} \Longrightarrow \alpha_{,x} = \beta_{,y} \;\&\; \alpha_{,y} = -\beta_{,x}$$

    - And the functions satisfy the Laplace equation

$$\alpha_{,xx} = \beta_{,xy} = -\alpha_{,yy} \;\&\; \beta_{,xx} = -\alpha_{,xy} = -\beta_{,yy} \Longrightarrow \nabla^2\alpha = \nabla^2\beta = 0$$

- Method for linear and elastic materials (2)
  - What becomes of the bi-harmonic equation?
    - Change of variables: $x, y \to \zeta, \bar{\zeta}$

$$\begin{cases} \zeta = x + iy \\ \bar{\zeta} = x - iy \end{cases} \Longrightarrow \begin{cases} \partial_x = \partial_\zeta \zeta_{,x} + \partial_{\bar{\zeta}} \bar{\zeta}_{,x} = \partial_\zeta + \partial_{\bar{\zeta}} \\ \partial_y = \partial_\zeta \zeta_{,y} + \partial_{\bar{\zeta}} \bar{\zeta}_{,y} = i\partial_\zeta - i\partial_{\bar{\zeta}} \end{cases}$$

    - The Airy function $\Phi(x, y) = \Phi(\zeta, \bar{\zeta})$
      - Let us define Ψ such that: $\nabla^2\Phi\left(\zeta, \bar{\zeta}\right) = \Psi\left(\zeta, \bar{\zeta}\right)$
      - The bi-harmonic equation is rewritten $\nabla^2\nabla^2\Phi\left(\zeta, \bar{\zeta}\right) = \nabla^2\Psi\left(\zeta, \bar{\zeta}\right) = 0$
    - Since Ψ satisfies the Laplace equation, it is the real part of a function $z(\zeta)$:
      - $\Psi\left(\zeta, \bar{\zeta}\right) = \mathcal{R}z = \dfrac{z + \bar{z}}{2}$ with $\bar{z} = z(\bar{\zeta})$ a function of $\bar{\zeta}$ only
      - But

$$\begin{cases} \Phi_{,xx} = \partial_x\left(\Phi_{,\zeta} + \Phi_{,\bar{\zeta}}\right) = \Phi_{,\zeta\zeta} + 2\Phi_{,\zeta\bar{\zeta}} + \Phi_{,\bar{\zeta}\bar{\zeta}} \\ \Phi_{,yy} = \partial_y\left(i\Phi_{,\zeta} - i\Phi_{,\bar{\zeta}}\right) = -\Phi_{,\zeta\zeta} + 2\Phi_{,\zeta\bar{\zeta}} - \Phi_{,\bar{\zeta}\bar{\zeta}} \end{cases}$$

$$\Longrightarrow \nabla^2\Phi = \Phi_{,xx} + \Phi_{,yy} = 4\Phi_{,\zeta\bar{\zeta}} = \Psi$$

- Method for linear and elastic materials (3)
  - What becomes of the bi-harmonic equation?
    - Change of variables: $x, y \rightarrow \zeta, \bar{\zeta}$

$$\begin{cases} \zeta = x + iy \\ \bar{\zeta} = x - iy \end{cases} \Longrightarrow \begin{cases} \partial_x = \partial_\zeta \zeta_{,x} + \partial_{\bar{\zeta}} \bar{\zeta}_{,x} = \partial_\zeta + \partial_{\bar{\zeta}} \\ \partial_y = \partial_\zeta \zeta_{,y} + \partial_{\bar{\zeta}} \bar{\zeta}_{,y} = i\partial_\zeta - i\partial_{\bar{\zeta}} \end{cases}$$

    - The Airy function $\Phi(x, y) = \Phi(\zeta, \bar{\zeta})$ with $\nabla^2 \Phi\left(\zeta, \bar{\zeta}\right) = \Psi\left(\zeta, \bar{\zeta}\right)$

$$\Longrightarrow \begin{cases} \Psi\left(\zeta, \bar{\zeta}\right) = \mathcal{R}z = \dfrac{z + \bar{z}}{2} \\ \nabla^2 \Phi = \Phi_{,xx} + \Phi_{,yy} = 4\Phi_{,\zeta\bar{\zeta}} = \Psi \end{cases}$$

    - Let $4\Omega'(\zeta) = z(\zeta)$ and $\omega(\zeta)$ be the unknown functions

$$\Longrightarrow 4\Phi_{,\zeta\bar{\zeta}} = \frac{z + \bar{z}}{2} = 2(\Omega' + \bar{\Omega}')$$

$$4\Phi_{,\zeta} = 2(\Omega'\bar{\zeta} + \bar{\Omega}) + f(\zeta)$$

$$4\Phi = 2(\Omega\bar{\zeta} + \bar{\Omega}\zeta) + \int f(\zeta) + 2\omega(\bar{\zeta})$$

$$\Longrightarrow \boxed{\Phi = \frac{(\Omega\bar{\zeta} + \bar{\Omega}\zeta) + \omega + \bar{\omega}}{2}}$$

    - $\Omega(\zeta)$ and $\omega(\zeta)$ are the new and only unknown functions

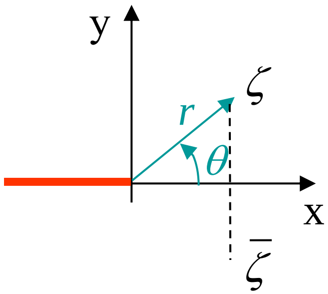
- Method for linear and elastic materials (4)
  - What become of the stresses?
    - Since $\Phi_{,\zeta\zeta} = \frac{\bar{\zeta}\Omega'' + \omega''}{2}$, $\Phi_{,\zeta\bar{\zeta}} = \frac{\Omega' + \bar{\Omega}'}{2}$ & $\Phi_{,\bar{\zeta}\bar{\zeta}} = \frac{\zeta\bar{\Omega}'' + \bar{\omega}''}{2}$
    - The stresses are rewritten

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \Phi_{,yy} = -\Phi_{,\zeta\zeta} + 2\Phi_{,\zeta\bar{\zeta}} - \Phi_{,\bar{\zeta}\bar{\zeta}} = \Omega' + \bar{\Omega}' - \frac{\bar{\zeta}\Omega'' + \omega'' + \zeta\bar{\Omega}'' + \bar{\omega}''}{2} \\ \boldsymbol{\sigma}_{yy} = \Phi_{,xx} = \Phi_{,\zeta\zeta} + 2\Phi_{,\zeta\bar{\zeta}} + \Phi_{,\bar{\zeta}\bar{\zeta}} = \Omega' + \bar{\Omega}' + \frac{\bar{\zeta}\Omega'' + \omega'' + \zeta\bar{\Omega}'' + \bar{\omega}''}{2} \\ \boldsymbol{\sigma}_{xy} = -\Phi_{,xy} = -\partial_y \left(\Phi_{,\zeta} + \Phi_{,\bar{\zeta}}\right) = -i\Phi_{,\zeta\zeta} + i\Phi_{,\bar{\zeta}\bar{\zeta}} = i\frac{\zeta\bar{\Omega}'' + \bar{\omega}'' - \bar{\zeta}\Omega'' - \omega''}{2} \end{cases}$$

- Method for linear and elastic materials (5)
  - Displacements can be deduced from the stresses
    - Complex form: $\boldsymbol{u} = \boldsymbol{u}_x\left(x,\,y\right) + i\boldsymbol{u}_y\left(x,\,y\right) = \boldsymbol{u}\left(\zeta,\,\bar{\zeta}\right)$
    - Using Hooke's law (see appendix 2)
      - $\boldsymbol{u} = -\dfrac{1+\nu}{E}\left(\zeta\bar{\Omega}' + \bar{\omega}' - \kappa\Omega\left(\zeta\right)\right)$
      - Plane σ: $\kappa = \dfrac{3-\nu}{1+\nu}$
      - Plane ε: $\kappa = 3 - 4\nu$

- Static & 2D assumptions: Summary
  - Plane σ ($\boldsymbol{\sigma}_{xz} = \boldsymbol{\sigma}_{yz} = \boldsymbol{\sigma}_{zz} = 0$) or plane ε ($\boldsymbol{\varepsilon}_{xz} = \boldsymbol{\varepsilon}_{yz} = \boldsymbol{\varepsilon}_{zz} = 0$ & $\frac{\partial}{\partial z} = 0$)
  - Hooke's law & linear momentum equations replaced by
    - $$\begin{cases} \boxed{\boldsymbol{\sigma}_{\alpha\beta} = -\Phi_{,\alpha\beta} + \delta_{\alpha\beta}\Phi_{,\gamma\gamma}} \\ \boxed{\nabla^2\nabla^2\Phi = 0} \end{cases}$$
    - One unknown function: finding Φ and defining so that BCs are satisfied
  - Complex variable
    - No more differential equation if we use

$$\boxed{\Phi = \frac{(\Omega\bar{\zeta} + \bar{\Omega}\zeta) + \omega + \bar{\omega}}{2}}$$

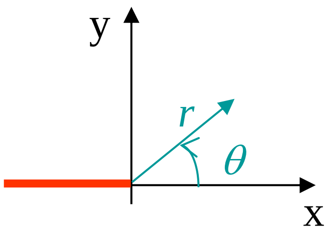


    - $\Omega(\zeta)$ and $\omega(\zeta)$ are the unknown functions which should satisfy the BCs

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \Omega' + \bar{\Omega}' - \dfrac{\bar{\zeta}\Omega'' + \omega'' + \zeta\bar{\Omega}'' + \bar{\omega}''}{2} \\ \boldsymbol{\sigma}_{yy} = \Omega' + \bar{\Omega}' + \dfrac{\bar{\zeta}\Omega'' + \omega'' + \zeta\bar{\Omega}'' + \bar{\omega}''}{2} \\ \boldsymbol{\sigma}_{xy} = i\dfrac{\zeta\bar{\Omega}'' + \bar{\omega}'' - \bar{\zeta}\Omega'' - \omega''}{2} \end{cases}$$

$$\begin{cases} \boldsymbol{u} = -\dfrac{1+\nu}{E}\left(\zeta\bar{\Omega}' + \bar{\omega}' - \kappa\Omega\left(\zeta\right)\right) \\ \kappa = \dfrac{3-\nu}{1+\nu} \quad \text{Plane } \sigma\text{:} \\ \kappa = 3 - 4\nu \quad \text{Plane } \varepsilon\text{:} \end{cases}$$

- Application: cracked body
  - Choice of complex functions
    - Owing to the discontinuity for $\theta = \pi$ we choose



$$\begin{cases} \Omega(\zeta) = \sum_\lambda \left(C_1^{(\lambda)} + iC_2^{(\lambda)}\right) \zeta^{\lambda+1} = \sum_\lambda \left(C_1^{(\lambda)} + iC_2^{(\lambda)}\right) r^{\lambda+1} \exp\left(i\theta(\lambda+1)\right) \\ \omega'(\zeta) = \sum_\lambda \left(C_3^{(\lambda)} + iC_4^{(\lambda)}\right) \zeta^{\lambda+1} = \sum_\lambda \left(C_3^{(\lambda)} + iC_4^{(\lambda)}\right) r^{\lambda+1} \exp\left(i\theta(\lambda+1)\right) \end{cases}$$

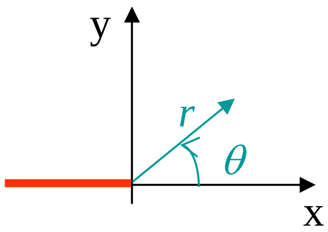
Indeed, for λ=-1/2 the functions are discontinuous

  - Conditions on λ
    - Stress σ in Ω' $\Rightarrow$ displacement $\boldsymbol{u}$ in Ω $\Rightarrow$ $\boldsymbol{u} \propto r^{\lambda+1}$

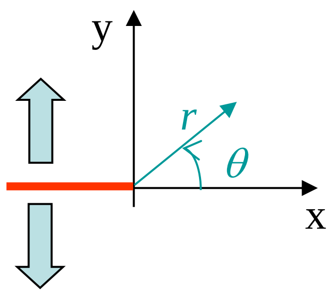
Displacement $\boldsymbol{u}$ remains finite $\Rightarrow$ λ >-1

    - Crack is stress free

$$\begin{cases} \boldsymbol{\sigma}_{yy}\left(\theta = \pm\pi\right) = 0 \\ \boldsymbol{\sigma}_{xy}\left(\theta = \pm\pi\right) = 0 \end{cases}$$



- Application to mode I (opening)
  - New constraints
    - $$\begin{cases} \Omega(\zeta) = \sum\limits_{\lambda \succ -1} \left(C_1^{(\lambda)} + iC_2^{(\lambda)}\right) \zeta^{\lambda+1} = \sum\limits_{\lambda \succ -1} \left(C_1^{(\lambda)} + iC_2^{(\lambda)}\right) r^{\lambda+1} \exp\left(i\theta(\lambda+1)\right) \\ \omega'(\zeta) = \sum\limits_{\lambda \succ -1} \left(C_3^{(\lambda)} + iC_4^{(\lambda)}\right) \zeta^{\lambda+1} = \sum\limits_{\lambda \succ -1} \left(C_3^{(\lambda)} + iC_4^{(\lambda)}\right) r^{\lambda+1} \exp\left(i\theta(\lambda+1)\right) \end{cases}$$
    - Symmetry:



$$\begin{cases} \boldsymbol{u}_x\left(\theta > 0\right) = \boldsymbol{u}_x\left(\theta < 0\right) \\ \boldsymbol{u}_y\left(\theta > 0\right) = -\boldsymbol{u}_y\left(\theta < 0\right) \end{cases}$$

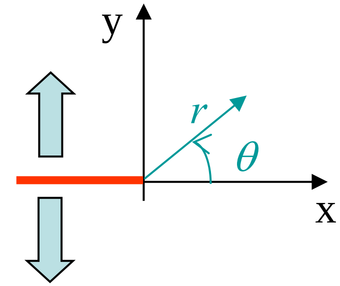
  - $$\boldsymbol{u} = -\frac{1+\nu}{E}\left(\zeta\bar{\Omega}' + \bar{\omega}' - \kappa\Omega\left(\zeta\right)\right)$$

⇒ $\Re\Omega(\zeta)\,,\Re\overline{\omega}'$ in $\cos(\theta)$ and $\Im\Omega(\zeta)\,,\Im\overline{\omega}'$ in $\sin(\theta)$

⇒ $C_2^{(\lambda)} = C_4^{(\lambda)} = 0$

Asymptotic solution

- Application to mode I (opening) (2)
  - New constraints (2)



- $$\begin{cases} \Omega(\zeta) = \sum_{\lambda > -1} C_1^{(\lambda)} \zeta^{\lambda+1} = \sum_{\lambda > -1} C_1^{(\lambda)} r^{\lambda+1} \exp\big(i\theta(\lambda+1)\big) \\ \omega'(\zeta) = \sum_{\lambda > -1} C_3^{(\lambda)} \zeta^{\lambda+1} = \sum_{\lambda > -1} C_3^{(\lambda)} r^{\lambda+1} \exp\big(i\theta(\lambda+1)\big) \end{cases}$$

- Stress field:

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \Omega' + \bar{\Omega}' - \dfrac{\bar{\zeta}\Omega'' + \omega'' + \zeta\bar{\Omega}'' + \bar{\omega}''}{2} \\ \boldsymbol{\sigma}_{yy} = \Omega' + \bar{\Omega}' + \dfrac{\bar{\zeta}\Omega'' + \omega'' + \zeta\bar{\Omega}'' + \bar{\omega}''}{2} \\ \boldsymbol{\sigma}_{xy} = i\dfrac{\zeta\bar{\Omega}'' + \bar{\omega}'' - \bar{\zeta}\Omega'' - \omega''}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \boldsymbol{\sigma}_{xx} = \sum_{\lambda > -1} (\lambda+1) \left[C_1^{(\lambda)}\left(\zeta^{\lambda} + \bar{\zeta}^{\lambda}\right) - \dfrac{C_1^{(\lambda)}\lambda\left(\bar{\zeta}\zeta^{\lambda-1} + \zeta\bar{\zeta}^{\lambda-1}\right) + C_3^{(\lambda)}\left(\zeta^{\lambda} + \bar{\zeta}^{\lambda}\right)}{2} \right] \\ \boldsymbol{\sigma}_{yy} = \sum_{\lambda > -1} (\lambda+1) \left[C_1^{(\lambda)}\left(\zeta^{\lambda} + \bar{\zeta}^{\lambda}\right) + \dfrac{C_1^{(\lambda)}\lambda\left(\bar{\zeta}\zeta^{\lambda-1} + \zeta\bar{\zeta}^{\lambda-1}\right) + C_3^{(\lambda)}\left(\zeta^{\lambda} + \bar{\zeta}^{\lambda}\right)}{2} \right] \\ \boldsymbol{\sigma}_{xy} = \sum_{\lambda > -1} (\lambda+1) \left[-i\dfrac{C_1^{(\lambda)}\lambda\left(\bar{\zeta}\zeta^{\lambda-1} - \zeta\bar{\zeta}^{\lambda-1}\right) + C_3^{(\lambda)}\left(\zeta^{\lambda} - \bar{\zeta}^{\lambda}\right)}{2} \right] \end{cases}$$

- Application to mode I (opening) (3)
  - New constraints (3)
    - Stress field:

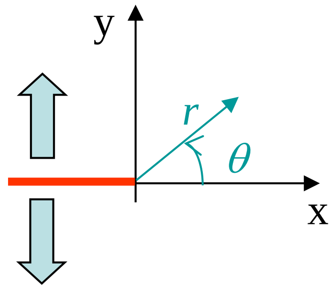
$$\begin{cases} \zeta = r\exp(i\theta) \\ \bar{\zeta} = r\exp(-i\theta) \end{cases}$$

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \sum\limits_{\lambda \succ -1} (\lambda+1)\left[C_1^{(\lambda)}\left(\zeta^\lambda + \bar{\zeta}^\lambda\right) - \dfrac{C_1^{(\lambda)}\lambda\left(\bar{\zeta}\zeta^{\lambda-1} + \zeta\bar{\zeta}^{\lambda-1}\right) + C_3^{(\lambda)}\left(\zeta^\lambda + \bar{\zeta}^\lambda\right)}{2}\right] \\ \boldsymbol{\sigma}_{yy} = \sum\limits_{\lambda \succ -1} (\lambda+1)\left[C_1^{(\lambda)}\left(\zeta^\lambda + \bar{\zeta}^\lambda\right) + \dfrac{C_1^{(\lambda)}\lambda\left(\bar{\zeta}\zeta^{\lambda-1} + \zeta\bar{\zeta}^{\lambda-1}\right) + C_3^{(\lambda)}\left(\zeta^\lambda + \bar{\zeta}^\lambda\right)}{2}\right] \\ \boldsymbol{\sigma}_{xy} = \sum\limits_{\lambda \succ -1} (\lambda+1)\left[-i\dfrac{C_1^{(\lambda)}\lambda\left(\bar{\zeta}\zeta^{\lambda-1} - \zeta\bar{\zeta}^{\lambda-1}\right) + C_3^{(\lambda)}\left(\zeta^\lambda - \bar{\zeta}^\lambda\right)}{2}\right] \end{cases}$$

$$\Rightarrow \begin{cases} \boldsymbol{\sigma}_{xx} = \sum\limits_{\lambda \succ -1} (\lambda+1) r^\lambda \left[C_1^{(\lambda)}\, 2\cos(\lambda\theta) - \dfrac{2C_1^{(\lambda)}\lambda\, \cos\big((\lambda-2)\theta\big) + 2C_3^{(\lambda)}\cos(\lambda\theta)}{2}\right] \\ \boldsymbol{\sigma}_{yy} = \sum\limits_{\lambda \succ -1} (\lambda+1) r^\lambda \left[C_1^{(\lambda)}\, 2\cos(\lambda\theta) + \dfrac{2C_1^{(\lambda)}\lambda\, \cos\big((\lambda-2)\theta\big) + 2C_3^{(\lambda)}\cos(\lambda\theta)}{2}\right] \\ \boldsymbol{\sigma}_{xy} = \sum\limits_{\lambda \succ -1} (\lambda+1) r^\lambda \left[-i\dfrac{2C_1^{(\lambda)}\lambda\, i\sin\big((\lambda-2)\theta\big) + 2C_3^{(\lambda)} i\sin(\lambda\theta)}{2}\right] \end{cases}$$

- Application to mode I (opening) (4)
  - New constraints (4)
    - $$\begin{cases} \boldsymbol{\sigma}_{yy} = \sum_{\lambda > -1} (\lambda+1) r^{\lambda} \left[2C_1^{(\lambda)} \cos(\lambda\theta) + C_1^{(\lambda)} \lambda \cos\big((\lambda-2)\theta\big) + C_3^{(\lambda)} \cos(\lambda\theta) \right] \\ \boldsymbol{\sigma}_{xy} = \sum_{\lambda > -1} (\lambda+1) r^{\lambda} \left[C_1^{(\lambda)} \lambda \sin\big((\lambda-2)\theta\big) + C_3^{(\lambda)} \sin(\lambda\theta) \right] \end{cases}$$
    - Stress free crack:

$$\begin{cases} \boldsymbol{\sigma}_{yy}\,(\theta = \pm\pi) = 0 \\ \boldsymbol{\sigma}_{xy}\,(\theta = \pm\pi) = 0 \end{cases}$$



$$\Rightarrow \begin{cases} 2C_1^{(\lambda)} \cos(\pm\lambda\pi) + C_1^{(\lambda)} \lambda \cos(\pm\lambda\pi) + C_3^{(\lambda)} \cos(\pm\lambda\pi) = 0 \\ C_1^{(\lambda)} \lambda \sin(\pm\lambda\pi) + C_3^{(\lambda)} \sin(\pm\lambda\pi) = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} (2+\lambda)\cos(\lambda\pi) & \cos(\lambda\pi) \\ \lambda \sin(\lambda\pi) & \sin(\lambda\pi) \end{pmatrix} \begin{pmatrix} C_1^{(\lambda)} \\ C_3^{(\lambda)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Application to mode I (opening) (5)
  - Solution
    - System of equations

$$\begin{pmatrix} (2+\lambda)\cos(\lambda\pi) & \cos(\lambda\pi) \\ \lambda\sin(\lambda\pi) & \sin(\lambda\pi) \end{pmatrix} \begin{pmatrix} C_1^{(\lambda)} \\ C_3^{(\lambda)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

    - Non trivial solution

$$0 = \begin{vmatrix} (2+\lambda)\cos(\lambda\pi) & \cos(\lambda\pi) \\ \lambda\sin(\lambda\pi) & \sin(\lambda\pi) \end{vmatrix} = \sin(2\lambda\pi)$$

$\Rightarrow$ $\lambda = n/2$, with n = -1, 0, 1, …. since λ >-1

    - Displacements in $r^{\lambda+1}$ and stresses in r^{λ}

$\Rightarrow$ the dominant term **near the crack tip** is obtained for λ=-1/2

$\Rightarrow$ only this term is considered in the **asymptotic** solution

    - Determination of the constants:

$$\boldsymbol{\sigma}_{xy}(\theta \pm \pi) = 0 \Rightarrow C_1^{(\lambda)}\lambda\sin(\pm\lambda\pi) + C_3^{(\lambda)}\sin(\pm\lambda\pi) = 0 \text{ for } \lambda = \frac{n}{2}$$

$$\Rightarrow -\lambda C_1^{(\lambda)} = C_3^{(\lambda)} \Rightarrow C_1^{\left(-\frac{1}{2}\right)} = 2C_3^{\left(-\frac{1}{2}\right)}$$

- Application to mode I (opening) (6)
  - Stress field:

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \sum_{\lambda > -1} (\lambda+1) r^{\lambda} \left[2C_1^{(\lambda)} \cos(\lambda\theta) - C_1^{(\lambda)} \lambda \cos\big((\lambda-2)\theta\big) - C_3^{(\lambda)} \cos(\lambda\theta) \right] \\ \boldsymbol{\sigma}_{yy} = \sum_{\lambda > -1} (\lambda+1) r^{\lambda} \left[2C_1^{(\lambda)} \cos(\lambda\theta) + C_1^{(\lambda)} \lambda \cos\big((\lambda-2)\theta\big) + C_3^{(\lambda)} \cos(\lambda\theta) \right] \\ \boldsymbol{\sigma}_{xy} = \sum_{\lambda > -1} (\lambda+1) r^{\lambda} \left[C_1^{(\lambda)} \lambda \sin\big((\lambda-2)\theta\big) + C_3^{(\lambda)} \sin(\lambda\theta) \right] \end{cases}$$

$$\cos a - \cos b = -2 \sin \frac{a-b}{2} \sin \frac{a+b}{2}$$

$$\lambda = -\frac{1}{2}, 0, \frac{1}{2}, \dots$$

$$-\lambda C_1^{(\lambda)} = C_3^{(\lambda)}$$

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \sum_{\lambda = -\frac{1}{2}, 0, \frac{1}{2}, \dots} (\lambda+1) C_1^{(\lambda)} r^{\lambda} \left[2\cos(\lambda\theta) - \lambda \left(\cos\big((\lambda-2)\theta\big) - \cos(\lambda\theta) \right) \right] \\ \boldsymbol{\sigma}_{yy} = \sum_{\lambda = -\frac{1}{2}, 0, \frac{1}{2}, \dots} (\lambda+1) C_1^{(\lambda)} r^{\lambda} \left[2\cos(\lambda\theta) + \lambda \left(\cos\big((\lambda-2)\theta\big) - \cos(\lambda\theta) \right) \right] \\ \boldsymbol{\sigma}_{xy} = \sum_{\lambda = -\frac{1}{2}, 0, \frac{1}{2}, \dots} (\lambda+1) r^{\lambda} C_1^{(\lambda)} \lambda \left[\sin\big((\lambda-2)\theta\big) - \sin(\lambda\theta) \right] \end{cases}$$

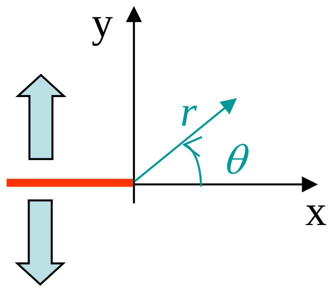
$$2\cos(\lambda\theta) + 2\lambda \sin(-\theta) \sin\big((\lambda-1)\theta\big)$$

$$2\cos(\lambda\theta) - 2\lambda \sin(-\theta) \sin\big((\lambda-1)\theta\big)$$

$$2\sin(-\theta)\cos\big((\lambda-1)\theta\big)$$

$$\sin a - \sin b = 2 \sin \frac{a-b}{2} \cos \frac{a+b}{2}$$

- Application to mode I (opening) (7)
  - Stress field summary:
    - Symmetry:

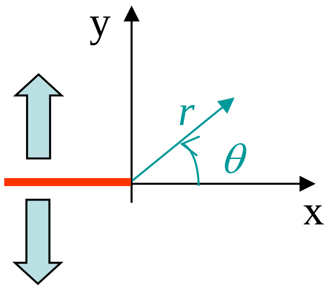


    - Leads to $C_2^{(\lambda)} = C_4^{(\lambda)} = 0$ & relation between $C_1^{(\lambda)}$ & $-\lambda C_1^{(\lambda)} = C_3^{(\lambda)}$

$$\begin{cases} \Omega(\zeta) = \sum\limits_{\lambda=-\frac{1}{2},0,\frac{1}{2},\dots} C_1^{(\lambda)}\, \zeta^{\lambda+1} = \sum\limits_{\lambda=-\frac{1}{2},0,\frac{1}{2},\dots} C_1^{(\lambda)}\, r^{\lambda+1} \exp\big(i\theta(\lambda+1)\big) \\ \omega'(\zeta) = \sum\limits_{\lambda=-\frac{1}{2},0,\frac{1}{2},\dots} -\lambda C_1^{(\lambda)}\, \zeta^{\lambda+1} = \sum\limits_{\lambda=-\frac{1}{2},0,\frac{1}{2},\dots} -\lambda C_1^{(\lambda)}\, r^{\lambda+1} \exp\big(i\theta(\lambda+1)\big) \end{cases}$$

$$\begin{cases} \sigma_{xx} = \frac{C_1}{\sqrt{r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + A(\theta)r^0 + B(\theta)\sqrt{r}\ \dots \\ \sigma_{yy} = \frac{C_1}{\sqrt{r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + C(\theta)r^0 + D(\theta)\sqrt{r}\ \dots \\ \sigma_{xy} = \frac{C_1}{\sqrt{r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + E(\theta)r^0 + F(\theta)\sqrt{r}\ \dots \end{cases}$$

- Application to mode I (opening) (8)
  - Asymptotic stress field
    - $\sigma_{yy} = \frac{C_1}{\sqrt{r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + C(\theta)r^0 + D(\theta)\sqrt{r}$ ...



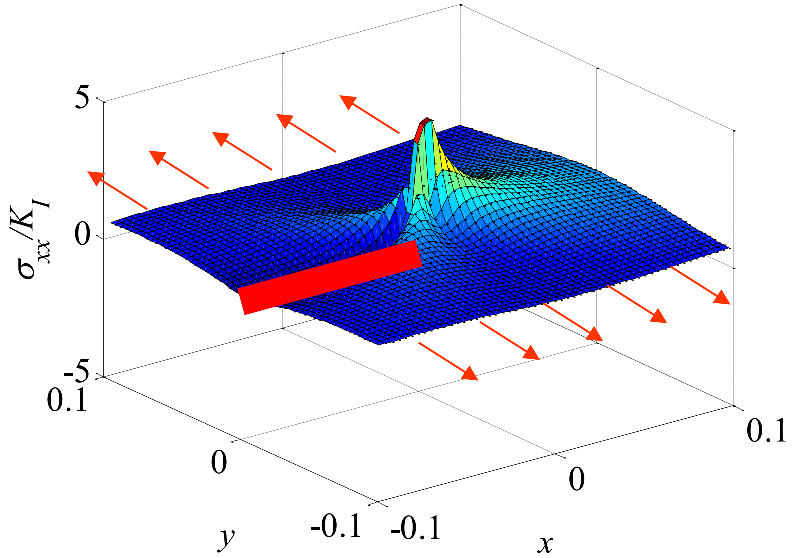
    - For mode I, the stress concentration is obtained for $\sigma_{yy}(\theta = 0)$
    - As the stress tends to infinity, the Stress Intensity Factor (SIF) is defined as

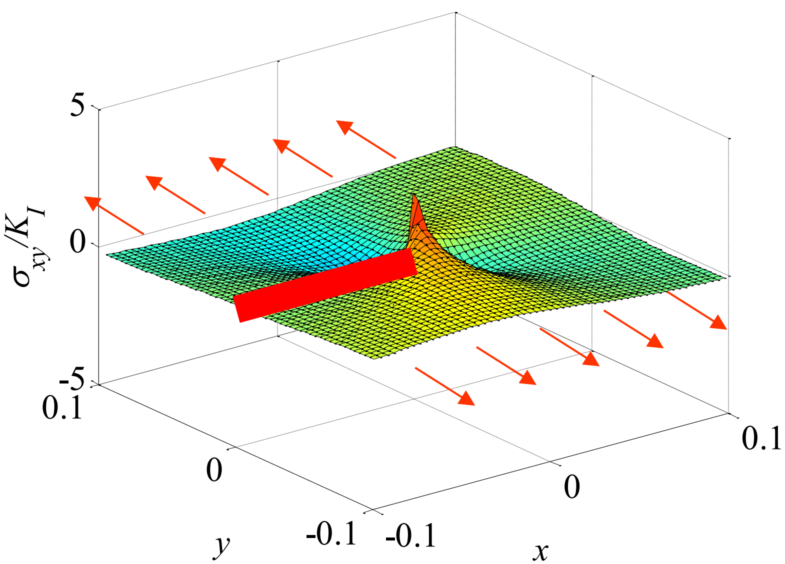
$$\Rightarrow \boxed{K_I = \lim_{r\to 0}\left(\sqrt{2\pi r}\,\boldsymbol{\sigma}_{yy}^{\text{mode I}}\Big|_{\theta=0}\right) = C_1\sqrt{2\pi}}$$

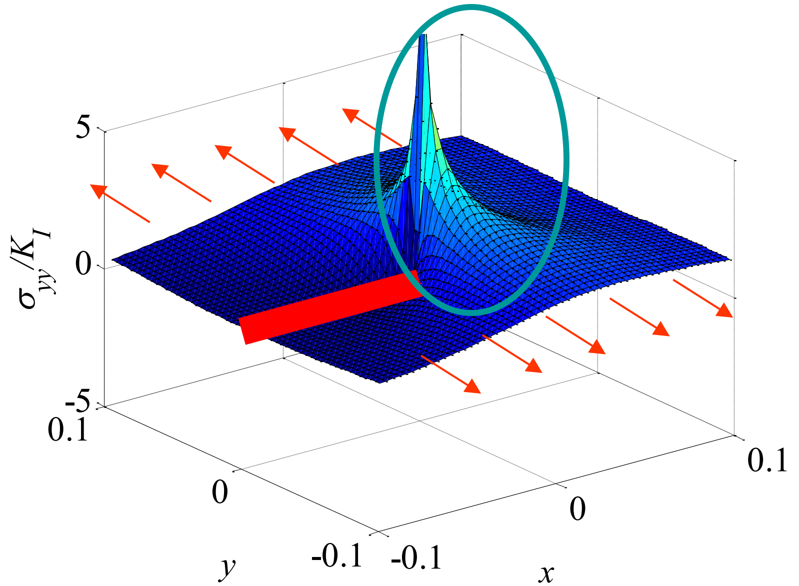
    - C_1 & so K_I depend on both the geometry and loading condition
    - Unit of the SIFs in Pa $m^{1/2}$

- Application to mode I (opening) (9)
  - Asymptotic stress field (dominant terms)

$$\left\{\begin{array}{l}\boldsymbol{\sigma}_{xx} = \dfrac{K_I}{\sqrt{2\pi r}} \cos\dfrac{\theta}{2}\left[1 - \sin\dfrac{\theta}{2}\sin\dfrac{3\theta}{2}\right] \\ \boldsymbol{\sigma}_{yy} = \dfrac{K_I}{\sqrt{2\pi r}} \cos\dfrac{\theta}{2}\left[1 + \sin\dfrac{\theta}{2}\sin\dfrac{3\theta}{2}\right] \\ \boldsymbol{\sigma}_{xy} = \dfrac{K_I}{\sqrt{2\pi r}} \cos\dfrac{\theta}{2}\sin\dfrac{\theta}{2}\cos\dfrac{3\theta}{2}\end{array}\right.$$







- Application to mode I (opening) (10)
  - Displacement field:

    - $$\begin{cases} \Omega(\zeta) = \sum_{\lambda=-\frac{1}{2},0,\frac{1}{2},\dots} C_1^{(\lambda)}\, \zeta^{\lambda+1} = \sum_{\lambda=-\frac{1}{2},0,\frac{1}{2},\dots} C_1^{(\lambda)}\, r^{\lambda+1} \exp\big(i\theta(\lambda+1)\big) \\ \omega'(\zeta) = \sum_{\lambda=-\frac{1}{2},0,\frac{1}{2},\dots} -\lambda C_1^{(\lambda)}\, \zeta^{\lambda+1} = \sum_{\lambda=-\frac{1}{2},0,\frac{1}{2},\dots} -\lambda C_1^{(\lambda)}\, r^{\lambda+1} \exp\big(i\theta(\lambda+1)\big) \end{cases}$$

    - $$\boldsymbol{u} = -\frac{1+\nu}{E}\left(\zeta\bar{\Omega}' + \bar{\omega}' - \kappa\Omega\left(\zeta\right)\right) \qquad \begin{cases} \kappa = \dfrac{3-\nu}{1+\nu} & \text{Plane } \sigma: \\ \kappa = 3 - 4\nu & \text{Plane } \varepsilon: \end{cases}$$

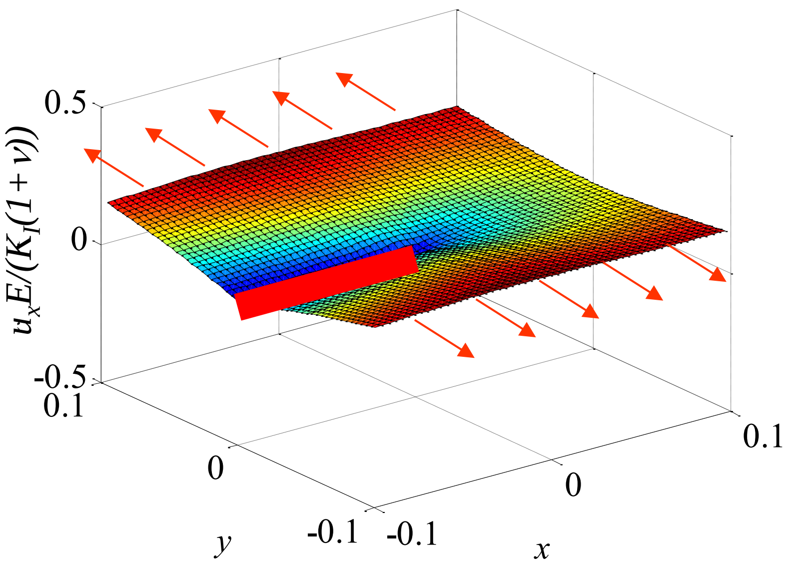
$$\boldsymbol{u_x} + i\boldsymbol{u_y} = \frac{1+\nu}{E} \sum_{\lambda=-\frac{1}{2},0,\frac{1}{2},\dots} C_1^{(\lambda)} r^{\lambda+1}\left[\kappa \exp(i(\lambda+1)\theta) - (\lambda+1)\exp(i(1-\lambda)\theta) + \lambda\exp(-i(\lambda+1)\theta)\right]$$

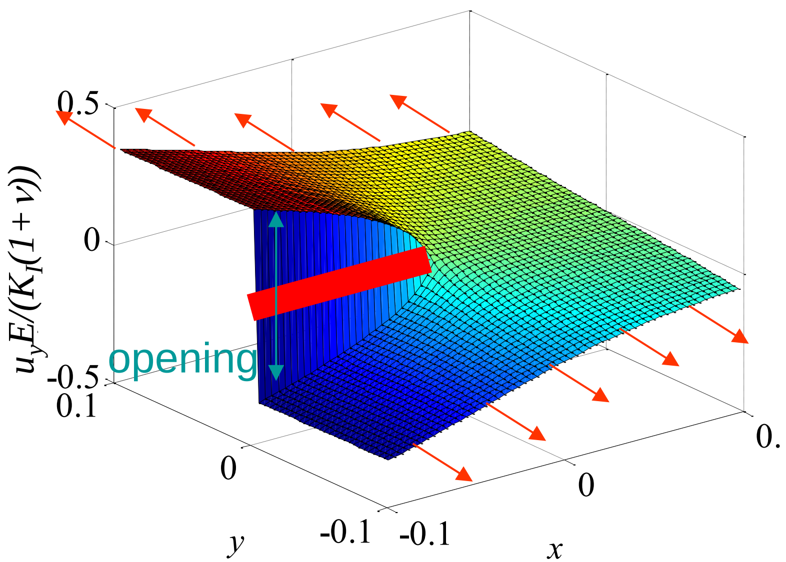
$$\& \quad C_1^{\left(-\frac{1}{2}\right)} = \frac{K_I}{\sqrt{2\pi}}$$

- Application to mode I (opening) (11)
  - Asymptotic displacements (dominant terms)

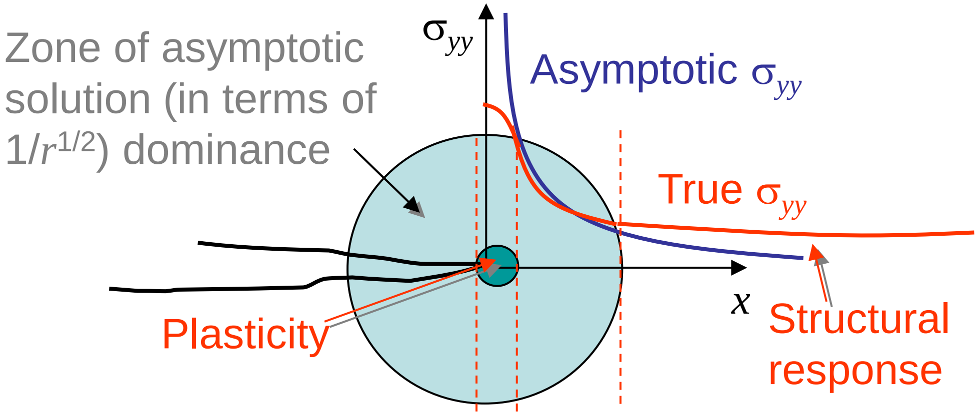
$$\begin{cases} \boldsymbol{u}_x = K_I \dfrac{1+\nu}{E}\sqrt{\dfrac{r}{2\pi}}\cos\dfrac{\theta}{2}\left[\kappa - 1 + 2\sin^2\dfrac{\theta}{2}\right] \\ \boldsymbol{u}_y = K_I \dfrac{1+\nu}{E}\sqrt{\dfrac{r}{2\pi}}\sin\dfrac{\theta}{2}\left[\kappa + 1 - 2\cos^2\dfrac{\theta}{2}\right] \end{cases}$$

- With for plane σ: $\kappa = \dfrac{3-\nu}{1+\nu}$ & for plane ε: $\kappa = 3 - 4\nu$

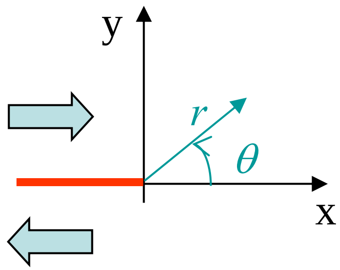




- Application to mode I (opening) (12)
  - Validity of the asymptotic solution
    - This solution is only valid in a restricted region



- Application to mode II (sliding)
  - New constraints:
    - Symmetry:



$$\begin{cases} \boldsymbol{\sigma}_{zz} = 0 \text{ or } \boldsymbol{\varepsilon}_{zz} = 0 \\ \boldsymbol{\sigma}_{yy}(\theta = \pm\pi) = 0 \\ \boldsymbol{\sigma}_{xy}(\theta = \pm\pi) = 0 \\ \boldsymbol{u}_x(\theta > 0) = -\boldsymbol{u}_x(\theta < 0) \\ \boldsymbol{u}_y(\theta > 0) = \boldsymbol{u}_y(\theta < 0) \end{cases}$$

- Leads to $C_1^{(\lambda)} = C_3^{(\lambda)} = 0$ & relation between $C_2^{(\lambda)}$ & $C_4^{(\lambda)}$

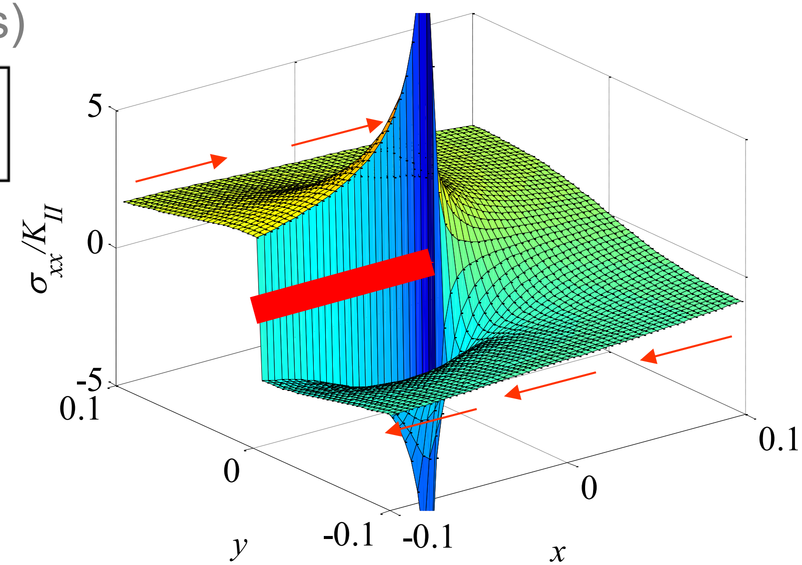
$$\Rightarrow \begin{cases} \Omega(\zeta) = \sum_{\lambda=-\frac{1}{2},0,\frac{1}{2},\dots} iC_2^{(\lambda)}\,\zeta^{\lambda+1} = \sum_{\lambda=-\frac{1}{2},0,\frac{1}{2},\dots} iC_2^{(\lambda)}\,r^{\lambda+1}\exp\big(i\theta(\lambda+1)\big) \\ \omega'(\zeta) = \sum_{\lambda=-\frac{1}{2},0,\frac{1}{2},\dots} iC_4^{(\lambda)}\,\zeta^{\lambda+1} = \sum_{\lambda=-\frac{1}{2},0,\frac{1}{2},\dots} iC_4^{(\lambda)}\,r^{\lambda+1}\exp\big(i\theta(\lambda+1)\big) \end{cases}$$

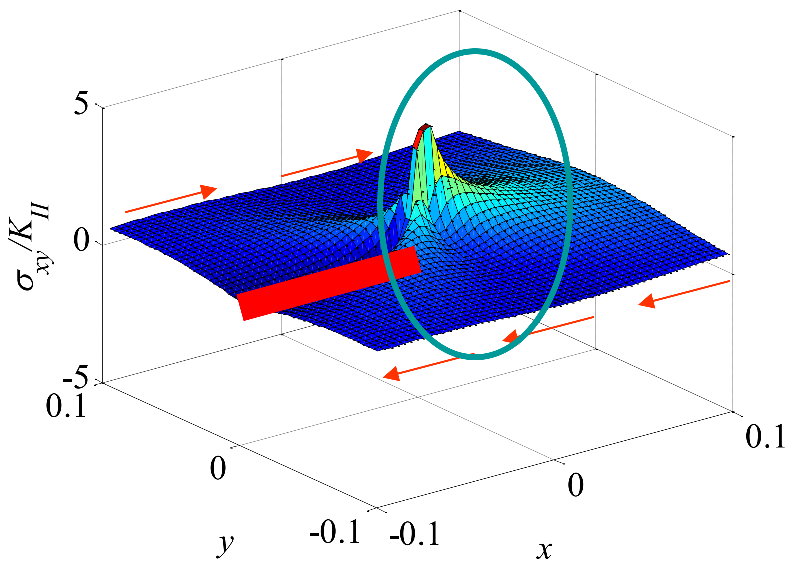
$$\Rightarrow \begin{cases} \sigma_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left[2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right] + G(\theta)r^0 + H(\theta)\sqrt{r}\ \dots \\ \sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} + I(\theta)r^0 + J(\theta)\sqrt{r}\ \dots \\ \sigma_{xy} = \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + K(\theta)r^0 + L(\theta)\sqrt{r}\ \dots \end{cases}$$

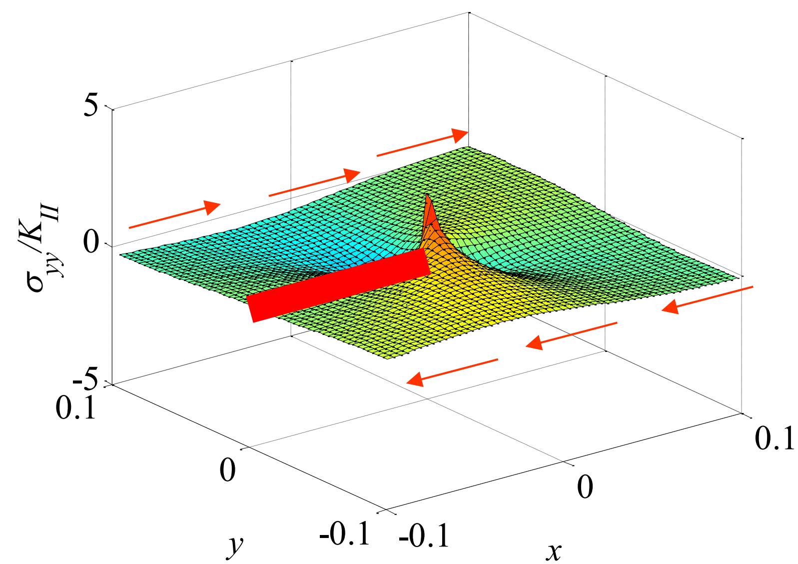
Asymptotic solution

- Application to mode II (sliding) (2)
  - Asymptotic stress field (dominant terms)

$$
\begin{cases}
\boldsymbol{\sigma}_{xx} = -\dfrac{K_{II}}{\sqrt{2\pi r}} \sin\dfrac{\theta}{2}\left[2 + \cos\dfrac{\theta}{2}\cos\dfrac{3\theta}{2}\right] \\
\boldsymbol{\sigma}_{yy} = \dfrac{K_{II}}{\sqrt{2\pi r}} \sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\cos\dfrac{3\theta}{2} \\
\boldsymbol{\sigma}_{xy} = \dfrac{K_{II}}{\sqrt{2\pi r}} \cos\dfrac{\theta}{2}\left[1 - \sin\dfrac{\theta}{2}\sin\dfrac{3\theta}{2}\right]
\end{cases}
$$





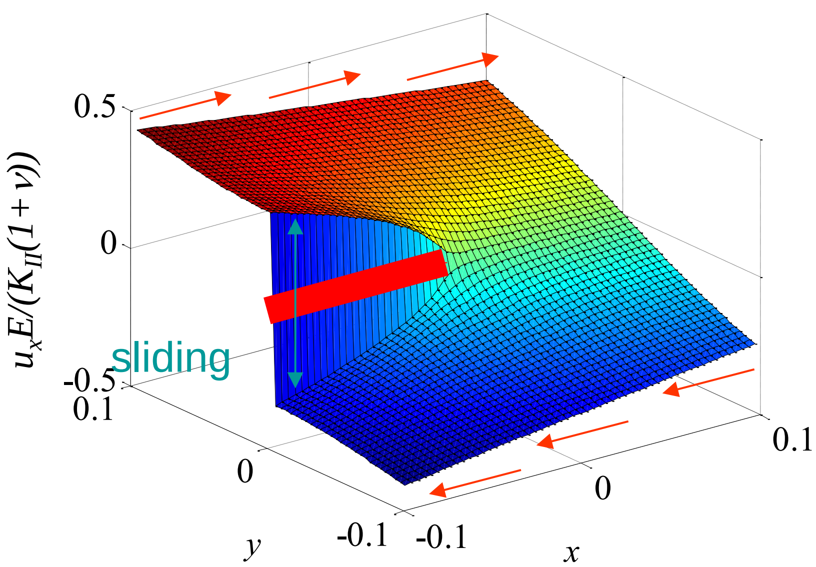


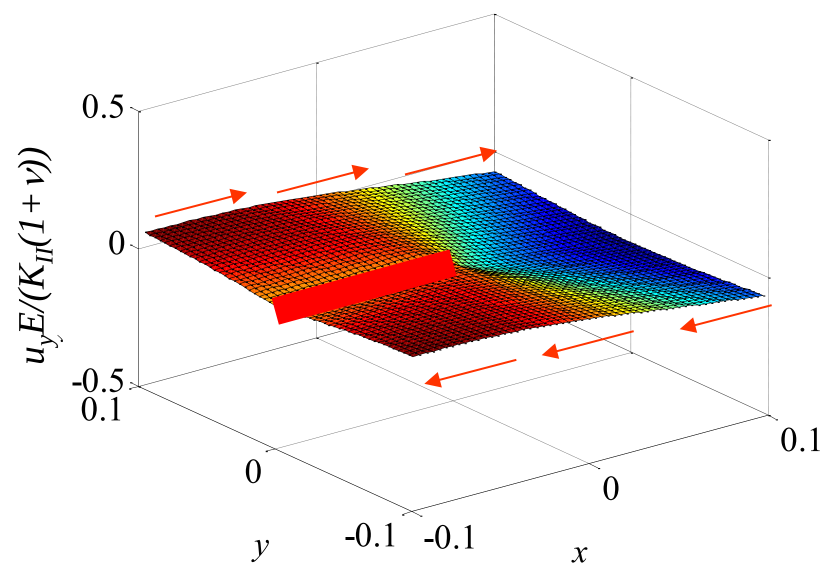
Asymptotic solution

- Application to mode II (sliding) (3)
  - Asymptotic displacements (dominant terms)

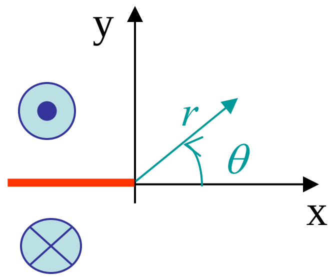
$$\begin{cases} \boldsymbol{u}_x = K_{II}\dfrac{1+\nu}{E}\sqrt{\dfrac{r}{2\pi}}\sin\dfrac{\theta}{2}\left[\kappa+1+2\cos^2\dfrac{\theta}{2}\right] \\ \boldsymbol{u}_y = K_{II}\dfrac{1+\nu}{E}\sqrt{\dfrac{r}{2\pi}}\cos\dfrac{\theta}{2}\left[1-\kappa+2\sin^2\dfrac{\theta}{2}\right] \end{cases}$$

- With for plane σ: $\kappa = \dfrac{3-\nu}{1+\nu}$ & for plane ε: $\kappa = 3-4\nu$





- Application to mode III (shearing)
  - Mode III:



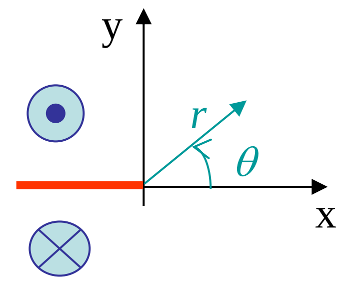
$$\begin{cases} \boldsymbol{\sigma}_{xx} = \boldsymbol{\sigma}_{xy} = \boldsymbol{\sigma}_{yy} = \boldsymbol{\sigma}_{zz} = 0 \\ \boldsymbol{u}_y = \boldsymbol{u}_x = 0 \\ \boldsymbol{u}_z\left(\theta > 0\right) = -\boldsymbol{u}_z\left(\theta < 0\right) \end{cases}$$

  - Equations
    - Hooke's law $\boldsymbol{\varepsilon}_{\alpha z} = \frac{1}{2}\boldsymbol{u}_{z,\alpha} \Longrightarrow \boldsymbol{\sigma}_{\alpha z} = \frac{E}{1+\nu}\boldsymbol{\varepsilon}_{\alpha z} = \frac{E}{2\left(1+\nu\right)}\boldsymbol{u}_{z,\alpha}$
    - Equilibrium $\boldsymbol{\sigma}_{\alpha z,\alpha} = 0 \Longrightarrow \nabla^2 \boldsymbol{u}_z = 0$
  - Laplace equation satisfied $\Longrightarrow$ $\boldsymbol{u}_z$ is the imaginary part of a function $z(\zeta)$
    - Choice of function $z = \sum_\lambda C^{(\lambda)} \zeta^\lambda = \sum_\lambda C^{(\lambda)} r^\lambda \exp(i\lambda\theta)$
    - $\boldsymbol{u}_z\left(\theta > 0\right) = -\boldsymbol{u}_z\left(\theta < 0\right)$

      $\Longrightarrow$ $\boldsymbol{u}_z$ is the imaginary part of z, and $C^{(\lambda)}$ is real

      $$\Longrightarrow \boldsymbol{u}_z = \sum_\lambda C^{(\lambda)} r^\lambda \sin(\lambda\theta)$$

- Application to mode III (shearing) (2)
  - Stress field:
    - $$\begin{cases} \boldsymbol{u}_z = \sum_\lambda C^{(\lambda)} r^\lambda \sin(\lambda\theta) \\ \boldsymbol{\sigma}_{\alpha z} = \frac{E}{1+\nu}\boldsymbol{\varepsilon}_{\alpha z} = \frac{E}{2(1+\nu)}\boldsymbol{u}_{z,\alpha} \end{cases}$$
    - Cylindrical CV

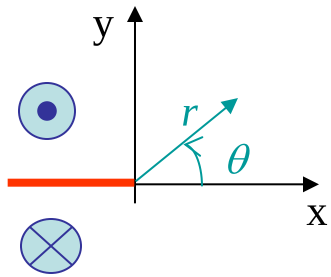


$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta \end{pmatrix}$$

$$\Rightarrow \begin{cases} \boldsymbol{\sigma}_{xz} = \frac{E}{2(1+\nu)} \sum_\lambda C^{(\lambda)} r^{\lambda-1} [\lambda \cos(\theta)\sin(\lambda\theta) - \lambda\sin(\theta)\cos(\lambda\theta)] \\ \boldsymbol{\sigma}_{yz} = \frac{E}{2(1+\nu)} \sum_\lambda C^{(\lambda)} r^{\lambda-1} [\lambda \sin(\theta)\sin(\lambda\theta) + \lambda\cos(\theta)\cos(\lambda\theta)] \end{cases}$$

$$\Rightarrow \begin{cases} \boldsymbol{\sigma}_{xz} = \frac{E}{2(1+\nu)} \sum_\lambda C^{(\lambda)} \lambda r^{\lambda-1} \sin\big((\lambda-1)\theta\big) \\ \boldsymbol{\sigma}_{yz} = \frac{E}{2(1+\nu)} \sum_\lambda C^{(\lambda)} \lambda r^{\lambda-1} \cos\big((\lambda-1)\theta\big) \end{cases}$$

- Application to mode III (shearing) (3)
  - Mode III:



$$\begin{cases} \boldsymbol{\sigma}_{xx} = \boldsymbol{\sigma}_{xy} = \boldsymbol{\sigma}_{yy} = \boldsymbol{\sigma}_{zz} = 0 \\ \boldsymbol{u}_y = \boldsymbol{u}_x = 0 \\ \boldsymbol{u}_z\,(\theta > 0) = -\boldsymbol{u}_z\,(\theta < 0) \end{cases}$$

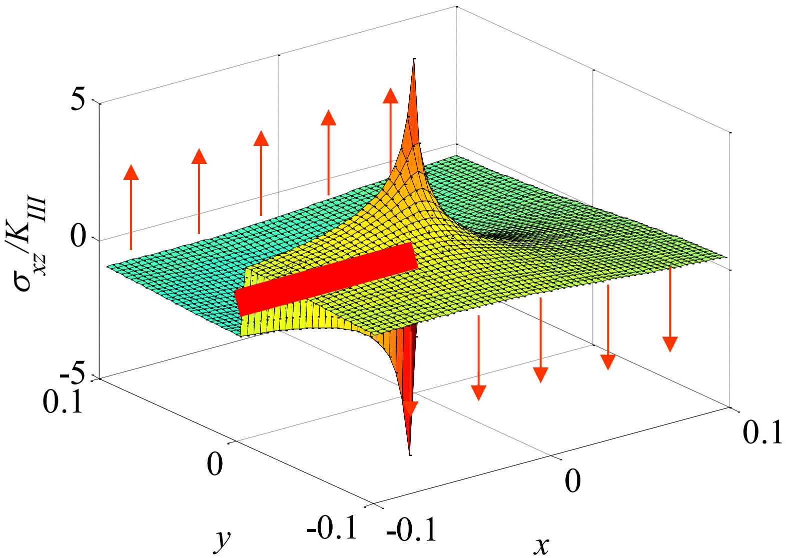
  - Resolution (Hooke's law & equilibrium)
    - $\sigma_{xz} = \sum_\lambda \frac{EC^{(\lambda)}}{2(1+\nu)} \lambda r^{\lambda-1} \sin[(\lambda-1)\theta]$ & $\sigma_{yz} = \sum_\lambda \frac{EC^{(\lambda)}}{2(1+\nu)} \lambda r^{\lambda-1} \cos[(\lambda-1)\theta]$
  - Constraints
    - Displacement field should be finite ⟹ $\lambda > 0$
    - Crack is stress free: $\cos[\pm(\lambda-1)\pi]$ ⟹ $\lambda = n/2$, with $n = 1, 2, \ldots.$ since $\lambda > 0$

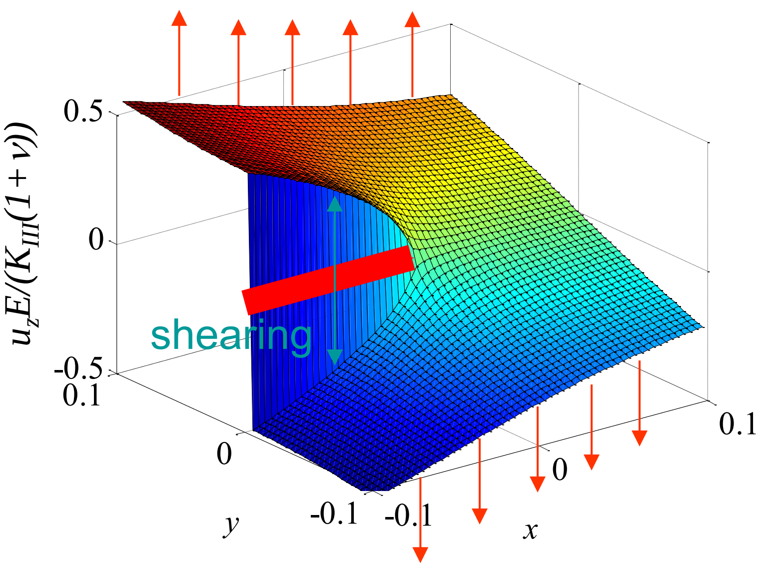
$$\Rightarrow \begin{cases} \sigma_{xz} = -\frac{EC}{4(1+\nu)\sqrt{r}} \sin\frac{\theta}{2} + M(\theta) r^0 + N(\theta)\sqrt{r} \ldots \\ \sigma_{yz} = \frac{EC}{4(1+\nu)\sqrt{r}} \cos\frac{\theta}{2} + O(\theta) r^0 + P(\theta)\sqrt{r} \ldots \end{cases}$$

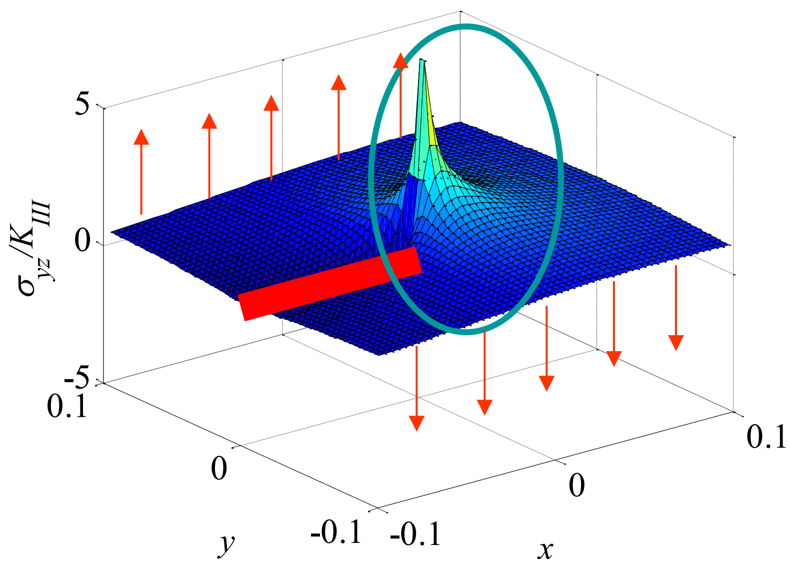
  - SIF: $K_{III} = \lim_{r\to 0} \left(\sqrt{2\pi r}\; \boldsymbol{\sigma}_{yz}^{\text{mode III}} \big|_{\theta=0} \right) = \frac{CE}{2(1+\nu)} \sqrt{\frac{\pi}{2}}$

- Application to mode III (shearing) (4)
  - Asymptotic fields

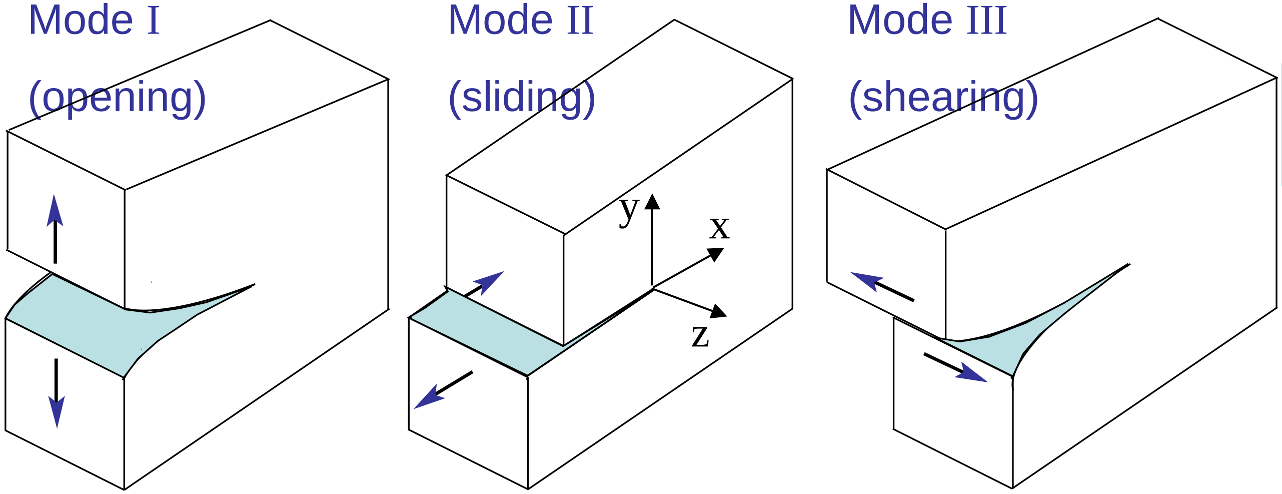
$$\begin{cases} \boldsymbol{u}_z = \dfrac{2K_{III}(1+\nu)}{E}\sqrt{\dfrac{2r}{\pi}}\sin\dfrac{\theta}{2} \\ \boldsymbol{\sigma}_{xz} = -\dfrac{K_{III}}{\sqrt{2\pi r}}\sin\dfrac{\theta}{2} \\ \boldsymbol{\sigma}_{yz} = \dfrac{K_{III}}{\sqrt{2\pi r}}\cos\dfrac{\theta}{2} \end{cases}$$







- Summary in 2D



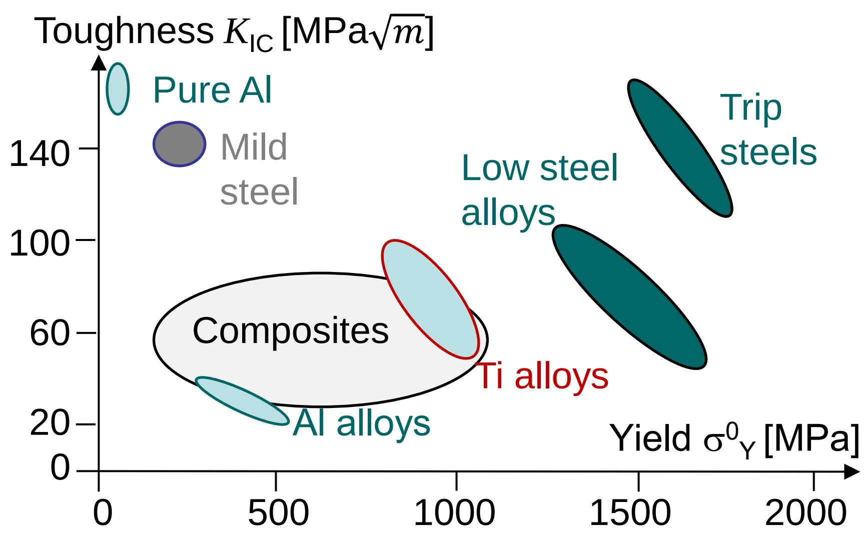
$$\boldsymbol{\sigma}^{\text{mode i}} = \frac{K_i}{\sqrt{2\pi r}} \mathbf{f}^{\text{mode i}}(\theta)$$

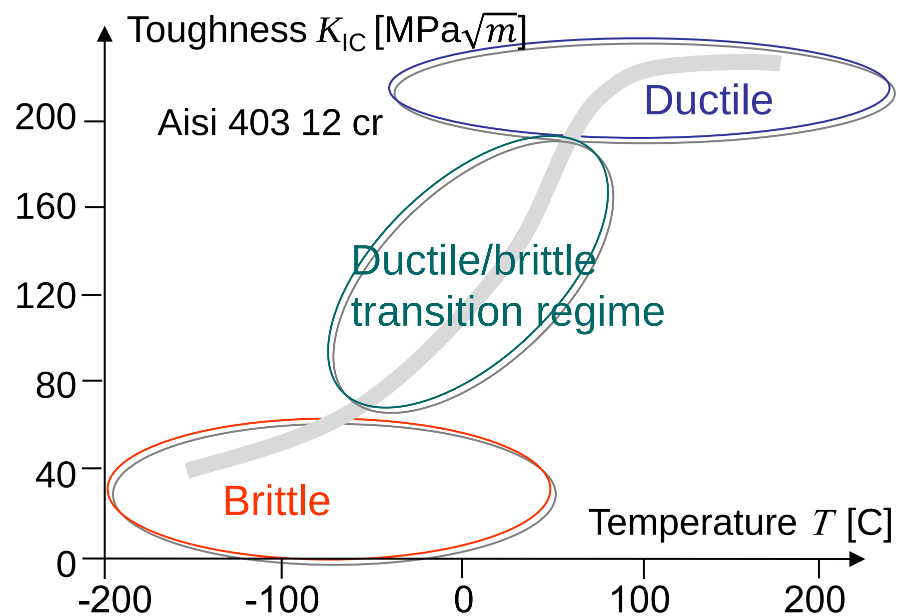
$$\boldsymbol{u}^{\text{mode i}} = K_i \sqrt{\frac{r}{2\pi}} \boldsymbol{g}^{\text{mode i}}(\theta)$$

– Principle of superposition holds as linear responses have been assumed
  - K_i depends on
    – The geometry and
    – The loading
  - ***u***, **σ** can be added for
    – ≠ modes: $\boldsymbol{u} = \boldsymbol{u}^{\text{mode I}} + \boldsymbol{u}^{\text{mode II}}$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\text{mode I}} + \boldsymbol{\sigma}^{\text{mode II}}$
    – ≠ loadings $\boldsymbol{u} = \boldsymbol{u}^{\text{loading 1}} + \boldsymbol{u}^{\text{loading 1}}$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\text{loading 1}} + \boldsymbol{\sigma}^{\text{loading 2}}$
  - K_i can be added
    – For ≠ loadings of the same mode $K_i = K_i^{\text{loading 1}} + K_i^{\text{loading 2}}$
  - But since ***f*** and **g** depend on the mode $K \neq K_{\text{mode 1}} + K_{\text{mode 2}}$

Linear Elastic Fracture Mechanics (LEFM)

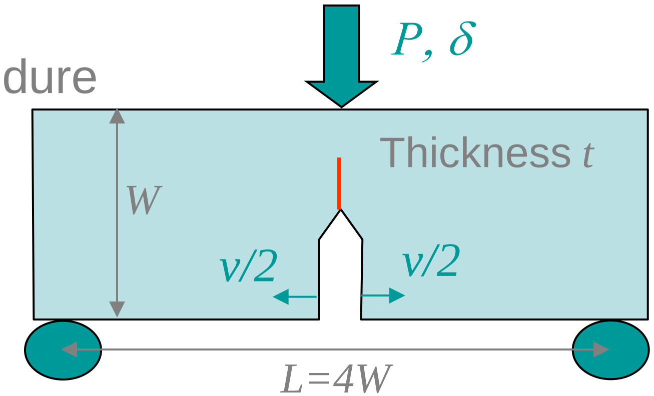
- 1957, Irwin, new failure criterion
  - $\sigma_{max} \rightarrow \infty$ ⟹ σ is irrelevant
  - Compare the SIFs (dependent on loading and geometry) to a new material property: the toughness
    - If $K_i = K_{iC}$ ⟹ crack growth
    - Toughness (ténacité) K_{Ic}
      - Steel, Al, … : see figures
      - Concrete: 0.2 - 1.4 $\mathrm{MPa\ m}^{1/2}$

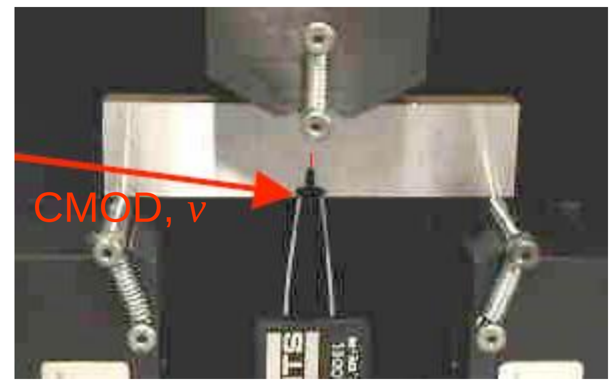




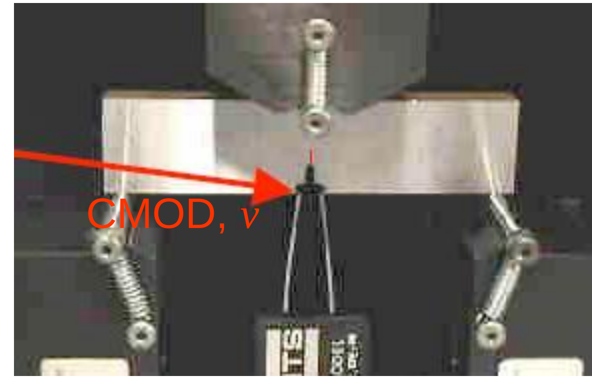
- Measuring K_{Ic}
  - Done by strictly following the ASTM E399 procedure
  - Specimen
    - Normalized, e.g. Single Notched Bend (SENB)
    - Plane strain constraint (thick enough)

      ⇒ conservative (see next slide)
    - Specimen machined with a V-notch
  - Crack initiation
    - Cyclic loading to initiate a fatigue crack
    - Crack length from compliance
      - Crack Mouth Opening Displacement ($CMOD=v$) measured with a clipped gauge
      - Calibrated using FEM

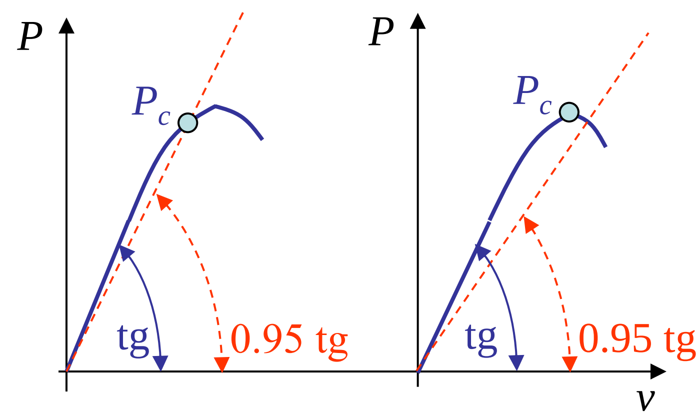




- Measuring K_{ic} (2)
  - Done by strictly following the ASTM E399 procedure
  - Toughness test
    - Calibrated P, δ recording equipment
    - Crack Mouth Opening Displacement (CMOD=v) measured with a clipped gauge



    - P_c is obtained on $P - v$ curves
      - Either the 95% offset value or
      - The maximal value reached before



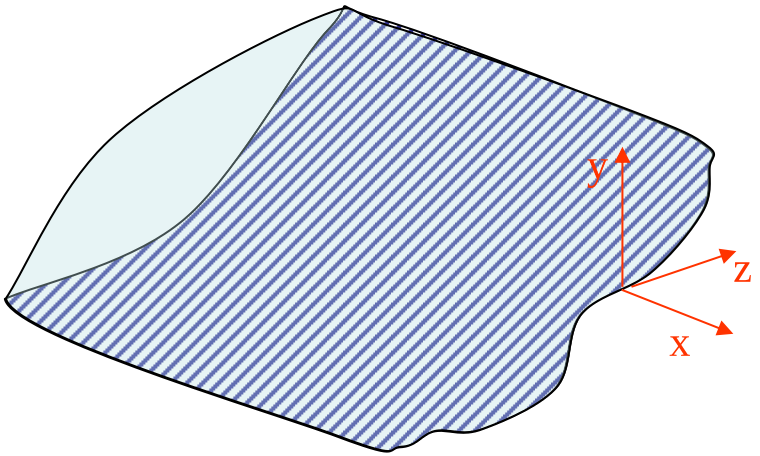
    - K_{Ic} is deduced from P_c using

$$K_I = \frac{PL}{tW^{\frac{3}{2}}} f\left(\frac{a}{W}\right)$$

      - $f(a/W)$ depends on the test (SENB, …)
      - $f(a/W)$ calibrated using FEM etc, in the norm
  - Check the constraint once you have K_{Ic}
    - Plane strain constraint (thick enough)

3D problems

- Only 2D solutions have been considered but a real crack is clearly 3D
  - At any point of the crack line
    - A local referential can be defined
    - Since the asymptotic solutions hold for $r \rightarrow 0$, at this distance the crack line seems straight, and the problem is locally 2D
    - The crack tip field can be broken into 3 2D problems (3 2D modes)

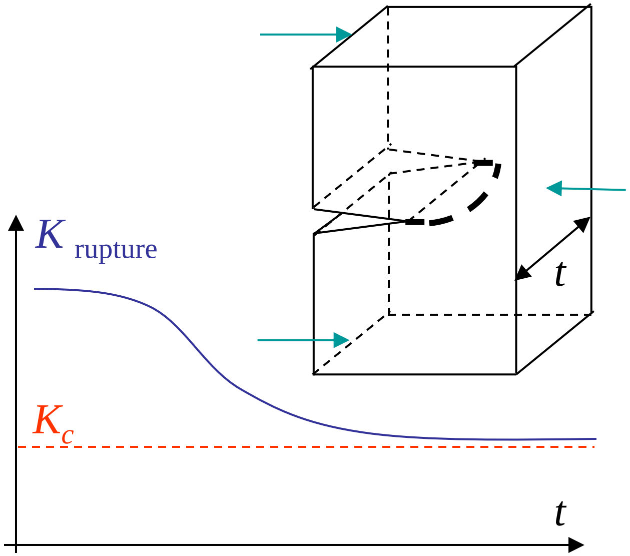


- 3D effects
  - Near the border of a specimen the problem is plane σ, while it is plane ε near the center

    ⟹ at the center there is a triaxial state
    - The SIF is larger at the center as no lateral deformations are possible (see next lecture)
    - 2 consequences
      - The front will first propagate at the center
      - The toughness decreases with the thickness
        - » Crude approximation

$$K_C\left(t\right) \simeq K_C\left(t\rightarrow\infty\right)\left[1+\frac{1.4}{t^2}\left(\frac{K_C\left(t\rightarrow\infty\right)}{\sigma_p^0}\right)^4\right]^{\frac{1}{2}}$$

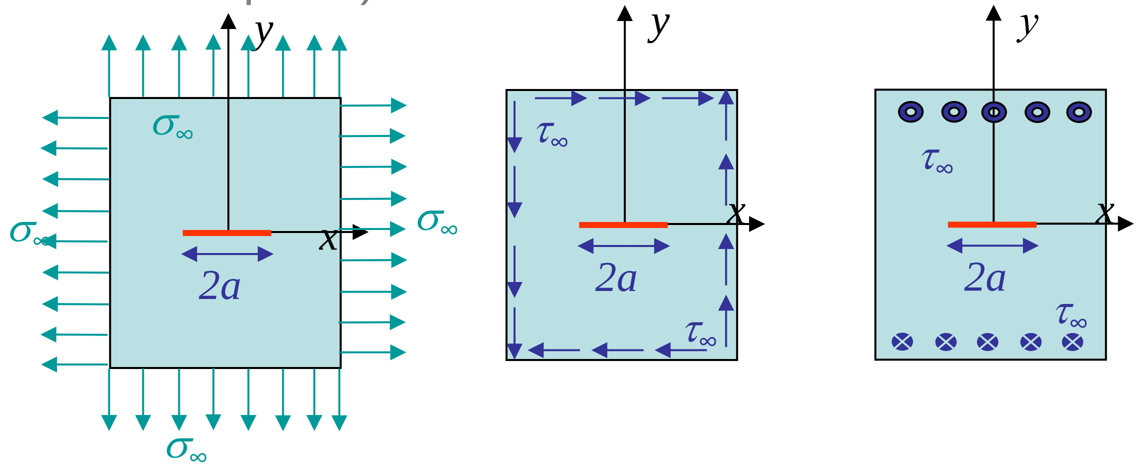
        - » Later on we will see how to evaluate this effect



  - The practical toughness K_c is the plane strain one

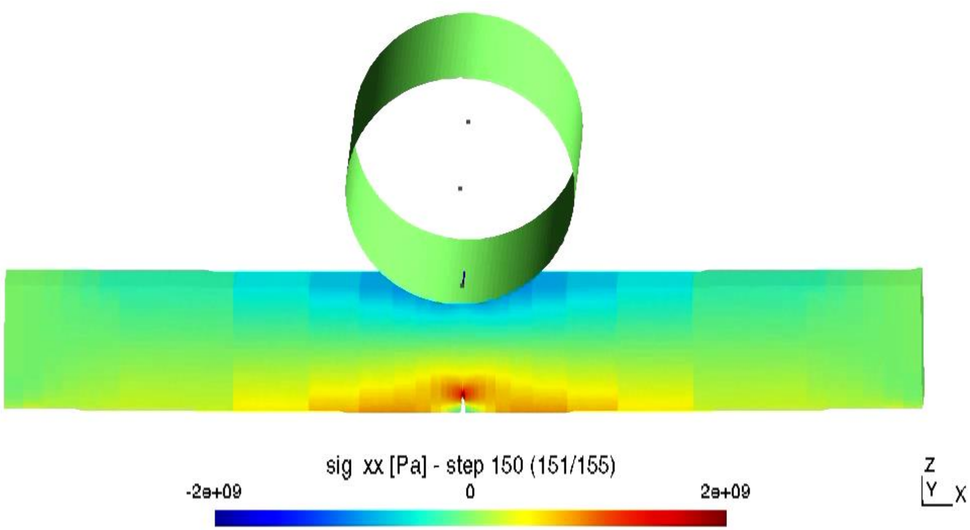
- Evaluation of the stress Intensity Factor (SIF)
  - Analytical (crack $2a$ in an infinite plane)

$$\begin{cases} K_I = \sigma_\infty \sqrt{\pi a} \\ K_{II} = \tau_\infty \sqrt{\pi a} \\ K_{III} = \tau_\infty \sqrt{\pi a} \end{cases}$$



  - Numerical

$$\begin{cases} K_I = \beta_I \sigma_\infty \sqrt{\pi a} \\ K_{II} = \beta_{II} \tau_\infty \sqrt{\pi a} \\ K_{III} = \beta_{III} \tau_\infty \sqrt{\pi a} \end{cases}$$



    - β_i depends on
      - Geometry
      - Crack length

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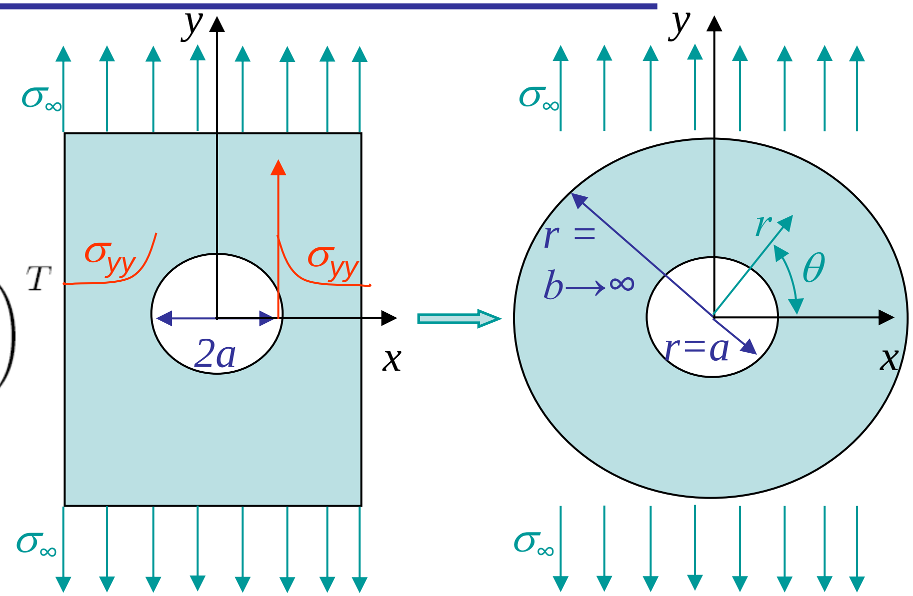
- Lecture notes
  - Lecture Notes on Fracture Mechanics, Alan T. Zehnder, Cornell University, Ithaca, http://hdl.handle.net/1813/3075
  - Fracture Mechanics Online Class, L. Noels, ULg, http://www.ltas-cm3.ulg.ac.be/FractureMechanics
- Book
  - Fracture Mechanics: Fundamentals and applications, D. T. Anderson. CRC press, 1991.
  - S. Suresh, Fatigue of Materials, Cambridge University Press, 2001

- Cylindrical coordinates

$$\begin{pmatrix} \boldsymbol{\sigma}_{rr} & \boldsymbol{\sigma}_{r\theta} \\ \boldsymbol{\sigma}_{r\theta} & \boldsymbol{\sigma}_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_{xx} & \boldsymbol{\sigma}_{xy} \\ \boldsymbol{\sigma}_{xy} & \boldsymbol{\sigma}_{yy} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^T$$



  - Boundary conditions
    - At $r = a$

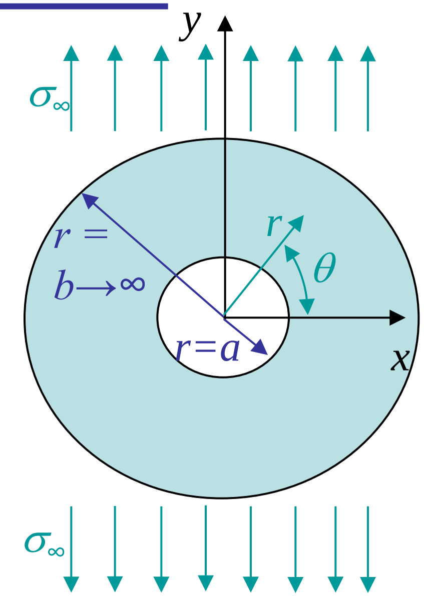
$$\boldsymbol{\sigma}_{rr}(r=a,\theta) = \boldsymbol{\sigma}_{r\theta}(r=a,\theta) = 0$$

    - At $r = b\to\infty$: only $\sigma_{yy} = \sigma_\infty$ is non zero

$$\begin{cases} \boldsymbol{\sigma}_{rr}(r=b,\theta) = \sigma_\infty \sin^2\theta = \frac{1}{2}\sigma_\infty - \frac{1}{2}\sigma_\infty \cos 2\theta \\ \boldsymbol{\sigma}_{r\theta}(r=b,\theta) = \sigma_\infty \cos\theta\sin\theta = \frac{1}{2}\sigma_\infty \sin 2\theta \end{cases}$$

Load case 1 independent from θ

Load case 2 dependent on $\cos 2\theta$ (symmetry with respect to x)



- Cylindrical coordinates (2)
  - Bi-harmonic equation $\nabla^2 \nabla^2 \Phi = 0$

with $\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta \end{pmatrix}$

$$\Longrightarrow \left(\partial^2_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial^2_{\theta\theta}\right)\left(\Phi_{,rr} + \frac{1}{r}\Phi_{,r} + \frac{1}{r^2}\Phi_{,\theta\theta}\right) = 0$$

  - Stresses from Airy function

$$\begin{cases} \boldsymbol{\sigma}_{rr} = \boldsymbol{\sigma}_{xx}\cos^2\theta + \boldsymbol{\sigma}_{yy}\sin^2\theta + 2\boldsymbol{\sigma}_{xy}\sin\theta\cos\theta \\ \boldsymbol{\sigma}_{\theta\theta} = \boldsymbol{\sigma}_{yy}\cos^2\theta + \boldsymbol{\sigma}_{xx}\sin^2\theta - 2\boldsymbol{\sigma}_{xy}\sin\theta\cos\theta \\ \boldsymbol{\sigma}_{r\theta} = (\boldsymbol{\sigma}_{yy} - \boldsymbol{\sigma}_{xx})\sin\theta\cos\theta + \boldsymbol{\sigma}_{xy}\left(\cos^2\theta - \sin^2\theta\right) \end{cases}$$

with $\boldsymbol{\sigma}_{\alpha\beta} = -\Phi_{,\alpha\beta} + \delta_{\alpha\beta}\Phi_{,\gamma\gamma} \Longrightarrow \boldsymbol{\sigma}_{xx} = \Phi_{,yy}$, $\boldsymbol{\sigma}_{yy} = \Phi_{,xx}$ & $\boldsymbol{\sigma}_{xy} = -\Phi_{,xy}$

- Eventually, after substitutions $\begin{cases} \boldsymbol{\sigma}_{rr} = \frac{1}{r}\Phi_{,r} + \frac{1}{r^2}\Phi_{,\theta\theta} \\ \boldsymbol{\sigma}_{\theta\theta} = \Phi_{,rr} \\ \boldsymbol{\sigma}_{r\theta} = -\frac{1}{r}\Phi_{,r\theta} + \frac{1}{r^2}\Phi_{,\theta} \end{cases}$

- Load case 1
  - Independent from θ $\Longrightarrow$ $\Phi(r)$
    - Bi-harmonic equation $\Longrightarrow$ $\left(\partial^2_{rr} + \frac{1}{r}\partial_r\right)\left(\Phi_{,rr} + \frac{1}{r}\Phi_{,r}\right) = 0$
    - General solution: $\Phi = C_1 \ln r + C_2 r^2 \ln r + C_3 r^2 + C_4$
    - Stresses: $\begin{cases} \boldsymbol{\sigma}_{rr} = \frac{1}{r}\Phi_{,r} = \frac{C_1}{r^2} + C_2\left(1 + 2\ln r\right) + 2C_3 \\ \boldsymbol{\sigma}_{\theta\theta} = \Phi_{,rr} = -\frac{C_1}{r^2} + C_2\left(3 + 2\ln r\right) + 2C_3 \\ \boldsymbol{\sigma}_{r\theta} = 0 \end{cases}$
    - Boundary conditions: $\begin{cases} \boldsymbol{\sigma}_{rr}\left(r = a, \theta\right) = \boldsymbol{\sigma}_{r\theta}\left(r = a, \theta\right) = 0 \\ \boldsymbol{\sigma}_{rr}\left(r = b, \theta\right) = \frac{1}{2}\sigma_\infty \end{cases}$

$\Longrightarrow$ $\frac{C_1}{a^2} + C_2\left(1 + 2\ln a\right) + 2C_3 = 0$ & $\lim_{b\to\infty}\left(\frac{C_1}{b^2} + C_2\left(1 + 2\ln b\right) + 2C_3\right) = \frac{1}{2}\sigma_\infty$

$\Longrightarrow$ $C_1 = -\frac{a^2}{2}\sigma_\infty$, $C_2 = 0$ & $C_3 = \frac{1}{4}\sigma_\infty$

- Solution: $\boxed{\boldsymbol{\sigma}_{rr} = \frac{1}{2}\sigma_\infty\left(1 - \frac{a^2}{r^2}\right)}$, $\boxed{\boldsymbol{\sigma}_{\theta\theta} = \frac{1}{2}\sigma_\infty\left(\frac{a^2}{r^2} + 1\right)}$ & $\boxed{\boldsymbol{\sigma}_{r\theta} = 0}$

- Load case 2
  - Dependent on $\cos 2\theta$ $\Longrightarrow$ $\Phi(r, \cos 2\theta) = g(r) \cos 2\theta$
    - Bi-harmonic equation

$$\left(\partial^2_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial^2_{\theta\theta}\right)\left[\left(g'' + \frac{g'}{r} - \frac{4g}{r^2}\right)\cos 2\theta\right] = 0$$

$$\Longrightarrow \left(g'''' + \frac{2g'''}{r} - \frac{9g''}{r^2} + \frac{9g'}{r^3}\right)\cos 2\theta = 0$$

    - General solution: $\Phi = \left(C_1 r^4 + C_2 r^2 + \frac{C_3}{r^2} + C_4\right)\cos 2\theta$

    - Stresses:

$$\begin{cases} \boldsymbol{\sigma}_{rr} = \left(\frac{g'}{r} - \frac{4g}{r^2}\right)\cos 2\theta = \left(-2C_2 - \frac{6C_3}{r^4} - \frac{4C_4}{r^2}\right)\cos 2\theta \\ \boldsymbol{\sigma}_{\theta\theta} = g''\cos 2\theta = \left(12C_1 r^2 + 2C_2 + \frac{6C_3}{r^4}\right)\cos 2\theta \\ \boldsymbol{\sigma}_{r\theta} = \left(\frac{2g'}{r} - \frac{2g}{r^2}\right)\sin 2\theta = \left(6C_1 r^2 + 2C_2 - \frac{6C_3}{r^4} - \frac{2C_4}{r^2}\right)\sin 2\theta \end{cases}$$

- Load case 2 (2)
  - Stresses

$$\begin{cases} \boldsymbol{\sigma}_{rr} = \left(\frac{g'}{r} - \frac{4g}{r^2}\right)\cos 2\theta = \left(-2C_2 - \frac{6C_3}{r^4} - \frac{4C_4}{r^2}\right)\cos 2\theta \\ \boldsymbol{\sigma}_{\theta\theta} = g''\cos 2\theta = \left(12C_1 r^2 + 2C_2 + \frac{6C_3}{r^4}\right)\cos 2\theta \\ \boldsymbol{\sigma}_{r\theta} = \left(\frac{2g'}{r} - \frac{2g}{r^2}\right)\sin 2\theta = \left(6C_1 r^2 + 2C_2 - \frac{6C_3}{r^4} - \frac{2C_4}{r^2}\right)\sin 2\theta \end{cases}$$

  - Boundary conditions:

$$\begin{cases} \boldsymbol{\sigma}_{rr}\left(r=a,\theta\right) = \boldsymbol{\sigma}_{r\theta}\left(r=a,\theta\right) = 0 \\ \boldsymbol{\sigma}_{rr}\left(r=b,\theta\right) = -\frac{1}{2}\sigma_\infty \cos 2\theta \\ \boldsymbol{\sigma}_{r\theta}\left(r=b,\theta\right) = \frac{1}{2}\sigma_\infty \sin 2\theta \end{cases}$$

$$\Longrightarrow \quad C_1 = 0 \ , C_2 = \frac{\sigma_\infty}{4} , C_3 = \frac{\sigma_\infty a^4}{4} \ \ \& \ \ C_4 = -\frac{\sigma_\infty a^2}{2}$$

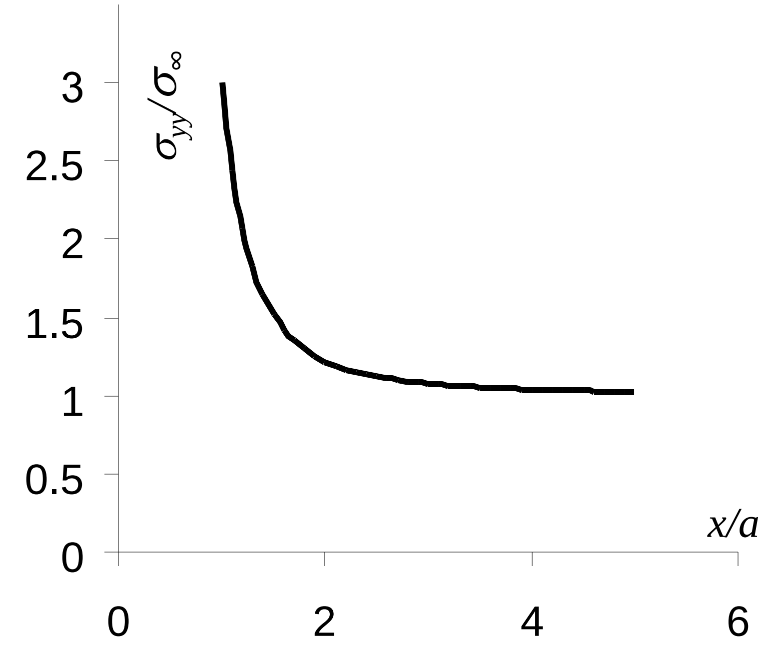
- Load case 2 (2)
  - Solution:

$$\left\{\begin{array}{l}\boldsymbol{\sigma}_{rr} = \frac{\sigma_\infty}{2}\left(\frac{4a^2}{r^2} - 1 - \frac{3a^4}{r^4}\right)\cos 2\theta \\ \boldsymbol{\sigma}_{\theta\theta} = \frac{\sigma_\infty}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta \\ \boldsymbol{\sigma}_{r\theta} = \frac{\sigma_\infty}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta\end{array}\right.$$

- Superposition of case 1 and 2
  - Stresses:
$$\begin{cases} \boldsymbol{\sigma}_{rr} = \dfrac{\sigma_\infty}{2}\left[\left(\dfrac{4a^2}{r^2} - 1 - \dfrac{3a^4}{r^4}\right)\cos 2\theta + 1 - \dfrac{a^2}{r^2}\right] \\ \boldsymbol{\sigma}_{\theta\theta} = \dfrac{\sigma_\infty}{2}\left[\left(1 + \dfrac{3a^4}{r^4}\right)\cos 2\theta + \dfrac{a^2}{r^2} + 1\right] \\ \boldsymbol{\sigma}_{r\theta} = \dfrac{\sigma_\infty}{2}\left(1 - \dfrac{3a^4}{r^4} + \dfrac{2a^2}{r^2}\right)\sin 2\theta \end{cases}$$
  - For $\theta = 0$: $\sigma_{yy} = \sigma_{\theta\theta}$

$$\Longrightarrow \boldsymbol{\sigma}_{yy}(x, y = 0) = \frac{\sigma_\infty}{2}\left(2 + \frac{3a^4}{x^4} + \frac{a^2}{x^2}\right)$$

  - Stress intensity factor: $K_{\text{circle}} = 3$



- Displacements for linear and elastic materials
  - Complex form: $\boldsymbol{u} = \boldsymbol{u}_x(x,\, y) + i\boldsymbol{u}_y(x,\, y) = \boldsymbol{u}(\zeta,\, \bar{\zeta})$
  - In terms of strains:

$$\boldsymbol{u}_{,\bar{\zeta}} = (\boldsymbol{u}_{x,x} + i\boldsymbol{u}_{y,x})\,\partial_{\bar{\zeta}}x + (\boldsymbol{u}_{x,y} + i\boldsymbol{u}_{y,y})\,\partial_{\bar{\zeta}}y = \frac{1}{2}(\boldsymbol{u}_{x,x} + i\boldsymbol{u}_{y,x} + i\boldsymbol{u}_{x,y} - \boldsymbol{u}_{y,y})$$

$$\Longrightarrow \boldsymbol{u}_{,\bar{\zeta}} = \frac{1}{2}(\boldsymbol{\varepsilon}_{xx} - \boldsymbol{\varepsilon}_{yy}) + i\boldsymbol{\varepsilon}_{xy}$$

  - Hooke's law (Plane σ or ε):

$$\boldsymbol{u}_{,\bar{\zeta}} = \frac{1+\nu}{2E}(\boldsymbol{\sigma}_{xx} - \boldsymbol{\sigma}_{yy} + 2i\boldsymbol{\sigma}_{xy}) = -\frac{1+\nu}{E}\left(\zeta\bar{\Omega}'' + \bar{\omega}''\right)$$

$$\Longrightarrow \boldsymbol{u} = -\frac{1+\nu}{E}\left(\zeta\bar{\Omega}' + \bar{\omega}' + \mu(\zeta)\right)$$

    - With $\mu(\zeta)$ such that the plane σ or ε condition is satisfied:

$$\begin{cases} \boldsymbol{u}_{,\zeta} = (\boldsymbol{u}_{x,x} + i\boldsymbol{u}_{y,x})\,\partial_{\zeta}x + (\boldsymbol{u}_{x,y} + i\boldsymbol{u}_{y,y})\,\partial_{\zeta}y = \frac{1}{2}(\boldsymbol{u}_{x,x} + i\boldsymbol{u}_{y,x} - i\boldsymbol{u}_{x,y} + \boldsymbol{u}_{y,y}) \\ \bar{\boldsymbol{u}}_{,\bar{\zeta}} = (\boldsymbol{u}_{x,x} - i\boldsymbol{u}_{y,x})\,\partial_{\bar{\zeta}}x + (\boldsymbol{u}_{x,y} - i\boldsymbol{u}_{y,y})\,\partial_{\bar{\zeta}}y = \frac{1}{2}(\boldsymbol{u}_{x,x} - i\boldsymbol{u}_{y,x} + i\boldsymbol{u}_{x,y} + \boldsymbol{u}_{y,y}) \end{cases}$$

$$\Longrightarrow \boldsymbol{u}_{,\zeta} + \bar{\boldsymbol{u}}_{,\bar{\zeta}} = \boldsymbol{u}_{x,x} + \boldsymbol{u}_{yy} = \boldsymbol{\varepsilon}_{xx} + \boldsymbol{\varepsilon}_{yy} = \boldsymbol{\varepsilon}_{\gamma\gamma}$$

    - But $\boldsymbol{u}_{,\zeta} + \bar{\boldsymbol{u}}_{,\bar{\zeta}} = -\frac{1+\nu}{E}\left(\bar{\Omega}' + \mu' + \Omega' + \bar{\mu}'\right)$

$$\Longrightarrow \boxed{\boldsymbol{\varepsilon}_{\gamma\gamma} = -\frac{1+\nu}{E}\left(\bar{\Omega}' + \mu' + \Omega' + \bar{\mu}'\right)}$$

- Displacements for linear and elastic materials (2)
  - Out-of-plane state effect

$$\varepsilon_{\gamma\gamma} = -\frac{1+\nu}{E}\left(\bar{\Omega}' + \mu' + \Omega' + \bar{\mu}'\right)$$

- Plane σ : $\varepsilon_{\gamma\gamma} = \frac{1-\nu}{E}\boldsymbol{\sigma}_{\gamma\gamma} = \frac{2\left(1-\nu\right)}{E}\left(\Omega' + \bar{\Omega}'\right)$

$\Longrightarrow$ $\mu\left(\zeta\right) = -\frac{3-\nu}{1+\nu}\Omega = -\kappa\Omega\left(\zeta\right)$ with $\kappa = \frac{3-\nu}{1+\nu}$

- Plane ε: $\varepsilon_{\gamma\gamma} = \frac{\left(1+\nu\right)\left(1-2\nu\right)}{E}\boldsymbol{\sigma}_{\gamma\gamma} = \frac{2\left(1+\nu\right)\left(1-2\nu\right)}{E}\left(\Omega' + \bar{\Omega}'\right)$

$\Longrightarrow$ $\mu\left(\zeta\right) = -\left(3-4\nu\right)\Omega = -\kappa\Omega\left(\zeta\right)$ with $\kappa = 3 - 4\nu$

  - For both plane states: $\boxed{\boldsymbol{u} = -\frac{1+\nu}{E}\left(\zeta\bar{\Omega}' + \bar{\omega}' - \kappa\Omega\left(\zeta\right)\right)}$