Fracture mechanics, Damage & Fatigue (MECA0058-1)
January 2023

First question

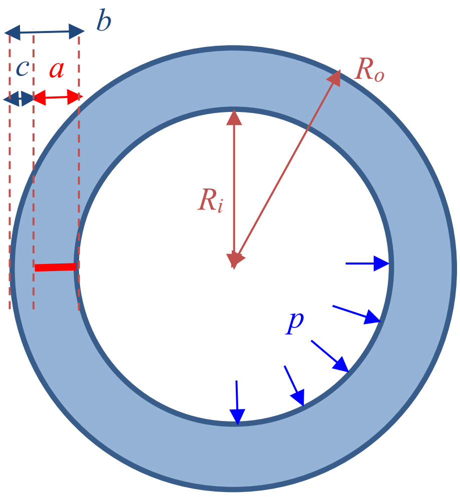

Figure 1: Axially cracked cylinder

Table 1 : Properties of the axially cracked cylinder

Properties	Values
Internal radius R_i	1. m
External radius R_o	1.05 m
Crack a	0.0125 m
Young E	210 GPa
Yield σ_p^0	350 MPa
Poisson ν [-]	0.3
Hardening exponent n	10
Hardening parameter α	1

One wants to assess the safety of a pressurized cylinder in which an extended internal axial crack has been found, see Figure 1. The cylinder is assumed to work in plane strain condition along its axis. The material follows a power law

$$\sigma_e = \sigma_p^0 \left(\frac{E\varepsilon}{\alpha\sigma_p^0}\right)^{\frac{1}{n}} \tag{1}$$

The geometry and material parameters are reported in Table 1.

The formula related to the computation of the elastic SIF reads

$$K_I = \frac{2pR_o^2\sqrt{\pi a}}{(R_o^2 - R_i^2)} F\left(\frac{a}{b}, \frac{R_i}{R_o}\right) \tag{2}$$

where the correction F is tabulated in the non-linear fracture mechanics handbook, see also Table 2.

In the elasto-plastic regime, the limit pressure, is given by

$$p_0 = \frac{2}{\sqrt{3}} \frac{c\sigma_p^0}{R_i + a} \tag{3}$$

The ratio $\frac{p}{p_0}$, with p the applied pressure, is used to evaluate the fraction

$$\eta = \frac{1}{2} \frac{1}{1 + \left(\frac{p}{p_0}\right)^2}, \tag{4}$$

of the plastic zone

$$r_p = \frac{1}{3\pi} \left[\frac{n-1}{n+1}\right] \left(\frac{K_I}{\sigma_p^0}\right)^2 \quad \text{for plane strain state,} \tag{5}$$

that is used to evaluate the effective crack length

$$a_{\mathrm{eff}} = a + \eta r_p. \tag{6}$$

The ratio p over the limit pressure (3) is used to compute the plastic part of the J-integral

$$J_p = \frac{\alpha \sigma_p^{0\,2}}{E} c \frac{a}{b} \; h_1\left(\frac{a}{b}, n; \frac{R_i}{R_o}\right) \left(\frac{p}{p_0}\right)^{n+1} \tag{7}$$

The function h_1 is tabulated in the non-linear fracture mechanics handbook, see also Table 3 to Table 5.

The material exhibits a Ductile Brittle Temperature Transition as shown in Figure 2.

The safety of the pressurized cylinder has to be assessed in two cases:

A) At temperatures lower than -50 degree C;
B) At temperatures higher than 120 degree C.

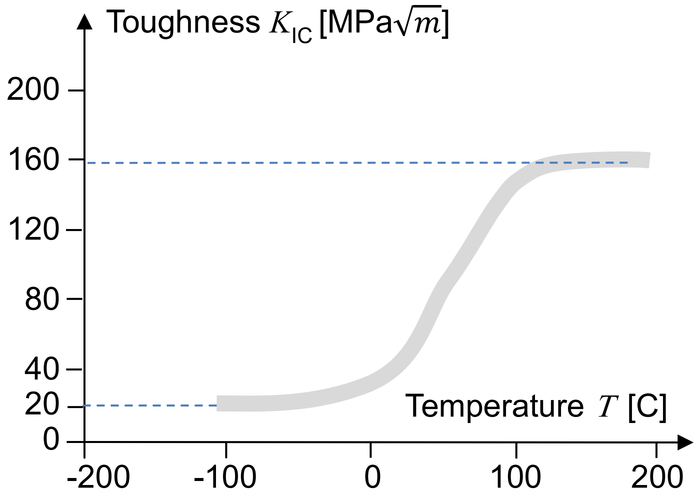

Figure 2: Toughness evolution with temperature

To this end, you are requested for both cases:

A) To evaluate the pressure p leading to crack propagation considering the Linear Elastic Fracture Mechanics framework;
B) To evaluate the pressure p leading to crack propagation considering the Small Scale Yielding solution (Linear Elastic Fracture Mechanics framework corrected with the effective crack size (6)), if needed;
C) Considering that the pressure p found in B) is applied to the sample, using the Non-Linear Fracture Mechanics framework if needed, to evaluate the total J integral and to conclude on the crack initiation;
D) If needed, and based on the solution of C), to correct the pressure found in B) and to reevaluate the total J integral to get closer to the limit pressure (you are not requested to find the exact limit pressure, only one iteration on the pressure is enough);
E) To comment on the validity of the developments.

Table 2 : Tabulated F and V_1 for an axially cracked cylinder

		a/b = 1/8	a/b = 1/4	a/b = 1/2	a/b = 3/4
$b/R_i = 1/5$	F	1.19	1.38	2.10	3.30
	V_1	1.51	1.83	3.44	7.50
$b/R_i = 1/10$	F	1.20	1.44	2.36	4.23
	V_1	1.54	1.91	3.96	10.4
$b/R_i = 1/20$	F	1.20	1.45	2.51	5.25
	V_1	1.54	1.92	4.23	13.5

Table 3 : Tabulated h_1 and h_2 for an axially cracked cylinder with $\frac{b}{R_i} = \frac{1}{5}$

		n = 1	n = 2	n = 3	n = 5	n = 7	n = 10
a/b = 1/8	h_1	6.32	7.93	9.32	11.5	13.12	14.94
	h_2	5.83	7.01	7.96	9.49	10.67	11.96
a/b = 1/4	h_1	7.00	8.34	9.03	9.59	9.71	9.45
	h_2	5.92	8.72	7.07	7.26	7.14	6.71
a/b = 1/2	h_1	9.79	10.37	9.07	5.61	3.52	2.11
	h_2	7.05	6.97	6.01	3.70	2.28	1.25
a/b = 3/4	h_1	11.00	5.54	2.84	1.24	0.83	0.493
	h_2	7.35	3.86	1.86	0.556	0.261	0.129

Table 4 : Tabulated h_1 and h_2 for an axially cracked cylinder with $\frac{b}{R_i} = \frac{1}{10}$

		n = 1	n = 2	n = 3	n = 5	n = 7	n = 10
a/b = 1/8	h_1	5.22	6.64	7.59	8.76	9.34	9.55
	h_2	5.31	6.25	6.88	7.65	8.02	8.09
a/b = 1/4	h_1	6.16	7.49	7.96	8.08	7.78	6.98
	h_2	5.56	6.31	6.52	6.40	6.01	5.27
a/b = 1/2	h_1	10.5	11.6	10.7	6.47	3.95	2.27
	h_2	7.48	7.72	7.01	4.29	2.58	1.37
a/b = 3/4	h_1	16.1	8.19	3.87	1.46	1.05	0.787
	h_2	9.57	5.40	2.57	0.706	0.370	0.232

Table 5 : Tabulated h_1 and h_2 for an axially cracked cylinder with $\frac{b}{R_i} = \frac{1}{20}$

		n = 1	n = 2	n = 3	n = 5	n = 7	n = 10
a/b = 1/8	h_1	4.50	5.79	6.62	7.65	8.07	7.75
	h_2	4.96	5.71	6.20	6.82	7.02	6.66
a/b = 1/4	h_1	5.57	6.91	7.37	7.47	7.21	6.53
	h_2	5.29	5.98	6.16	6.01	5.63	4.93
a/b = 1/2	h_1	10.8	12.8	12.8	8.16	4.88	2.62
	h_2	7.66	8.33	8.13	5.33	3.20	1.65
a/b = 3/4	h_1	23.1	13.1	5.87	1.90	1.23	0.883
	h_2	12.1	7.88	3.84	1.01	0.454	0.240

Second question

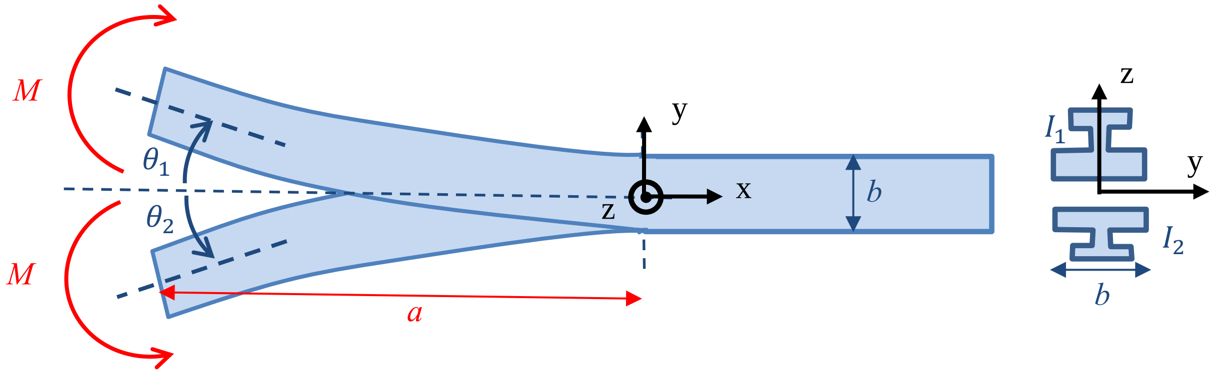

Figure 3: Two-beam system

Two beams of the same material, see Figure 3, with Young modulus E and Poisson ratio ν, and with respective symmetric cross sections of inertia I_1 and I_2, are welded on top of each other on their width b in the plane Oxy. The two beams deformations are unconstrained on a surface $a \times b$ and each beam extremity is submitted to a bending moment $\pm M$ yielding deflection angles θ_1 and $-\theta_2$, around the Oz axis, respectively for each beam, and displacements in the Oxy plane.

We define the J-integral in the Oxz plane for an abscise y as

$$J = \oint_\Gamma \left[U\, n_x - \boldsymbol{u}_{,x} \cdot \boldsymbol{T}\right] dl \tag{1}$$

where Γ is a contour in the Oxz plane encompassing the crack tip, n_x is the component along the x-axis of the outward unit normal to the contour Γ, U is the strain energy per unit volume and $\boldsymbol{T} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$ is the surface traction with $\boldsymbol{\sigma}$ the stress tensor and $\boldsymbol{n}$ the outward unit normal to the contour Γ.

You are requested to evaluate in terms of I_1, I_2, b, M, E, a the average J-integral along the crack front

$$\bar{J} = \frac{1}{b} \int_{\text{width}} J(y)\, dy \tag{2}$$

where the width is along the crack front. To this end, you can proceed as follows

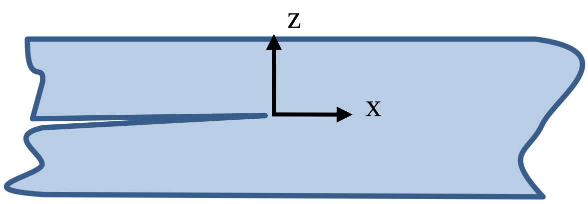

Figure 4: Section for given y-coordinate

A) Consider a coordinate y along the width of the beam, see Figure 4, and define a contour Γ embedding the crack tip.

B) Along the different parts of the contour Γ for a coordinate y, express the internal energy times the normal component along x, Un_x, and the contributions $\sigma_{km}\, n_m$, and $u_{k,x}\, \sigma_{km} n_m$, all in terms of $I_1,\ I_2, b, M, E, a$.
C) Evaluate the generic expression (1) for a coordinate y as integral(s) of parts in terms of $I_1,\ I_2, b, E, a$ (you do not need to perform the integration in this point C).
D) Evaluate the average J-integral (2) in terms of $I_1,\ I_2, b, M, E, a$ (here the integrals should be resolved).
E) To discuss the stability of the crack.
F) In the case of rectangular cross sections $h_1 \times b = 0.01 \times 0.02\ \mathrm{m}^2$ and $h_2 \times b = 0.005 \times 0.02\ \mathrm{m}^2$, for the values of $E = 28$ GPa, $\nu = 0.22$, $a = 0.25$ mm and a critical stress intensity factor of $30\ \mathrm{MPa} \cdot \mathrm{m}^{\frac{1}{2}}$, to evaluate the maximum bending moment M that can be applied.

Reminder of beam bending theory

For a beam under pure bending (constant bending moment M) in the referential α, β, see Figure 5:

1) The curvature κ is constant and is related to the neutral axis deflection u and to the bending moment M through
$$\frac{1}{\kappa} = \frac{\partial^2 u(\alpha)}{\partial \alpha^2} = \frac{M}{EI} = cst \qquad (1)$$
where E is the Young's modulus and I the cross-section inertia.
2) The deflection angle reads
$$\theta(\alpha) = \frac{\partial u(\alpha)}{\partial \alpha} \qquad (2)$$
3) The inertia with the **origin** taken at the **section center of inertia,** reads
$$I = \int_A \beta^2 dA; \qquad (3)$$
4) At any location of the beam, the stress state is uniaxial along the direction α ;
5) The stress distribution of a symmetric profile of uniform Young's modulus, with the **origin** taken at the **section center of inertia,** reads
$$\sigma_{\alpha\alpha} = \frac{M\beta}{I} \qquad (4)$$

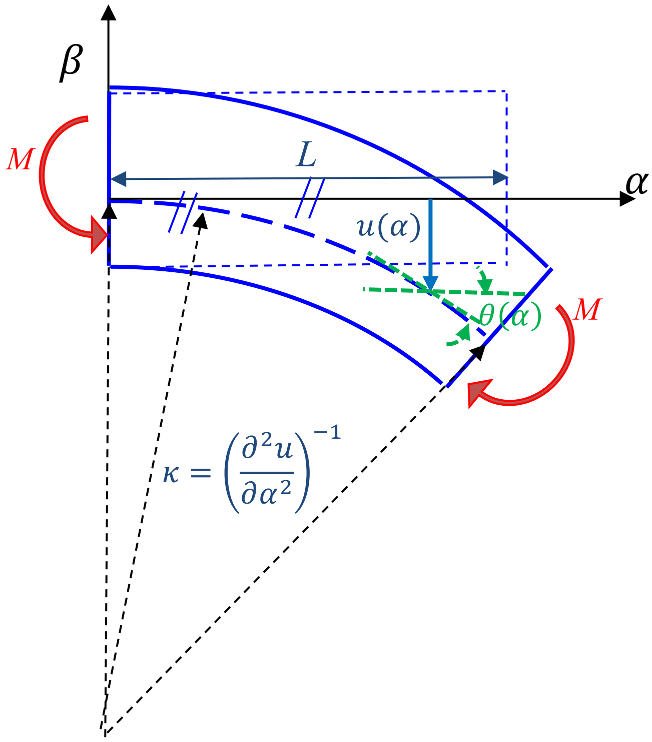

Figure 5: Beam under pure bending

6) The internal energy **per unit beam length** of a symmetric profile of uniform Young's modulus reads
$$e_{int} = \frac{M^2}{2EI} \qquad (5)$$