University of Liège
Aerospace & Mechanical Engineering

Fracture Mechanics, Damage and Fatigue: Composites

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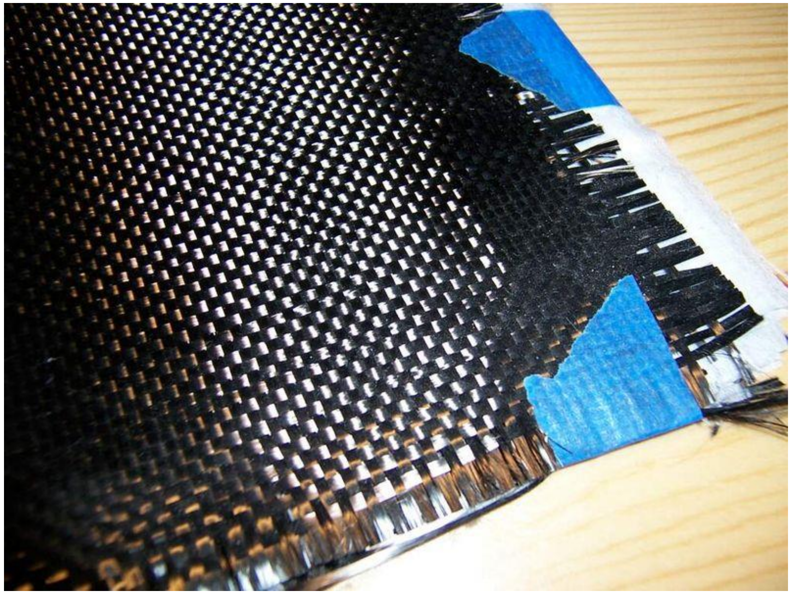
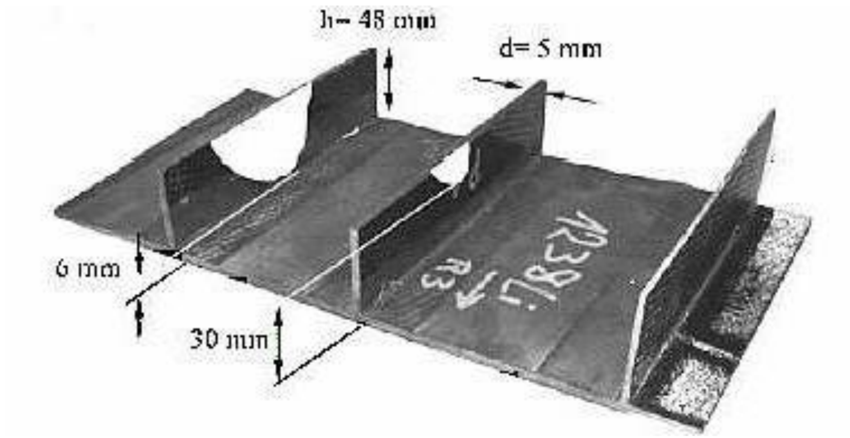
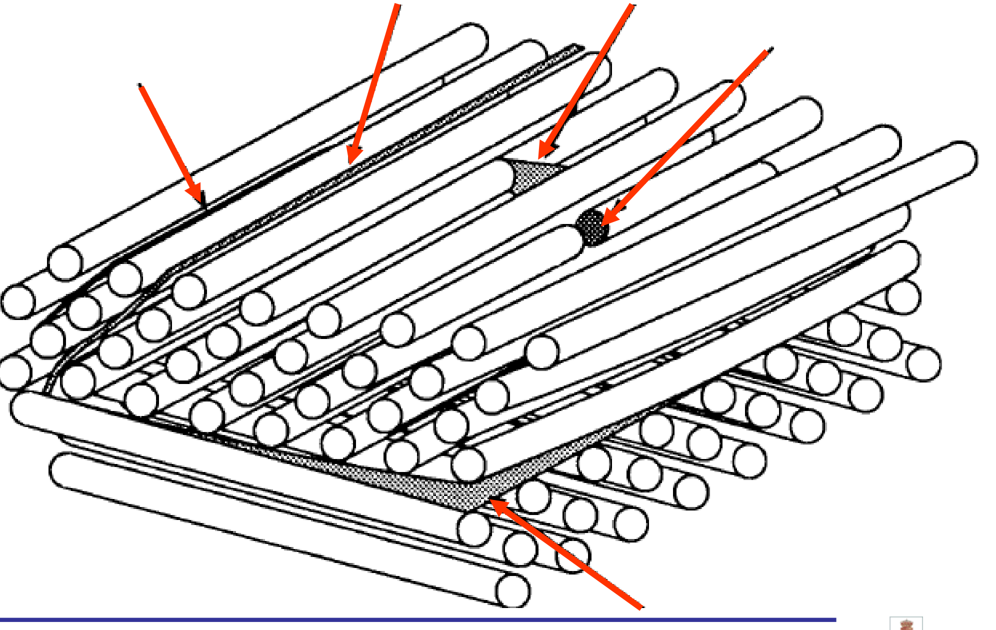
- Composite
  - Fibers in a matrix
    - Fibers: polymers, metals or ceramics
    - Matrix: polymers, metals or ceramics
    - Fibers orientation: unidirectional, woven, random
  - Carbon Fiber Reinforced Plastic
    - Carbon woven fibers in epoxy resin
      - Picture: carbon fibers
    - Theoretical tensile strength: 1400 MPa
    - Density: 1800 $kg{\cdot}m^{-3}$
    - A laminate is a stack of CFRP plies
      - Picture: skin with stringers



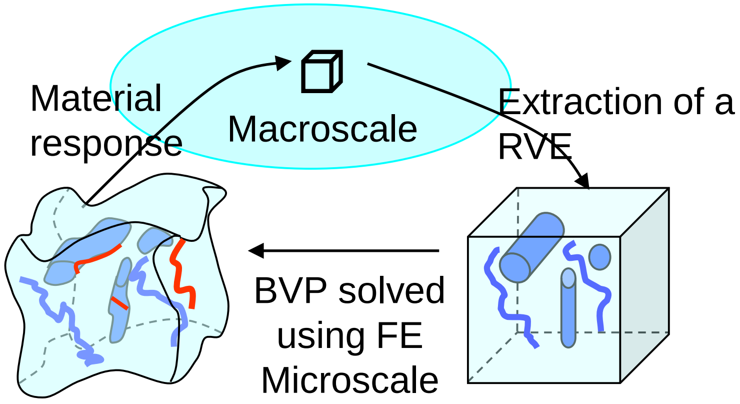


- Composite (2)
  - Drawbacks
    - “Brittle” rupture mode
    - Impact damage
    - Resin can absorb moisture
  - Complex failure modes
    - Transverse matrix fracture
    - Longitudinal matrix fracture
    - Fiber rupture
    - Fiber debonding
    - Delamination
    - Macroscopically: no plastic deformation

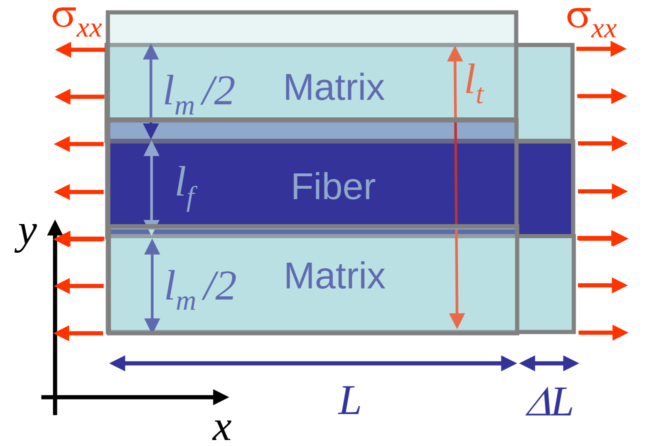


- Composite (3)
  - Wing, fuselage, …
  - Typhoon: CFRP
    - 70% of the skin
    - 40% of total weight
  - B787:
    - Fuselage all in CFRP

- Approaches in analyzing composite materials
  - Micromechanics
    - Composite is considered as an heterogeneous material
    - Material properties change from one point to the other
      - Resin
      - Fiber
      - Ply
    - Method used to study composite properties
  - Macromechanics
    - Composite is seen as an homogenized material
    - Material properties are constant in each direction
      - They change from one direction to the other
    - Method used in preliminary design
  - Multiscale
    - Combining both approaches



- Ply (lamina) mechanics: E_x
  - Symmetrical piece of lamina
    - Matrix-Fiber-Matrix
  - Constraint (small) longitudinal displacement ΔL
    - Small strain $\varepsilon_{xx} = \frac{\Delta L}{L}$
    - Microscopic stresses
      - Fiber $\sigma_{xx\,f} = E_f \varepsilon_{xx}$
      - Matrix $\sigma_{xx\,m} = E_m \varepsilon_{xx}$
    - Resultant stress $\sigma_{xx} = E_x \varepsilon_{xx}$
    - Compatibility $\sigma_{xx} l_t = \sigma_{xx\,f} l_f + \sigma_{xx\,m} l_m$

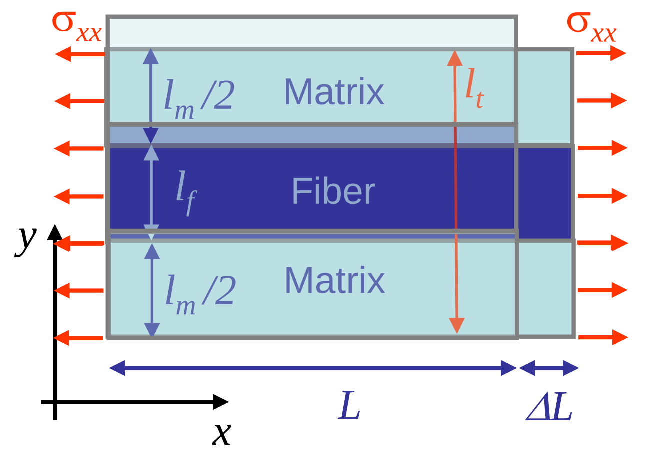


$$\Longrightarrow E_x \varepsilon_{xx} l_t = E_f \varepsilon_{xx} l_f + E_m \varepsilon_{xx} l_m$$

$$\Longrightarrow E_x = \frac{l_f}{l_t} E_f + E_m \frac{l_m}{l_t} = v_f E_f + v_m E_m$$

  - The mixture law gives the longitudinal Young modulus of a unidirectional fiber lamina from the matrix and fiber volume ratio
    - As $E_f >> E_m$, in general $E_x \sim v_f E_f$

- Ply (lamina) mechanics: ν_{xy}
  - Constraint (small) longitudinal displacement ΔL

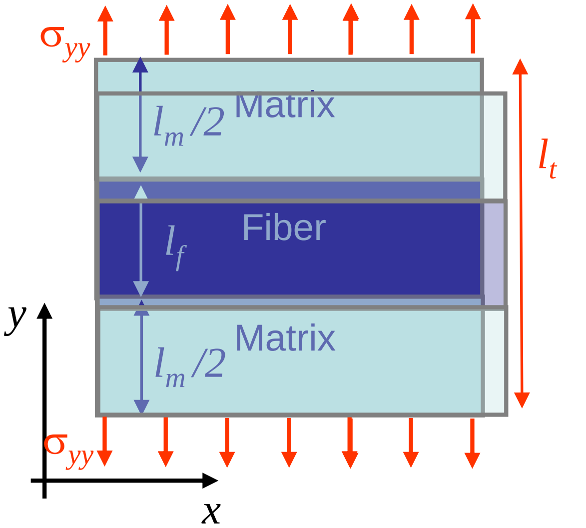


- Transverse displacement $\Delta l_t = \Delta l_f + \Delta l_m$
- Microscopic strains
  - Fiber $\Delta l_f = -\nu_f \varepsilon_{xx\,f} l_f = -\nu_f l_f \varepsilon_{xx}$
  - Matrix $\Delta l_m = -\nu_m \varepsilon_{xx\,m} l_m = -\nu_m l_m \varepsilon_{xx}$
- Resultant strain $\Delta l_t = -\nu_{xy} \varepsilon_{xx} l_t$
- Compatibility $-\nu_{xy}\varepsilon_{xx} l_t = \Delta l_t = \Delta l_f + \Delta l_m = -\nu_f \varepsilon_{xx} l_f - \nu_m \varepsilon_{xx} l_m$

$$\Longrightarrow \nu_{xy} = \nu_f \frac{l_f}{l_t} + \nu_m \frac{l_m}{l_t} = \nu_f v_f + \nu_m v_m$$

- This coefficient ν_{xy} is called major Poisson's ratio of the lamina

- Ply (lamina) mechanics: E_y
  - Constraint (small) transversal displacement $\Delta\, l$



- Total displacement $\Delta l = \Delta l_m + \Delta l_f$
- Microscopic small strains
  - Fiber $\varepsilon_{yy\,f} = \dfrac{\Delta l_f}{l_f}$
  - Matrix $\varepsilon_{yy\,m} = \dfrac{\Delta l_m}{l_m}$
- Small resultant strain
  - $\varepsilon_{yy} = \dfrac{\Delta l}{l_t} = \dfrac{\Delta l_m}{l_t} + \dfrac{\Delta l_f}{l_t} \Longrightarrow \varepsilon_{yy} = \varepsilon_{yy\,f}\dfrac{l_f}{l_t} + \varepsilon_{yy\,m}\dfrac{l_m}{l_t}$
- Resultant stresses = microscopic stresses
  - $\boldsymbol{\sigma}_{yy} = E_y\varepsilon_{yy} = E_f\varepsilon_{yy\,f} = E_m\varepsilon_{yy\,m}$
  - Relation $\varepsilon_{yy} = \varepsilon_{yy\,f}\dfrac{l_f}{l_t} + \varepsilon_{yy\,m}\dfrac{l_m}{l_t}$

$$\Longrightarrow \frac{\boldsymbol{\sigma}_{yy}}{E_y} = \frac{\boldsymbol{\sigma}_{yy}}{E_f}\frac{l_f}{l_t} + \frac{\boldsymbol{\sigma}_{yy}}{E_m}\frac{l_m}{l_t} \Longrightarrow \frac{1}{E_y} = \frac{v_f}{E_f} + \frac{v_m}{E_m}$$

- As $E_f >> E_m$, in general $E_y \sim E_m / v_m$

- Ply (lamina) mechanics: ν_{yx}
  - Constraint (small) transversal displacement Δl (2)
    - Longitudinal strains are equal by compatibility
      - Resultant $\varepsilon_{xx} = \frac{\Delta L}{L} = -\nu_{yx}\varepsilon_{yy}$
      - Fiber $\varepsilon_{xx\,f} = -\nu_f \varepsilon_{yy\,f} = \varepsilon_{xx}$
      - Matrix $\varepsilon_{xx\,m} = -\nu_m \varepsilon_{yy\,m} = \varepsilon_{xx}$
    - But from previous analysis

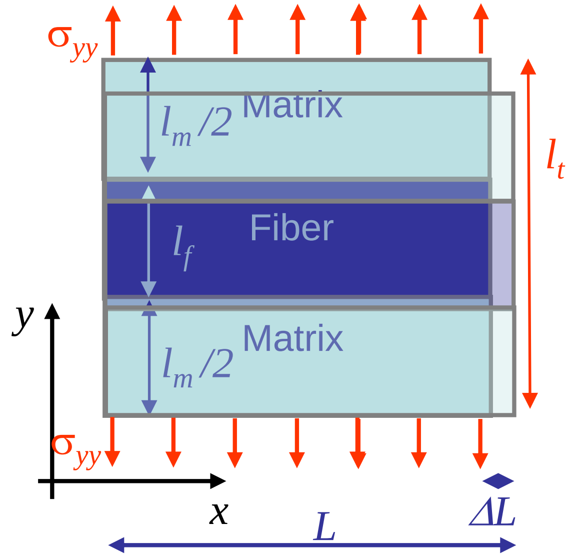
$$\varepsilon_{yy} = \varepsilon_{yy\,f} v_f + \varepsilon_{yy\,m} v_m$$

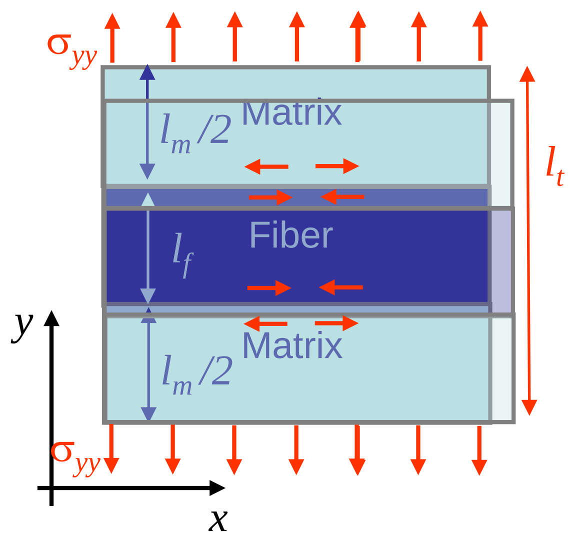
$$\Longrightarrow \nu_{yx}\left(\varepsilon_{yy\,f} v_f + \varepsilon_{yy\,m} v_m\right) = \nu_f \varepsilon_{yy\,f} = \nu_m \varepsilon_{yy\,m}$$

$$\Longrightarrow \nu_{yx}\varepsilon_{yy\,f}\left(v_f + \frac{v_m \nu_f}{\nu_m}\right) = \nu_f \varepsilon_{yy\,f}$$

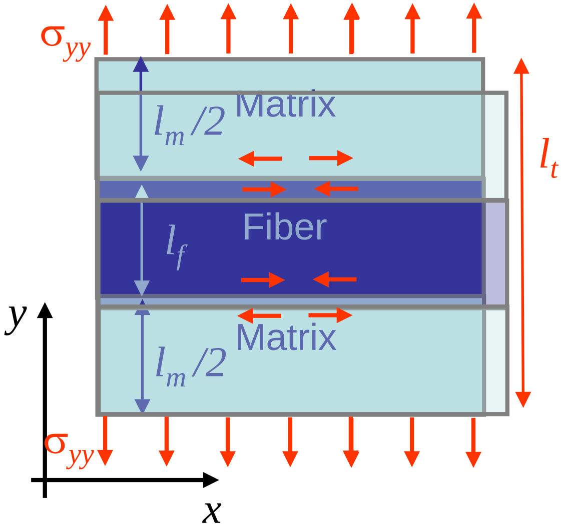
$$\Longrightarrow \nu_{yx} = \frac{\nu_f \nu_m}{v_f \nu_m + v_m \nu_f}$$

    - But this is wrong as there are microscopic stresses to constrain the compatibility, so relation $\varepsilon_{xx\,f} = -\nu_f \varepsilon_{yy\,f} = \varepsilon_{xx}$ is wrong





- Ply (lamina) mechanics: ν_{yx} (2)
  - Constraint (small) transversal displacement Δl (3)



- Resultant longitudinal strain
  - $\boldsymbol{\varepsilon}_{xx} = \frac{\Delta L}{L} = -\nu_{yx}\boldsymbol{\varepsilon}_{yy} = -\nu_{yx}\frac{\boldsymbol{\sigma}_{yy}}{E_y}$
- Microscopic strains & compatibility
  - Fiber

$$\boldsymbol{\varepsilon}_{xx\,f} = \frac{1}{E_f}\left(\boldsymbol{\sigma}_{xx\,f} - \nu_f\boldsymbol{\sigma}_{yy\,f}\right) = \frac{1}{E_f}\left(\boldsymbol{\sigma}_{xx\,f} - \nu_f\boldsymbol{\sigma}_{yy}\right)$$

  - Matrix

$$\boldsymbol{\varepsilon}_{xx\,m} = \frac{1}{E_m}\left(\boldsymbol{\sigma}_{xx\,m} - \nu_m\boldsymbol{\sigma}_{yy\,m}\right) = \frac{1}{E_m}\left(\boldsymbol{\sigma}_{xx\,m} - \nu_m\boldsymbol{\sigma}_{yy}\right)$$

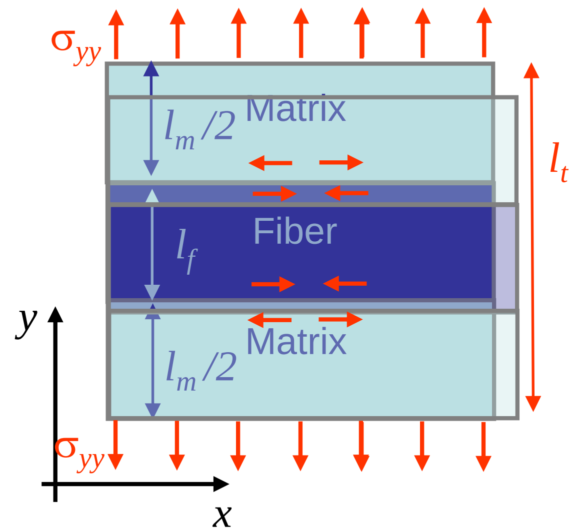
- Resultant stress along x is equal to zero

$$l_f\boldsymbol{\sigma}_{xx\,f} + l_m\boldsymbol{\sigma}_{xx\,m} = 0 \Longrightarrow v_f\boldsymbol{\sigma}_{xx\,f} = -v_m\boldsymbol{\sigma}_{xx\,m}$$

- Using compatibility of strains

$$\frac{1}{E_m}\left(\boldsymbol{\sigma}_{xx\,m} - \nu_m\boldsymbol{\sigma}_{yy}\right) = \frac{1}{E_f}\left(-\frac{v_m}{v_f}\boldsymbol{\sigma}_{xx\,m} - \nu_f\boldsymbol{\sigma}_{yy}\right) = -\nu_{yx}\frac{\boldsymbol{\sigma}_{yy}}{E_y}$$

- Ply (lamina) mechanics: ν_{yx} (3)
  - Constraint (small) transversal displacement Δl (4)



$$\frac{1}{E_m}\left(\boldsymbol{\sigma}_{xxm} - \nu_m \boldsymbol{\sigma}_{yy}\right) =$$

$$\frac{1}{E_f}\left(-\frac{v_m}{v_f}\boldsymbol{\sigma}_{xxm} - \nu_f \boldsymbol{\sigma}_{yy}\right) = -\nu_{yx}\frac{\boldsymbol{\sigma}_{yy}}{E_y}$$

$$\Rightarrow v_m\left(\frac{1}{v_m E_m} + \frac{1}{E_f v_f}\right)\boldsymbol{\sigma}_{xxm} = \boldsymbol{\sigma}_{yy}\left(\frac{\nu_m}{E_m} - \frac{\nu_f}{E_f}\right)$$

$$\Rightarrow \boldsymbol{\sigma}_{xxm} = \boldsymbol{\sigma}_{yy}\frac{\nu_m E_f - \nu_f E_m}{v_m E_f E_m}\,\frac{v_m E_m v_f E_f}{E_f v_f + E_m v_m} = v_f \boldsymbol{\sigma}_{yy}\frac{\nu_m E_f - \nu_f E_m}{E_f v_f + E_m v_m}$$

$$\Rightarrow -\nu_{yx}\frac{\boldsymbol{\sigma}_{yy}}{E_y} = \frac{1}{E_m}\left(v_f\frac{\nu_m E_f - \nu_f E_m}{E_f v_f + E_m v_m} - \nu_m\right)\boldsymbol{\sigma}_{yy}$$

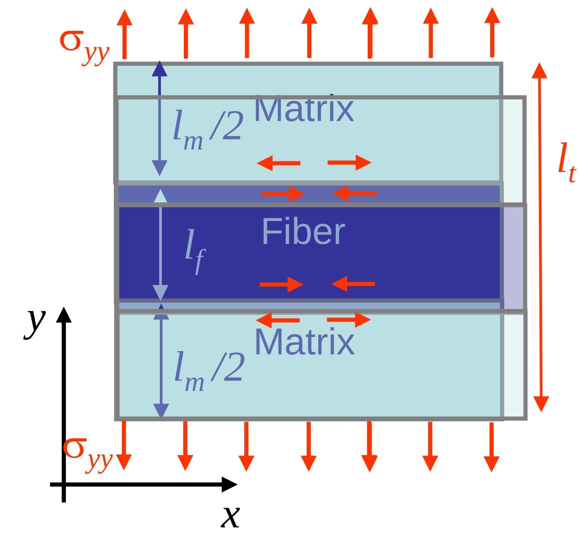
$$\Rightarrow \frac{\nu_{yx}}{E_y} = \frac{1}{E_m}\,\frac{v_f \nu_f E_m + v_m \nu_m E_m}{E_f v_f + E_m v_m}$$

$$\Rightarrow \nu_{yx} = E_y\frac{v_f \nu_f + v_m \nu_m}{E_f v_f + E_m v_m} = \frac{E_y \nu_{xy}}{E_x}$$

- Ply (lamina) mechanics: ν_{yx} (4)
  - Constraint (small) transversal displacement $\Delta\, l$ (5)
    - Minor Poisson coefficient

$$\nu_{yx} = E_y \frac{v_f \nu_f + v_m \nu_m}{E_f v_f + E_m v_m} = \frac{E_y \nu_{xy}}{E_x}$$

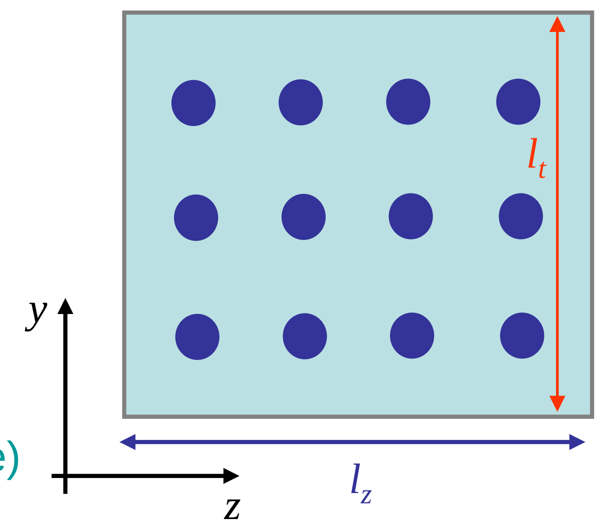
    - This is called the minor one as usually $E_m << E_f \Longrightarrow E_x >> E_y \Longrightarrow \nu_{yx} < \nu_{xy}$



- Remarks
  - The stresses in the matrix and in the fiber

$$v_f \boldsymbol{\sigma}_{xx\,f} = -v_m \boldsymbol{\sigma}_{xx\,m}$$

    can lead to fiber debonding
  - In all the previous developments we have assumed zero-stress along z-axis
    - This is justified as the behaviors in z and y directions are similar.
    - This will not be true in a stack of laminas (laminate)

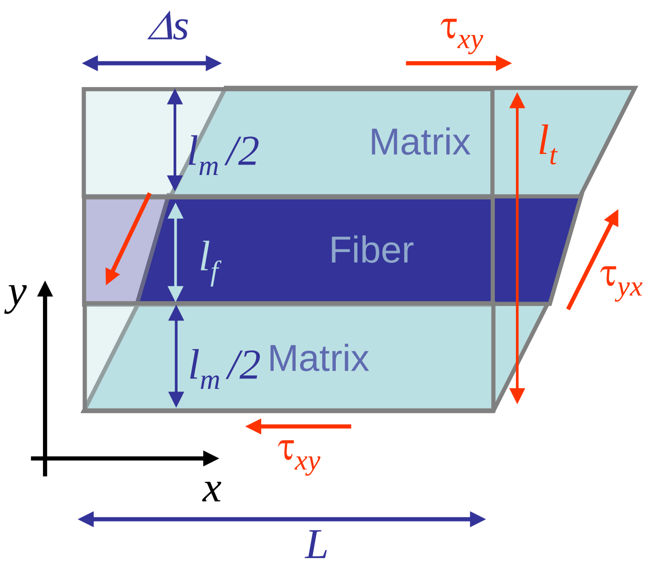


- Ply (lamina) mechanics: μ_{xy}
  - Constraint (small) shearing $\gamma = \Delta s / l_t$
    - Assumption: fiber and matrix are subjected to the same shear stress $\tau_{xy} = \tau_{yx}$
    - Resultant shear sliding
      - $\mu_{xy} = \mu_{yx} = \frac{\tau_{xy} l_t}{\Delta s} = \frac{\tau_{yx} l_t}{\Delta s}$
    - Microscopic shearing
      - Fiber $\Delta s_f = \frac{\tau_{xy}}{\mu_f} l_f$
      - Matrix $\Delta s_m = \frac{\tau_{xy}}{\mu_m} l_m$
    - Compatibility

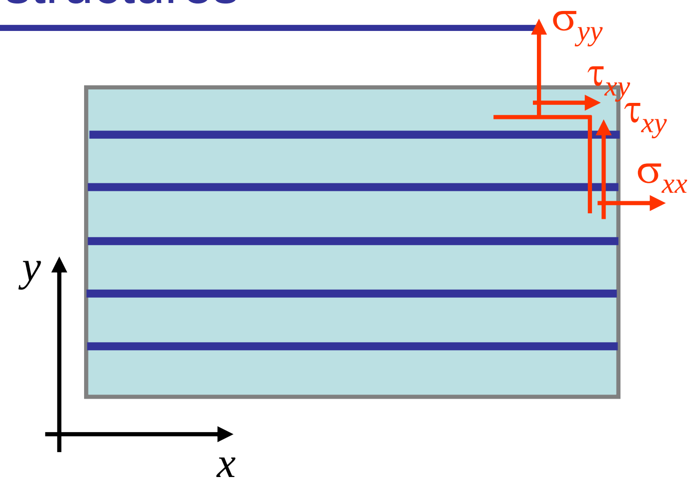
$$\Delta s = \Delta s_f + \Delta s_m \Rightarrow \frac{\tau_{xy}}{\mu_{xy}} l_t = \frac{\tau_{xy}}{\mu_f} l_f + \frac{\tau_{xy}}{\mu_m} l_m$$

$$\Rightarrow \frac{1}{\mu_{xy}} = \frac{1}{\mu_f} v_f + \frac{1}{\mu_m} v_m$$

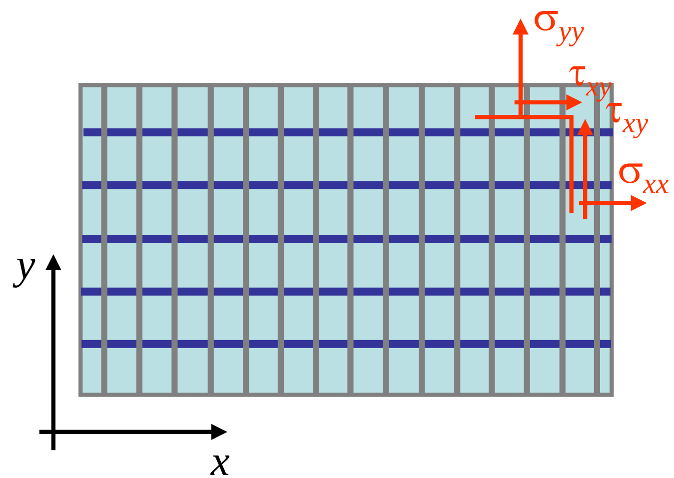
    - As $\mu_f >> \mu_m$, in general $\mu_{xy} = \mu_m / v_m$
  - Unlike isotropic materials, shear modulus is **NOT** related to E and ν



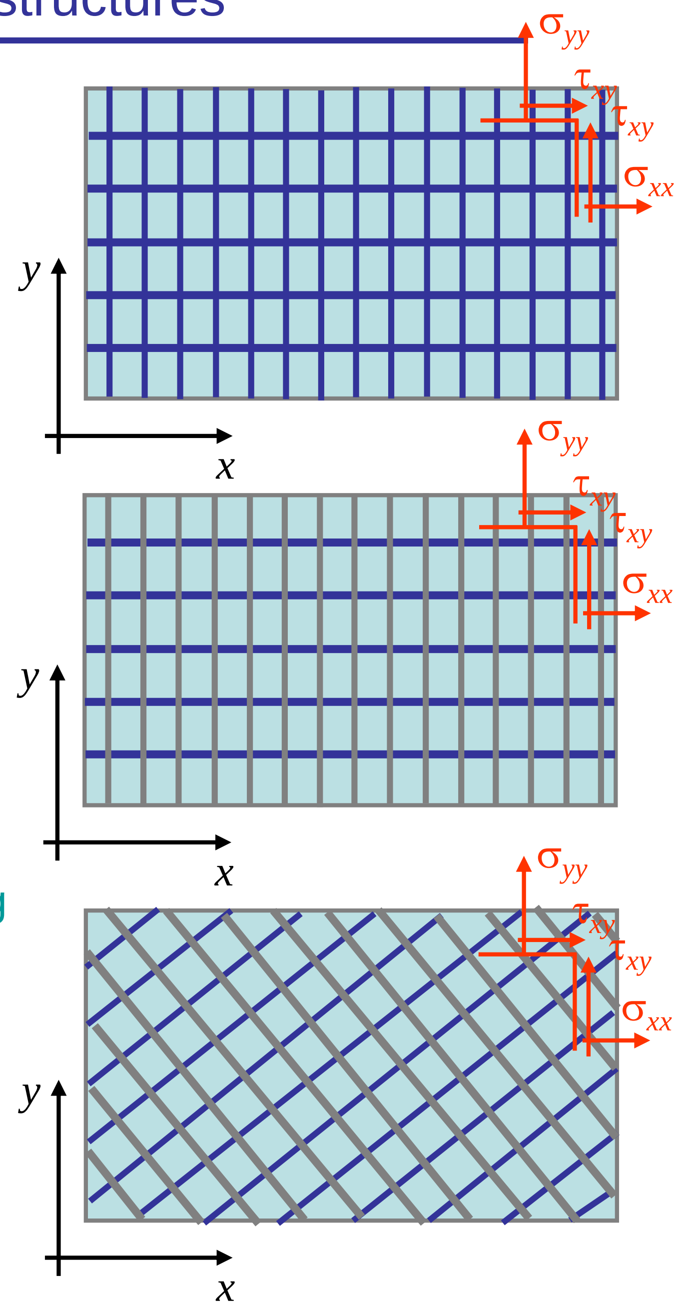
- Orthotropic ply mechanics
  - Single sheet of composite with
    - Fibers aligned in one direction: unidirectional ply or lamina



    - Fibers in perpendicular direction: woven ply



- Orthotropic ply mechanics (2)
  - Woven ply
    - Transversally isotropic
      - Fiber reinforcements the same in both directions
      - Same material properties in the 2 fiber directions
    - Orthotropic
      - Fiber reinforcements not the same in both directions
      - Different material properties in the 2 directions
      - Specially orthotropic: Applied loading in the directions of the plies
      - Generally orthotropic: Applied loading not in the directions of the plies



- Specially orthotropic ply mechanics
  - Plane stress (Plane-σ) state
    - Isotropic materials

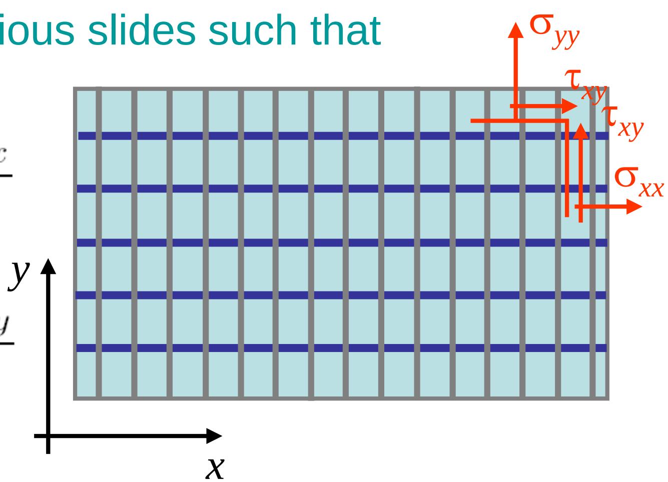
$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{2\mu} \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{xy} \end{pmatrix}$$

    - New resultant material properties defined in previous slides such that
      - For uniaxial tension along x

$$\varepsilon_{xx} = \frac{\boldsymbol{\sigma}_{xx}}{E_x} \quad \& \quad \varepsilon_{yy} = \nu_{xy}\varepsilon_{xx} = -\frac{\nu_{xy}\boldsymbol{\sigma}_{xx}}{E_x}$$

      - For uniaxial tension along y

$$\varepsilon_{yy} = \frac{\boldsymbol{\sigma}_{yy}}{E_y} \quad \& \quad \varepsilon_{xx} = \nu_{yx}\varepsilon_{yy} = -\frac{\nu_{yx}\boldsymbol{\sigma}_{yy}}{E_y}$$



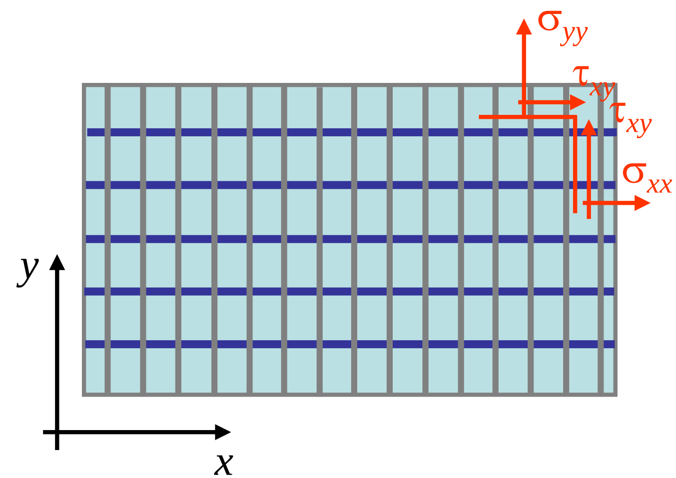
    - Superposition leads to the orthotropic law

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & 0 \\ 0 & 0 & \frac{1}{2\mu_{xy}} \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{xy} \end{pmatrix}$$

      - Be careful μ_{xy} cannot be computed from $E_{x/y}$, $\nu_{xy/yx}$

- Specially orthotropic ply mechanics (2)
  - Plane stress (Plane-σ) state (2)
    - Reciprocal stress-strain relationship

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & 0 \\ 0 & 0 & \frac{1}{2\mu_{xy}} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}$$



$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{E_x}{1-\nu_{xy}\nu_{yx}} & \frac{\nu_{yx}E_x}{1-\nu_{xy}\nu_{yx}} & 0 \\ \frac{\nu_{xy}E_y}{1-\nu_{xy}\nu_{yx}} & \frac{E_y}{1-\nu_{xy}\nu_{yx}} & 0 \\ 0 & 0 & 2\mu_{xy} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix}$$

    - To be compared to stress-strain relationship for isotropic materials

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & 2\mu \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix}$$

- Specially orthotropic ply mechanics (3)
  - General 3D expression
    - Hooke's law $\boldsymbol{\sigma} = \mathcal{C} : \boldsymbol{\varepsilon}$ or $\boldsymbol{\sigma}_{ij} = \mathcal{C}_{ijkl}\boldsymbol{\varepsilon}_{kl}$
    - Can be rewritten under the form

$$\begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{zz} \\ \boldsymbol{\sigma}_{yz} \\ \boldsymbol{\sigma}_{zx} \\ \boldsymbol{\sigma}_{xy} \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{xxxx} & \mathcal{C}_{xxyy} & \mathcal{C}_{xxzz} & 0 & 0 & 0 \\ \mathcal{C}_{yyxx} & \mathcal{C}_{yyyy} & \mathcal{C}_{yyzz} & 0 & 0 & 0 \\ \mathcal{C}_{zzxx} & \mathcal{C}_{zzyy} & \mathcal{C}_{zzzz} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mathcal{C}_{yzyz} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mathcal{C}_{xzxz} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mathcal{C}_{xyxy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_{xx} \\ \boldsymbol{\varepsilon}_{yy} \\ \boldsymbol{\varepsilon}_{zz} \\ \boldsymbol{\varepsilon}_{yz} \\ \boldsymbol{\varepsilon}_{zx} \\ \boldsymbol{\varepsilon}_{xy} \end{pmatrix}$$

$$\boldsymbol{\sigma}_{yz} = \mathcal{C}_{yzkl}\boldsymbol{\varepsilon}_{kl} = \mathcal{C}_{yzyz}\boldsymbol{\varepsilon}_{yz} + \mathcal{C}_{yzzy}\boldsymbol{\varepsilon}_{zy} = 2\mathcal{C}_{yzyz}\boldsymbol{\varepsilon}_{yz}$$

- Specially orthotropic ply mechanics (4)
  - General 3D expression (2)
    - Hooke's law $\boldsymbol{\sigma} = \mathcal{C} : \boldsymbol{\varepsilon}$ or $\boldsymbol{\sigma}_{ij} = \mathcal{C}_{ijkl}\boldsymbol{\varepsilon}_{kl}$
    - With the 21 non-zero components of the fourth-order tensor being
      - $\mathcal{C}_{xxxx} = \frac{1-\nu_{yz}\nu_{zy}}{E_y E_z\, D}$, $\mathcal{C}_{xxyy} = \frac{\nu_{yx}+\nu_{zx}\nu_{yz}}{E_y E_z\, D}$ & $\mathcal{C}_{xxzz} = \frac{\nu_{zx}+\nu_{yx}\nu_{zy}}{E_y E_z\, D}$
      - $\mathcal{C}_{yyxx} = \frac{\nu_{xy}+\nu_{zy}\nu_{xz}}{E_x E_z\, D}$, $\mathcal{C}_{yyyy} = \frac{1-\nu_{xz}\nu_{zx}}{E_x E_z\, D}$ & $\mathcal{C}_{yyzz} = \frac{\nu_{zy}+\nu_{xy}\nu_{zx}}{E_x E_z\, D}$
      - $\mathcal{C}_{zzxx} = \frac{\nu_{xz}+\nu_{xy}\nu_{yz}}{E_y E_x\, D}$, $\mathcal{C}_{zzyy} = \frac{\nu_{yz}+\nu_{xz}\nu_{yx}}{E_y E_x\, D}$ & $\mathcal{C}_{zzzz} = \frac{1-\nu_{yx}\nu_{xy}}{E_y E_x\, D}$
      - $\mathcal{C}_{yzyz} = \mathcal{C}_{yzzy} = \mathcal{C}_{zyzy} = \mathcal{C}_{zyyz} = \mu_{yz}$
      - $\mathcal{C}_{xyxy} = \mathcal{C}_{xyyx} = \mathcal{C}_{yxyx} = \mathcal{C}_{yxxy} = \mu_{xy}$
      - $\mathcal{C}_{xzxz} = \mathcal{C}_{xzzx} = \mathcal{C}_{zxzx} = \mathcal{C}_{zxxz} = \mu_{xz}$
    - And the denominator $D = \frac{1-\nu_{xy}\nu_{yx}-\nu_{zy}\nu_{yz}-\nu_{xz}\nu_{zx}-2\nu_{xy}\nu_{yz}\nu_{zx}}{E_x E_y E_z}$

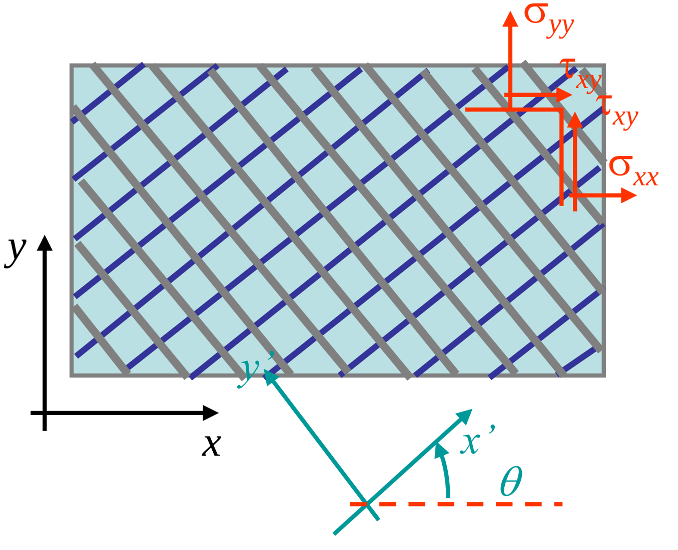
- Generally orthotropic ply mechanics
  - Stress-strain relationship
    - Stress-strain relationship in the axes *O'x'y'* is known for plane-σ state

$$\begin{pmatrix} \boldsymbol{\sigma}_{x'x'} \\ \boldsymbol{\sigma}_{y'y'} \\ \boldsymbol{\sigma}_{x'y'} \end{pmatrix} = \begin{pmatrix} \frac{E_{x'}}{1-\nu_{x'y'}\nu_{y'x'}} & \frac{\nu_{y'x'}E_{x'}}{1-\nu_{x'y'}\nu_{y'x'}} & 0 \\ \frac{\nu_{x'y'}E_{y'}}{1-\nu_{x'y'}\nu_{y'x'}} & \frac{E_{y'}}{1-\nu_{x'y'}\nu_{y'x'}} & 0 \\ 0 & 0 & 2\mu_{x'y'} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_{x'x'} \\ \boldsymbol{\varepsilon}_{y'y'} \\ \boldsymbol{\varepsilon}_{x'y'} \end{pmatrix}$$

or in tensorial form $\boldsymbol{\sigma}' = \mathcal{C}' : \boldsymbol{\varepsilon}'$

- If θ is the angle between *Ox* & *O'x'*
  - $\begin{cases} \boldsymbol{\sigma}' = \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^T \\ \boldsymbol{\varepsilon}' = \mathbf{R}\boldsymbol{\varepsilon}\mathbf{R}^T \end{cases}$

  with $\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$



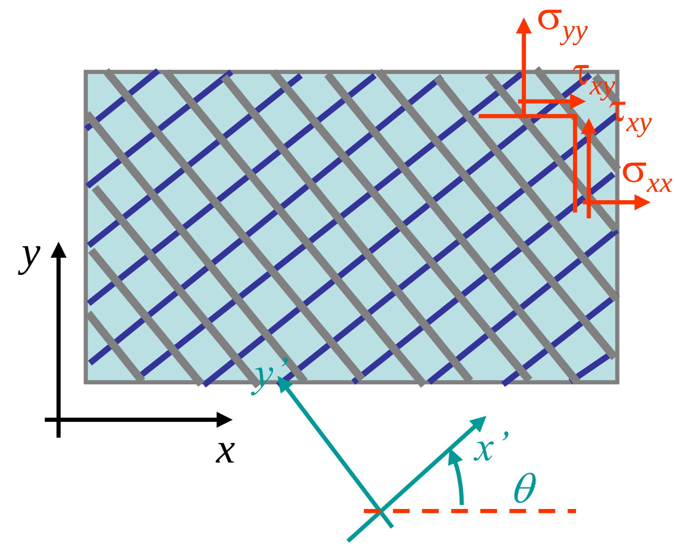
- From there we can get $\mathcal{C}$ such that

$$\boldsymbol{\sigma} = \mathcal{C} : \boldsymbol{\varepsilon}$$

- Generally orthotropic ply mechanics (2)
  - Stress-strain relationship (2)
    - Equation $\boldsymbol{\sigma}' = \mathcal{C}' : \boldsymbol{\varepsilon}'$

      with $\boldsymbol{\sigma}' = \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^T$, $\boldsymbol{\varepsilon}' = \mathbf{R}\boldsymbol{\varepsilon}\mathbf{R}^T$

      & in 2D $\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$



    - Solution

$$\Longrightarrow \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^T = \mathcal{C}' : \mathbf{R}\boldsymbol{\varepsilon}\mathbf{R}^T$$

$$\Longrightarrow \boldsymbol{\sigma}_{ij} = \mathbf{R}_{ki}\mathcal{C}'_{klmn}\mathbf{R}_{lj}\mathbf{R}_{mp}\boldsymbol{\varepsilon}_{pq}\mathbf{R}_{nq}$$

$$\Longrightarrow \boldsymbol{\sigma}_{ij} = \mathbf{R}_{ki}\mathbf{R}_{lj}\mathcal{C}'_{klmn}\mathbf{R}_{mp}\mathbf{R}_{nq}\boldsymbol{\varepsilon}_{pq} = \mathcal{C}_{ijpq}\boldsymbol{\varepsilon}_{pq}$$

    - Or again $\boldsymbol{\sigma} = \mathcal{C} : \boldsymbol{\varepsilon}$

      with $\mathcal{C}_{ijkl} = \mathbf{R}_{mi}\mathbf{R}_{nj}\mathcal{C}'_{mnpq}\mathbf{R}_{pk}\mathbf{R}_{ql}$

- Generally orthotropic ply mechanics (3)
  - Plane σ state
    - From $\begin{pmatrix} \boldsymbol{\sigma}_{x'x'} \\ \boldsymbol{\sigma}_{y'y'} \\ \boldsymbol{\sigma}_{x'y'} \end{pmatrix} = \begin{pmatrix} \frac{E_{x'}}{1-\nu_{x'y'}\nu_{y'x'}} & \frac{\nu_{y'x'}E_{x'}}{1-\nu_{x'y'}\nu_{y'x'}} & 0 \\ \frac{\nu_{x'y'}E_{y'}}{1-\nu_{x'y'}\nu_{y'x'}} & \frac{E_{y'}}{1-\nu_{x'y'}\nu_{y'x'}} & 0 \\ 0 & 0 & 2\mu_{x'y'} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_{x'x'} \\ \boldsymbol{\varepsilon}_{y'y'} \\ \boldsymbol{\varepsilon}_{x'y'} \end{pmatrix}$

  - The non-zero components are
    - » $\mathcal{C}'_{x'x'x'x'} = \frac{E_{x'}}{1 - \nu_{x'y'}\nu_{y'x'}}$
    - » $\mathcal{C}'_{y'y'y'y'} = \frac{E_{y'}}{1 - \nu_{x'y'}\nu_{y'x'}}$
    - » $\mathcal{C}'_{x'x'y'y'} = \mathcal{C}'_{y'y'x'x'} = \frac{\nu_{y'x'}E_{x'}}{1 - \nu_{x'y'}\nu_{y'x'}} = \frac{\nu_{x'y'}E_{y'}}{1 - \nu_{x'y'}\nu_{y'x'}}$
    - » $\mathcal{C}'_{x'y'x'y'} = \mathcal{C}'_{x'y'y'x'} = \mathcal{C}'_{y'x'x'y'} = \mathcal{C}'_{y'x'y'x'} = \mu_{x'y'}$
  - Let $c = \cos\theta$, $s = \sin\theta$,
    - » $\mathbf{R}_{x'x} = \mathbf{R}_{y'y} = c$
    - » $\mathbf{R}_{x'y} = -\mathbf{R}_{y'x} = s$

- Generally orthotropic ply mechanics (4)
  - Plane σ state (2)
    - Using $\mathbf{R}_{x'x} = \mathbf{R}_{y'y} = c$ & $\mathbf{R}_{x'y} = -\mathbf{R}_{y'x} = s$ expression $\mathcal{C}_{ijkl} = \mathbf{R}_{mi}\mathbf{R}_{nj}\mathcal{C}'_{mnpq}\mathbf{R}_{pk}\mathbf{R}_{ql}$ leads to

$$\begin{aligned}\mathcal{C}_{xxxx} &= \mathbf{R}_{mx}\mathbf{R}_{nx}\mathcal{C}'_{mnpq}\mathbf{R}_{px}\mathbf{R}_{qx} \\ &= \mathbf{R}_{x'x}\mathbf{R}_{x'x}\mathcal{C}'_{x'x'x'x'}\mathbf{R}_{x'x}\mathbf{R}_{x'x} + \mathbf{R}_{x'x}\mathbf{R}_{x'x}\mathcal{C}'_{x'x'y'y'}\mathbf{R}_{y'x}\mathbf{R}_{y'x} + \\ &\quad \mathbf{R}_{x'x}\mathbf{R}_{y'x}\mathcal{C}'_{x'y'x'y'}\mathbf{R}_{x'x}\mathbf{R}_{y'x} + \mathbf{R}_{x'x}\mathbf{R}_{y'x}\mathcal{C}'_{x'y'y'x'}\mathbf{R}_{y'x}\mathbf{R}_{x'x} + \\ &\quad \mathbf{R}_{y'x}\mathbf{R}_{x'x}\mathcal{C}'_{y'x'x'y'}\mathbf{R}_{x'x}\mathbf{R}_{y'x} + \mathbf{R}_{y'x}\mathbf{R}_{x'x}\mathcal{C}'_{y'x'y'x'}\mathbf{R}_{y'x}\mathbf{R}_{x'x} + \\ &\quad \mathbf{R}_{y'x}\mathbf{R}_{y'x}\mathcal{C}'_{y'y'x'x'}\mathbf{R}_{x'x}\mathbf{R}_{x'x} + \mathbf{R}_{y'x}\mathbf{R}_{y'x}\mathcal{C}'_{y'y'y'y'}\mathbf{R}_{y'x}\mathbf{R}_{y'x}\end{aligned}$$

$$\Longrightarrow \mathcal{C}_{xxxx} = c^4\mathcal{C}'_{x'x'x'x'} + c^2s^2\left(\mathcal{C}'_{x'x'y'y'} + \mathcal{C}'_{x'y'x'y'} + \mathcal{C}'_{x'y'y'x'} + \mathcal{C}'_{y'x'x'y'} + \mathcal{C}'_{y'x'y'x'} + \mathcal{C}'_{y'y'x'x'}\right) + s^4\mathcal{C}'_{y'y'y'y'}$$

    - Eventually, using minor & major symmetry of material tensor

$$\mathcal{C}_{xxxx} = c^4\mathcal{C}'_{x'x'x'x'} + 2c^2s^2\left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'}\right) + s^4\mathcal{C}'_{y'y'y'y'}$$

- Generally orthotropic ply mechanics (5)
  - Plane σ state (3)
    - Doing the same for the other components leads to

$$\begin{cases} \mathcal{C}_{xxxx} = c^4 \mathcal{C}'_{x'x'x'x'} + 2c^2 s^2 \left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'} \right) + s^4 \mathcal{C}'_{y'y'y'y'} \\ \mathcal{C}_{yyyy} = s^4 \mathcal{C}'_{x'x'x'x'} + 2c^2 s^2 \left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'} \right) + c^4 \mathcal{C}'_{y'y'y'y'} \\ \mathcal{C}_{xxyy} = \mathcal{C}_{yyxx} = \left(c^4 + s^4 \right) \mathcal{C}'_{x'x'y'y'} + c^2 s^2 \left(\mathcal{C}'_{x'x'x'x'} + \mathcal{C}'_{y'y'y'y'} - 4\mathcal{C}'_{x'y'x'y'} \right) \\ \mathcal{C}_{xyxy} = \mathcal{C}_{xyyx} = \mathcal{C}_{yxxy} = \mathcal{C}_{yxyx} = \\ \quad \left(c^2 - s^2 \right)^2 \mathcal{C}'_{x'y'x'y'} + c^2 s^2 \left(\mathcal{C}'_{x'x'x'x'} + \mathcal{C}'_{y'y'y'y'} - 2\mathcal{C}'_{x'x'y'y'} \right) \end{cases}$$

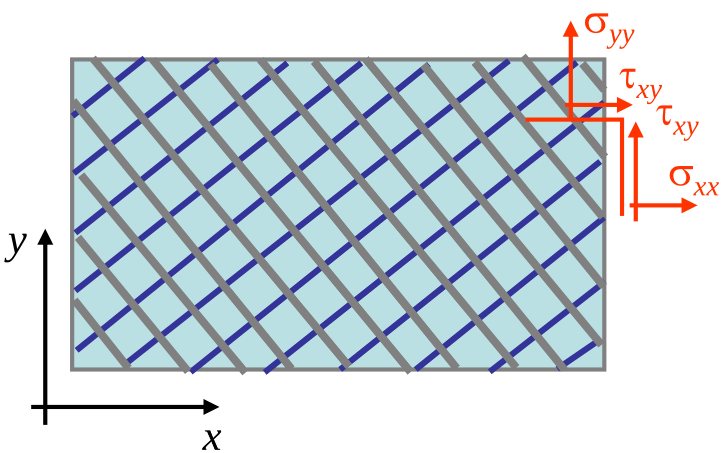
    - These are the 8 non-zero components
    - In the *O'x'y'* there were 8 non-zero components

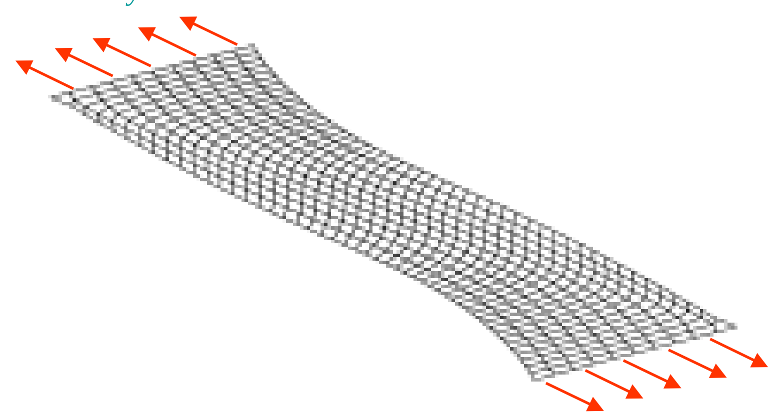
- Generally orthotropic ply mechanics (6)
  - Plane σ state (4)
    - But due to the rotation: a coupling between tension and shearing appears

$$\begin{aligned}\mathcal{C}_{xxxy} &= \mathbf{R}_{mx}\mathbf{R}_{nx}\mathcal{C}'_{mnpq}\mathbf{R}_{px}\mathbf{R}_{qy}\\ &= \mathbf{R}_{x'x}\mathbf{R}_{x'x}\mathcal{C}'_{x'x'x'x'}\mathbf{R}_{x'x}\mathbf{R}_{x'y} + \mathbf{R}_{x'x}\mathbf{R}_{x'x}\mathcal{C}'_{x'x'y'y'}\mathbf{R}_{y'x}\mathbf{R}_{y'y} +\\ &\quad \mathbf{R}_{x'x}\mathbf{R}_{y'x}\mathcal{C}'_{x'y'x'y'}\mathbf{R}_{x'x}\mathbf{R}_{y'y} + \mathbf{R}_{x'x}\mathbf{R}_{y'x}\mathcal{C}'_{x'y'y'x'}\mathbf{R}_{y'x}\mathbf{R}_{x'y} +\\ &\quad \mathbf{R}_{y'x}\mathbf{R}_{x'x}\mathcal{C}'_{y'x'x'y'}\mathbf{R}_{x'x}\mathbf{R}_{y'y} + \mathbf{R}_{y'x}\mathbf{R}_{x'x}\mathcal{C}'_{y'x'y'x'}\mathbf{R}_{y'x}\mathbf{R}_{x'y} +\\ &\quad \mathbf{R}_{y'x}\mathbf{R}_{y'x}\mathcal{C}'_{y'y'x'x'}\mathbf{R}_{x'x}\mathbf{R}_{x'y} + \mathbf{R}_{y'x}\mathbf{R}_{y'x}\mathcal{C}'_{y'y'y'y'}\mathbf{R}_{y'x}\mathbf{R}_{y'y}\end{aligned}$$

$$\Rightarrow \mathcal{C}_{xxxy} = c^3 s\left(\mathcal{C}'_{x'x'x'x'} - \mathcal{C}'_{x'x'y'y'} - \mathcal{C}'_{x'y'x'y'} - \mathcal{C}'_{y'x'x'y'}\right) + cs^3\left(\mathcal{C}'_{x'y'y'x'} + \mathcal{C}'_{y'x'y'x'} + \mathcal{C}'_{y'y'x'x'} - \mathcal{C}'_{y'y'y'y'}\right)$$

    - A traction σ_{xx} along Ox induces a shearing ε_{xy} due to the fiber orientation





- Generally orthotropic ply mechanics (7)
  - Plane σ state (5)
    - All the non-zero components are

$$\left\{\begin{array}{l} \mathcal{C}_{xxxx} = c^4\mathcal{C}'_{x'x'x'x'} + 2c^2s^2\left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'}\right) + s^4\mathcal{C}'_{y'y'y'y'} \\ \mathcal{C}_{yyyy} = s^4\mathcal{C}'_{x'x'x'x'} + 2c^2s^2\left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'}\right) + c^4\mathcal{C}'_{y'y'y'y'} \\ \mathcal{C}_{xxyy} = \mathcal{C}_{yyxx} = \left(c^4 + s^4\right)\mathcal{C}'_{x'x'y'y'} + c^2s^2\left(\mathcal{C}'_{x'x'x'x'} + \mathcal{C}'_{y'y'y'y'} - 4\mathcal{C}'_{x'y'x'y'}\right) \\ \mathcal{C}_{xyxy} = \mathcal{C}_{xyyx} = \mathcal{C}_{yxxy} = \mathcal{C}_{yxyx} = \\ \quad \left(c^2 - s^2\right)^2\mathcal{C}'_{x'y'x'y'} + c^2s^2\left(\mathcal{C}'_{x'x'x'x'} + \mathcal{C}'_{y'y'y'y'} - 2\mathcal{C}'_{x'x'y'y'}\right) \\ \mathcal{C}_{xxxy} = \mathcal{C}_{xyxx} = \mathcal{C}_{xxyx} = \mathcal{C}_{yxxx} = \\ \quad c^3s\left(\mathcal{C}'_{x'x'x'x'} - \mathcal{C}'_{x'x'y'y'} - 2\mathcal{C}'_{x'y'x'y'}\right) + cs^3\left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'} - \mathcal{C}'_{y'y'y'y'}\right) \\ \mathcal{C}_{yyxy} = \mathcal{C}_{xyyy} = \mathcal{C}_{yyyx} = \mathcal{C}_{yxyy} = \\ \quad cs^3\left(\mathcal{C}'_{x'x'x'x'} - \mathcal{C}'_{x'x'y'y'} - 2\mathcal{C}'_{x'y'x'y'}\right) + c^3s\left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'} - \mathcal{C}'_{y'y'y'y'}\right) \end{array}\right.$$

- Generally orthotropic ply mechanics (8)
  - Plane σ state (6)
    - Can be rewritten under the form

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{xxxx} & \mathcal{C}_{xxyy} & 2\mathcal{C}_{xxxy} \\ \mathcal{C}_{yyxx} & \mathcal{C}_{yyyy} & 2\mathcal{C}_{yyxy} \\ \mathcal{C}_{xyxx} & \mathcal{C}_{xyyy} & 2\mathcal{C}_{xyxy} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix}$$

$$\sigma_{xx} = \mathcal{C}_{xxxx}\varepsilon_{xx} + \mathcal{C}_{xxyy}\varepsilon_{yy} + \mathcal{C}_{xxxy}\varepsilon_{xy} + \mathcal{C}_{xxyx}\varepsilon_{yx}$$

$$\sigma_{xy} = \mathcal{C}_{xyxx}\varepsilon_{xx} + \mathcal{C}_{xyyy}\varepsilon_{yy} + \mathcal{C}_{xyxy}\varepsilon_{xy} + \mathcal{C}_{xyyx}\varepsilon_{yx}$$

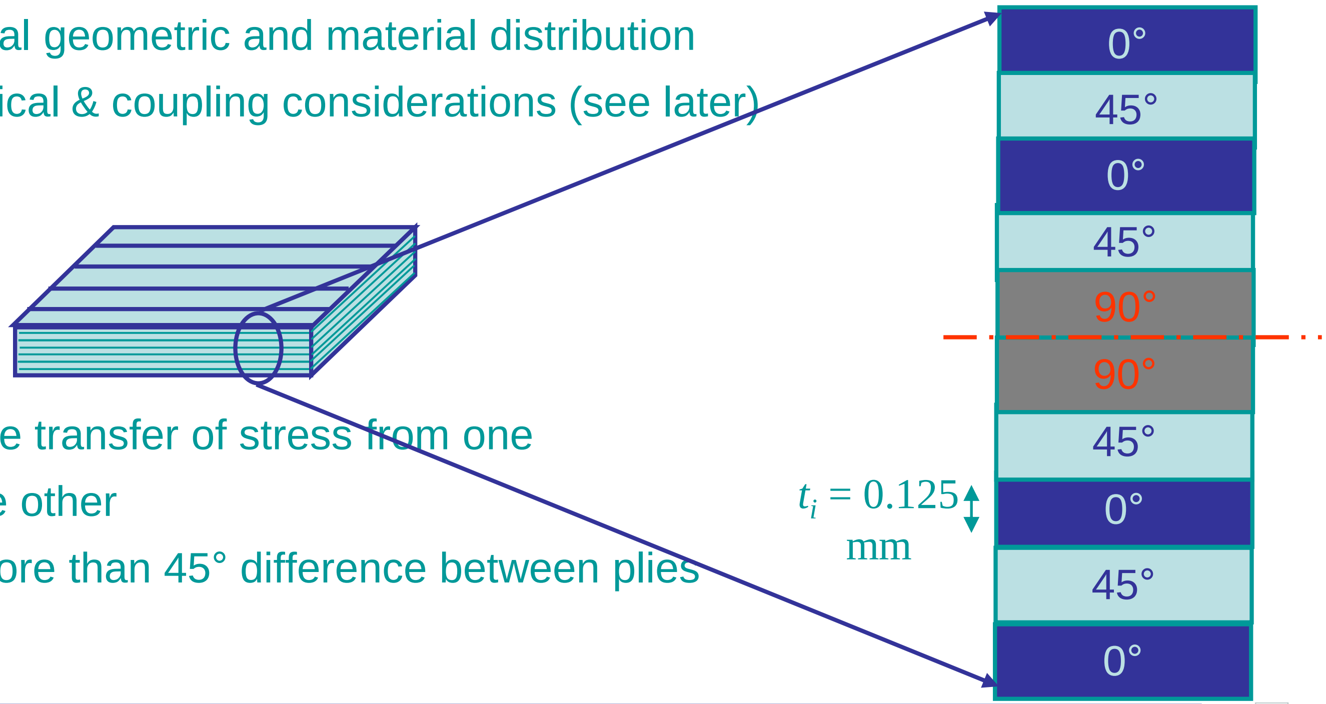
    - Remark: a symmetric matrix (not a tensor) can be recovered by using
      - The shear angle $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\,\varepsilon_{xy}$
      - $$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{xxxx} & \mathcal{C}_{xxyy} & \mathcal{C}_{xxxy} \\ \mathcal{C}_{yyxx} & \mathcal{C}_{yyyy} & \mathcal{C}_{yyxy} \\ \mathcal{C}_{xyxx} & \mathcal{C}_{xyyy} & \mathcal{C}_{xyxy} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix}$$

Tension/shearing coupling

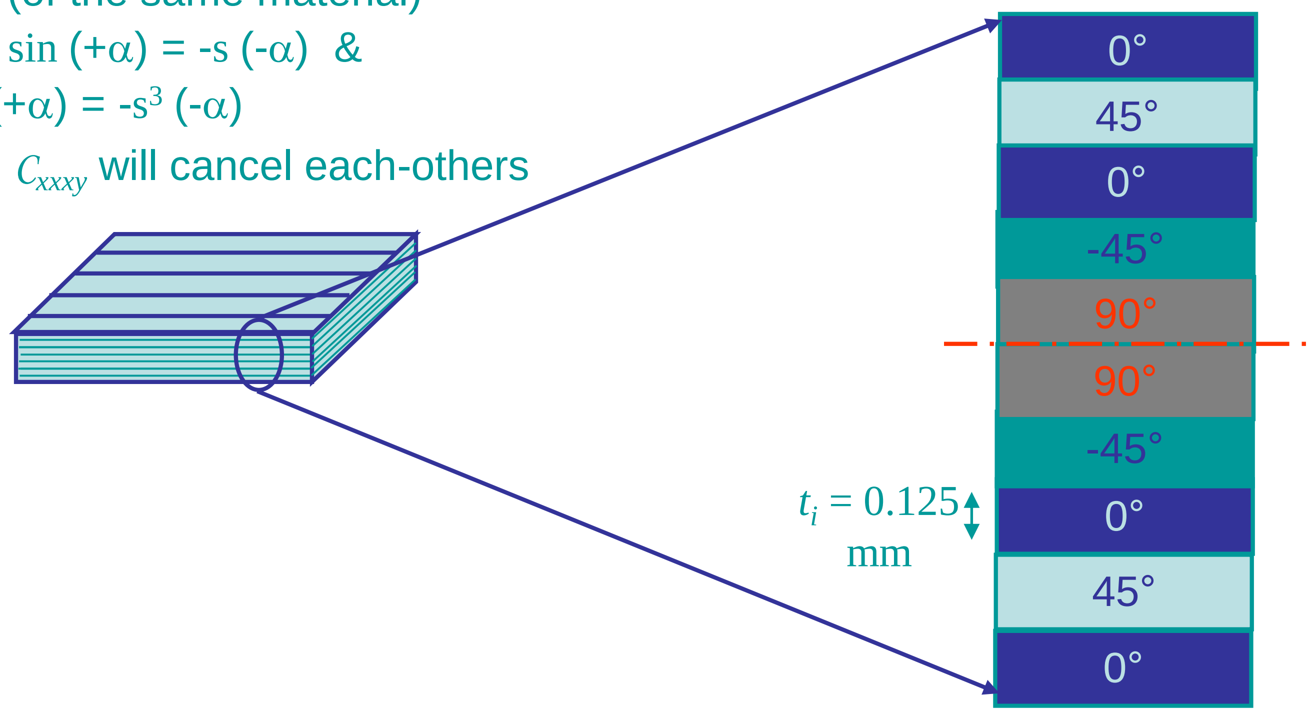
- Laminated composite
  - A laminate is the superposition of different plies
    - For a ply i of general orientation θ_i, there is a coupling between tension and shearing

$$\begin{pmatrix} \boldsymbol{\sigma}^i_{xx} \\ \boldsymbol{\sigma}^i_{yy} \\ \boldsymbol{\sigma}^i_{xy} \end{pmatrix} = \begin{pmatrix} \mathcal{C}^i_{xxxx} & \mathcal{C}^i_{xxyy} & \mathcal{C}^i_{xxxy} \\ \mathcal{C}^i_{yyxx} & \mathcal{C}^i_{yyyy} & \mathcal{C}^i_{yyxy} \\ \mathcal{C}^i_{xyxx} & \mathcal{C}^i_{xyyy} & \mathcal{C}^i_{xyxy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}^i_{xx} \\ \boldsymbol{\varepsilon}^i_{yy} \\ \gamma^i_{xy} \end{pmatrix}$$

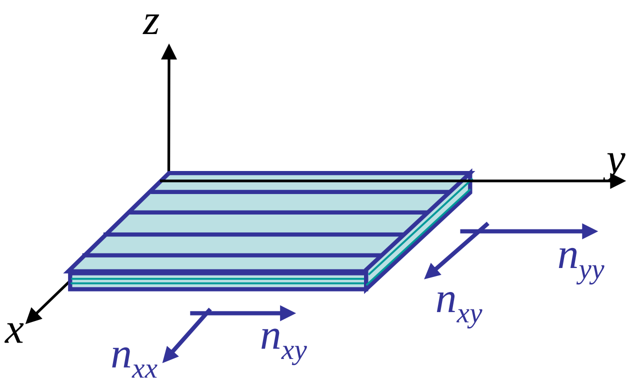
  - Symmetrical laminate
    - Symmetrical geometric and material distribution
    - Technological & coupling considerations (see later)
    - To ease the transfer of stress from one layer to the other
      - No more than 45° difference between plies



- Laminated composite (2)
  - Supression of tensile/shearing coupling
    - $$\mathcal{C}_{xxxy} = c^3 s \left(\mathcal{C}'_{x'x'x'x'} - \mathcal{C}'_{x'x'y'y'} - \mathcal{C}'_{x'y'x'y'} - \mathcal{C}'_{y'x'x'y'}\right) + cs^3 \left(\mathcal{C}'_{x'y'y'x'} + \mathcal{C}'_{y'x'y'x'} + \mathcal{C}'_{y'y'x'x'} - \mathcal{C}'_{y'y'y'y'}\right)$$
    - Supression of tensile/shearing coupling requires
      - Same proportion in +α° and –α° oriented laminas (of the same material)
    - Then $s(+\alpha) = \sin(+\alpha) = -s(-\alpha)$ & $s^3(+\alpha) = \sin^3(+\alpha) = -s^3(-\alpha)$
    - So two terms $\mathcal{C}_{xxxy}$ will cancel each-others



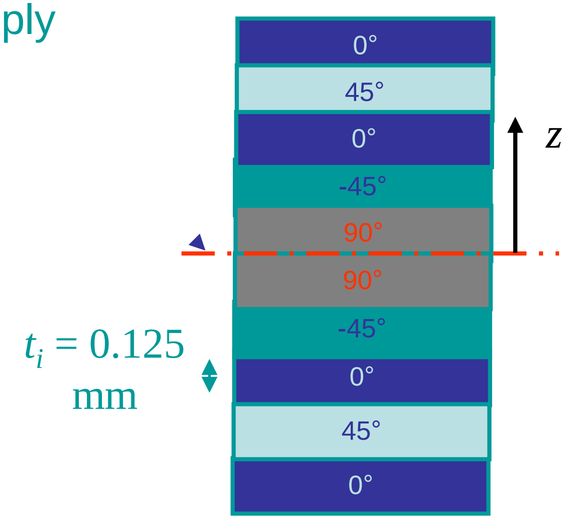
- Laminated composite (3)
  - Resulting elastic properties of a laminate can be deduced
  - Deformations of the laminate assumed to correspond to a plate
    - Membrane mode & resultant membrane stresses



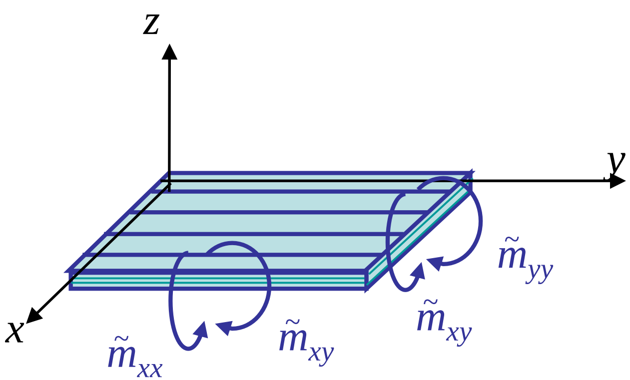
$$\begin{cases} n_{xx} = \int_h \boldsymbol{\sigma}_{xx} dz = \boldsymbol{n}_x^x = \left(\int_h \boldsymbol{\sigma} \cdot \boldsymbol{E}^x dz \right)_x \\ n_{yy} = \int_h \boldsymbol{\sigma}_{yy} dz = \boldsymbol{n}_y^y = \left(\int_h \boldsymbol{\sigma} \cdot \boldsymbol{E}^y dz \right)_y \\ n_{xy} = \int_h \boldsymbol{\sigma}_{xy} dz = \boldsymbol{n}_y^x = \boldsymbol{n}_x^y = \left(\int_h \boldsymbol{\sigma} \cdot \boldsymbol{E}^y dz \right)_x \end{cases}$$

    - For a laminate the integration is performed on each ply

$$\begin{cases} n_{xx} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{xx} dz \\ n_{yy} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{yy} dz \\ n_{xy} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{xy} dz \end{cases}$$



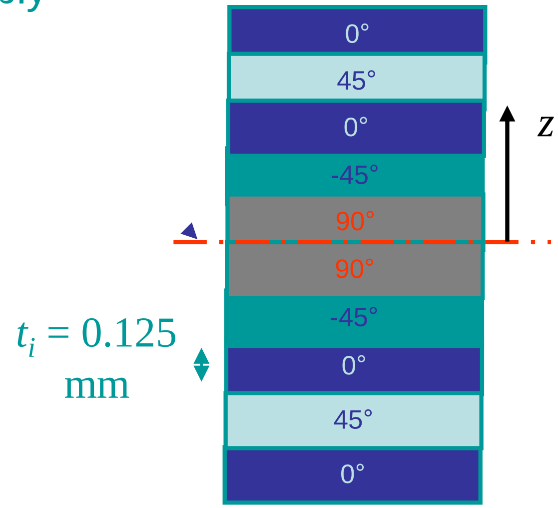
- Laminated composite (4)
  - Deformations of the laminate assumed to correspond to a plate (2)
    - Bending mode & resultant bending stresses



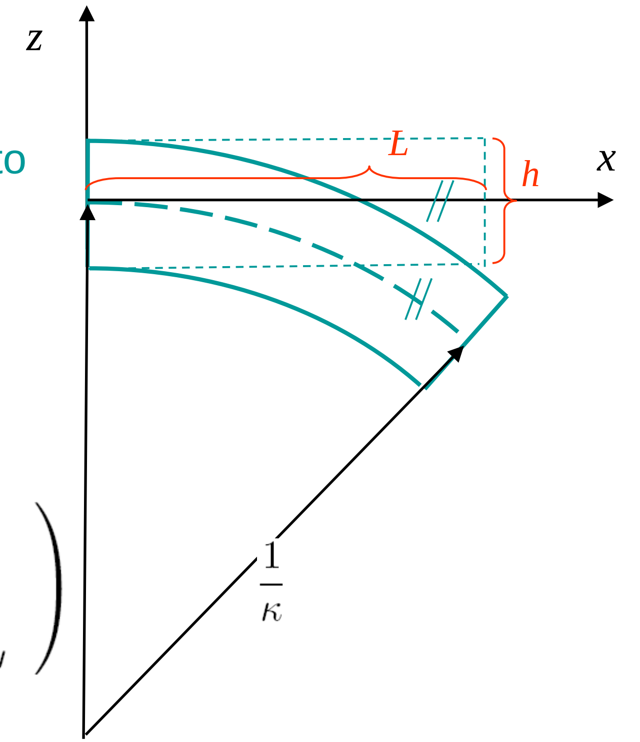
$$\begin{cases} \tilde{m}_{xx} = \int_h \boldsymbol{\sigma}_{xx} z dz = \tilde{\boldsymbol{m}}_x^x = \left(\int_h \boldsymbol{\sigma} \cdot \boldsymbol{E}^x z dz \right)_x \\ \tilde{m}_{yy} = \int_h \boldsymbol{\sigma}_{yy} z dz = \tilde{\boldsymbol{m}}_y^y = \left(\int_h \boldsymbol{\sigma} \cdot \boldsymbol{E}^y z dz \right)_y \\ \tilde{m}_{xy} = \int_h \boldsymbol{\sigma}_{xy} z dz = \tilde{\boldsymbol{m}}_y^x = \tilde{\boldsymbol{m}}_x^y = \left(\int_h \boldsymbol{\sigma} \cdot \boldsymbol{E}^y z dz \right)_x \end{cases}$$

    - For a laminate the integration is performed on each ply

$$\begin{cases} \tilde{m}_{xx} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{xx} z dz \\ \tilde{m}_{yy} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{yy} z dz \\ \tilde{m}_{xy} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{xy} z dz \end{cases}$$



- Laminated composite (5)
  - Stress-strain relationship
    - Deformation of the laminate can be separated into
      - Deformation of the neutral plane
      - Deformation due to bending
        - » See picture for beam analogy
    - Strains can then be expressed as



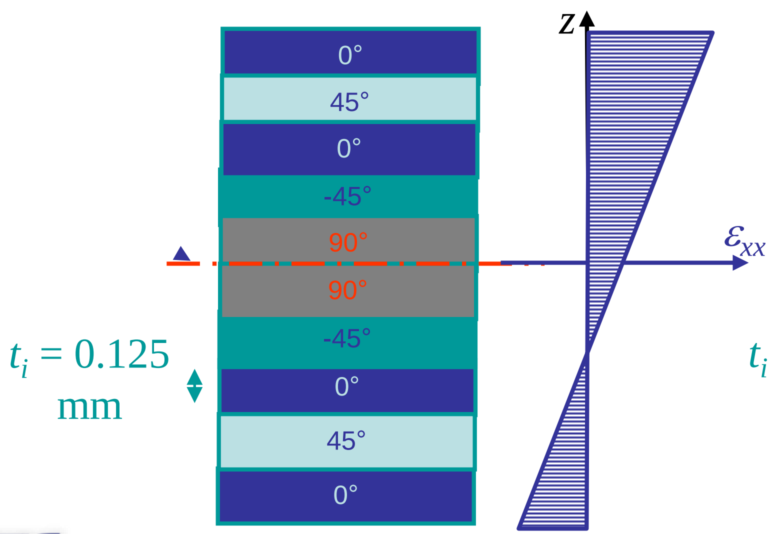
$$\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{pmatrix} = \begin{pmatrix}\varepsilon_{xx}(z=0)\\ \varepsilon_{yy}(z=0)\\ \gamma_{xy}(z=0)\end{pmatrix} + z\begin{pmatrix}-\boldsymbol{u}_{z,xx}\\ -\boldsymbol{u}_{z,yy}\\ -2\boldsymbol{u}_{z,xy}\end{pmatrix}$$

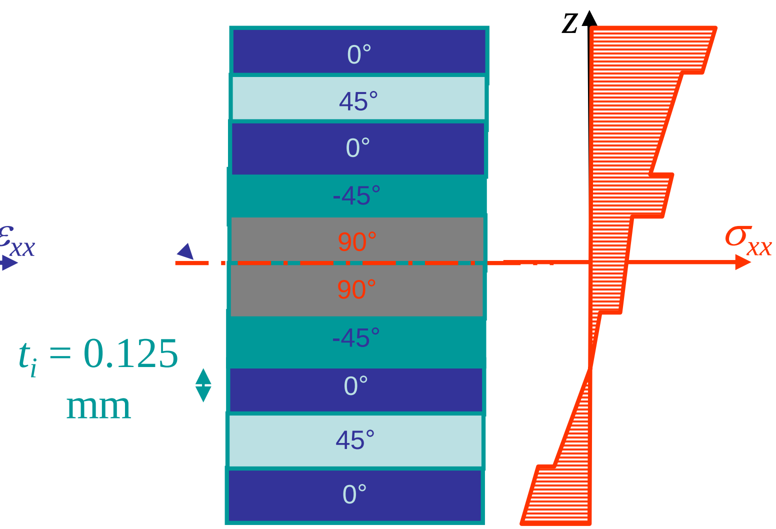
$$= \begin{pmatrix}\varepsilon^0_{xx}\\ \varepsilon^0_{yy}\\ \gamma^0_{xy}\end{pmatrix} + z\begin{pmatrix}\kappa^0_{xx}\\ \kappa^0_{yy}\\ \kappa^0_{xy}\end{pmatrix}$$

  - Exponent zero refers to neutral plane
    - » Assumed to be at $z = 0$
    - » In case of symmetric laminate it is located at the mid-plane
  - Classe on shells for rigourous demonstation

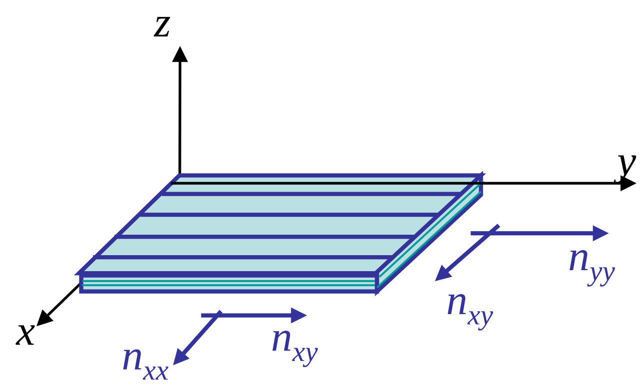
- Laminated composite (6)
  - Stress-strain relationship (2)
    - In each ply $\left(\begin{array}{c}\boldsymbol{\sigma}^i_{xx}\\ \boldsymbol{\sigma}^i_{yy}\\ \boldsymbol{\sigma}^i_{xy}\end{array}\right) = \left(\begin{array}{ccc}\mathcal{C}^i_{xxxx} & \mathcal{C}^i_{xxyy} & \mathcal{C}^i_{xxxy}\\ \mathcal{C}^i_{yyxx} & \mathcal{C}^i_{yyyy} & \mathcal{C}^i_{yyxy}\\ \mathcal{C}^i_{xyxx} & \mathcal{C}^i_{xyyy} & \mathcal{C}^i_{xyxy}\end{array}\right)\left(\begin{array}{c}\boldsymbol{\varepsilon}^i_{xx}\\ \boldsymbol{\varepsilon}^i_{yy}\\ \gamma^i_{xy}\end{array}\right)$

      With $\left(\begin{array}{c}\boldsymbol{\varepsilon}_{xx}\\ \boldsymbol{\varepsilon}_{yy}\\ \gamma_{xy}\end{array}\right) = \left(\begin{array}{c}\boldsymbol{\varepsilon}^0_{xx}\\ \boldsymbol{\varepsilon}^0_{yy}\\ \gamma^0_{xy}\end{array}\right) + z\left(\begin{array}{c}\kappa^0_{xx}\\ \kappa^0_{yy}\\ \kappa^0_{xy}\end{array}\right)$
    - So, using tensorial notation $\boldsymbol{\sigma}^i_{\alpha\beta} = \mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{\varepsilon}^0_{\gamma\delta} - z\mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{u}^0_{z,\gamma\delta}$
    - As properties change in each ply, this theoretically leads to discontinuous stress





- Laminated composite (7)
  - Stress-strain relationship (3)
    - Membrane resultant stress
      - As in each ply



$$\boldsymbol{\sigma}^i_{\alpha\beta} = \mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{\varepsilon}^0_{\gamma\delta} - z\mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{u}^0_{z,\gamma\delta}$$

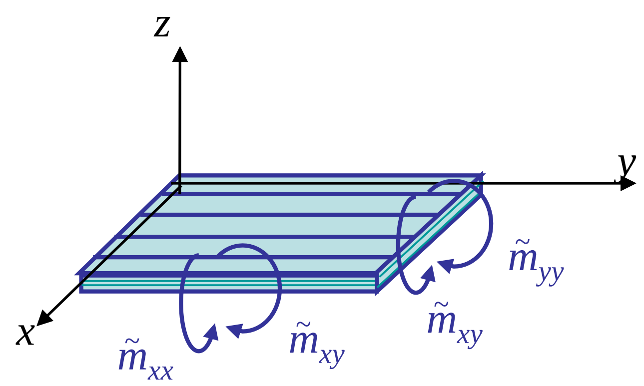
$$\Rightarrow n_{\alpha\beta} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{\alpha\beta} dz = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \int_{z_i}^{z_{i+1}} \mathcal{C}^i_{\alpha\beta\gamma\delta} dz - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \int_{z_i}^{z_{i+1}} z\mathcal{C}^i_{\alpha\beta\gamma\delta} dz$$

$$\Rightarrow n_{\alpha\beta} = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} \frac{z^2_{i+1} - z^2_i}{2}$$

$$\Rightarrow n_{\alpha\beta} = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i$$

- With the position of the neutral plane of each ply $\bar{z}_i = \frac{z_{i+1} + z_i}{2}$

- Laminated composite (8)
  - Stress-strain relationship (4)
    - Bending resultant stress
      - As in each ply



$$\boldsymbol{\sigma}^i_{\alpha\beta} = \mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{\varepsilon}^0_{\gamma\delta} - z\mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{u}^0_{z,\gamma\delta}$$

$$\Rightarrow \tilde{m}_{\alpha\beta} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{\alpha\beta} z dz = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \int_{z_i}^{z_{i+1}} z\mathcal{C}^i_{\alpha\beta\gamma\delta} dz - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \int_{z_i}^{z_{i+1}} z^2 \mathcal{C}^i_{\alpha\beta\gamma\delta} dz$$

$$\Rightarrow \tilde{m}_{\alpha\beta} = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} \frac{z^3_{i+1} - z^3_i}{3}$$

$$\Rightarrow \tilde{m}_{\alpha\beta} = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \frac{z^2_{i+1} + z^2_i + z_i z_{i+1}}{3}$$

$$\Rightarrow \tilde{m}_{\alpha\beta} = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \left[\left(\frac{z_{i+1} + z_i}{2}\right)^2 + \frac{(z_{i+1} - z_i)^2}{12}\right]$$

$$\Rightarrow \tilde{m}_{\alpha\beta} = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} \left(t_i \bar{z}^2_i + \frac{t^3_i}{12}\right)$$

- Laminated composite (9)
  - Stress-strain relationship (5)
    - The two equations are

$$\begin{cases} n_{\alpha\beta} = \boldsymbol{\varepsilon}^0_{\gamma\delta} \underbrace{\sum_i C^i_{\alpha\beta\gamma\delta} t_i}_{A_{\alpha\beta\gamma\delta}} - \boldsymbol{u}^0_{z,\gamma\delta} \underbrace{\sum_i C^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i}_{B_{\alpha\beta\gamma\delta}} \\ \tilde{m}_{\alpha\beta} = \boldsymbol{\varepsilon}^0_{\gamma\delta} \underbrace{\sum_i C^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i}_{B_{\alpha\beta\gamma\delta}} - \boldsymbol{u}^0_{z,\gamma\delta} \underbrace{\sum_i C^i_{\alpha\beta\gamma\delta} \left(t_i \bar{z}_i^2 + \frac{t_i^3}{12} \right)}_{D_{\alpha\beta\gamma\delta}} \end{cases}$$

    - Which can be rewritten under the form

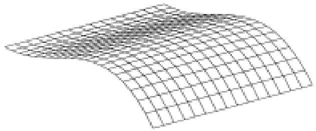
$$\begin{pmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \\ \tilde{m}_{xx} \\ \tilde{m}_{yy} \\ \tilde{m}_{xy} \end{pmatrix} = \begin{pmatrix} A_{xxxx} & A_{xxyy} & 2A_{xxxy} & B_{xxxx} & B_{xxyy} & 2B_{xxxy} \\ A_{yyxx} & A_{yyyy} & 2A_{yyxy} & B_{yyxx} & B_{yyyy} & 2B_{yyxy} \\ A_{xyxx} & A_{xyyy} & 2A_{xyxy} & B_{xyxx} & B_{xyyy} & 2B_{xyxy} \\ B_{xxxx} & B_{xxyy} & 2B_{xxxy} & D_{xxxx} & D_{xxyy} & 2D_{xxxy} \\ B_{yyxx} & B_{yyyy} & 2B_{yyxy} & D_{yyxx} & D_{yyyy} & 2D_{yyxy} \\ B_{xyxx} & B_{xyyy} & 2B_{xyxy} & D_{xyxx} & D_{xyyy} & 2D_{xyxy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}^0_{xx} \\ \boldsymbol{\varepsilon}^0_{yy} \\ \boldsymbol{\varepsilon}^0_{xy} \\ -\boldsymbol{u}^0_{z,xx} \\ -\boldsymbol{u}^0_{z,yy} \\ -\boldsymbol{u}^0_{z,xy} \end{pmatrix}$$

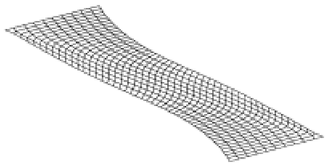
$$n_{xx} = A_{xxxx}\boldsymbol{\varepsilon}^0_{xx} + A_{xxyy}\boldsymbol{\varepsilon}^0_{yy} + A_{xxxy}\boldsymbol{\varepsilon}^0_{xy} + A_{xxyx}\boldsymbol{\varepsilon}^0_{yx} - B_{xxxx}\boldsymbol{u}^0_{z,xx} - B_{xxyy}\boldsymbol{u}^0_{z,yy} - B_{xxxy}\boldsymbol{u}^0_{z,xy} - B_{xxyx}\boldsymbol{u}^0_{z,yx}$$

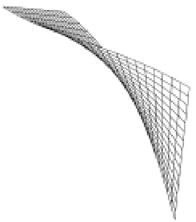
- Laminated composite (10)
  - Stress-strain relationship (6)
    - As $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx}$ & $\kappa_{xy} = -u_{z,xy} - u_{z,yx}$

$$\begin{pmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \\ \tilde{m}_{xx} \\ \tilde{m}_{yy} \\ \tilde{m}_{xy} \end{pmatrix} = \begin{pmatrix} A_{xxxx} & A_{xxyy} & A_{xxxy} & B_{xxxx} & B_{xxyy} & B_{xxxy} \\ A_{yyxx} & A_{yyyy} & A_{yyxy} & B_{yyxx} & B_{yyyy} & B_{yyxy} \\ A_{xyxx} & A_{xyyy} & A_{xyxy} & B_{xyxx} & B_{xyyy} & B_{xyxy} \\ B_{xxxx} & B_{xxyy} & B_{xxxy} & D_{xxxx} & D_{xxyy} & D_{xxxy} \\ B_{yyxx} & B_{yyyy} & B_{yyxy} & D_{yyxx} & D_{yyyy} & D_{yyxy} \\ B_{xyxx} & B_{xyyy} & B_{xyxy} & D_{xyxx} & D_{xyyy} & D_{xyxy} \end{pmatrix} \begin{pmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \\ \kappa^0_{xx} \\ \kappa^0_{yy} \\ \kappa^0_{xy} \end{pmatrix}$$

- Terms B are responsible for traction/bending coupling
  - With $B_{\alpha\beta\gamma\delta} = \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i$
  - A symmetrical stack prevents this coupling
    - » 2 identical C^i at z^i opposite
- Terms A_{xxxy} are responsible for tensile/shearing coupling
  - Can be avoided by using the same proportion of +α and -α plies
- Terms D_{xxxy} are responsible for torsion/bending coupling







- Symmetrical laminated composite
  - Stress-strain relationship
    - Terms B vanish

$$\Longrightarrow \begin{pmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{pmatrix} = \begin{pmatrix} A_{xxxx} & A_{xxyy} & A_{xxxy} \\ A_{yyxx} & A_{yyyy} & A_{yyxy} \\ A_{xyxx} & A_{xyyy} & A_{xyxy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}^0_{xx} \\ \boldsymbol{\varepsilon}^0_{yy} \\ \gamma^0_{xy} \end{pmatrix}$$

- If h is the laminate thickness

$$\Longrightarrow \begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{xy} \end{pmatrix} = \frac{1}{h} \begin{pmatrix} A_{xxxx} & A_{xxyy} & A_{xxxy} \\ A_{yyxx} & A_{yyyy} & A_{yyxy} \\ A_{xyxx} & A_{xyyy} & A_{xyxy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}^0_{xx} \\ \boldsymbol{\varepsilon}^0_{yy} \\ \gamma^0_{xy} \end{pmatrix}$$

- As $A_{\alpha\beta\gamma\delta} = \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i$ with

$$\mathcal{C}_{xxxy} = \mathcal{C}_{xyxx} = \mathcal{C}_{xxyx} = \mathcal{C}_{yxxx} = \\ c^3 s \left(\mathcal{C}'_{x'x'x'x'} - \mathcal{C}'_{x'x'y'y'} - 2\mathcal{C}'_{x'y'x'y'}\right) + cs^3 \left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'} - \mathcal{C}'_{y'y'y'y'}\right)$$

  - Supression of tensile/shearing coupling requires same proportion in $+\alpha°$ and $-\alpha°$ oriented laminas (of the same material)
  - Then $s(+\alpha) = \sin(+\alpha) = -s(-\alpha)$ & $s^3(+\alpha) = \sin^3(+\alpha) = -s^3(-\alpha)$
  - So two terms $\mathcal{C}_{xxxy}$ will cancel each-others

- Symmetrical laminated composite without tensile/shearing coupling
  - Stress-strain relationship
    - Terms B & A_{xxxy} vanish

$$\Rightarrow \begin{pmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{pmatrix} = \begin{pmatrix} A_{xxxx} & A_{xxyy} & A_{xxxy} \\ A_{yyxx} & A_{yyyy} & A_{yyxy} \\ A_{xyxx} & A_{xyyy} & A_{xyxy} \end{pmatrix} \begin{pmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{pmatrix}$$

    - To be compared with an orthotropic material

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{E_x}{1-\nu_{xy}\nu_{yx}} & \frac{\nu_{yx}E_x}{1-\nu_{xy}\nu_{yx}} & 0 \\ \frac{\nu_{xy}E_y}{1-\nu_{xy}\nu_{yx}} & \frac{E_y}{1-\nu_{xy}\nu_{yx}} & 0 \\ 0 & 0 & 2\mu_{xy} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix}$$

  - Homogenized orthotropic material

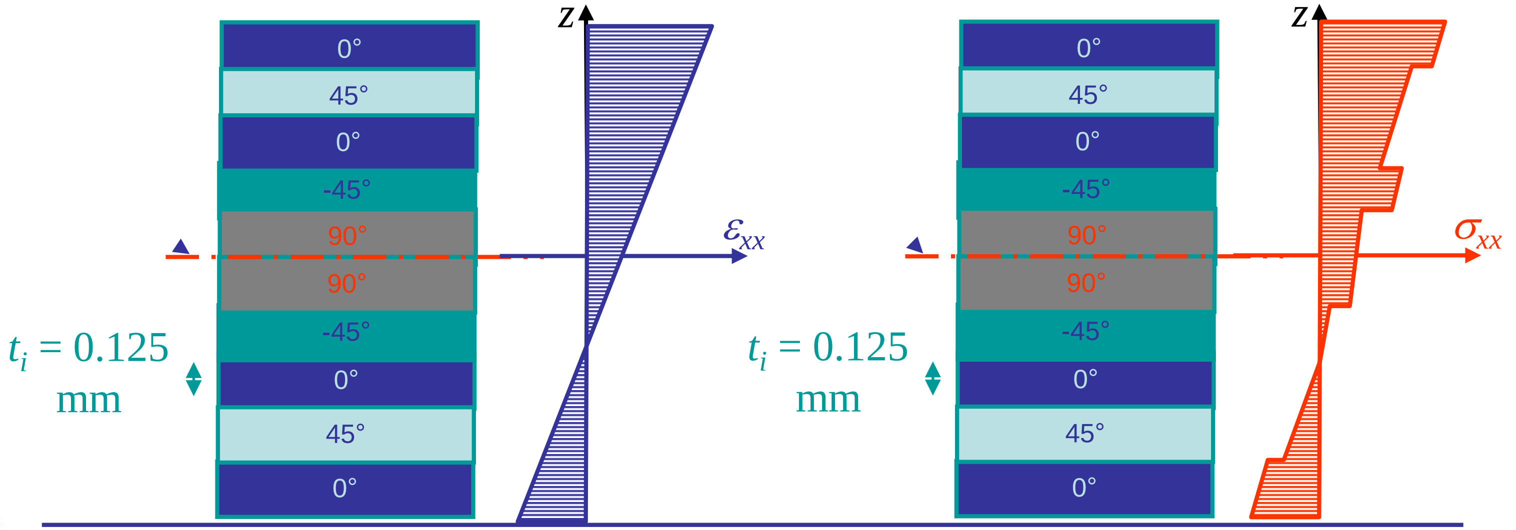
$$\begin{cases} E_x = \dfrac{A_{xxxx}A_{yyyy} - A^2_{xxyy}}{hA_{yyyy}} \\ E_y = \dfrac{A_{xxxx}A_{yyyy} - A^2_{xxyy}}{hA_{xxxx}} \\ \nu_{xy} = \dfrac{A_{xxyy}}{A_{yyyy}} \quad \& \quad \nu_{yx} = \dfrac{A_{xxyy}}{A_{xxxx}} \end{cases}$$

- Methodology
  - Finite element problem solved using *ABD* matrix of the laminated structure

$$\begin{pmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \\ \tilde{m}_{xx} \\ \tilde{m}_{yy} \\ \tilde{m}_{xy} \end{pmatrix} = \begin{pmatrix} A_{xxxx} & A_{xxyy} & A_{xxxy} & B_{xxxx} & B_{xxyy} & B_{xxxy} \\ A_{yyxx} & A_{yyyy} & A_{yyxy} & B_{yyxx} & B_{yyyy} & B_{yyxy} \\ A_{xyxx} & A_{xyyy} & A_{xyxy} & B_{xyxx} & B_{xyyy} & B_{xyxy} \\ B_{xxxx} & B_{xxyy} & B_{xxxy} & D_{xxxx} & D_{xxyy} & D_{xxxy} \\ B_{yyxx} & B_{yyyy} & B_{yyxy} & D_{yyxx} & D_{yyyy} & D_{yyxy} \\ B_{xyxx} & B_{xyyy} & B_{xyxy} & D_{xyxx} & D_{xyyy} & D_{xyxy} \end{pmatrix} \begin{pmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \\ \kappa^0_{xx} \\ \kappa^0_{yy} \\ \kappa^0_{xy} \end{pmatrix}$$

With $\gamma_{ij} = \varepsilon_{ij} + \varepsilon_{ji}$ & $\kappa_{ij} = -u_{z,ij} - u_{z,ji}$

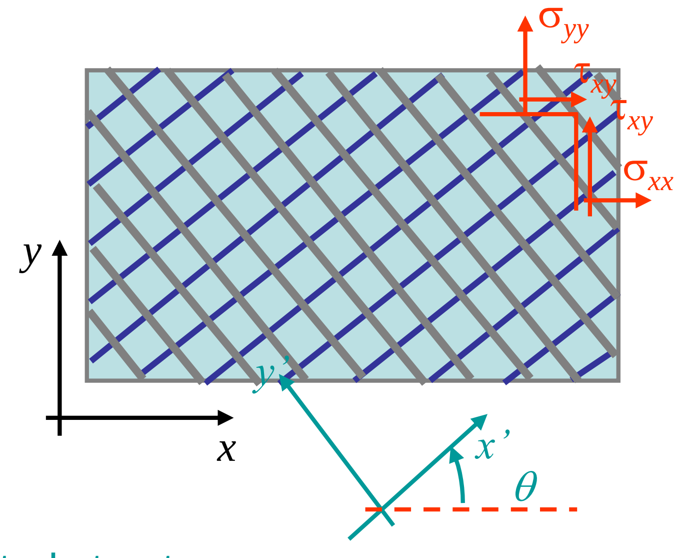
  - In each ply i, field σ^i_{xx}, σ^i_{yy}, σ^i_{xy} in laminated axes: $\boldsymbol{\sigma}^i_{\alpha\beta} = \mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{\varepsilon}^0_{\gamma\delta} - z\mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{u}^0_{z,\gamma\delta}$



- Methodology (2)
  - In each ply i, field σ^i_{xx}, σ^i_{yy}, σ^i_{xy} in laminated axes:
    - $\boldsymbol{\sigma}^i_{\alpha\beta} = \mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{\varepsilon}^0_{\gamma\delta} - z\mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{u}^0_{z,\gamma\delta}$
  - In each ply i, field $\sigma^i_{x'x'}$, $\sigma^i_{y'y'}$, $\sigma^i_{x'y'}$ in the laminate axes
    - $\boldsymbol{\sigma}^i_{\alpha'\beta'} = \mathbf{R}^i_{\alpha'\alpha}\boldsymbol{\sigma}^i_{\alpha\beta}\mathbf{R}^i_{\beta'\beta}$

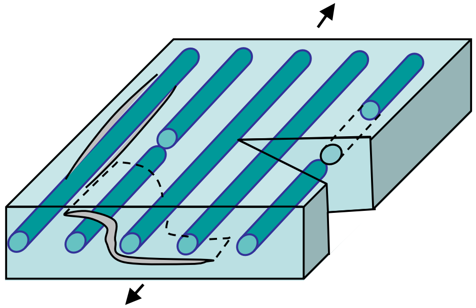
      with $\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$
    - The strain can be deduced from the stress

      using $\boldsymbol{\sigma}' = \mathcal{C}' : \boldsymbol{\varepsilon}'$

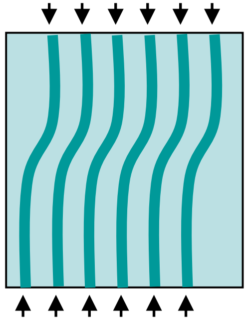


  - We have access to
    - Homogenized resultant stress/strain in laminated structure
    - Homogenized stress/strain in each ply (in the ply main directions)
      - Analyses can predict stress/strain in the fibers/matrix
  - How can we predict failure of laminated structure?

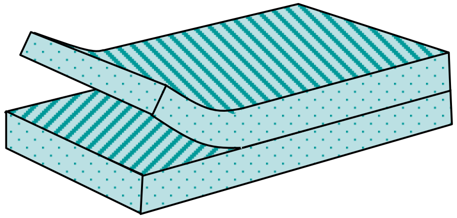
- Heterogeneous structure of composites
  - Failure mechanisms depend on the loading
    - Tensile loading
      - Matrix or fiber cracking, debonding …



    - Compressive loading
      - (Micro-)buckling



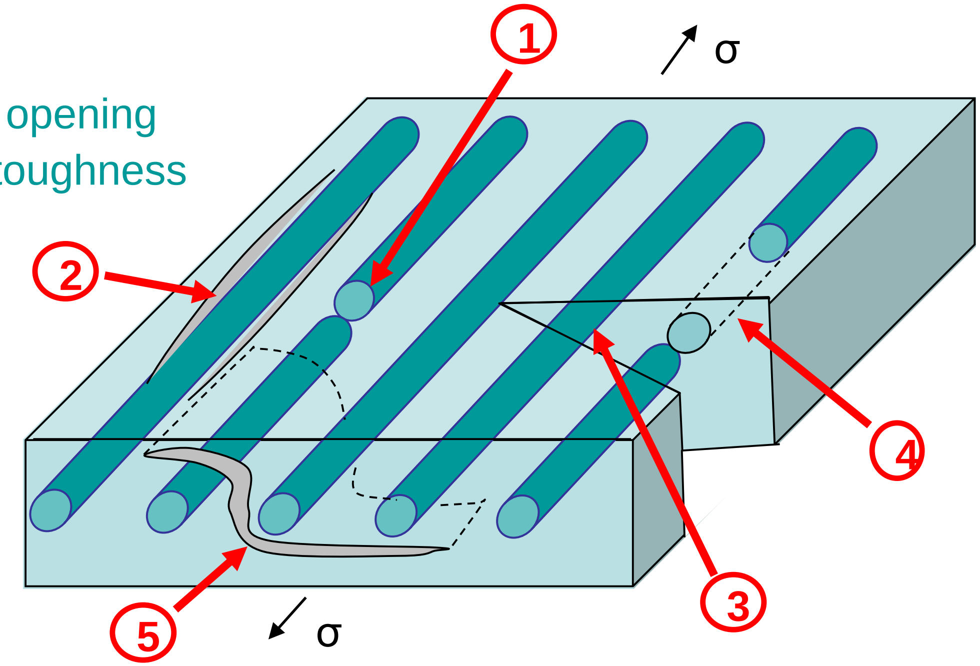
    - Out-of-plane loading
      - Delamination



  - Several of these mechanisms may be simultaneously involved

Failure mechanisms of composites

- Tensile loading
  - Fiber rupture (1)
    - If no matrix
      - Fiber would not be able to carry any loading
      - Fiber would become useless
    - In reality
      - Matrix transmits the load between the two broken parts
      - Fiber can still (partially) carry the loading
  - Fiber/matrix debonding (2)
  - Fiber bridging (3)
    - Prevents the crack from further opening
    - Corresponds to an increase of toughness
  - Fiber Pullout (4)
  - Matrix cracking (5)
    - Facilitates moisture absorption
    - May initiate delamination between plies
  - Ultimate tensile failure
    - Several of these mechanisms



Failure mechanisms of composites

- Tensile loading: Strength of unidirectional fiber reinforced composite
  - A simple model
    - To be applied **in each ply**
      - Study in the longitudinal direction
      - For clarity $\varepsilon^{i}_{x'x'} \to \varepsilon$, $\sigma^{i}_{x'x'} \to \sigma$
    - Strain compatibility for fiber and matrix

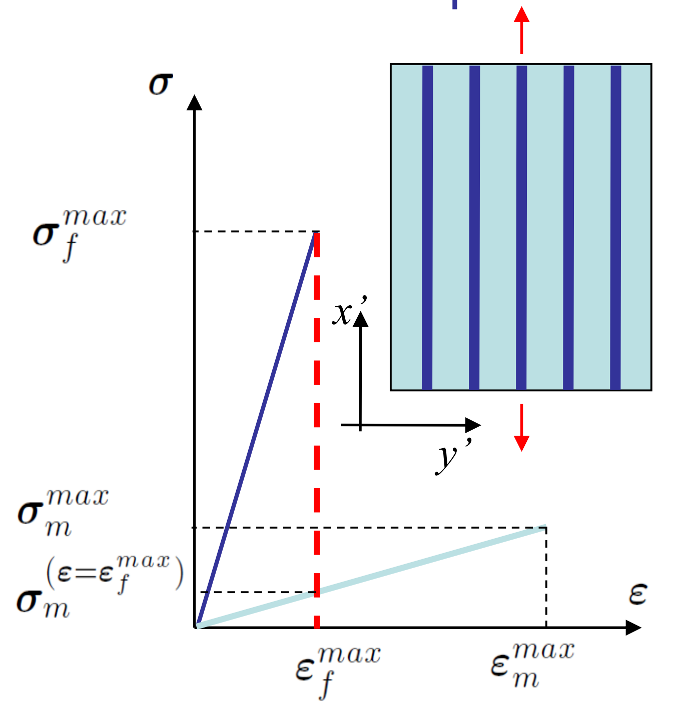
$$\frac{\Delta L}{L} = \varepsilon = \varepsilon_f = \varepsilon_m$$

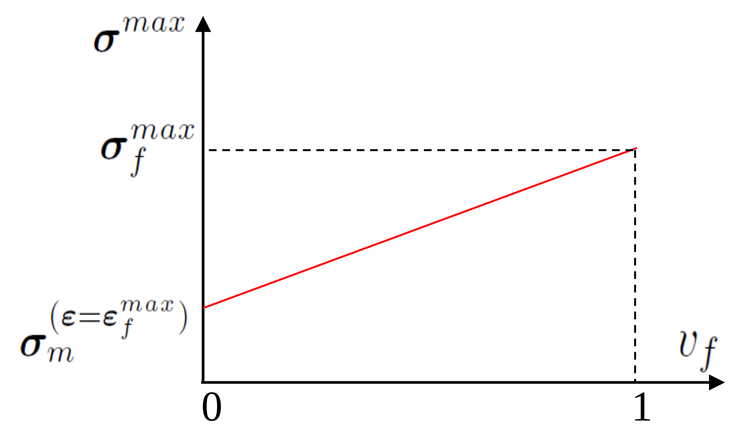
    - Since the fiber is more brittle than the matrix

$$\varepsilon_f^{max} < \varepsilon_m^{max} \to \varepsilon^{max} = \varepsilon_f^{max}$$

    - Fracture stress along x' of ply i

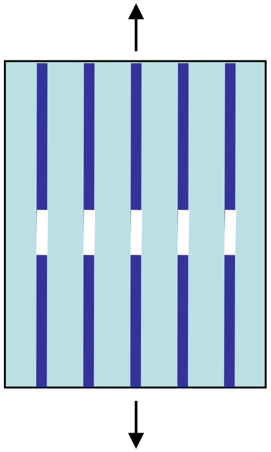
$$\boldsymbol{\sigma}^{max} = \boldsymbol{\sigma}_f^{max} v_f + \boldsymbol{\sigma}_m^{(\varepsilon=\varepsilon_f^{max})} (1 - v_f)$$





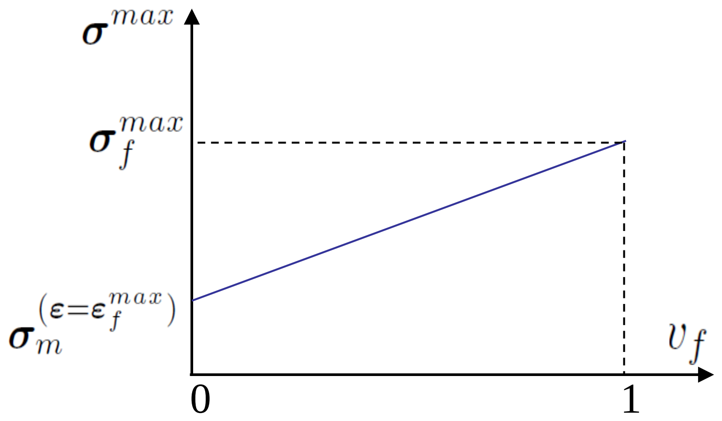
  - Resulting strength curve **of a ply**
  - What happens if a fiber breaks ?

- Tensile loading: Strength of unidirectional fiber reinforced composite (2)
  - A Simple model (2)
    - What happens if $\boldsymbol{\sigma} > \boldsymbol{\sigma}^{max}$?
      - Fiber will break
      - Matrix may still have a load carrying capacity
    - Assume that all fibers break simultaneously
      - Matrix carries the whole load
      - Fracture strain is now $\boldsymbol{\varepsilon}_m^{max}$
      - Fracture strength $\boldsymbol{\sigma}_{broken\ fibers}^{max} = \boldsymbol{\sigma}_m^{max}(1 - v_f)$



- Tensile loading: Strength of unidirectional fiber reinforced composite (3)
  - A Simple model (3)
    - Fiber dominated failure

$$\boldsymbol{\sigma}^{max} = \boldsymbol{\sigma}_f^{max} v_f + \boldsymbol{\sigma}_m^{(\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_f^{max})}(1 - v_f)$$

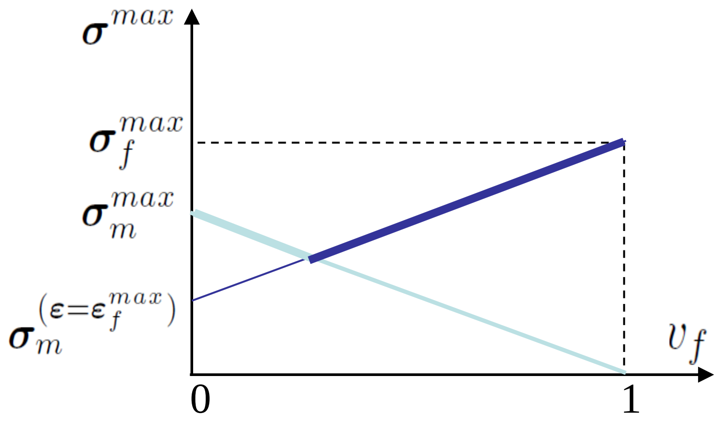


- Tensile loading: Strength of unidirectional fiber reinforced composite (3)
  - A Simple model (3)
    - Fiber dominated failure

$$\boldsymbol{\sigma}^{max} = \boldsymbol{\sigma}_f^{max} v_f + \boldsymbol{\sigma}_m^{(\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_f^{max})} (1 - v_f)$$

    - Matrix dominated failure

$$\boldsymbol{\sigma}_{broken\ fibers}^{max} = \boldsymbol{\sigma}_m^{max} (1 - v_f)$$

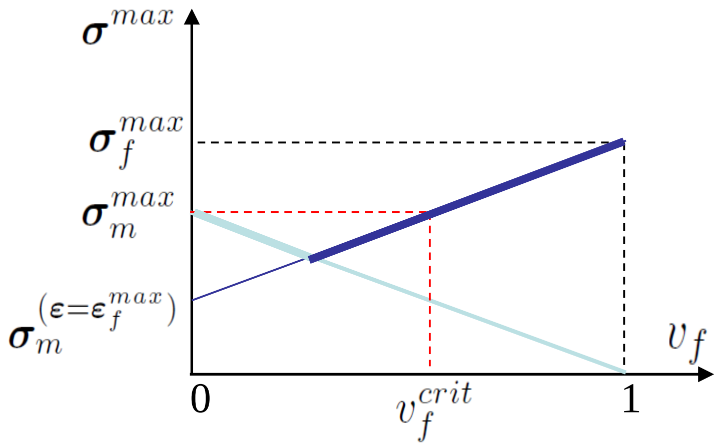


- Tensile loading: Strength of unidirectional fiber reinforced composite (3)
  - A Simple model (3)
    - Fiber dominated failure

$$\boldsymbol{\sigma}^{max} = \boldsymbol{\sigma}_f^{max} v_f + \boldsymbol{\sigma}_m^{(\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_f^{max})}(1 - v_f)$$

    - Matrix dominated failure

$$\boldsymbol{\sigma}_{broken\ fibers}^{max} = \boldsymbol{\sigma}_m^{max}(1 - v_f)$$



    - Critical fiber volume ratio
      - v_f below which the composite strength is lower than matrix strength

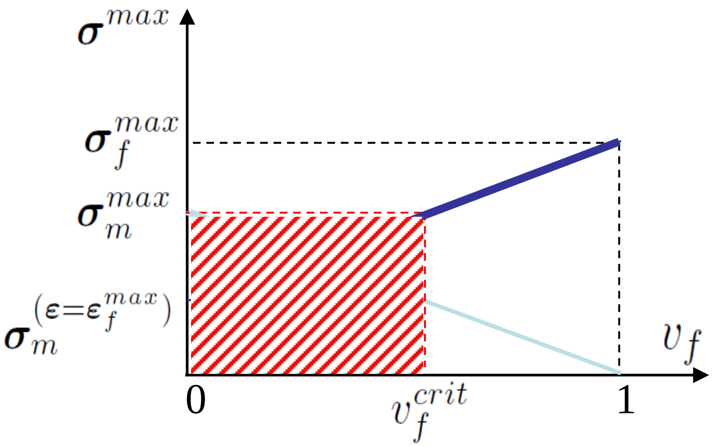
$$v_f^{crit} = \frac{\boldsymbol{\sigma}_m^{max} - \boldsymbol{\sigma}_m^{(\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_f^{max})}}{\boldsymbol{\sigma}_f^{max} - \boldsymbol{\sigma}_m^{(\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_f^{max})}}$$

- Tensile loading: Strength of unidirectional fiber reinforced composite (3)
  - A Simple model (3)
    - Fiber dominated failure

$$\boldsymbol{\sigma}^{max} = \boldsymbol{\sigma}_f^{max} v_f + \boldsymbol{\sigma}_m^{(\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_f^{max})}(1 - v_f)$$

    - Matrix dominated failure

$$\boldsymbol{\sigma}_{broken\ fibers}^{max} = \boldsymbol{\sigma}_m^{max}(1 - v_f)$$



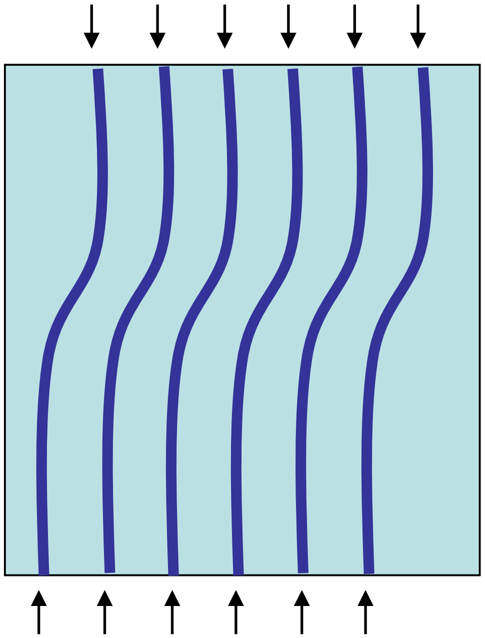
    - Critical fiber volume ratio
      - v_f below which the composite strength is lower than matrix strength

$$v_f^{crit} = \frac{\boldsymbol{\sigma}_m^{max} - \boldsymbol{\sigma}_m^{(\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_f^{max})}}{\boldsymbol{\sigma}_f^{max} - \boldsymbol{\sigma}_m^{(\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_f^{max})}}$$

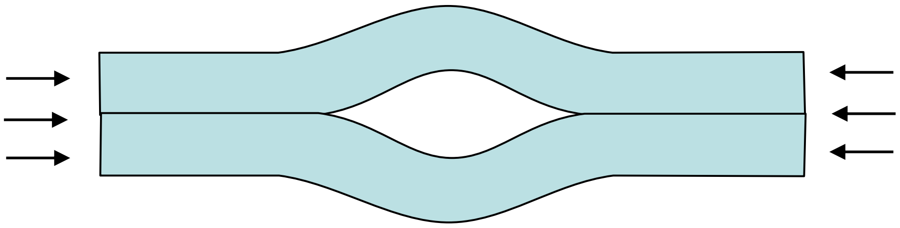
    - Reinforce a matrix with a stiffer and more brittle fiber
      - Always leads to an increase in stiffness
      - But not necessarily to an increase in strength

Failure mechanisms of composites

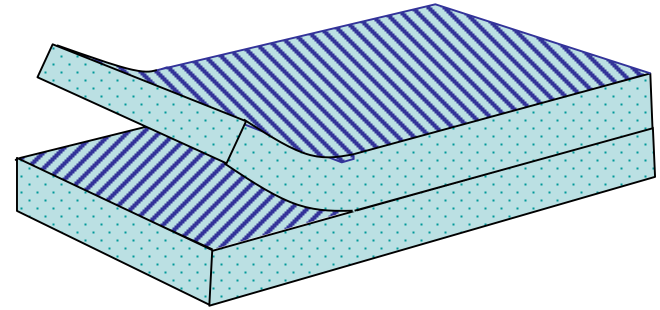
- Compressive loading
  - Microbuckling
    - Fibers
      - Long and thin
      - Unstable in compression
    - Never perfectly straight in the matrix
      - Fiber waviness
      - Increases the buckling risk

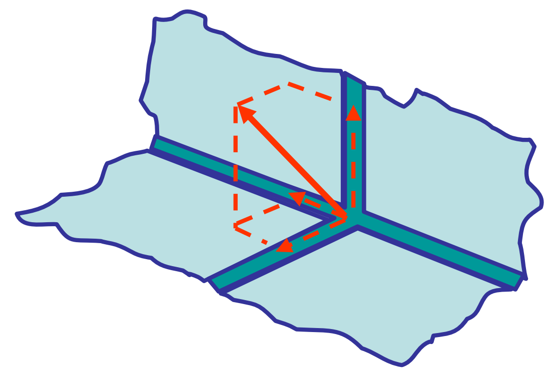


  - Macroscopic delamination buckling
    - Especially if the material contains a pre-existing delaminated region



- Out-of-plane stress: Delamination
  - Fibers cannot carry out-of-plane stress
    - Failure between plies
  - Out-of-plane stress can result from
    - Structural geometry
      - 2 panels joined in a « T » configuration
      - Should be reinforced by stringers
    - Free edge effect (see next slides)
  - Delamination can also be caused by impact loadings
    - Accidental drop of a tool during manufacturing
    - Bird strike on aircraft structures
  - Damage not always apparent
    - Dangerous
    - Ultrasonic inspection



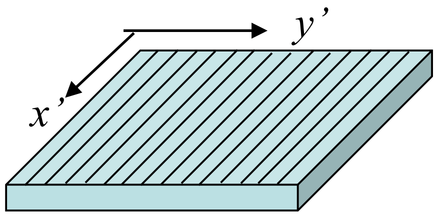


- Delamination – Free edge effect
  - Classical laminated theory assumes plane-σ state of each plies

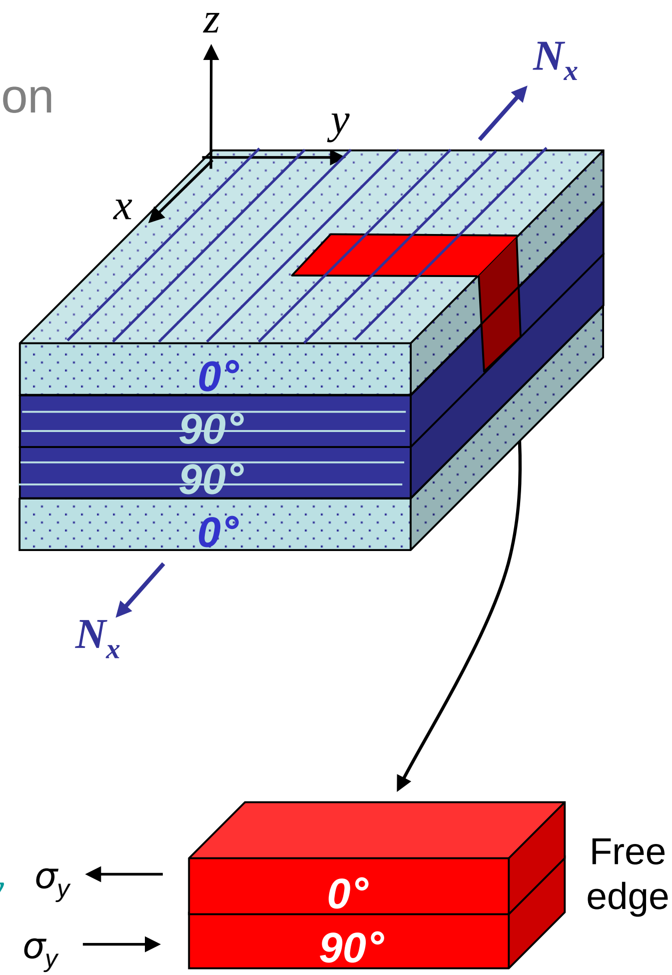
$$\boldsymbol{\sigma}_{zz} = \boldsymbol{\sigma}_{xz} = \boldsymbol{\sigma}_{yz} = 0$$

  - BUT
    - Significant out-of-plane interlaminar stress may appear
      - In small zones
      - Close to the free edges
    - Even for in-plane external loading only
    - This is the *free edge effect* which can initiate delamination
  - Plane-σ laminated theory fails to predict interlaminar stresses
    - 3D laminated theory and correct boundary conditions should be used
    - Involves complex differential equations
    - Requires numerical solving methods (e.g.: finite differences)

- Delamination – Free edge effect (2)
  - Example : $[0°/90°]_s$ laminated structure in tension
    - Let $x'y'$ be the local axes in each ply

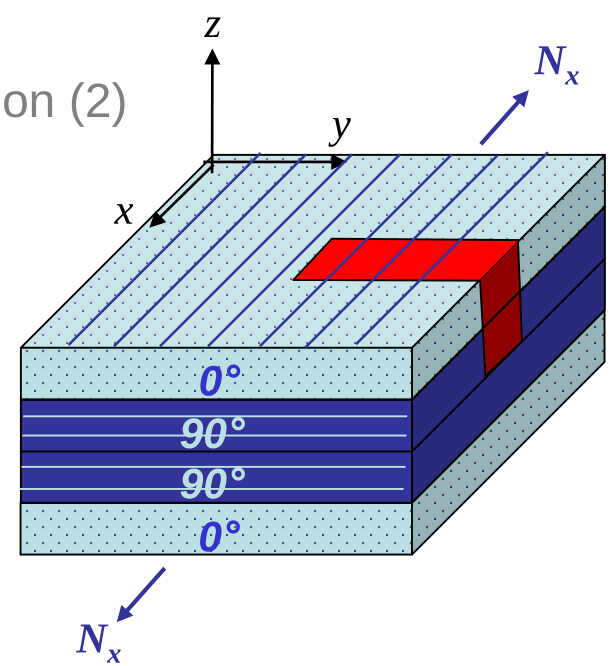


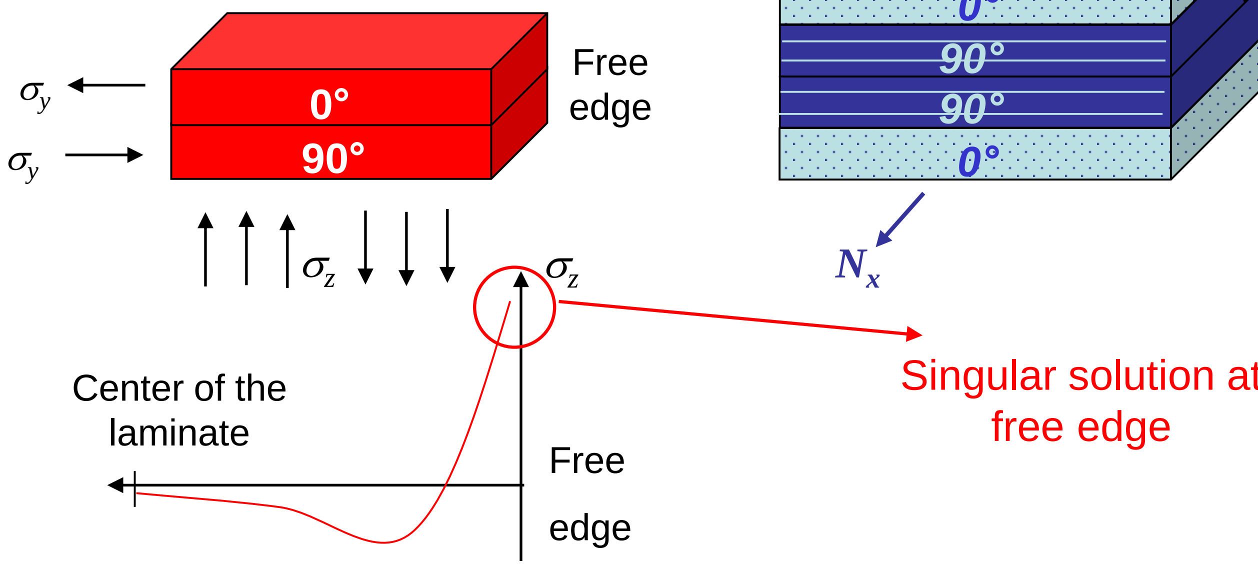
- In the global axes $\begin{cases} \nu_{xy}^{0°} = \nu_{x'y'} \\ \nu_{xy}^{90°} = \nu_{y'x'} \end{cases}$
- Bonding compatibility
  - Same strain along y
  - Since $\nu_{y'x'} < \nu_{x'y'}$
    - » 0°-ply: tension along y
    - » 90°-ply: compression along y
  - This leads to this stress distribution along y



- Rotational equilibrium is not satisfied !!
  - There should be a restoring moment

- Delamination – Free edge effect (3)
  - Example : $[0°/90°]_s$ laminated structure in tension (2)
    - Restoring moment
      - Free edge and the upper faces stress-free
      - There should be a σ_z distribution at the 90°/90° interface





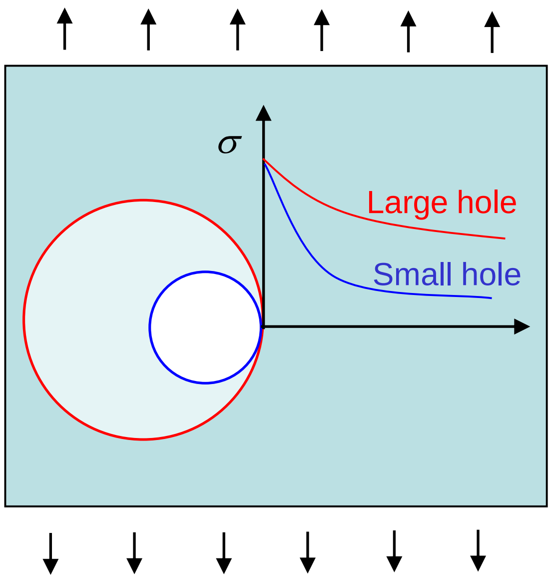
- At each interface
  - Interlaminar stress distribution (mode I)
- At each interface except symmetrical one
  - Shearing (mode II)
  - But obviously vanishes at free edge

- Delamination – Free edge effect (4)
  - Interlaminar stresses can initiate delamination at the edge of a laminate
  - These stresses strongly depend on the stacking sequence
  - Free edge effect can be reduced by
    - Modifying the stacking sequence
    - Using edge reinforcements
    - Modifying edge geometry

- Heterogeneous vs isotropic homogenous materials
  - Notch strength: assume plate large compared to hole
    - Isotropic homogenous materials
      - Stress profile (lecture 2)

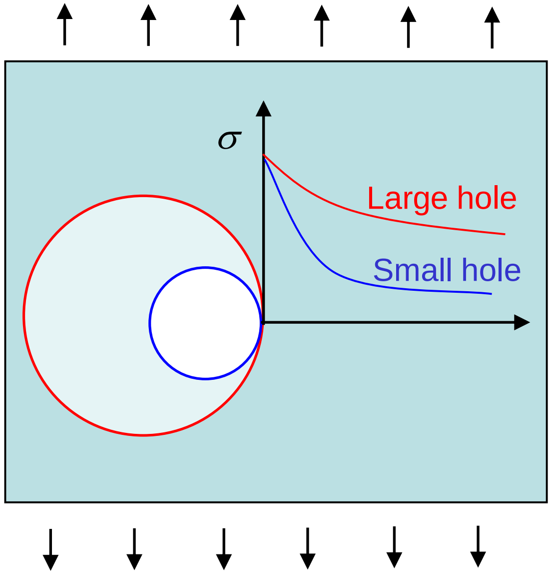
$$\boldsymbol{\sigma}_{yy}(x, y=0) = \frac{\sigma_\infty}{2}\left(2 + \frac{3a^4}{x^4} + \frac{a^2}{x^2}\right)$$

      - Stress concentration factor equals 3
        - » Whatever the radius of the hole
        - » Thus, for a stress-based criterion strength independent of radius
      - However the distance where the stress concentration acts depends of the hole radius
    - Composite
      - Measurements show a radius dependence on material strength
      - Increasing radius lowers the strength
      - Volume over which the stress acts is important

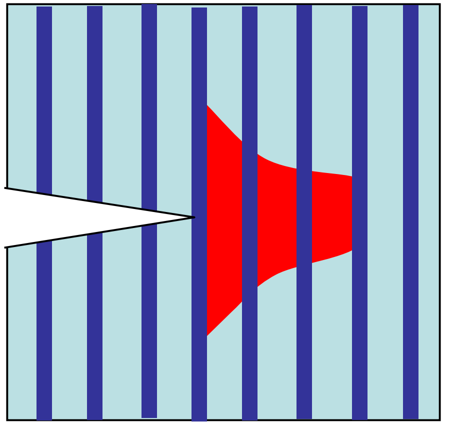


Failure study methodology

- Heterogeneous vs isotropic homogenous materials (2)
  - Notch strength: assume plate large compared to hole (2)
    - Composite (2)
      - Increasing radius lowers the strength
      - Volume over which the stress acts is important
    - Whitney-Nuismer criterion for failure in notched composites
      - Failure will occur if the stress exceeds the un-notched strength σ^f over a critical distance d
      - This parameter is obtained experimentally
      - Criterion $\boldsymbol{\sigma}_y(R+d,0) > \boldsymbol{\sigma}^f$
    - This is not a rigorous approach
  - Due to the heterogeneity, there is a scale-effect
  - What happens for sharp crack?



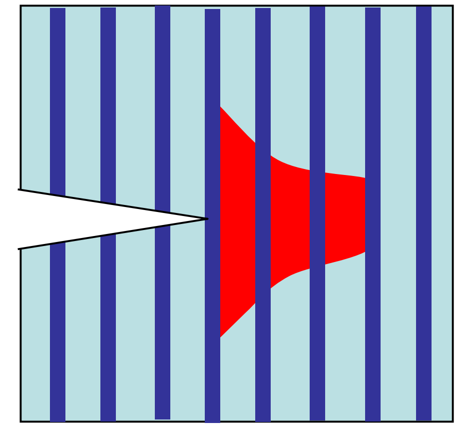
- Heterogeneous vs isotropic homogenous materials (3)
  - Sharp crack
    - (Brittle) homogenous material
      - Asymptotic solution $\boldsymbol{\sigma}^{\text{mode i}} = \frac{K_i}{\sqrt{2\pi r}} \mathbf{f}^{\text{mode i}}(\theta)$
      - Outside singularity zone
        - » Solution completed by terms in r^0, $r^{1/2}$, …
        - » Geometry dependant
      - K-only-based fracture criterion
        - » Only if all non-linear behavior in singularity zone

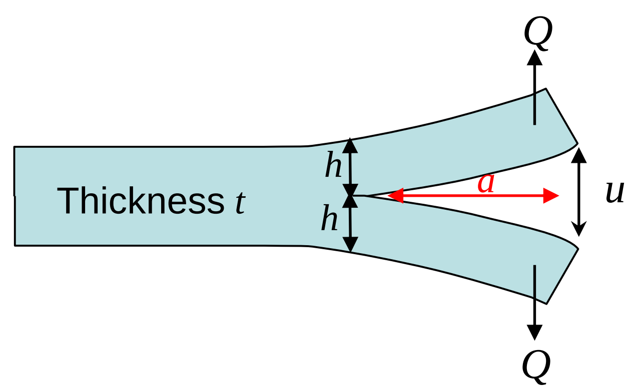


    - This model is based on continuum mechanics assumption
      - Theoretical concept that is verified or not depending on which scale a material is studied
      - For LEFM: micro structural constituents small compared to singularity zone
        - » Non-damaged metals
        - » Ceramics
        - » Plastics
      - What about composites?

Failure study methodology

- Heterogeneous vs isotropic homogenous materials (4)
  - Sharp crack (2)
    - Composite
      - LEFM valid if continuum mechanics is valid
        - » Fiber spacing small compared to the size of the singularity zone (continuity condition)
        - » Nonlinear damage (debonding, matrix cracking …) must be confined to a small region within the singularity zone
      - However, anisotropy has to be taken into account
        - » Asymptotic solutions will be different
        - » SIF's now depend on geometry, loading AND anisotropic parameters
    - Interlaminar failure: delamination
      - Crack is usually confined to the matrix between plies
      - Continuum theory is applicable
      - LEFM can be used

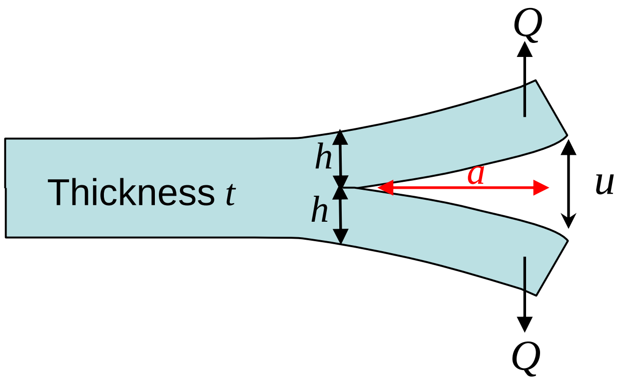




Failure study methodology

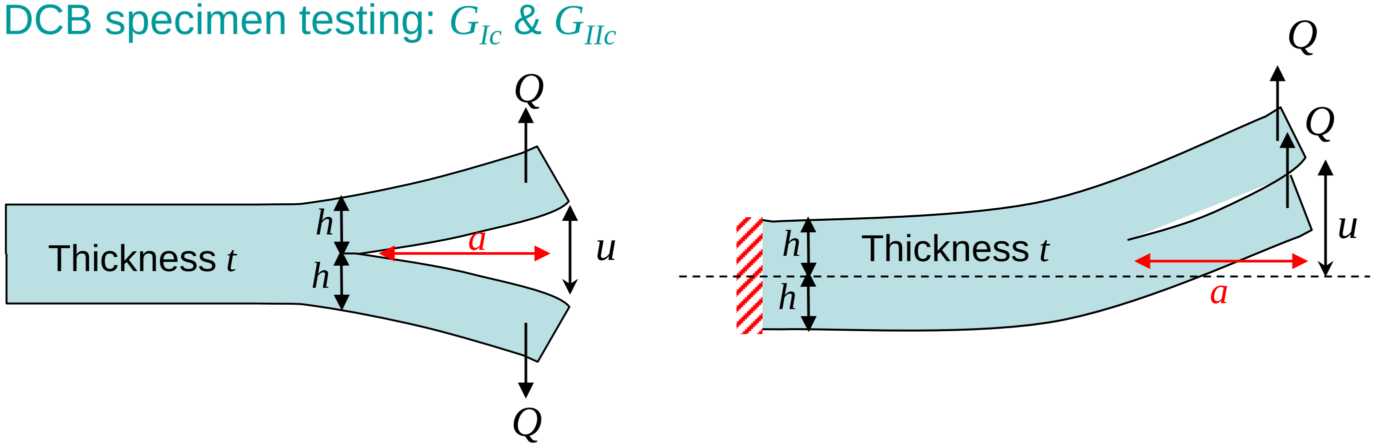
- Heterogeneous vs isotropic homogenous materials (5)
  - Sharp crack (3)
    - Composite (2)
      - In some cases, LEFM is valid
        » See previous slide
      - BUT for composites
        » These conditions are not always met, and when met
        » Several complex fracture mechanisms are involved
        » Failure is often controlled by micro-cracks distributed throughout the material instead of a single macroscopic crack

- Failure prediction of composite materials
  - Interlaminar failure: delamination
    - Analytical: LEFM
    - Numerical point of view
      - Crack path is known
      - Cohesive elements can be used
      - With appropriate traction-separation law



  - Intralaminar failure
    - Often controlled by micro-cracks distributed throughout the material
    - This can be better handled by damage mechanics for example
      - The ply is homogenized
      - Loss of integrity in the ply is introduced through damage variables
      - Damage affects the stiffness and the strength of the material in a continuous way (see lecture on numerical methods)
    - Numerical point of vue
      - Verify a failure criterion on each ply for fracture initiation
      - Can be completed with damage theory

- Interlaminar fracture toughness
  - Assuming continuum mechanics & SSY hold: LEFM
  - Due to anisotropy, G_c is not the same in the two directions
    - The fracture energy will be different in mode I and mode II
    - DCB specimen testing: G_{Ic} & G_{IIc}

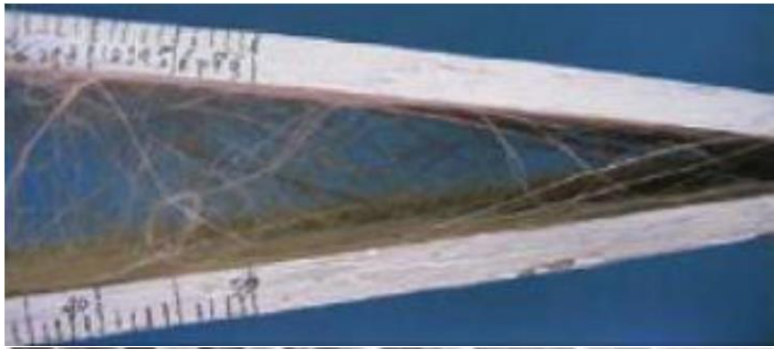


- Mixed mode fracture criterion

$$\left(\frac{G_I}{G_{Ic}}\right)^m + \left(\frac{G_{II}}{G_{IIc}}\right)^n = 1$$

where m & n are empirical parameters

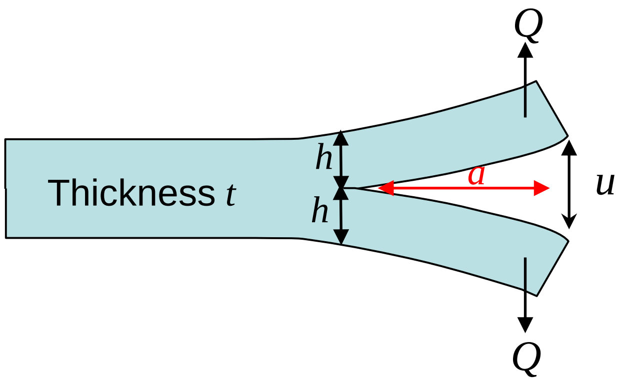
- Interlaminar fracture toughness: Mode I
  - Crack propagate in the matrix (resin)
    - $G_{Ic} = G_c$ of resin?
  - Due to the presence of the fibers
    - $G_{Ic} \neq G_c$ of the pure resin
    - Fiber bridging
      - Increases toughness
    - Fiber/matrix debonding
      - Brittle matrix
        - » Crack surface is not straight as it follows the fibers
        - » More surface created
        - » Higher toughness
      - Tough matrix
        - » Fibers may prevent the damage zone in the matrix from extending far away
        - » Smaller surface created
        - » Lower toughness

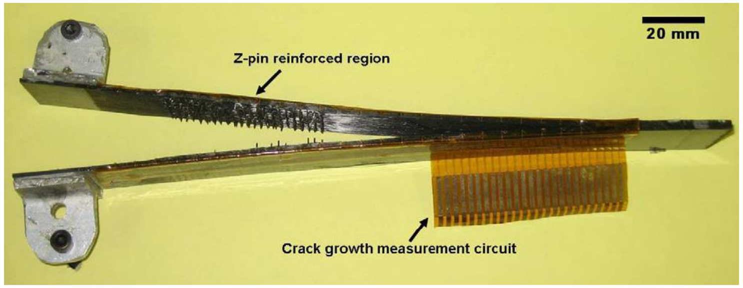


- Interlaminar fracture toughness: Mode I (2)
  - Measure of G_{Ic}
    - DCB (see previous lecture)

$$\begin{cases} u = \dfrac{8Qa^3}{Eth^3} \\ G = \dfrac{12Q^2a^2}{Et^2h^3} = \dfrac{3u^2Eh^3}{16a^4} = \dfrac{3uQ}{2at} \end{cases}$$

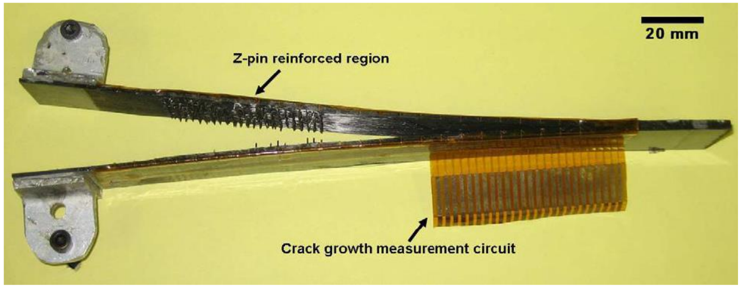
    - At fracture $G_{Ic} = \dfrac{3u_cQ_c}{2at}$
    - The initial delaminated zone is introduced by placing a non-adhesive insert between plies prior to molding



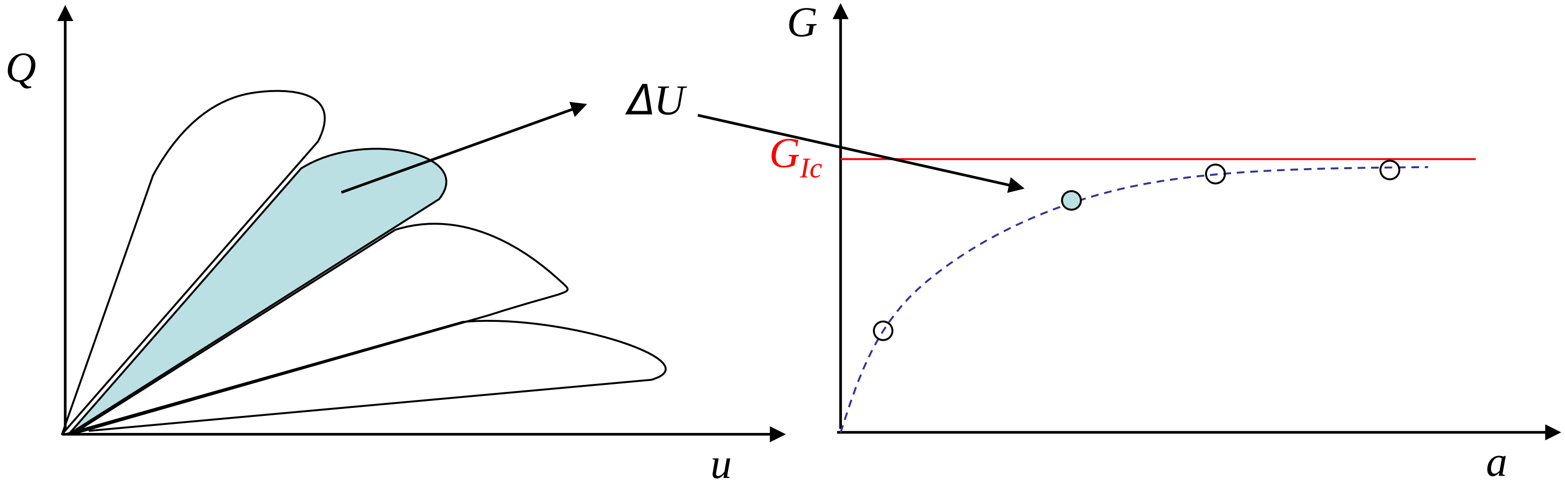


Paul Tihon, coexpair

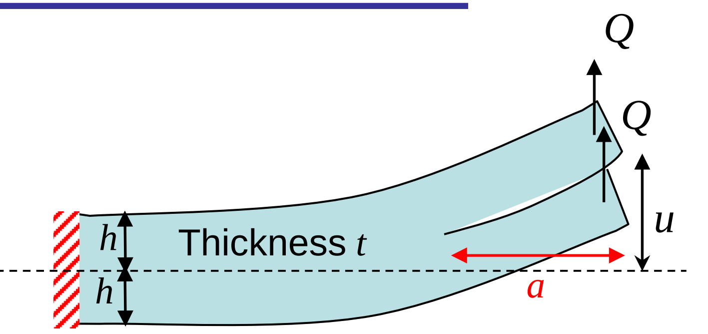
- Interlaminar fracture toughness: Mode I (3)
  - Measure of G_{Ic} (2)
    - Linear beam theory may give wrong estimates of energy release rate
  - The area method is an alternative solution
    - Periodic loading with small crack propagation increments
      - The loading part is usually nonlinear prior to fracture
    - Since G is the energy released per unit area of crack advance : $G = \frac{\Delta U}{t \Delta a}$

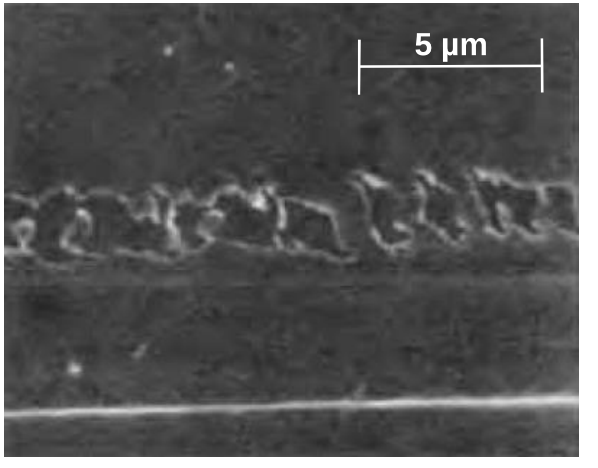


Paul Tihon, coexpair



- Interlaminar fracture toughness: Mode II
  - G_{IIc}
    - Usually 2-10 times higher than G_{Ic}
      - Especially for brittle matrix
    - In mode II loading
      - Extended damage zone, containing micro-cracks, forms ahead of the crack tip
      - The formation of this damaged zone is energy consuming
        - » High relative toughness in mode II





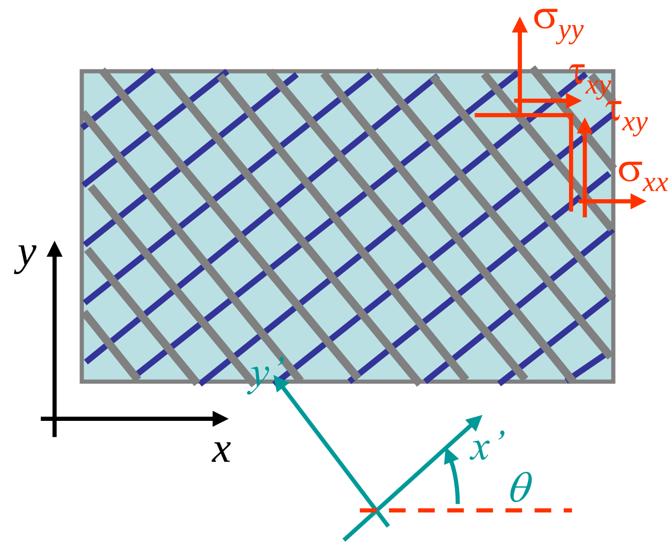
- Note that micro-cracks are 45°-kinked
  - Since pure shearing is involved, this is the direction of maximal tensile stress
  - Thus, the micro-cracks are loaded in mode I

- Intralaminar failure prediction
  - Aim: Predict if a composite will break or not for a given loading
    - Failure often controlled by micro-cracks distributed throughout the material
    - Numerically
      - Verify a failure criterion on each ply for fracture initiation
      - Can be completed with damage theory for failure evolution
  - Practically
    - Proceed on each ply
    - Use of homogenized properties of the ply
    - Extract stress in the main direction of the ply

$$\begin{cases} \boldsymbol{\sigma}^i_{\alpha\beta} = \mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{\varepsilon}^0_{\gamma\delta} - z\mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{u}^0_{z,\gamma\delta} \\ \boldsymbol{\sigma}^i_{\alpha'\beta'} = \mathbf{R}^i_{\alpha'\alpha}\boldsymbol{\sigma}^i_{\alpha\beta}\mathbf{R}^i_{\beta'\beta} \end{cases}$$

    - Consider a fracture surface F (σ', Material parameters) ≤ 1
      - If $F > 1$, then the composite breaks
      - The material parameters are determined experimentally
      - Microscopic failure mechanisms are hidden behind these parameters

- Intralaminar failure criteria
  - Homogenized stress state
    - For conciseness, rename stresses
    - Longitudinal stress $\sigma_{x'x'} \rightarrow \sigma_1$
    - Transverse stress $\sigma_{y'y'} \rightarrow \sigma_2$
    - Shear stress $\sigma_{x'y'} \rightarrow \tau_{12}$
  - Failure criterion for an orthotropic ply in plane-σ state should consider
    - Longitudinal (along Ox') tension and compression strengths: X_t & X_c
    - Transverse tension and compression strengths: Y_t & Y_c
    - In-plane shearing strength: S
  - This means at least 5 material parameters
    - 3 if no distinction between tension and compression
  - Criteria
    - Maximum stress
    - Considering a surface (Tsai-Hill & Tsai-Wu)

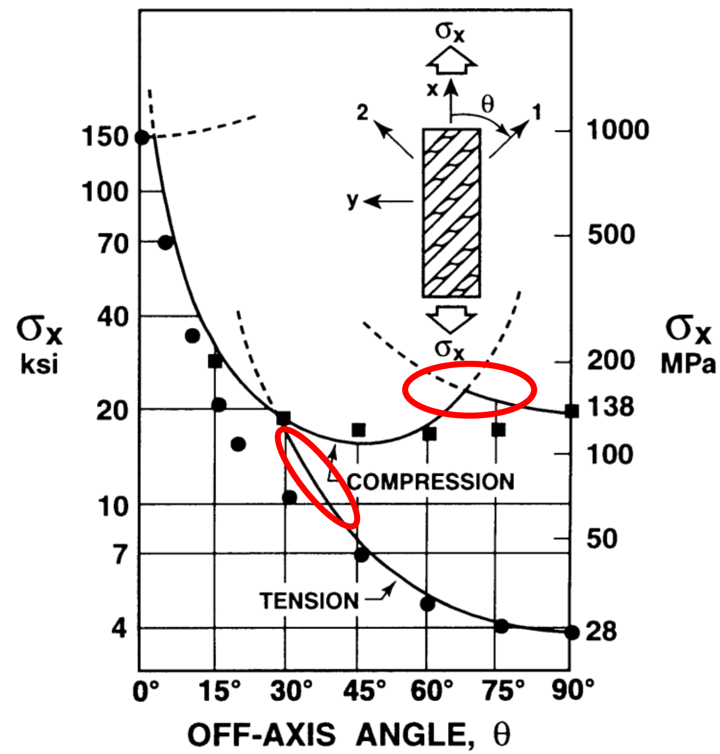


- Intralaminar failure: Maximum stress criterion
  - Interactions between stresses is neglected

$$\begin{cases} \sigma_1 < X_t & \text{if} \quad \sigma_1 \geq 0 \\ \sigma_1 > X_c & \text{if} \quad \sigma_1 < 0 \\ \sigma_2 < Y_t & \text{if} \quad \sigma_2 \geq 0 \\ \sigma_2 > Y_c & \text{if} \quad \sigma_2 < 0 \\ |\tau_{12}| < S & \end{cases}$$

  - Maximum strain criterion
    - Same formulation, but in terms of strains
  - Experiments
    - Discrepancy for biaxial stress state
    - Biaxial strength criterion to be sought

E-Glass-Epoxy



S Tsai, Strength theories of filamentary structures, in Fundamentals Aspects of FRPC, 1968, Wiley, New-York

- Tsai-Hill criterion
  - Aim: consider biaxial stress state
    - Inspired from yield von-Mises surface
  - Hill proposed a yield surface for orthotropic materials in 3D

$$(G+H)\boldsymbol{\sigma}_1^2 + (F+H)\boldsymbol{\sigma}_2^2 + (F+G)\boldsymbol{\sigma}_3^2 - 2H\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 - 2G\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 - 2F\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 + 2L\tau_{23}^2 + 2M\tau_{13}^2 + 2N\tau_{12}^2 < 1$$

    - 6 parameters: $F\ G\ H\ L\ M\ N$
  - Tsai modified these parameters for composite failure
    - Relate these parameters to the failure of an orthotropic ply in composites
      - X: longitudinal strength,
      - Y: Transverse strength
      - S: Shear strength
    - Consider separate critical loadings to find the parameters
      - Only $\tau_{12} = S \Longrightarrow 2N = \frac{1}{S^2}$
      - Only $\tau_{13} = S_{13} \Longrightarrow 2M = \frac{1}{S_{13}^2}$
      - Only $\tau_{23} = S_{23} \Longrightarrow 2L = \frac{1}{S_{23}^2}$

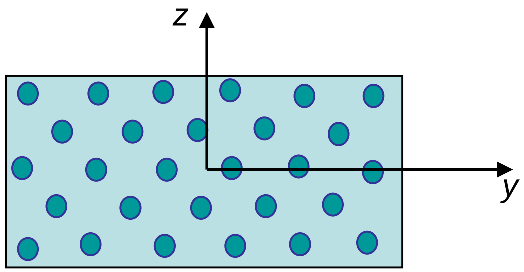
- Tsai-Hill criterion (2)
  - Tsai modified these parameters for composite (2)
    - Consider separate critical loadings to find the parameters (2)

$$(G+H)\boldsymbol{\sigma}_1^2 + (F+H)\boldsymbol{\sigma}_2^2 + (F+G)\boldsymbol{\sigma}_3^2 - 2H\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 - 2G\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 - 2F\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 + 2L\tau_{23}^2 + 2M\tau_{13}^2 + 2N\tau_{12}^2 < 1$$

- Only $\boldsymbol{\sigma}_1 = X \quad \Longrightarrow \quad G + H = \frac{1}{X^2}$
- Only $\boldsymbol{\sigma}_2 = Y \quad \Longrightarrow \quad F + H = \frac{1}{Y^2}$
- Only $\boldsymbol{\sigma}_3 = Z \quad \Longrightarrow \quad F + G = \frac{1}{Z^2}$
  - » General 3D case: strength Z in the third direction
- Resolution of the system:

$$2G = \frac{1}{X^2} + \frac{1}{Z^2} - \frac{1}{Y^2} \qquad 2F = \frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2} \qquad 2H = \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}$$

- For unidirectional fibers in the x-direction, the strength is the same in y and z: $Y = Z$



- Tsai-Hill criterion (3)
  - Hill criterion

$$(G+H)\boldsymbol{\sigma}_1^2 + (F+H)\boldsymbol{\sigma}_2^2 + (F+G)\boldsymbol{\sigma}_3^2 - 2H\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 - 2G\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 - 2F\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 + 2L\tau_{23}^2 + 2M\tau_{13}^2 + 2N\tau_{12}^2 < 1$$

With

$$\begin{cases} G+H = \frac{1}{X^2} \quad \& \quad F+H = \frac{1}{Y^2} \quad \& \quad F+G = \frac{1}{Z^2} \\ 2G = \frac{1}{X^2} + \frac{1}{Z^2} - \frac{1}{Y^2} \quad \& \quad 2F = \frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2} \quad \& \quad 2H = \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2} \\ 2N = \frac{1}{S^2} \quad \& \quad 2M = \frac{1}{S_{13}^2} \quad \& \quad 2L = \frac{1}{S_{23}^2} \end{cases}$$

  - Assuming plane stress, and $Y=Z$, the Hill criterion becomes the Tsai-Hill criterion for unidirectional composite ply

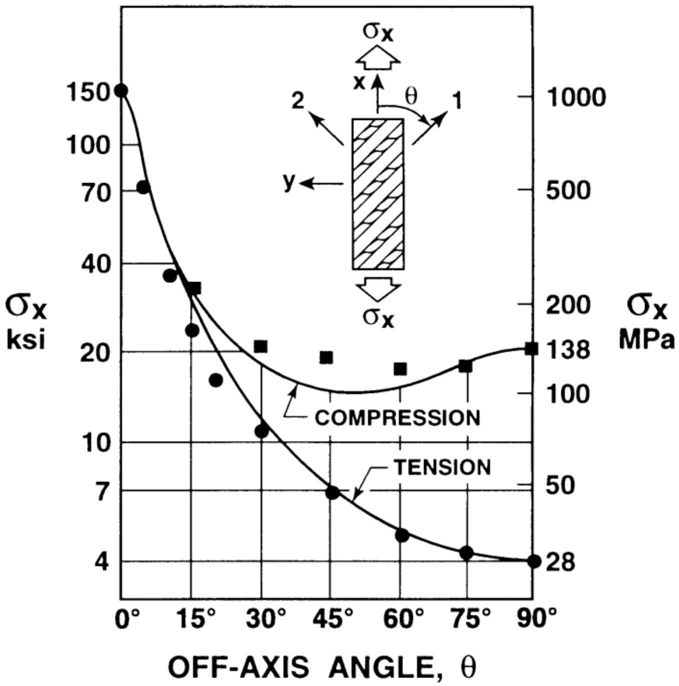
$$\frac{\boldsymbol{\sigma}_1^2}{X^2} - \frac{\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2}{X^2} + \frac{\boldsymbol{\sigma}_2^2}{Y^2} + \frac{\tau_{12}^2}{S^2} < 1$$

    - The values for X and Y are taken depending on the sign of σ_1 and σ_2

$$\begin{cases} X = \begin{cases} X_t & \text{if} \quad \boldsymbol{\sigma}_1 \geq 0 \\ X_c & \text{if} \quad \boldsymbol{\sigma}_1 < 0 \end{cases} \\ Y = \begin{cases} Y_t & \text{if} \quad \boldsymbol{\sigma}_2 \geq 0 \\ Y_c & \text{if} \quad \boldsymbol{\sigma}_2 < 0 \end{cases} \end{cases}$$

- Tsai-Hill criterion (4)
  - Tsai-Hill criterion vs Maximum stress criterion
  - Depending on composite
    - Tsai-hill criterion may or may not give better results than maximum stress
    - One way to improve the criteria is to add more terms

E-Glass-Epoxy



S Tsai, Strength theories of filamentary structures, in Fundamentals Aspects of FRPC, 1968, Wiley, New-York

- Tsai-Wu tensor failure criterion
  - Add terms to the surface
    - Strength parameters in a tensor form
    - Failure surface: $F_i \boldsymbol{\sigma}_i + F_{ij} \boldsymbol{\sigma}_i \boldsymbol{\sigma}_j < 1 \quad (i, j = 1, ..., 6)$
    - Subscripts i,j correspond to Voight notation

$$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_{11},\ ...,\ \boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_{33},\ \boldsymbol{\sigma}_4 = \tau_{23},\ \boldsymbol{\sigma}_5 = \tau_{13},\ \boldsymbol{\sigma}_6 = \tau_{12}$$

    - More experimental parameters required in the general case
  - For an orthotropic composite ply in plane-σ state
    - Plane-σ state: F_3, F_4, F_5, F_{i3}, F_{i4}, F_{i5}, F_{3i}, F_{4i}, F_{5i} disappear
    - Assume no coupling between tensile and shear stress failure parameters
      - $F_{16} = F_{26} = 0$
      - Otherwise the criterion would depend on the sign of shear stress
      - See the remark on F_6 on the next slide

$$F_1 \boldsymbol{\sigma}_1 + F_2 \boldsymbol{\sigma}_2 + F_6 \tau_{12} + F_{11} \boldsymbol{\sigma}_1^2 + F_{22} \boldsymbol{\sigma}_2^2 + F_{66} \tau_{12}^2 + 2 F_{12} \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 < 1$$

    - Linear terms are useful to distinguish traction and compression failure

- Tsai-Wu tensor failure criterion (2)
  - Criterion
    - $F_1\boldsymbol{\sigma}_1 + F_2\boldsymbol{\sigma}_2 + F_6\tau_{12} + F_{11}\boldsymbol{\sigma}_1^2 + F_{22}\boldsymbol{\sigma}_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 < 1$
  - Identification of parameters
    - Only $\boldsymbol{\sigma}_1 = X_t \implies F_1 X_t + F_{11} X_t^2 = 1$ (traction)
    - Only $\boldsymbol{\sigma}_1 = X_c \implies F_1 X_c + F_{11} X_c^2 = 1$ (compression)

$$\implies F_1 = \frac{1}{X_t} + \frac{1}{X_c} \quad \& \quad F_{11} = -\frac{1}{X_t X_c}$$

    - Same for transverse stress

$$\implies F_2 = \frac{1}{Y_t} + \frac{1}{Y_c} \quad \& \quad F_{22} = -\frac{1}{Y_t Y_c}$$

    - Only $\tau_{12} = S \implies F_6 S + F_{66} S^2 = 1$
    - Shear criterion should be independent of the sign of $\tau_{12} \implies F_6 = 0$

$$\implies F_{66} = \frac{1}{S^2}$$

    - F_{12} ?

- Tsai-Wu tensor failure criterion (3)
  - Criterion
    - $F_1\boldsymbol{\sigma}_1 + F_2\boldsymbol{\sigma}_2 + F_6\tau_{12} + F_{11}\boldsymbol{\sigma}_1^2 + F_{22}\boldsymbol{\sigma}_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 < 1$
  - F_{12}:
    - A bi-axial loading is required.
    - Let's choose $\sigma_1 = \sigma_2 = \sigma$: $(F_1 + F_2)\boldsymbol{\sigma} + (F_{11} + F_{22} + 2F_{12})\boldsymbol{\sigma}^2 = 1$
      - As $\begin{cases} F_1 = \frac{1}{X_t} + \frac{1}{X_c} \quad \& \quad F_{11} = -\frac{1}{X_t X_c} \\ F_2 = \frac{1}{Y_t} + \frac{1}{Y_c} \quad \& \quad F_{22} = -\frac{1}{Y_t Y_c} \end{cases}$

$$\Longrightarrow F_{12} = \frac{1}{2\boldsymbol{\sigma}^2}\left[1 - \left(\frac{1}{X_t} + \frac{1}{X_c} + \frac{1}{Y_t} + \frac{1}{Y_c}\right)\boldsymbol{\sigma} + \left(\frac{1}{X_t X_c} + \frac{1}{Y_t Y_c}\right)\boldsymbol{\sigma}^2\right]$$

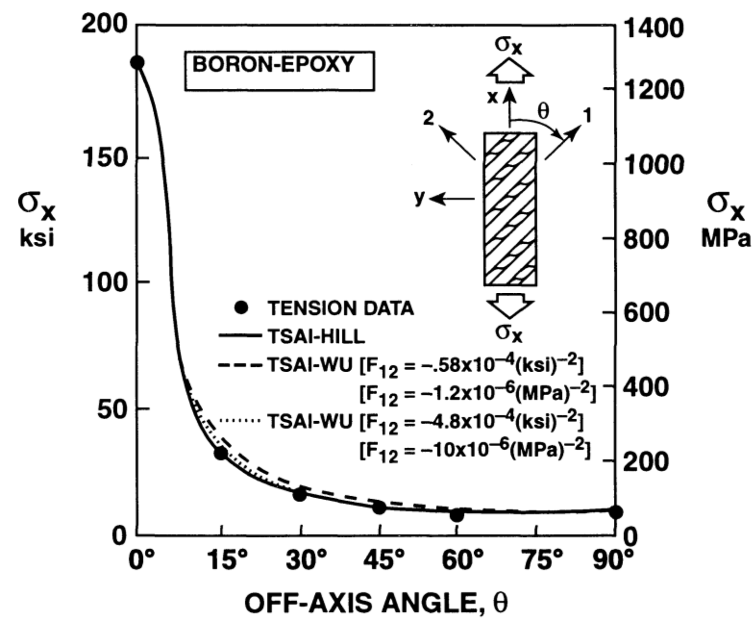
  - F_{12} depends on
    - Tension/compression strength parameters
    - AND σ
    - How to determine it?

- Tsai-Wu tensor failure criterion (4)
  - Criterion
    - $F_1\boldsymbol{\sigma}_1 + F_2\boldsymbol{\sigma}_2 + F_6\tau_{12} + F_{11}\boldsymbol{\sigma}_1^2 + F_{22}\boldsymbol{\sigma}_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 < 1$

with $F_{12} = \frac{1}{2\boldsymbol{\sigma}^2}\left[1 - \left(\frac{1}{X_t} + \frac{1}{X_c} + \frac{1}{Y_t} + \frac{1}{Y_c}\right)\boldsymbol{\sigma} + \left(\frac{1}{X_tX_c} + \frac{1}{Y_tY_c}\right)\boldsymbol{\sigma}^2\right]$

  - Value of $\sigma_1 = \sigma_2 = \sigma$ leading to failure
    - Can be determined experimentally
      - Such tests are expensive
    - Criterion not really sensitive to F_{12}
      - Approximate solution

$$F_{12} \approx -\frac{1}{2}\sqrt{\frac{1}{X_tX_c}\frac{1}{Y_tY_c}}$$



  - Final Tsai-Wu criterion is

$$\left(\frac{1}{X_t} + \frac{1}{X_c}\right)\boldsymbol{\sigma}_1 + \left(\frac{1}{Y_t} + \frac{1}{Y_c}\right)\boldsymbol{\sigma}_2 - \frac{\boldsymbol{\sigma}_1^2}{X_tX_c} - \frac{\boldsymbol{\sigma}_2^2}{Y_tY_c} + \frac{\tau_{12}^2}{S^2} - \sqrt{\frac{1}{X_tX_c}\frac{1}{Y_tY_c}}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 < 1$$

Failure study: Intralaminar fracture

- Remarks on failure criteria
  - The choice of a criterion is not an easy task
    - No one is universal
    - Can lead to good or inaccurate results depending on the loading and on the composite
  - The fracture envelope is constructed by a curve fitting procedure
    - Some fracture points of the envelope are experimentally measured
    - The whole fracture envelope is then fitted assuming (for example) a polynomial shape
    - However, the experimentally measured points correspond to different physical mechanisms, e.g. for a unidirectional ply
      - X_t corresponds to fiber fracture
      - X_c corresponds to fiber buckling
      - Y_t and Y_c correspond to matrix fracture
    - So there is no physical reason to connect these points with a continuous curve
  - However, such criteria are
    - Very simple to use in practice
    - Can give good results

- Extension to 3D
  - To include transverse shearing and normal out of plane stress (if any)
    - Let S_{12}, S_{13}, S_{23} be the shear strength along 12, 13, 23 respectively.
    - The methodology is exactly the same as before
  - Tsai-Hill

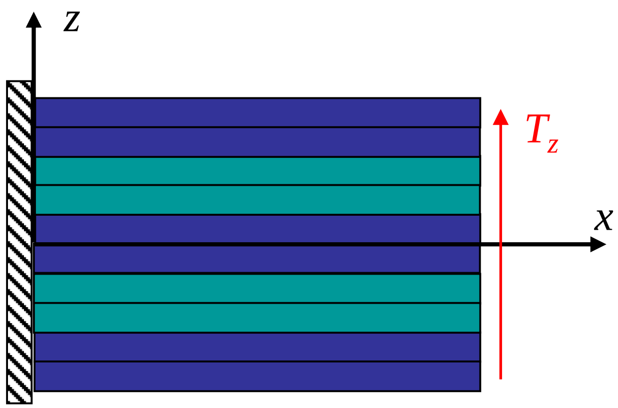
$$\frac{\boldsymbol{\sigma}_1^2}{X^2} + \frac{\boldsymbol{\sigma}_2^2}{Y^2} + \frac{\boldsymbol{\sigma}_3^2}{Z^2} - \left(\frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}\right)\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 - \left(\frac{1}{X^2} + \frac{1}{Z^2} - \frac{1}{Y^2}\right)\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 - \left(\frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2}\right)\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 + \frac{\tau_{23}^2}{S_{23}^2} + \frac{\tau_{13}^2}{S_{13}^2} + \frac{\tau_{12}^2}{S_{12}^2} < 1$$

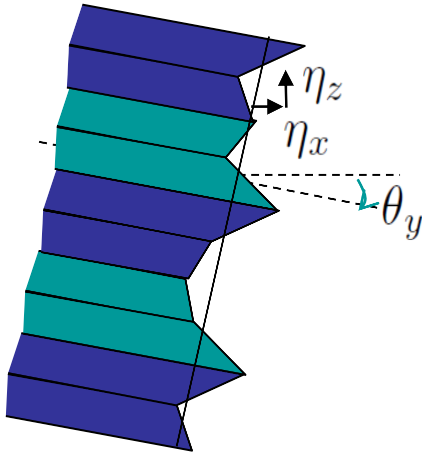
  - Tsai-Wu

$$\left(\frac{1}{X_t} + \frac{1}{X_c}\right)\boldsymbol{\sigma}_1 + \left(\frac{1}{Y_t} + \frac{1}{Y_c}\right)\boldsymbol{\sigma}_2 + \left(\frac{1}{Z_t} + \frac{1}{Z_c}\right)\boldsymbol{\sigma}_3 - \frac{\boldsymbol{\sigma}_1^2}{X_tX_c} - \frac{\boldsymbol{\sigma}_2^2}{Y_tY_c} - \frac{\boldsymbol{\sigma}_3^2}{Z_tZ_c} + \frac{\tau_{23}^2}{S_{23}^2} + \frac{\tau_{13}^2}{S_{13}^2} + \frac{\tau_{12}^2}{S_{12}^2} + 2F_{12}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 + 2F_{13}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 + 2F_{23}\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 < 1$$

    - With $F_{12} \approx -\frac{1}{2}\sqrt{\frac{1}{X_tX_c}\frac{1}{Y_tY_c}}$ $F_{13} \approx -\frac{1}{2}\sqrt{\frac{1}{X_tX_c}\frac{1}{Z_tZ_c}}$ $F_{23} \approx -\frac{1}{2}\sqrt{\frac{1}{Y_tY_c}\frac{1}{Z_tZ_c}}$
  - But how to evaluate the transverse shear σ_{13}, σ_{23} in each ply?

- Transverse shear stress
  - Plane-σ state was considered in each ply of the laminated structure
  - Transverse shear stress
    - Exists
    - Can lead to mode II debonding
    - Should also be considered in failure criterion
  - Example: consider
    - Laminated structure
    - Submit to shear resultant T_z



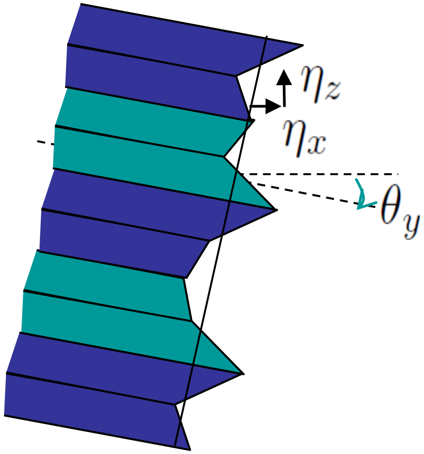


    - Cross section will rotate of a mean angle θ_y
    - Each ply will warp differently (increments η_x and η_z)

Shear stress

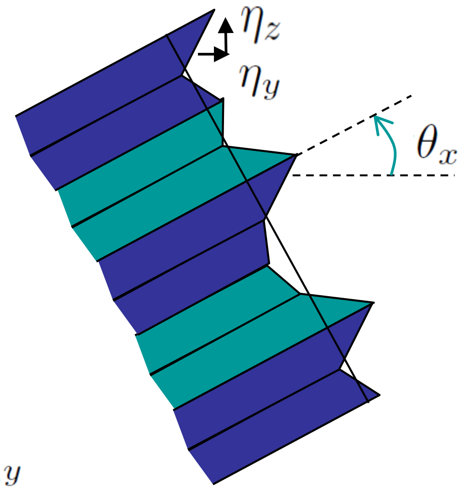
- Transverse shear stress (2)
  - Displacement field

$$\begin{cases} \boldsymbol{u}_x(x,\,y,\,z) = \boldsymbol{u}_x^0 + z\theta_y(x,\,y) + \eta_x(x,\,y,\,z) \\ \boldsymbol{u}_y(x,\,y,\,z) = \boldsymbol{u}_y^0 - z\theta_x(x,\,y) + \eta_y(x,\,y,\,z) \\ \boldsymbol{u}_z(x,\,y,\,z) = \boldsymbol{u}_z^0 + \eta_z(x,\,y,\,z) \end{cases}$$



  - Strain field

$$\begin{cases} \boldsymbol{\varepsilon}_{xx} = \boldsymbol{\varepsilon}_{xx}^0 + z\theta_{y,x} + \cancel{\eta_{x,x}} \\ \boldsymbol{\varepsilon}_{yy} = \boldsymbol{\varepsilon}_{yy}^0 - z\theta_{x,y} + \cancel{\eta_{y,y}} \\ \gamma_{yz} = \underbrace{\boldsymbol{u}_{z,y}^0 - \theta_x}_{\gamma_{yz}^0} + \eta_{y,z} + \cancel{\eta_{z,y}} \\ \gamma_{zx} = \underbrace{\boldsymbol{u}_{z,x}^0 + \theta_y}_{\gamma_{zx}^0} + \eta_{x,z} + \cancel{\eta_{z,x}} \\ \gamma_{xy} = \underbrace{\boldsymbol{u}_{x,y}^0 + \boldsymbol{u}_{y,x}^0}_{\gamma_{xy}^0} + z\left(\theta_{y,y} - \theta_{x,x}\right) + \cancel{\eta_{y,x} + \eta_{x,y}} \end{cases}$$

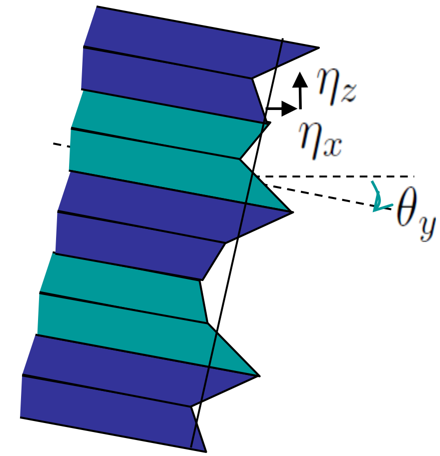


Assume negligible in-plane variation or warping

- Transverse shear stress (3)
  - Strain field (2)

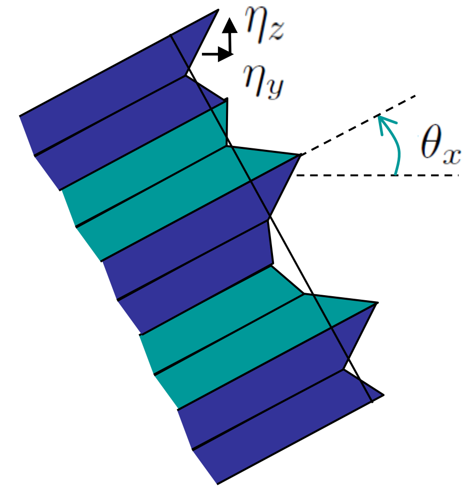
$$\left\{\begin{array}{lllll} \varepsilon_{xx} & = & \varepsilon_{xx}^0 & + & z\theta_{y,x} \\ \varepsilon_{yy} & = & \varepsilon_{yy}^0 & - & z\theta_{x,y} \\ \gamma_{yz} & = & \gamma_{yz}^0 & + & \eta_{y,z} \\ \gamma_{zx} & = & \gamma_{zx}^0 & + & \eta_{x,z} \\ \gamma_{xy} & = & \gamma_{xy}^0 & + & z\left(\theta_{y,y} - \theta_{x,x}\right) \end{array}\right.$$

$\kappa^0{}_{xx}$ $\quad -\kappa^0{}_{yy}$ $\quad \kappa^0{}_{xy}$



$$\Rightarrow \left\{\begin{array}{lllll} \varepsilon_{xx} & = & \varepsilon_{xx}^0 & + & z\kappa_{xx}^0 \\ \varepsilon_{yy} & = & \varepsilon_{yy}^0 & + & z\kappa_{yy}^0 \\ \gamma_{yz} & = & \gamma_{yz}^0 & + & \eta_{y,z} \\ \gamma_{zx} & = & \gamma_{zx}^0 & + & \eta_{x,z} \\ \gamma_{xy} & = & \gamma_{xy}^0 & + & z\kappa_{xy}^0 \end{array}\right.$$

As without shearing



- Transverse shear stress (4)
  - In each plane-σ ply i, we found

$$\begin{pmatrix} \boldsymbol{\sigma}^i_{xx} \\ \boldsymbol{\sigma}^i_{yy} \\ \boldsymbol{\sigma}^i_{xy} \end{pmatrix} = \begin{pmatrix} \mathcal{C}^i_{xxxx} & \mathcal{C}^i_{xxyy} & \mathcal{C}^i_{xxxy} \\ \mathcal{C}^i_{yyxx} & \mathcal{C}^i_{yyyy} & \mathcal{C}^i_{yyxy} \\ \mathcal{C}^i_{xyxx} & \mathcal{C}^i_{xyyy} & \mathcal{C}^i_{xyxy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}^i_{xx} \\ \boldsymbol{\varepsilon}^i_{yy} \\ \gamma^i_{xy} \end{pmatrix}$$

    - Assuming only 0 and 90° plies, and adding out-of-plane shear effect, ply k reads

$$\begin{pmatrix} \boldsymbol{\sigma}^k_{xx} \\ \boldsymbol{\sigma}^k_{yy} \\ \boldsymbol{\sigma}^k_{yz} \\ \boldsymbol{\sigma}^k_{zx} \\ \boldsymbol{\sigma}^k_{xy} \end{pmatrix} = \begin{pmatrix} \mathcal{C}^k_{xxxx} & \mathcal{C}^k_{xxyy} & 0 & 0 & 0 \\ \mathcal{C}^k_{yyxx} & \mathcal{C}^k_{yyyy} & 0 & 0 & 0 \\ 0 & 0 & \mathcal{C}^k_{yzyz} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{C}^k_{xzxz} & 0 \\ 0 & 0 & 0 & 0 & \mathcal{C}^k_{xyxy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}^k_{xx} \\ \boldsymbol{\varepsilon}^k_{yy} \\ \gamma^k_{yz} \\ \gamma^k_{zx} \\ \gamma^k_{xy} \end{pmatrix}$$

    - Stress obtained using

$$\left\{ \begin{array}{lcccl} \boldsymbol{\varepsilon}_{xx} & = & \boldsymbol{\varepsilon}^0_{xx} & + & z\kappa^0_{xx} \\ \boldsymbol{\varepsilon}_{yy} & = & \boldsymbol{\varepsilon}^0_{yy} & + & z\kappa^0_{yy} \\ \gamma_{yz} & = & \gamma^0_{yz} & + & \eta_{y,z} \\ \gamma_{zx} & = & \gamma^0_{zx} & + & \eta_{x,z} \\ \gamma_{xy} & = & \gamma^0_{xy} & + & z\kappa^0_{xy} \end{array} \right.$$

$$\Longrightarrow \boldsymbol{\sigma}^k_{xx} = \mathcal{C}^k_{xxxx}\boldsymbol{\varepsilon}_{xx} + \mathcal{C}^k_{xxyy}\boldsymbol{\varepsilon}_{yy} = \mathcal{C}^k_{xxxx}\left(\boldsymbol{\varepsilon}^0_{xx} + z\kappa^0_{xx}\right) + \mathcal{C}^k_{xxyy}\left(\boldsymbol{\varepsilon}^0_{yy} + z\kappa^0_{yy}\right)$$

- Transverse shear stress (5)
  - Recall plane-σ state relations for a laminated structure

$$\begin{pmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \\ \tilde{m}_{xx} \\ \tilde{m}_{yy} \\ \tilde{m}_{xy} \end{pmatrix} = \begin{pmatrix} A_{xxxx} & A_{xxyy} & A_{xxxy} & B_{xxxx} & B_{xxyy} & B_{xxxy} \\ A_{yyxx} & A_{yyyy} & A_{yyxy} & B_{yyxx} & B_{yyyy} & B_{yyxy} \\ A_{xyxx} & A_{xyyy} & A_{xyxy} & B_{xyxx} & B_{xyyy} & B_{xyxy} \\ B_{xxxx} & B_{xxyy} & B_{xxxy} & D_{xxxx} & D_{xxyy} & D_{xxxy} \\ B_{yyxx} & B_{yyyy} & B_{yyxy} & D_{yyxx} & D_{yyyy} & D_{yyxy} \\ B_{xyxx} & B_{xyyy} & B_{xyxy} & D_{xyxx} & D_{xyyy} & D_{xyxy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}^0_{xx} \\ \boldsymbol{\varepsilon}^0_{yy} \\ \gamma^0_{xy} \\ \kappa^0_{xx} \\ \kappa^0_{yy} \\ \kappa^0_{xy} \end{pmatrix}$$

  - Assume symmetric pile up **AND** only 0 and 90° plies

$$\mathbf{A} = \begin{pmatrix} A_{xxxx} & A_{xxyy} & 0 \\ A_{yyxx} & A_{yyyy} & 0 \\ 0 & 0 & A_{xyxy} \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} D_{xxxx} & D_{xxyy} & 0 \\ D_{yyxx} & D_{yyyy} & 0 \\ 0 & 0 & D_{xyxy} \end{pmatrix}$$

$$\Rightarrow$$

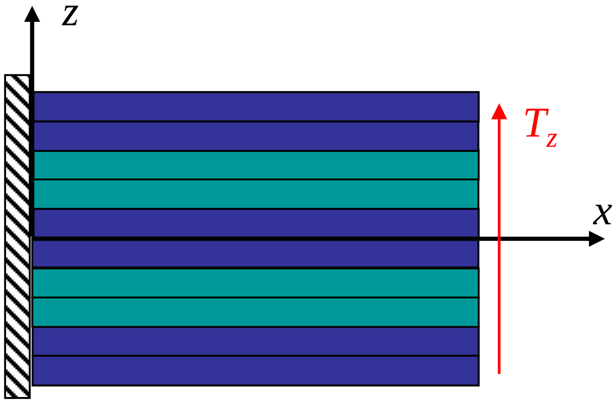
$$\begin{pmatrix} \boldsymbol{\varepsilon}^0_{xx} \\ \boldsymbol{\varepsilon}^0_{yy} \\ \gamma^0_{xy} \end{pmatrix} = \begin{pmatrix} A_{xxxx} & A_{xxyy} & 0 \\ A_{yyxx} & A_{yyyy} & 0 \\ 0 & 0 & A_{xyxy} \end{pmatrix}^{-1} \begin{pmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{pmatrix}$$

$$\begin{pmatrix} \kappa^0_{xx} \\ \kappa^0_{yy} \\ \kappa^0_{xy} \end{pmatrix} = \begin{pmatrix} D_{xxxx} & D_{xxyy} & 0 \\ D_{yyxx} & D_{yyyy} & 0 \\ 0 & 0 & D_{xyxy} \end{pmatrix}^{-1} \begin{pmatrix} \tilde{m}_{xx} \\ \tilde{m}_{yy} \\ \tilde{m}_{xy} \end{pmatrix}$$

- Transverse shear stress (6)
  - Ply k
    - $\boldsymbol{\sigma}^k_{xx} = \mathcal{C}^k_{xxxx}\varepsilon_{xx} + \mathcal{C}^k_{xxyy}\varepsilon_{yy} = \mathcal{C}^k_{xxxx}\left(\varepsilon^0_{xx} + z\kappa^0_{xx}\right) + \mathcal{C}^k_{xxyy}\left(\varepsilon^0_{yy} + z\kappa^0_{yy}\right)$
    - With adequate assumptions (symmetric pile up **AND** only 0 and 90° plies)

$$\begin{pmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{pmatrix} = \begin{pmatrix} A_{xxxx} & A_{xxyy} & 0 \\ A_{yyxx} & A_{yyyy} & 0 \\ 0 & 0 & A_{xyxy} \end{pmatrix}^{-1} \begin{pmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{pmatrix}$$

      - Couple does not introduced tension
    - For problem under consideration
      - No tension
      - No coupling bending/tension
      - Could be added later by superposition



$\Rightarrow$ $\varepsilon^0_{xx} = \varepsilon^0_{yy} = 0 \quad \rightarrow \quad \boldsymbol{\sigma}^k_{xx} = z\left(\mathcal{C}^k_{xxxx}\kappa^0_{xx} + \mathcal{C}^k_{xxyy}\kappa^0_{yy}\right)$

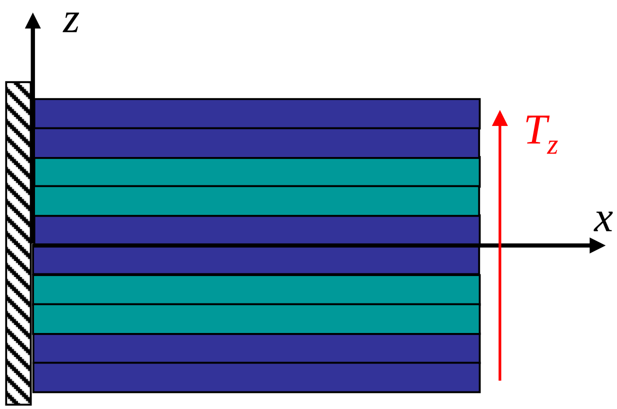
- Transverse shear stress (7)
  - Ply k
    - $\boldsymbol{\sigma}_{xx}^{k} = z\left(\mathcal{C}_{xxxx}^{k}\kappa_{xx}^{0} + \mathcal{C}_{xxyy}^{k}\kappa_{yy}^{0}\right)$
    - As

$$\begin{pmatrix} \kappa_{xx}^{0} \\ \kappa_{yy}^{0} \\ \kappa_{xy}^{0} \end{pmatrix} = \begin{pmatrix} D_{xxxx} & D_{xxyy} & 0 \\ D_{yyxx} & D_{yyyy} & 0 \\ 0 & 0 & D_{xyxy} \end{pmatrix}^{-1} \begin{pmatrix} \tilde{m}_{xx} \\ \tilde{m}_{yy} \\ \tilde{m}_{xy} \end{pmatrix} \Rightarrow \begin{cases} \kappa_{xx}^{0} = \dfrac{D_{yyyy}\tilde{m}_{xx} - D_{xxyy}\tilde{m}_{yy}}{D_{xxxx}D_{yyyy} - D_{xxyy}^{2}} \\ \kappa_{yy}^{0} = \dfrac{D_{xxxx}\tilde{m}_{yy} - D_{xxyy}\tilde{m}_{xx}}{D_{xxxx}D_{yyyy} - D_{xxyy}^{2}} \end{cases}$$

$$\Rightarrow \begin{aligned} \boldsymbol{\sigma}_{xx}^{k} &= z\left(\mathcal{C}_{xxxx}^{k}\kappa_{xx}^{0} + \mathcal{C}_{xxyy}^{k}\kappa_{yy}^{0}\right) \\ &= z\frac{(\mathcal{C}_{xxxx}^{k}D_{yyyy} - \mathcal{C}_{xxyy}^{k}D_{xxyy})\tilde{m}_{xx} + (\mathcal{C}_{xxyy}^{k}D_{xxxx} - \mathcal{C}_{xxxx}^{k}D_{xxyy})\tilde{m}_{yy}}{D_{xxxx}D_{yyyy} - D_{xxyy}^{2}} \end{aligned}$$

- In case of no applied bending $\tilde{m}_{yy}$

$$\boldsymbol{\sigma}_{xx}^{k} = z\frac{\mathcal{C}_{xxxx}^{k}D_{yyyy} - \mathcal{C}_{xxyy}^{k}D_{xxyy}}{D_{xxxx}D_{yyyy} - D_{xxyy}^{2}}\tilde{m}_{xx}$$



- Transverse shear stress (8)
  - Ply k (2)
    - $\boldsymbol{\sigma}_{xx}^{k} = z\frac{\mathcal{C}_{xxxx}^{k} D_{yyyy} - \mathcal{C}_{xxyy}^{k} D_{xxyy}}{D_{xxxx} D_{yyyy} - D_{xxyy}^{2}} \tilde{m}_{xx}$
    - As linear momentum balance reads $\frac{\partial \boldsymbol{\sigma}_{ij}}{\partial x_j} = 0$

$$\Rightarrow \frac{\partial \boldsymbol{\sigma}_{xx}}{\partial x} + \frac{\partial \boldsymbol{\sigma}_{xy}}{\partial y} + \frac{\partial \boldsymbol{\sigma}_{xz}}{\partial z} = 0$$

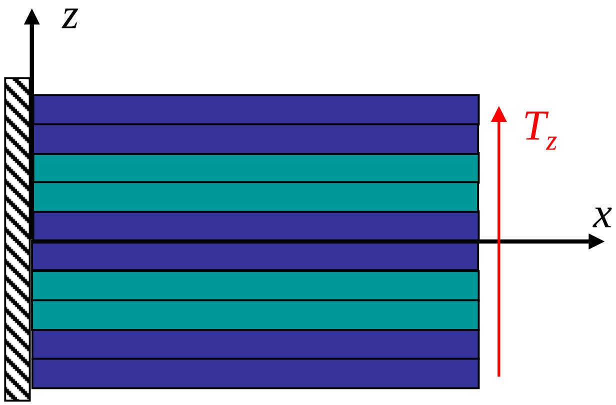
    - Assuming

$$\boldsymbol{\sigma}_{xy} = 0 \Rightarrow \frac{\partial \boldsymbol{\sigma}_{xz}}{\partial z} = -\frac{\partial \boldsymbol{\sigma}_{xx}}{\partial x} = z\frac{\mathcal{C}_{xxyy}^{k} D_{xxyy} - \mathcal{C}_{xxxx}^{k} D_{yyyy}}{D_{xxxx} D_{yyyy} - D_{xxyy}^{2}} \frac{\partial \tilde{m}_{xx}}{\partial x}$$

    - Finally

$$\frac{\partial \tilde{m}_{xx}}{\partial x} = -T_z$$

$$\Rightarrow \frac{\partial \boldsymbol{\sigma}_{xz}}{\partial z} = -z\frac{\mathcal{C}_{xxyy}^{k} D_{xxyy} - \mathcal{C}_{xxxx}^{k} D_{yyyy}}{D_{xxxx} D_{yyyy} - D_{xxyy}^{2}} T_z$$

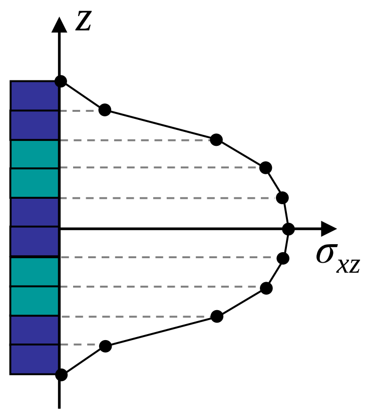


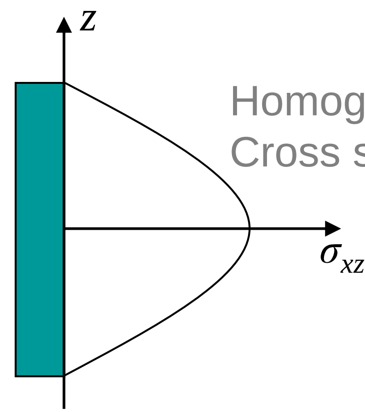
- Transverse shear stress (9)

$$\frac{\partial \boldsymbol{\sigma}_{xz}}{\partial z} = -z \frac{\mathcal{C}^k_{xxyy} D_{xxyy} - \mathcal{C}^k_{xxxx} D_{yyyy}}{D_{xxxx} D_{yyyy} - D^2_{xxyy}} T_z$$

  – Transverse shear distribution

    - By recursive integration on each ply
    - BCs $\begin{cases} \boldsymbol{\sigma}_{xz} = 0 & \text{On the lower and upper faces} \\ \boldsymbol{\sigma}^+_{xz} = \boldsymbol{\sigma}^-_{xz} & \text{Between plies (shear stress continuity)} \end{cases}$
    - Can be very different from the distribution expected for an homogeneous beam





$$\boldsymbol{\sigma}_{xz} = \frac{3 T_x}{2bh} \left[1 - \left(\frac{2z}{h} \right)^2 \right]$$

References

- Lecture notes
  - Lecture Notes on Fracture Mechanics, Alan T. Zehnder, Cornell University, Ithaca, http://hdl.handle.net/1813/3075
- Other references
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    - Fracture Mechanics: Fundamentals and applications, D. T. Anderson. CRC press, 1991
    - Mechanics of composite materials, R.M. Jones, 1999
    - Composite materials: design and applications, D. Gay, S.V. Hoa, S.W. Tsai, CRC press, 2003