Aircraft Structures
Structural & Loading Discontinuities

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Elasticity

- **Balance of body** $B$
  - Momenta balance
    - Linear
    - Angular
  - Boundary conditions
    - Neumann
    - Dirichlet

\[
\rho \ddot{x} = b + \nabla \cdot \sigma^T
\]
\[
\rho \ddot{x}_i = b_i + \frac{\partial}{\partial x_j} \sigma_{ij}
\]
\[
\sigma^T = \sigma
\]

- **Small deformations with linear elastic, homogeneous & isotropic material**
  - (Small) Strain tensor $\varepsilon = \frac{1}{2} (\nabla \otimes u + u \otimes \nabla)$, or
  \[
  \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} u_j + \frac{\partial}{\partial x_j} u_i \right)
  \]
  \[
  \varepsilon_{ij} = \frac{1}{2} (u_{j,i} + u_{i,j})
  \]
  - Hooke’s law $\sigma = \mathcal{H} : \varepsilon$, or $\sigma_{ij} = \mathcal{H}_{ijkl} \varepsilon_{kl}$
  \[
  \mathcal{H}_{ijkl} = \frac{E \nu}{(1 + \nu)(1 - 2\nu)} \delta_{ij} \delta_{kl} + \frac{E}{1 + \nu} \left( \frac{1}{2} \delta_{ik} \delta_{jl} + \frac{1}{2} \delta_{il} \delta_{jk} \right)
  \]
  - Inverse law $\varepsilon = \mathcal{G} : \sigma$
  $\lambda = K - 2\mu/3$
  $2\mu$
  \[
  \mathcal{G}_{ijkl} = \frac{1 + \nu}{E} \left( \frac{1}{2} \delta_{ik} \delta_{jl} + \frac{1}{2} \delta_{il} \delta_{jk} \right) - \frac{\nu}{E} \delta_{ij} \delta_{kl}
  \]
Pure bending: linear elasticity summary

- **General expression for unsymmetrical beams**
  - Stress \[ \sigma_{xx} = \kappa E z \cos \alpha - \kappa E y \sin \alpha \]
    
    With \[ \left( \begin{array}{c} \cos \alpha \\ \sin \alpha \end{array} \right) = \frac{\| M_{xx} \|}{\kappa E} \left( \begin{array}{cc} I_{yy} & -I_{yz} \\ -I_{yz} & I_{zz} \end{array} \right)^{-1} \left( \begin{array}{c} \sin \theta \\ -\cos \theta \end{array} \right) \]
  - Curvature
    
    \[ \left( \begin{array}{c} -u_{zx,xx} \\ u_{yy,xx} \end{array} \right) = \frac{\| M_{xx} \|}{E (I_{yy}I_{zz} - I_{yz}I_{yz})} \left( \begin{array}{cc} I_{zz} & I_{yz} \\ I_{yz} & I_{yy} \end{array} \right) \left( \begin{array}{c} \sin \theta \\ -\cos \theta \end{array} \right) \]
  - In the principal axes \( I_{yz} = 0 \)

- **Euler-Bernoulli equation in the principal axis**
  - \[ \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 u_z}{\partial x^2} \right) = f(x) \quad \text{for } x \in [0, L] \]
  - BCs
    
    \[ - \frac{\partial}{\partial x} \left( EI \frac{\partial^2 u_z}{\partial x^2} \right) \bigg|_{0, L} = \bar{T}_z \bigg|_{0, L} \]
    \[ - EI \frac{\partial^2 u_z}{\partial x^2} \bigg|_{0, L} = \bar{M}_{xx} \bigg|_{0, L} \]
  - Similar equations for \( u_y \)
• General relationships

\[
\begin{align*}
  f_z(x) &= -\partial_x T_z = -\partial_{xx} M_y \\
  f_y(x) &= -\partial_x T_y = \partial_{xx} M_z
\end{align*}
\]

• Two problems considered

  – Thick symmetrical section
    - Shear stresses are small compared to bending stresses if \( h/L << 1 \)

  – Thin-walled (unsymmetrical) sections
    - Shear stresses are not small compared to bending stresses
    - Deflection mainly results from bending stresses
    - 2 cases
      - Open thin-walled sections
        » Shear = shearing through the shear center + torque
      - Closed thin-walled sections
        » Twist due to shear has the same expression as torsion
Beam shearing: linear elasticity summary

- Shearing of symmetrical thick-section beams
  - Stress \( \sigma_{zx} = -\frac{T_z S_n(z)}{I_{yy} b(z)} \)
    - With \( S_n(z) = \int_{A^*} z \, dA \)
    - Accurate only if \( h > b \)
  - Energetically consistent averaged shear strain \( \tilde{\gamma} \)
    - \( \tilde{\gamma} = \frac{T_z}{A' \mu} \) with \( A' = \frac{1}{\int_A \frac{S_n^2}{T_{yy} b^2} \, dA} \)
    - Shear center on symmetry axes
  - Timoshenko equations
    - \( \tilde{\gamma} = 2 \tilde{\varepsilon}_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \theta_y + \partial_x u_z \) & \( \kappa = \frac{\partial \theta_y}{\partial x} \)
    - On [0 L]: \( \begin{cases} 
      \frac{\partial}{\partial x} \left( EI \frac{\partial \theta_y}{\partial x} \right) - \mu A' (\theta_y + \partial_x u_z) = 0 \\
      \frac{\partial}{\partial x} (\mu A' (\theta_y + \partial_x u_z)) = -f 
    \end{cases} \)
Beam shearing: linear elasticity summary

- Shearing of open thin-walled section beams
  - Shear flow
    \[ q = \frac{t\tau}{I_{zz}T_z - I_{yz}T_y} \int_0^s tz\,ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2} \int_0^s ty\,ds' \]
  - In the principal axes
    \[ q(s) = -\frac{T_z}{I_{yy}} \int_0^s tz\,ds' - \frac{T_y}{I_{zz}} \int_0^s ty\,ds' \]
  - Shear center \( S \)
    - On symmetry axes
    - At walls intersection
    - Determined by momentum balance
  - Shear loads correspond to
    - Shear loads passing through the shear center &
    - Torque
Shearing of closed thin-walled section beams

- **Shear flow**  \( q = t \tau \)
  
  - \( q(s) = q_o(s) + q(0) \)
  
  - **Open part (for anticlockwise of \( q, s \))**
    \[
    q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{uz}^2} \int_0^s t(s') z(s') \, ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2} \int_0^s t(s') y(s') \, ds'
    \]
  
  - **Constant twist part**
    \[
    q(s = 0) = \frac{y_T T_z - z_T T_y - \int p(s) \, q_o(s) \, ds}{2A_h}
    \]

- The \( q(0) \) is related to the closed part of the section, but there is a \( q_o(s) \) in the open part which should be considered for the shear torque \( \int p(s) \, q_o(s) \, ds \)
Shearing of closed thin-walled section beams

- Warping around twist center $R$
  \[ u_x(s) = u_x(0) + \int_0^s \frac{q}{\mu t} ds - \frac{1}{A_h} \int \frac{q}{\mu t} ds \left\{ A_{CP}(s) + \frac{z_R [y(s) - y(0)] - y_R [z(s) - z(0)]}{2} \right\} \]
  \[ u_x(0) = \frac{\int t u_x(s) ds}{\int t(s) ds} \]
  - $u_x(0) = 0$ for symmetrical section if origin on the symmetry axis

- Shear center $S$
  
  - Compute $q$ for shear passing thought $S$
  
  - Use
  \[ q(s = 0) = \frac{y_T T_z - z_T T_y - \int p(s) q_o(s) ds}{2 A_h} \]

  With point $S = T$
• Torsion of symmetrical thick-section beams
  
  – Circular section
    • $\tau = \mu \gamma = r \mu \theta, x$
    • $C = \frac{M_x}{\theta, x} = \int_A \mu r^2 dA$

  – Rectangular section
    • $\tau_{\text{max}} = \frac{M_x}{\alpha h b^2}$
    • $C = \frac{M_x}{\theta, x} = \beta h b^3 \mu$
    • If $h \gg b$
      - $\tau_{xy} = 0$ \& $\tau_{xz} = 2 \mu y \theta, x$
      - $\tau_{\text{max}} = \frac{3 M_x}{h b^2}$
      - $C = \frac{M_x}{\theta, x} = \frac{h b^3 \mu}{3}$

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<th>$h/b$</th>
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<td>$\beta$</td>
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<td>$\frac{1}{3}$</td>
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</tbody>
</table>
Beam torsion: linear elasticity summary

- Torsion of open thin-walled section beams
  - Approximated solution for twist rate
    - Thin curved section
      \[ \tau_{xs} = 2\mu n \theta_{,x} \]
      \[ C = \frac{M_x}{\theta_{,x}} = \frac{1}{3} \int \mu t^3 ds \]
    - Rectangles
      \[ \tau_{\text{max}_i} = \mu t_i \theta_{,x} \]
      \[ \frac{M_x}{\theta_{,x}} = \sum_i \frac{l_i t_i^3 \mu}{3} \]
  - Warping of \( s \)-axis
    - \( \mathbf{u}_x^s (s) = \mathbf{u}_x^s (0) - \theta_{,x} \int_0^s p_R ds' = \mathbf{u}_x^s (0) - 2A_{Rp} (s) \theta_{,x} \)
• **Torsion of closed thin-walled section beams**
  
  - Shear flow due to torsion \( M_x = 2A_h q \)
  
  - Rate of twist
    
    • \( \theta_x = \frac{M_x}{4A_h^2} \int \frac{1}{\mu t} \, ds \)
  
    • Torsion rigidity for constant \( \mu \)
      
      \[
      I_T = \frac{4A_h^2}{\int \frac{1}{\mu t} \, ds} \leq I_p = \int_A r^2 \, dA
      \]
    
  - Warping due to torsion
    
    • \( u_x(s) = u_x(0) + \frac{M_x}{2A_h} \left[ \int_0^s \frac{1}{\mu t} \, ds - \frac{A_{R_p}(s)}{A_h} \int \frac{1}{\mu t} \, ds \right] \)
    
    • \( A_{R_p} \) from twist center
Structure idealization summary

- Panel idealization
  - Booms’ area depending on loading
    - For linear direct stress distribution

\[
\begin{align*}
A_1 &= \frac{t_D b}{6} \left( 2 + \frac{\sigma_{xx}^2}{\sigma_{xx}^1} \right) \\
A_2 &= \frac{t_D b}{6} \left( 2 + \frac{\sigma_{xx}^1}{\sigma_{xx}^2} \right)
\end{align*}
\]
Consequence on bending

- If Direct stress due to bending is carried by booms only
  - The position of the neutral axis, and thus the second moments of area
    - Refer to the direct stress carrying area only
    - Depend on the loading case only

Consequence on shearing

- Open part of the shear flux
  - Shear flux for open sections

\[ q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2} \left[ \int_0^s t_{\text{direct}} \sigma z \, ds + \sum_{i: s_i \leq s} z_i A_i \right] - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2} \left[ \int_0^s t_{\text{direct}} \sigma y \, ds + \sum_{i: s_i \leq s} y_i A_i \right] \]

Consequence on torsion

- If no axial constraint
  - Torsion analysis does not involve axial stress
  - So torsion is unaffected by the structural idealization
**Virtual displacement**

- In linear elasticity the general formula of virtual displacement reads

\[
\int_0^L \int_A \sigma^{(1)} : \varepsilon dA dx = P^{(1)} \Delta_P
\]

- \( \sigma^{(1)} \) is the stress distribution corresponding to a (unit) load \( P^{(1)} \)
- \( \Delta_P \) is the energetically conjugated displacement to \( P \) in the direction of \( P^{(1)} \) that corresponds to the strain distribution \( \varepsilon \)

- Example bending of semi cantilever beam

\[
\int_0^L \int_A \sigma^{(1)}_{xx} \varepsilon_{xx} dA dx = \Delta_P u
\]

- In the principal axes

\[
\Delta_P u = \frac{1}{EI_{yy}I_{zz}} \int_0^L \left\{ I_{zz} M_y^{(1)} M_y + I_{yy} M_z^{(1)} M_z \right\} dx
\]

- Example shearing of semi-cantilever beam

\[
\int_0^L \int_s q^{(1)} \frac{q}{\mu t} ds dx = T^{(1)} \Delta u = \Delta_T u
\]
• **Previously developed equations**
  
  – Stresses & displacements produced by
    
    • Axial loads
    • Shear forces
    • Bending moments
    • Torsion
  
  – No allowance for constrained warping
    
    • Due to structural or loading discontinuities
    • Example torsion of a built-in beam
      
      – No warping allowed at clamping
  
  – Coupling shearing-bending neglected
    
    • Effect of shear strains on the direct stress
    • Shear strains prevent cross section to remain plane
    • Direct stress predicted by pure bending theory not correct anymore
    • For wing box, shear strains can be important
Limitations of these theories

• These effects can be analyzed on simple problems
  – Problem of axial constraint divided in two parts
    • Shear stress distribution calculated at the built-in section
    • Stress distribution calculated on the beam length for the separate loading cases of bending & torsion
  – Problem related to instabilities as buckling
    • See later

• For more complex problems
  – Finite element simulations required
**Closed-section beam**

- **Shear stress distribution at a built-in end**
  - Idealized or not cross-sections
  - Assume a beam with closed cross-section
    - Center of twist \( R \)
    - Undistorted section of the beam
    - Shear flow, displacements and rotation of the section were found to be
      
      \[
      - \frac{q}{\mu t} = \frac{\partial u_x}{\partial s} + \left[ p - y_R \sin \Psi + z_R \cos \Psi \right] \frac{\partial \theta}{\partial x}
      \]
      
      - With
        \[
        \begin{align*}
        y_R &= -\frac{\partial_x u^C_z}{\partial x \theta} \\
        z_R &= \frac{\partial_x u^C_y}{\partial x \theta}
        \end{align*}
        \]
      
      - At built-in this relation simplifies into
        \[
        \frac{q}{\mu t} = p \frac{\partial \theta}{\partial x} + \frac{\partial u^C_z}{\partial x} \sin \Psi + \frac{\partial u^C_y}{\partial x} \cos \Psi
        \]
Closed-section beam

- Shear stress distribution at a built-in end (2)
  - At built-in shear flux is written
    \[ \frac{q}{\mu t} = p \frac{\partial \theta}{\partial x} + \frac{\partial u^C_y}{\partial x} \sin \Psi + \frac{\partial u^C_z}{\partial x} \cos \Psi \]
  - By equilibrium
    \[ T_y = \int q \cos \Psi \, ds \]
    \[ T_z = \int q \sin \Psi \, ds \]
    \[ y_T T_z - z_T T_y = \int p q \, ds \]
  - After substitution of shear flux
    \[
    \begin{aligned}
    T_y &= \frac{\partial \theta}{\partial x} \int \mu t p \cos \Psi \, ds + \frac{\partial u^C_y}{\partial x} \int \mu t \cos^2 \Psi \, ds + \frac{\partial u^C_z}{\partial x} \int \mu t \cos \Psi \sin \Psi \, ds \\
    T_z &= \frac{\partial \theta}{\partial x} \int \mu t p \sin \Psi \, ds + \frac{\partial u^C_y}{\partial x} \int \mu t \cos \Psi \sin \Psi \, ds + \frac{\partial u^C_z}{\partial x} \int \mu t \sin^2 \Psi \, ds \\
    y_T T_z - z_T T_y &= \frac{\partial \theta}{\partial x} \int \mu t p^2 \, ds + \frac{\partial u^C_y}{\partial x} \int \mu t p \cos \Psi \, ds + \frac{\partial u^C_z}{\partial x} \int \mu t p \sin \Psi \, ds
    \end{aligned}
    \]
• Shear stress distribution at a built-in end (3)
  – New system of 3 equations and 3 unknowns

\[
\begin{align*}
T_y &= \frac{\partial \theta}{\partial x} \int \mu t \cos \Psi ds + \frac{\partial u_y^C}{\partial x} \int \mu t \cos^2 \Psi ds + \frac{\partial u_z^C}{\partial x} \int \mu t \cos \Psi \sin \Psi ds \\
T_z &= \frac{\partial \theta}{\partial x} \int \mu t \sin \Psi ds + \frac{\partial u_y^C}{\partial x} \int \mu t \cos \Psi \sin \Psi ds + \frac{\partial u_z^C}{\partial x} \int \mu t \sin^2 \Psi ds \\
y_T T_z - z_T T_y &= \frac{\partial \theta}{\partial x} \int \mu t p^2 ds + \frac{\partial u_y^C}{\partial x} \int \mu t p \cos \Psi ds + \frac{\partial u_z^C}{\partial x} \int \mu t p \sin \Psi ds
\end{align*}
\]

• Solution of the system: \(\frac{\partial \theta}{\partial x}, \frac{\partial u_y^C}{\partial x}\) & \(\frac{\partial u_z^C}{\partial x}\)

  – This solution is then substituted into

\[
\frac{q}{\mu t} = p \frac{\partial \theta}{\partial x} + \frac{\partial u_y^C}{\partial x} \sin \Psi + \frac{\partial u_y^C}{\partial x} \cos \Psi
\]

• Shear flow and shear stress are then defined

• Remains true for any choice of \(C\) as long as \(p\) is computed from there
Closed-section beam

• Example
  – Built-in end
    • Section with constant shear modulus
  – Shear stress distribution?
  – Center of twist?

\[
\begin{array}{|c|c|c|}
\hline
\text{Wall} & \text{Length (m)} & \text{Thickness (mm)} \\
\hline
\text{AB} & 0.375 & 1.6 \\
\text{BC} & 0.500 & 1.0 \\
\text{CD} & 0.125 & 1.2 \\
\text{DA} & 1.0 & 1.0 \\
\hline
\end{array}
\]

\[T_z = 22 \text{ kN}\]
• **Deformation**
  
  – Sign convention: >0 anticlockwise
  – Angle $\alpha$: $\sin \alpha = 0.25/0.5 \rightarrow \alpha = 30^\circ$
  – Coefficients

\[
\int tp \cos \Psi \, ds = \int_B^C p_A t^{BC} \cos \Psi^{BC} \, ds^{BC} + \int_C^D p_A t^{CD} \cos \Psi^{CD} \, ds^{CD}
\]

\[
\int tp \cos \Psi \, ds = l^{BC} t^{BC} l^{AB} \cos \frac{\pi}{6} \cos \frac{7\pi}{6} + l^{CD} t^{CD} l^{AD} \cos \frac{3\pi}{2}
\]

\[
\int tp \cos \Psi \, ds = -0.5 \times 10^{-3} \times 0.375 \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = -0.14 \times 10^{-3} \, m^3
\]

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</tr>
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</table>
Closed-section beam

• Deformation (2)
  – Coefficients (2)

\[
\int tp \sin \Psi ds = \int_B^C p_A t^{BC} \sin \Psi^{BC} ds^{BC} + \int_C^D p_A t^{CD} \sin \Psi^{CD} ds^{CD}
\]

\[
\int tp \sin \Psi ds = l^{BC} t^{BC} l^{AB} \cos \frac{\pi}{6} \sin \frac{7\pi}{6} + l^{CD} t^{CD} l^{BC} \cos \frac{\pi}{6} \sin \frac{3\pi}{2}
\]

\[
\int tp \sin \Psi ds = -0.5 \times 10^{-3} \times 0.375 \times \frac{\sqrt{3}}{2} \times \frac{1}{2} - 0.125 \times 1.2 \times 10^{-3} \times 0.5 \times \frac{\sqrt{3}}{2}
\]

\[
= -0.15 \times 10^{-3} \text{ m}^3
\]

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</table>
Closed-section beam

- Deformation (3)
  - Coefficients (3)

\[
\int t \cos^2 \Psi ds = \int_A^{B} t^{AB} \cos^2 \Psi^{AB} ds^{AB} + \\
\int_B^{C} t^{BC} \cos^2 \Psi^{BC} ds^{BC} + \\
\int_C^{D} t^{CD} \cos^2 \Psi^{CD} ds^{CD} + \\
\int_D^{A} t^{DA} \cos^2 \Psi^{DA} ds^{DA}
\]

\[
\int t \cos^2 \Psi ds = l^{AB} t^{AB} \cos^2 \frac{\pi}{2} + l^{BC} t^{BC} \cos^2 \frac{7\pi}{6} + \\
l^{CD} t^{CD} \cos^2 \frac{3\pi}{2} + l^{DA} t^{DA} \cos^2 2\pi
\]

\[
= 0.5 10^{-3} \frac{3}{4} + 0.5 \frac{\sqrt{3}}{2} 10^{-3}
\]

\[
= 0.81 10^{-3} \text{ m}^2
\]

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<td>DA</td>
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Closed-section beam

- Deformation (4)
  - Coefficients (4)

\[
\int t \sin^2 \Psi \, ds = \int_A^B t^{AB} \sin^2 \Psi^{AB} \, ds^{AB} +
\int_B^C t^{BC} \sin^2 \Psi^{BC} \, ds^{BC} +
\int_C^D t^{CD} \sin^2 \Psi^{CD} \, ds^{CD} +
\int_D^A t^{DA} \sin^2 \Psi^{DA} \, ds^{DA}
\]

\[
\int t \sin^2 \Psi \, ds = t^{AB} t^{AB} \sin^2 \frac{\pi}{2} + t^{BC} t^{BC} \sin^2 \frac{7\pi}{6} +
-t^{CD} t^{CD} \sin^2 \frac{3\pi}{2} + t^{DA} t^{DA} \sin^2 2\pi
\]

\[
\int t \sin^2 \Psi \, ds = 0.375 \times 1.6 \times 10^{-3} + 0.5 \times 10^{-3} \times \frac{1}{4} +
0.125 \times 1.2 \times 10^{-3} = 0.88 \times 10^{-3} \text{ m}^2
\]

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\[T_z = 22 \text{ kN}\]
Closed-section beam

- Deformation (5)
  - Coefficients (5)

\[
\int t \sin \Psi \cos \Psi \, ds = \int_{A}^{B} t_{AB} \frac{\sin 2\Psi_{AB}}{2} ds_{AB} + \int_{B}^{C} t_{BC} \frac{\sin 2\Psi_{BC}}{2} ds_{BC} + \int_{C}^{D} t_{CD} \frac{\sin 2\Psi_{CD}}{2} ds_{CD} + \int_{D}^{A} t_{DA} \frac{\sin 2\Psi_{DA}}{2} ds_{DA}
\]

\[
\int t \sin \Psi \cos \Psi \, ds = l_{AB} t_{AB} \frac{\sin \pi}{2} + l_{BC} t_{BC} \frac{\sin \frac{7\pi}{3}}{3} + l_{CD} t_{CD} \frac{\sin \frac{3\pi}{2}}{2} + l_{DA} t_{DA} \frac{\sin 4\pi}{2}
\]

\[
\int t \sin \Psi \cos \Psi \, ds = 0.5 \times 10^{-3} \times \frac{\sqrt{3}}{4} = 0.22 \times 10^{-3} \text{ m}^2
\]

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Closed-section beam

- Deformation (6)
  - Coefficients (6)

$$\int tp^2 ds = \int_B^C p_A^2 t_{BC} ds_{BC} + \int_C^D p_A^2 t_{CD} ds_{CD}$$

$$\int tp^2 ds = l_{BC} t_{BC} \left( l_{AB} \cos \frac{\pi}{6} \right)^2 +$$

$$l_{CD} t_{CD} \left( l_{BC} \cos \frac{\pi}{6} \right)^2$$

$$\int tp^2 ds = 0.5 \times 10^{-3} \times 0.375^2 \times \frac{3}{4} +$$

$$0.125 \times 1.2 \times 10^{-3} \times 0.5^2 \times \frac{3}{4} = 0.081 \times 10^{-3} \text{ m}^4$$

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Closed-section beam

- **Deformation (7)**
  - System with origin of the axis at point $A$ ($C \leftrightarrow A$)

  \[ T_y = \frac{\partial \theta}{\partial x} \int \mu t p \cos \Psi \, ds + \frac{\partial u^C_y}{\partial x} \int \mu t \cos^2 \Psi \, ds + \frac{\partial u^C_z}{\partial x} \int \mu t \cos \Psi \sin \Psi \, ds \]

  \[ = -0.14 \times 10^{-3} \, m^3 \mu \frac{\partial \theta}{\partial x} + 0.81 \times 10^{-3} \, m^2 \mu \frac{\partial u^A_y}{\partial x} + 0.22 \times 10^{-3} \, m^2 \mu \frac{\partial u^A_z}{\partial x} = 0 \]

  \[ T_z = \frac{\partial \theta}{\partial x} \int \mu t p \sin \Psi \, ds + \frac{\partial u^C_y}{\partial x} \int \mu t \cos \Psi \sin \Psi \, ds + \frac{\partial u^C_z}{\partial x} \int \mu t \sin^2 \Psi \, ds \]

  \[ = -0.15 \times 10^{-3} \, m^3 \mu \frac{\partial \theta}{\partial x} + 0.22 \times 10^{-3} \, m^2 \mu \frac{\partial u^A_y}{\partial x} + 0.88 \times 10^{-3} \, m^2 \mu \frac{\partial u^A_z}{\partial x} = 22 \times 10^3 \, N \]

  \[ y_T T_z - z_T T_y = \frac{\partial \theta}{\partial x} \int \mu t p^2 \, ds + \frac{\partial u^C_y}{\partial x} \int \mu t p \cos \Psi \, ds + \frac{\partial u^C_z}{\partial x} \int \mu t p \sin \Psi \, ds \]

  \[ = 0.081 \times 10^{-3} \, m^4 \mu \frac{\partial \theta}{\partial x} - 0.14 \times 10^{-3} \, m^3 \mu \frac{\partial u^A_y}{\partial x} - 0.15 \times 10^{-3} \, m^3 \mu \frac{\partial u^A_z}{\partial x} = 2.2 \times 10^3 \, N \cdot m \]
Closed-section beam

- **Deformation (8)**
  - System (2)

  \[
  -0.14 \times 10^{-3} \, \text{m}^3 \mu \frac{\partial \theta}{\partial x} + 0.81 \times 10^{-3} \, \text{m}^2 \mu \frac{\partial u_y^A}{\partial x} + 0.22 \times 10^{-3} \, \text{m}^2 \mu \frac{\partial u_z^A}{\partial x} = 0
  \]

  \[
  \mu \frac{\partial \theta}{\partial x} = 5.79 \, \text{m}^{-1} \mu \frac{\partial u_y^A}{\partial x} + 1.57 \, \text{m}^{-1} \mu \frac{\partial u_z^A}{\partial x}
  \]

  \[
  -0.15 \times 10^{-3} \, \text{m}^3 \mu \frac{\partial \theta}{\partial x} + 0.22 \times 10^{-3} \, \text{m}^2 \mu \frac{\partial u_y^A}{\partial x} + 0.88 \times 10^{-3} \, \text{m}^2 \mu \frac{\partial u_z^A}{\partial x} = 22 \times 10^3 \, \text{N}
  \]

  \[
  -0.65 \times 10^{-3} \, \text{m}^2 \mu \frac{\partial u_y^A}{\partial x} + 0.64 \times 10^{-3} \, \text{m}^2 \mu \frac{\partial u_z^A}{\partial x} = 22 \times 10^3 \, \text{N}
  \]

  \[
  \mu \frac{\partial u_y^A}{\partial x} = 0.98 \mu \frac{\partial u_z^A}{\partial x} - 33.85 \times 10^6 \, \text{N} \cdot \text{m}^{-2}
  \]

  \[
  \mu \frac{\partial \theta}{\partial x} = 7.24 \, \text{m}^{-1} \mu \frac{\partial u_z^A}{\partial x} - 196 \times 10^6 \, \text{N} \cdot \text{m}^{-2}
  \]

  \[
  0.081 \times 10^{-3} \, \text{m}^4 \mu \frac{\partial \theta}{\partial x} - 0.14 \times 10^{-3} \, \text{m}^3 \mu \frac{\partial u_y^A}{\partial x} - 0.15 \times 10^{-3} \, \text{m}^3 \mu \frac{\partial u_z^A}{\partial x} = 2.2 \times 10^3 \, \text{N} \cdot \text{m}
  \]

  \[
  \mu \frac{\partial \theta}{\partial x} = 123 \times 10^6 \, \text{N} \cdot \text{m}^{-3}
  \]

  \[
  \mu \frac{\partial u_y^A}{\partial x} = 9.3 \times 10^6 \, \text{N} \cdot \text{m}^{-2}
  \]

  \[
  \mu \frac{\partial u_z^A}{\partial x} = 44 \times 10^6 \, \text{N} \cdot \text{m}^{-2}
  \]

---

2013-2014 Aircraft Structures - Structural & Loading Discontinuities 28
Shear flux

\[- \frac{q}{\mu t} = p_A \frac{\partial \theta}{\partial x} + \frac{\partial u_z^A}{\partial x} \sin \Psi + \frac{\partial u_y^A}{\partial x} \cos \Psi \]

- Wall AB

\[q^{AB} = t^{AB} \mu \frac{\partial u_z^A}{\partial x} \]

\[q^{AB} = 1.6 \times 10^{-3} \times 44 \times 10^6 = 70 \times 10^3 \text{ N} \cdot \text{m}^{-1} \]

\[\tau^{AB} = \frac{q^{AB}}{t^{AB}} = 44 \text{ MPa} \]

- Wall DA

\[q^{DA} = t^{DA} \mu \frac{\partial u_y^A}{\partial x} \cos 2\pi \]

\[q^{DA} = 10^{-3} \times 9.3 \times 10^6 = 9.3 \times 10^3 \text{ N} \cdot \text{m}^{-1} \]

\[\tau^{DA} = \frac{q^{DA}}{t^{DA}} = \frac{9.3 \times 10^3}{10^{-3}} = 9.3 \text{ MPa} \]
Closed-section beam

- Shear flux (2)

$$- \frac{q}{\mu t} = p_A \frac{\partial \theta}{\partial x} + \frac{\partial u_z^A}{\partial x} \sin \Psi + \frac{\partial u_y^A}{\partial x} \cos \Psi$$

- Wall BC

$$q^{BC} = p_A t^{BC} \mu \frac{\partial \theta}{\partial x} + t^{BC} \mu \frac{\partial u_z^A}{\partial x} \sin \Psi^{BC} + t^{BC} \mu \frac{\partial u_y^A}{\partial x} \cos \Psi^{BC}$$

$$q^{BC} = l^{AB} \cos \frac{\pi}{6} t^{BC} \mu \frac{\partial \theta}{\partial x} +$$

$$t^{BC} \sin \frac{7\pi}{6} \mu \frac{\partial u_z^A}{\partial x} + t^{BC} \cos \frac{7\pi}{6} \mu \frac{\partial u_y^A}{\partial x}$$

$$q^{BC} = 0.375 \sqrt{\frac{3}{2}} 10^{-3} \cdot 123 \cdot 10^6 -$$

$$10^{-3} \frac{1}{2} 44 \cdot 10^6 - 10^{-3} \frac{\sqrt{3}}{2} 9.3 \cdot 10^6$$

$$= 9.9 \cdot 10^3 \text{ N} \cdot \text{m}^{-1}$$

$$\tau^{BC} = \frac{q^{BC}}{t^{BC}} = \frac{9.9 \cdot 10^3}{10^{-3}} = 9.9 \text{ MPa}$$

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Closed-section beam

• Shear flux (3)

\[ \frac{q}{\mu t} = p_A \frac{\partial \theta}{\partial x} + \frac{\partial u^A_z}{\partial x} \sin \Psi + \frac{\partial u^A_y}{\partial x} \cos \Psi \]

– Wall CD

\[ q^{CD} = p_A t^{CD} \mu \frac{\partial \theta}{\partial x} + t^{CD} \mu \frac{\partial u^A_z}{\partial x} \sin \Psi^{CD} + t^{CD} \mu \frac{\partial u^A_y}{\partial x} \cos \Psi^{CD} \]

\[ q^{CD} = t^{BC} \cos \frac{\pi}{6} t^{CD} \mu \frac{\partial \theta}{\partial x} + t^{CD} \sin \frac{3\pi}{2} \mu \frac{\partial u^A_z}{\partial x} \]

\[ q^{CD} = 0.5 \frac{\sqrt{3}}{2} 1.2 \times 10^{-3} 123 \times 10^6 - 1.2 \times 10^{-3} 44 \times 10^6 = 11.1 \times 10^3 \text{ N} \cdot \text{m}^{-1} \]

\[ \tau^{CD} = \frac{q^{CD}}{t^{CD}} = \frac{11.1 \times 10^3}{1.2 \times 10^{-3}} = 9.3 \text{ MPa} \]

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Center of twist
- System linked to point A

\[ y_R = -\frac{\partial_x u_x}{\partial_x \theta} = -\frac{44 \times 10^6}{123 \times 10^6} = -0.36 \text{ m} \]

\[ z_R = \frac{\partial_x u_y}{\partial_x \theta} = \frac{9.3 \times 10^6}{123 \times 10^6} = 0.076 \text{ m} \]

Remarks
- The center of twist
  - Depends on loading \((y_T \text{ and } T)\)
  - Does not correspond to the center of shear
  - Due to the warping constrain
- Shear flux discontinuity at corners
  - Requires booms in order of avoiding stress concentrations

### Wall Specifications

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\(T_z = 22 \text{ kN}\)
Closed-section beam

- Thin walled rectangular-section beam subjected to torsion
  - In the case of free warping, we found
    \[
    \begin{align*}
    u^A_{x} &= u^C_{x} = \frac{M_x z}{8 \mu hb} \left( \frac{h}{t_h} - \frac{b}{t_b} \right) \\
    u^B_{x} &= u^D_{x} = \frac{M_x z}{8 \mu hb} \left( \frac{b}{t_b} - \frac{h}{t_h} \right) \\
    \theta_{,x} &= \frac{M_x}{2 \mu h^2 b^2} \left( \frac{h}{t_h} + \frac{b}{t_b} \right)
    \end{align*}
    \]
  - If warping is constrained (built-in end)
    - Direct stress are introduced
    - Different shear stress distribution
Closed-section beam

• Thin walled rectangular-section beam subjected to torsion (2)
  – Idealization
    • Warping to be suppressed is linear & symmetrical
      \[ \text{Direct stress also linear & symmetrical} \]
    • Idealization
      – Four identical booms carrying direct stress only
        \[ A = \frac{bt_b}{6} (2 - 1) + \frac{ht_h}{6} (2 - 1) \]
        \[ A = \frac{bt_b + ht_h}{6} \]
      – Panels carry shear flux only

\[ A_1 = \frac{t_D b}{6} \left( 2 + \frac{\sigma_{xx}^2}{\sigma_{xx}^1} \right) \]
\[ A_2 = \frac{t_D b}{6} \left( 2 + \frac{\sigma_{xx}^1}{\sigma_{xx}^2} \right) \]
Thin walled rectangular-section beam subjected to torsion (3)

- Warping at a given section
  - Shearing (see beam lecture)
    \[
    \left\{
    \begin{align*}
    q &= \tau t = \mu t \gamma \\
    \gamma &= 2\varepsilon_{xs} = \frac{\partial u_s}{\partial x} + \frac{\partial u_x}{\partial s}
    \end{align*}
    \right.
    \Rightarrow
    q = \mu t (u_{s,x} + u_{x,s})
    \]
  - Warping
    - If $u_x^m$ is the maximum warping
    - On webs
      \[u_{x,s} = u_{x,z} = \frac{u_x^m}{h/2}\]
    - On covers
      \[u_{x,s} = -u_{x,y} = -\frac{u_x^m}{b/2}\]
• **Thin walled rectangular-section beam subjected to torsion (4)**
  – Warping of a given section (2)
    • Kinematics
      – See lecture on beams
        \[ \delta u_s = p_R \delta \theta \]
    • As twist center is at section center (by symmetry)
      – On webs
        \[ u_{s,x} = \frac{b}{2} \theta_x \]
      – On covers
        \[ u_{s,x} = \frac{h}{2} \theta_x \]
  – Combining results
    • On webs
      \[ q_h = \mu t_h \left( \frac{b}{2} \theta_x + \frac{2}{h} u^m_x \right) \]
    • On covers
      \[ q_b = \mu t_b \left( \frac{h}{2} \theta_x - \frac{2}{b} u^m_x \right) \]
**Closed-section beam**

- **Thin walled rectangular-section beam subjected to torsion (5)**
  - **Torque**
    - From shear flow \( q_h \) & \( q_b \)
      
      \[
      M_x = \int q_p C \, ds = 2 \frac{h}{2} q_b b + 2 \frac{b}{2} q_h h
      \]
      
      \[
      \Rightarrow M_x = bh (q_b + q_h)
      \]
    - Using
      
      \[
      \begin{align*}
      q_h &= \mu t_h \left( \frac{b}{2} \theta_{,x} + \frac{2}{h} u^m_x \right) \\
      q_b &= \mu t_b \left( \frac{h}{2} \theta_{,x} - \frac{2}{b} u^m_x \right)
      \end{align*}
      \]
      
      \[
      \Rightarrow M_x = \mu t_h \left( \frac{b^2 h}{2} \theta_{,x} + 2 b u^m_x \right) + \mu t_b \left( \frac{bh^2}{2} \theta_{,x} - 2 h u^m_x \right)
      \]
  - Twist rate is directly obtained
    
    \[
    \theta_{,x} = \frac{2 M_x}{\mu t_h b^2 h + \mu t_b h^2 b} + \frac{4 u^m_x (t_b h - t_h b)}{t_h b^2 h + t_b h^2 b}
    \]
Closed-section beam

- Thin walled rectangular-section beam subjected to torsion (6)
  - Shear flows
    - From shear flow $q_h$ & $q_b$
      
      \[
      q_h = \mu t_h \left( \frac{b}{2} \theta_{,x} + \frac{2}{h} u^{m}_x \right) \\
      q_b = \mu t_b \left( \frac{h}{2} \theta_{,x} - \frac{2}{b} u^{m}_x \right)
      \]
    - Using
      
      \[
      \theta_{,x} = \frac{2M_x}{\mu t_h b^2 h + \mu t_b h^2 b} + \frac{4u^{m}_x (t_b h - t_h b)}{t_h b^2 h + t_b h^2 b}
      \]
      
      \[
      \begin{align*}
      q_h &= \frac{M_x t_h}{t_h b h + t_b h^2} + u^{m}_x \frac{4\mu t_h t_b}{t_h b + t_b h} \\
      q_b &= \frac{M_x t_b}{t_h b^2 + t_b h b} - u^{m}_x \frac{4\mu t_b t_h}{t_b h + t_h b}
      \end{align*}
      \]
  - Missing balance equation is obtained from boom balance
Closed-section beam

- Thin walled rectangular-section beam subjected to torsion (7)
  - Boom (of section A) balance equation
    
    \[ (\sigma_{xx} + \partial_x \sigma_{xx} \delta x) A - \sigma_{xx} A + q_b \delta x - q_h \delta x = 0 \]
    
    \[ A \partial_x \sigma_{xx} + q_b - q_h = 0 \]

- As boom carries direct stress only
  
  \[ \sigma_{xx} = E \partial_x u^m_x \]
  
  \[ E A \frac{\partial^2 u^m_x}{\partial x^2} + q_b - q_h = 0 \]

- With

  \[ q_h = \frac{M_x t_h}{t_h b h + t_b h^2} + u^m_x \frac{4 \mu t_h t_b}{t_h b + t_b h} \]

  \[ q_b = \frac{M_x t_b}{t_h b^2 + t_b h b} - u^m_x \frac{4 \mu t_b t_h}{t_b h + t_h b} \]

  \[ E A \frac{\partial^2 u^m_x}{\partial x^2} + \frac{M_x}{hb} \frac{t_b h - t_h b}{t_h b + t_b h} - \frac{8 \mu t_h t_b}{t_h b + t_b h} u^m_x = 0 \]
Closed-section beam

- Thin walled rectangular-section beam subjected to torsion (8)
  - Differential equation
    \[
    \frac{\partial^2 u_x^m}{\partial x^2} - w^2 u_x^m = -\frac{M_x}{EAhb} \frac{t_b h - t_h b}{t_h b + t_b h} \quad \text{with} \quad w^2 = \frac{1}{EA} \frac{8\mu t_h t_b}{t_h b + t_b h}
    \]
  - Solution
    - General form
      \[
      u_x^m(x) = C_1 \cosh wx + C_2 \sinh wx + \frac{M_x}{8\mu hb} \frac{t_b h - t_h b}{t_h t_b}
      \]
    - Boundary conditions at \( x = 0 \) (constraint warping)
      \[
      u_x^m(0) = C_1 + \frac{M_x}{8\mu hb} \frac{t_b h - t_h b}{t_h t_b} = 0 \quad \implies \quad C_1 = -\frac{M_x}{8\mu hb} \frac{t_b h - t_h b}{t_h t_b}
      \]
    - Boundary conditions at \( x = L \) (free edge)
      \[
      \partial_x u_x^m(L) = wC_1 \sinh wL + wC_2 \cosh wL = 0
      \]
      \[
      \implies C_2 = -C_1 \tanh wL = \frac{M_x}{8\mu hb} \frac{t_b h - t_h b}{t_h t_b} \tanh wL
      \]
    - Final form
      \[
      u_x^m(x) = \frac{M_x}{8\mu hb} \frac{t_b h - t_h b}{t_h t_b} (1 + \tanh wL \sinh wx - \cosh wx)
      \]
      \[
      \implies u_x^m(x) = \frac{M_x}{8\mu hb} \frac{t_b h - t_h b}{t_h t_b} \left( 1 - \frac{\cosh (wL - wx)}{\cosh wL} \right)
      \]
Closed-section beam

- Thin walled rectangular-section beam subjected to torsion (9)
  - Warping
    \[ u^m_x(x) = \frac{M_x}{8\mu hb} \frac{t_b h - t_h b}{t_h t_b} \left( 1 - \frac{\cosh (wL - wx)}{\cosh wL} \right) \]

- At free end: \[ u^L_x = u^m_x(L) = \frac{M_x}{8\mu hb} \frac{t_b h - t_h b}{t_h t_b} \left( 1 - \frac{1}{\cosh wL} \right) \]

- To be compared with the warping of the free-free beam

  - \[ u^A_x = u^C_x = \frac{M_x z}{8\mu hb} \left( \frac{h}{t_h} - \frac{b}{t_b} \right) \]
  - Same for \( L \to \infty \)
Closed-section beam

- Thin walled rectangular-section beam subjected to torsion (10)
  - Direct stress in booms
    \[ \sigma_{xx} = E \partial_x u_x^m = wE \frac{M_x}{8\mu hb} \frac{t_h h - t_h b}{t_h t_b} \frac{\sinh (wL - wx)}{\cosh wL} \]
  - Direct load in booms
    \[ P_x = A \sigma_{xx} = wEA \frac{M_x}{8\mu hb} \frac{t_h h - t_h b}{t_h t_b} \frac{\sinh (wL - wx)}{\cosh wL} \]
Closed-section beam

- Thin walled rectangular-section beam subjected to torsion (11)
  - Shear flow
    - Using
      \[ u^m_x(x) = \frac{M_x}{8\mu hb} \frac{t_b h - t_h b}{t_h t_b} \left( 1 - \frac{\cosh(wL - wx)}{\cosh wL} \right) \]
    - The shear flows becomes

\[
\begin{align*}
q_h &= \frac{M_x t_h}{t_h b h + t_b h^2} + u^m_x \frac{4\mu t_h t_b}{t_h b + t_b h} \\
q_b &= \frac{M_x t_b}{t_h b^2 + t_b h b} - u^m_x \frac{4\mu t_b t_h}{t_b h + t_h b} \\
q_h &= \frac{M_x}{2hb} \left( 1 + \frac{t_h b - t_b h}{t_h b + t_b h} \frac{\cosh(wL - wx)}{\cosh wL} \right) \\
q_b &= \frac{M_x}{2hb} \left( 1 + \frac{t_b h - t_h b}{t_h b + t_b h} \frac{\cosh(wL - wx)}{\cosh wL} \right)
\end{align*}
\]
Closed-section beam

- Thin walled rectangular-section beam subjected to torsion (12)

\[
\begin{align*}
\tau_h &= \frac{q_h}{t_h} = \frac{M_x}{2hbt_h} \left( 1 + \frac{t_h b - t_b h \cosh (wL - wx)}{t_h b + t_b h} \frac{\cosh wL}{\cosh wL} \right) \\
\tau_b &= \frac{q_b}{t_b} = \frac{M_x}{2hbt_b} \left( 1 + \frac{t_b h - t_h b \cosh (wL - wx)}{t_h b + t_b h} \frac{\cosh wL}{\cosh wL} \right)
\end{align*}
\]
Closed-section beam

- Thin walled rectangular-section beam subjected to torsion (13)
  - Rate of twist
    - Using
      \[ u^m_x(x) = \frac{M_x}{8\mu bh} \left( \frac{t_b h - t_h b}{t_h t_b} \right) \left( 1 - \frac{\cosh(wL - wx)}{\cosh wL} \right) \]
    - The rate of twist becomes
      \[ \theta_{,x} = \frac{2M_x}{\mu t_h b^2 h + \mu t_b h^2 b} + \frac{4u^m_x(t_b h - t_h b)}{t_h b^2 h + t_b h^2 b} \]
      \[ \theta_{,x} = \frac{M_x}{2\mu b^2 h^2 t_h t_b} \left[ t_b h + t_h b - \left( \frac{t_b h - t_h b}{t_b h + t_h b} \right)^2 \frac{\cosh(wL - wx)}{\cosh wL} \right] \]
    - To be compared with the unconstraint theory
      \[ \theta_{,x} = \frac{M_x}{2\mu h^2 b^2} \left( \frac{h}{t_h} + \frac{b}{t_b} \right) \]
    - Constraint reduces the twist rate
• **Problem of axial constraint**
  – In previous example the twist center was known by symmetry
  – In the general case
    • Twist center differs from shear center due to axial constraint
    • Proceed by increment of $\Delta L$
      – Shear stress distribution calculated at the built-in section
        » As in first example
        » Allows determination of the twist center AT THAT SECTION
      – Use the previously developed theory on $\Delta L$
      – New stress distribution on the new section
        » New twist center
        » …
Closed-section beam

- **Shear lag**
  - Beam shearing
    - Shear strain in cross-section
    - Deformation of cross-section
    - Elementary theory of bending
      - For pure bending
      - Not valid anymore due to cross section deformation
    - New distribution of direct stress
  - For wings
    - Wide & thin walled beam
    - Shear distortion of upper and lower skins causes redistribution of stress in the stringers
• Example
  – Assumptions
    • Doubly symmetrical 6-boom beam
    • Shear load through shear center $\rightarrow$ No twist $\rightarrow$ No warping due to twist
    • Uniform panel thickness $t$
    • Shear loads applied at corner booms
• **Shear lag (2)**
  
  - For a given section
    
    - Uniform shear flow between booms
    - Shear flow in web should balance the shear load
      
      \[ q_h = \frac{T_z}{2h} \]
    
    - Corner booms subjected to opposite loads \( P^1 \), with, by equilibrium
      
      \[ P^1 + \partial_x P^1 \delta x - P^1 - q_h \delta x + q_d \delta x = 0 \]
      
      \[ \partial_x P^1 = \frac{T_z}{2h} - q_d \]
    
    - Equilibrium of central boom
      
      - Due to symmetric distribution of \( q_d \)
      
      \[ P^2 + \partial_x P^2 \delta x - P^2 - 2q_d \delta x = 0 \]
      
      \[ \partial_x P^2 = 2q_d \]
• Shear lag (3)
  – For a given section (2)
    • Equilibrium of the cover
      – At the free end
        \[ 2P^1 + P^2 + 2qh (L - x) = 0 \]
        \[ \Rightarrow 2P^1 + P^2 = \frac{T_z}{h} (x - L) \]
    • Summary
      – \[ \partial_x P^1 = \frac{T_z}{2h} - q_d \]
      – \[ \partial_x P^2 = 2q_d \]
      – \[ 2P^1 + P^2 = \frac{T_z}{h} (x - L) \]
        – Third equation is the integration of the first two
        – 3 unknowns so one equation is missing
    • Compatibility
Closed-section beam

- Shear lag (4)
  - Deformations of top cover

\[
(1 + \varepsilon^1_{xx}) \delta x = (1 + \varepsilon^2_{xx}) \delta x + d(\gamma_{xy} + \partial_x \gamma_{xy} \delta x) - d\gamma_{xy}
\]

\[
\partial_x \gamma_{xy} = \frac{\varepsilon^1_{xx} - \varepsilon^2_{xx}}{d} = \frac{P^1}{dEA^1} - \frac{P^2}{dEA^2}
\]

- As \( q_d = -\mu t \gamma_{xy} \)

\[
-\frac{1}{\mu t} \partial_x q_d = \frac{P^1}{dEA^1} - \frac{P^2}{dEA^2}
\]
• Shear lag (5)

  – Equations

    • \[ \partial_x P^1 = \frac{T_z}{2h} - q_d \]

    • \[ \partial_x P^2 = 2q_d \]

    • \[ 2P^1 + P^2 = \frac{T_z}{h} (x - L) \]

    • \[-\frac{1}{\mu t} \partial_x q_d = \frac{P^1}{dEA^1} - \frac{P^2}{dEA^2} \]

    \[ -\frac{1}{2\mu t} \partial_{xx} P^2 = \frac{T_z}{h} (x - L) - P^2 \]

    \[ -\frac{1}{2\mu t} \partial_{xx} P^2 = \frac{T_z}{h} (x - L) - P^2 \]

    \[ -\frac{2\mu t}{dE} \left( \frac{1}{A^2} + \frac{1}{2A^1} \right) P^2 = \frac{2\mu t T_z (L - x)}{2hdEA^1} \]

  – General solution

    • \[ P^2 = C_1 \cosh w (L - x) + C_2 \sinh w (L - x) - \frac{T_z (L - x)}{h \left( \frac{2A^1}{A^2} + 1 \right)} \]

    with \[ w^2 = \frac{2\mu t}{dE} \left( \frac{1}{A^2} + \frac{1}{2A^1} \right) \]
• **Shear lag (6)**
  
  – **General solution**
    
    \[ P^2 = C_1 \cosh w(L - x) + C_2 \sinh w(L - x) - \frac{T_z(L - x)}{h\left(\frac{2A_1}{A^2} + 1\right)} \]

  – **Boundary conditions**
    
    • Zero axial load at \(x = L\) \(\implies C_1 = 0\)
    
    • Zero shear deformation at \(x = 0\)
    
    – As \(\partial_x P^2 = 2q_d\) & \(q_d = -\mu t \gamma_{xy}\)
      
      \[ \partial_x P^2 (x = 0) = -C_2 w \cosh wL + \frac{T_z}{h\left(\frac{2A_1}{A^2} + 1\right)} = 0 \]
      
      \[ \implies C_2 = \frac{T_z}{wh \cos wL \left(\frac{2A_1}{A^2} + 1\right)} \]

  – **Booms direct loadings**
    
    • \(P^2 = -\frac{T_z}{h\left(\frac{2A_1}{A^2} + 1\right)}\left(L - x - \frac{\sinh w(L - x)}{w \cosh wL}\right)\)

    • \(P^1 = \frac{T_z}{2h} (x - L) - \frac{P^2}{2} = \frac{T_z}{2h\left(\frac{2A_1}{A^2} + 1\right)}\left(\frac{2A_1}{A^2} (x - L) - \frac{\sinh w(L - x)}{w \cosh wL}\right)\)
Closed-section beam

- Shear lag (7)
  - Direct load in top cover
    - \( \sigma^2 = \frac{P^2}{A^2} = -\frac{T_z}{h \left(2A^1 + A^2\right)} \left(L - x - \frac{\sinh w (L - x)}{w \cosh wL}\right) \)
    - \( \sigma^1 = \frac{P^1}{A^1} = -\frac{T_z}{h \left(2A^1 + A^2\right)} \left((L - x) + \frac{A^2}{2A^1} \frac{\sinh w (L - x)}{w \cosh wL}\right) \)
  - Pure bending theory leads to
    - \( \sigma^1 = \sigma^2 = -\frac{T_z}{h \left(2A^1 + A^2\right)} (L - x) \)
  - Compared to pure bending theory
    - Compression in central boom is lower
    - Compression in corner boom is higher
Closed-section beam

- Shear lag (8)
  - Shearing of top cover
    - As $\partial_x P^2 = 2q_d$
    
    $$q_d = \frac{\partial_x P^2}{2} = \frac{T_z}{2h \left( \frac{2A_1}{A_2} + 1 \right)} \left( 1 - \frac{\cosh w(L - x)}{\cosh wL} \right)$$
  
  - Deformation of top cover
  
    $$\gamma_{xy} = -\frac{q_d}{\mu t} = -\frac{T_z}{2h \mu t \left( \frac{2A_1}{A_2} + 1 \right)} \left( 1 - \frac{\cosh w(L - x)}{\cosh wL} \right)$$
• Shear lag (9)
  - Remark
    • The solution depends on BCs
    • For a realistic wing structure, intermediate stringers have different BCs
Open-section beam

- **I-section beam subjected to torsion without built-in end**
  - Reminder
    - Shear
      - \( \tau_{xs} = 2\mu n \theta_{,x} \)
    - \( C = \frac{M_x}{\theta_{,x}} = \frac{1}{3} \int \mu t^3 ds \)
    - Or
      - \( \frac{M_x}{\theta_{,x}} = \sum_i \frac{l_i t_i^3 \mu}{3} \)
  - Warping
    - \( u_x^s(s) = u_x^s(0) - \theta_{,x} \int_0^{s} p_R ds' \)
  - Particular case of the I-Section beam
    - There is no shear stress at mid plane of flanges
    - They remain rectangular after torsion
**Open-section beam**

- **I-section beam subjected to torsion with built-in end**
  - Contrarily to the free/free beam
  - Presence of the built-end leads to deformation of the flanges

- The beam still twists but with a non-constant twist rate
- Method of solving: Combination of
  - Saint-Venant shear stress
  - Bending of flanges

\[ M_x = M_x^t + M_x^b \]
Open-section beam

- I-section beam subjected to torsion with built-in end (2)
  - Saint-Venant shear stress
    - $M^t_x = C\theta_x$
    - Where $\theta_x$ is not constant
Open-section beam

- I-section beam subjected to torsion with built-in end (3)
  - Bending of the flanges
    - For a given section
      - Angle of torsion $\theta$
      - Lateral displacement of lower flange
        $u_y = \frac{\theta h}{2}$
      - Bending moment in lower flange
        $M_{zz}^f = EI_{zz}^f u_{y,xx}$
      - With
        $I_{zz}^f = \frac{t_f b_f^3}{12}$
      - It has been assumed that displacement of the flange results from bending only
    - Shearing in the lower flange
      $T_y^f = -M_{zz,x}^f = -EI_{zz}^f u_{y,xxx}$
      $T_y^f = -\frac{hEI_{zz}^f}{2} \theta_{,xxx}$
Open-section beam

- I-section beam subjected to torsion with built-in end (4)
  - Bending of the flanges (2)
    - For a given section (2)
      - Shearing in the lower flange
        \[ T_y^f = -\frac{hEI_{zz}^f}{2} \theta,xxx \]
      - As shearing in top flange is in opposite direction, moment due to bending of the flange becomes
        \[ M_x^b = hT_y^f = -\frac{h^2EI_{zz}^f}{2} \theta,xxx \]
  - Total torque on the beam
    - \[ M_x = M_x^t + M_x^b \]
    - \[ M_x = C\theta,xx - \frac{h^2EI_{zz}^f}{2} \theta,xxx \]
Open-section beam

- Arbitrary-section beam subjected to torsion with built-in end
  - Wagner torsion theory
    - Assumptions
      - Length >> sectional dimensions
      - Undistorted cross-section
      - Shear stress at midsection negligible
        » But shear load not negligible
    - Under these assumptions, we can use the primary warping (of mid section) expression developed for torsion of free/free open-section beams
      $$- u_x^s(s) = u_x^s(0) - \theta_{,x} \int_0^s p_R ds'$$
      $$= u_x^s(0) - 2AR_p(s) \theta_{,x}$$
    - As twist rate is not constant
      - There is a direct induced stress
        $$\sigma_{xx}^\Gamma(s) = E u_{x,x} = E u_x^s(0) - 2EA_{R_p}(s) \theta_{,xx}$$
Open-section beam

- Arbitrary-section beam subjected to torsion with built-in end (2)
  - Wagner torsion theory (2)
    - Direct stress resulting from primary warping
      \[
      \sigma_{xx}^{\Gamma}(s) = E u_{x,x}^s(0) - 2EA_{R_p}(s) \theta_{,xx}
      \]
    - As only a torsion couple is applied
      - Integrating on the whole section \( C \times t \)
        should lead to 0

\[
\int_C t\sigma^{\Gamma} ds = 0
\]

\[
\int_C u_{x,x}^s(0) Et ds - \theta_{,xx} \int_C Et 2A_{R_p}(s) ds = 0
\]

\[
u_{x,x}^s(0) = \frac{\theta_{,xx} \int_C Et 2A_{R_p}(s) ds}{\int_C Et ds}
\]
Open-section beam

- Arbitrary-section beam subjected to torsion with built-in end (3)
  - Wagner torsion theory (3)
    - Direct stress resulting from primary warping (2)
      \[ \sigma_{xx}^r(s) = E u_{xx,x}(0) - 2EA_{R_p}(s) \theta_{,xx} \]
    - As only a torsion couple is applied (2)
      \[ u_{xx,x}^s(0) = \frac{\theta_{,xx} \int_C E t 2A_{R_p}(s) \, ds}{\int_C E t \, ds} \]
  - Direct stress is equilibrated by shear flow
    - See lecture on beams
      \[ (\sigma_{xx} + \partial_x \sigma_{xx} \delta x) t \delta s - \sigma_{xx} t \delta s + (q + \partial_s q \delta s) \delta x - q \delta x = 0 \]
      \[ t \partial_x \sigma_{xx} + \partial_s q = 0 \]
    - In this case
      \[ q_{,s}^r(s) = -t \sigma_{xx,x,x} \]
      \[ q_{,s}^r(s) = -Et u_{xx,x}(0) + 2EtA_{R_p}(s) \theta_{,xx} \]
Open-section beam

- Arbitrary-section beam subjected to torsion with built-in end (4)
  - Wagner torsion theory (4)
    - Equations
      - \( \sigma_{xx}^\Gamma (s) = E u_{x,x}^s (0) - 2EA_{R_p} (s) \theta_{,xx} \)
      - \( u_{x,x}^s (0) = \frac{\theta_{,xx} \int_C Et2A_{R_p} (s) \, ds}{\int_C Et \, ds} \)
      - \( q_{,s}^\Gamma (s) = -Etu_{x,xx}^s (0) + 2EtA_{R_p} (s) \theta_{,xxx} \)

- As for \( s = 0 \) (free edge) \( q(0) = 0 \)
  - \( q_{,s}^\Gamma (s) = \left( -\frac{\int_C Et2A_{R_p} (s) \, ds}{\int_C Et \, ds} + 2A_{R_p} (s) \right) Et \theta_{,xxx} \)
  - \( q^\Gamma (s) = \left( -\frac{\int_C Et2A_{R_p} (s) \, ds}{\int_C Et \, ds} Et s + \int_0^s 2EtA_{R_p} (s') \, ds' \right) \theta_{,xxx} \)
Open-section beam

- Arbitrary-section beam subjected to torsion with built-in end (5)
  - Wagner torsion theory (5)

- Torque

\[ M_x^b = \int_C p_R q^\Gamma (s) \, ds \]

- With

\[ q^\Gamma (s) = \left( -\frac{\int_C Et2A_{R_p} (s) \, ds}{\int_C Etds} Ets + \int_0^s 2EtA_{R_p} (s') \, ds' \right) \theta_{xxx} \]

\[ M_x^b = \left( -\frac{\int_C Et2A_{R_p} (s) \, ds}{\int_C Etds} \int_C p_R Ets ds + \int_C \left\{ p_R \int_0^s 2EtA_{R_p} (s') \, ds' \right\} ds \right) \theta_{xxx} \]
• **Arbitrary-section beam subjected to torsion with built-in end (6)**
  
  – Wagner torsion theory (6)

  • Torque (2)

  \[
  M_x^b = \left( -\frac{\int_C E I_2 A_{R_p} (s) \, ds}{\int_C E I_2 ds} \right) + \int_C p R E I_2 ds + \int_C \left\{ p R \int_0^s 2 E I A_{R_p} (s') \, ds' \right\} ds \right) \theta_{xxx}
  \]

  • Using \( p R = 2 A_{R_p,s} \), the second term becomes

  \[
  \int_C \left\{ 2 A_{R_p,s} \int_0^s E I_2 A_{R_p} (s') \, ds' \right\} ds = 2 A_{R_p} (s) \left[ \int_0^s E I_2 A_{R_p} (s') \, ds' \right]_0^L - \int_C 4 A_{R_p}^2 E I_2 ds
  \]

  – For \( s = 0 \), \( A_{R_p} = 0 \)

  – For \( s = L \), as the edge is free, there is no shear flux

  \[
  0 = q^T (L) = \left( -\frac{\int_C E I_2 A_{R_p} (s) \, ds}{\int_C E I_2 ds} \right) E I L + \int_C E I A_{R_p} (s') \, ds' \right) \theta_{xxx}
  \]

  – Using these two boundary conditions, second term is rewritten

  \[
  \int_C \left\{ 2 A_{R_p,s} \int_0^s E I_2 A_{R_p} (s') \, ds' \right\} ds = \frac{\int_C E I_2 A_{R_p} (s) \, ds}{\int_C E I_2 ds} 2 A_{R_p} (L) E I L - \int_C 4 A_{R_p}^2 E I_2 ds
  \]
Open-section beam

• **Arbitrary-section beam subjected to torsion with built-in end (7)**
  
  – Wagner torsion theory (7)

    • Torque (3)

      \[
      M_x^b = \left( - \frac{\int_C Et2A_{R_p}(s)\,ds}{\int_C Et\,ds} \right) \int_C p_REt\,ds + \int_C \left\{ p_R \int_0^s 2EtA_{R_p}(s')\,ds' \right\} ds \theta_{xxx}
      \]

    • Using \( p_R = 2A_{R_p,s} \) the integral of first term becomes

      \[
      \int_C 2A_{R_p,s}Et\,ds = 2A_{R_p}(s)EtL_0 - \int_C 2A_{R_p}Et\,ds
      \]

    • As for \( s = 0, A_{R_p} = 0 \), and using

      \[
      \int_C \left\{ 2A_{R_p,s} \int_0^s Et2A_{R_p}(s')\,ds' \right\} ds = \frac{\int_C Et2A_{R_p}(s)\,ds}{\int_C Et\,ds} 2A_{R_p}(L)EtL - \int_C 4A_{R_p}^2Et\,ds
      \]

    • The final expression reads

      \[
      M_x^b = \left( \frac{\left( \int_C Et2A_{R_p}(s)\,ds \right)^2}{\int_C Et\,ds} - \int_C 4A_{R_p}^2Et\,ds \right) \theta_{xxx}
      \]
Open-section beam

- **Arbitrary-section beam subjected to torsion with built-in end (8)**
  - General expression for torque
    - \[ M_x = M^t_x + M^b_x \longrightarrow M_x = C\theta_x - C^\Gamma \theta_{xxx} \]
  - With
    - \[ C^\Gamma = \int_C 4A^2_{Rp} Etds - \frac{\left(\int_C Et2A_{Rp}(s) \, ds\right)^2}{\int_C Etds} \]
  - Case of the I-section beam
    - Center of twist is the center of symmetry \( C \)
    - For the web: \( A_{Rp}(s) = 0 \longrightarrow \) no contribution to \( C^\Gamma \)
    - For lower flange
      - \[ A_{Rp}(s) = \frac{hs}{4} \]
      - \[ \begin{align*}
        \int_C Et2A_{Rp}(s) \, ds &= Et_f \frac{hb^2_f}{4} \\
        \int_C Et4A^2_{Rp}(s) \, ds &= Et_f \frac{h^2b^3_f}{12}
      \end{align*} \]
    - For the I-section
      - \[ C^\Gamma = 2 \left( Et_f \frac{h^2b^3_f}{12} - Et_f \frac{h^2b^3_f}{16} \right) = Et_f \frac{h^2b^3_f}{24} \]
Open-section beam

• Arbitary-section beam subjected to torsion with built-in end (9)
  – Case of the I-section beam (2)
    • Expression $M_x = C \theta_{,x} - C^\Gamma \theta_{,xxx}$
      – With $C^\Gamma = \int_C 4A^2 R_p E t ds - \left( \frac{\int_C Et A R_p (s) ds}{\int_C Et ds} \right)^2$
      \[C^\Gamma = 2 \left( Et_f \frac{h^2 b_f^3}{12} - Et_f \frac{h^2 b_f^3}{16} \right) = Et_f \frac{h^2 b_f^3}{24}\]

• To be compared with

$M_x = C \theta_{,x} - \frac{h^2 E I_{zz} f}{2} \theta_{,xxx}$
Open-section beam

- Idealized beam subjected to torsion with built-in end
  - For idealized sections with booms
    - In expression
      \[
      C^\Gamma = \int_C 4A^2_{R_p} Et ds - \left( \frac{\int_C Et 2A_{R_p} (s) ds}{\int_C Et ds} \right)^2
      \]
    - The direct stress is carried out by
      - \( t_{\text{direct}} \)
      - Booms of section \( A_i \)
    \[
    C^\Gamma = \int_C 4A^2_{R_p} Et_{\text{direct}} ds + \sum_i 4A^2_{R_p} (s^i) EA_i - \\
    \left( \frac{\int_C Et_{\text{direct}} 2A_{R_p} (s) ds + \sum_i 2A_{R_p} (s^i) EA_i}{\int_C Et_{\text{direct}} ds + \sum_i EA_i} \right)^2
    \]
Open-section beam

- Applications of beam subjected to torsion with built-in end
  - Solution for pure torque
    - $M_x = C \theta_{,x} - CT \theta_{,xxx}$
    - $\theta_{,xxx} - w^2 \theta_{,x} = -\frac{w^2}{C} M_x$ with $w^2 = \frac{C}{CT}$
  - Solution
    - $\theta_{,x} = C_1 \cosh wx + C_2 \sinh wx + \frac{M_x}{C}$
  - Boundary conditions
    - At built-in end $x = 0$: No warping, and as $u^s_x (s) = u^s_x (0) - \theta_{,x} \int_0^s p_R ds'$
      - $\theta_{,x} (0) = 0 \Rightarrow C_1 = -\frac{M_x}{C}$
    - At free end $x = L$: no direct load,
      - $\sigma_{xx}^\Gamma (s) = E u^s_{x,x} (0) - 2E A_{R_p} (s) \theta_{,xx}$
      - $u^s_{x,x} (0) = \frac{\theta_{,xx} \int_C Et2A_{R_p} (s) ds}{\int_C Et ds}$
      - $\theta_{,xx} (L) = 0 \Rightarrow C_2 = \frac{M_x}{C} \tanh wL$
Open-section beam

- Applications of beam subjected to torsion with built-in end (2)
  - Solution for pure torque (2)

- Twist rate

\[
\theta_{,x} = \frac{M_x}{C} \left( 1 - \cosh wx + \tanh wL \sinh wx \right)
\]

\[
\theta_{,x} = \frac{M_x}{C} \left( 1 - \frac{\cosh (wL - wx)}{\cosh wL} \right)
\]
Open-section beam

- Applications of beam subjected to torsion with built-in end (3)
  - Solution for pure torque (3)
    - Angle of twist
      - As \( \theta_{,x} = \frac{M_x}{C} \left( 1 - \frac{\cosh (wL - wx)}{\cosh wL} \right) \)
      \[ \theta (x) = \frac{M_x}{C} \left( x + \frac{\sinh (wL - wx)}{w \cosh wL} + C_3 \right) \]
    - Boundary condition at built end \( x = 0 \): No twist
      \[ 0 = \theta (0) = \frac{M_x}{C} \left( \frac{\sinh wL}{w \cosh wL} + C_3 \right) \]
      \[ \theta (x) = \frac{M_x}{C} \left( x + \frac{\sinh (wL - wx)}{w \cosh wL} - \frac{\sinh wL}{w \cosh wL} \right) \]
  - At free end
    \[ \theta (L) = \frac{M_x L}{C} \left( 1 - \frac{\tanh wL}{wL} \right) \]
    Reduction compared to free-free case
Applications of beam subjected to torsion with built-in end (4)

- Distributed torque loading \( m_x \)
  - Two contributions to torque
    \[
    M_x = M_x^t + M_x^b
    \]
  - Balance equation
    \[
    M_x^t + \partial_x M_x^t \delta x + M_x^b + \partial_x M_x^b \delta x + m_x \delta x = M_x^t + M_x^b
    \]
    \[
    \partial_x M_x = \partial_x M_x^t + \partial_x M_x^b = -m_x
    \]
    \[
    \begin{cases}
    M_x^t = C \theta, x \\
    M_x^b = -C^T \theta, xxx
    \end{cases}
    \]
    \[
    \partial_x \left( C^T \theta, xxx - C \theta, x \right) = m_x (x)
    \]
  - As
  - To be solved with adequate boundary conditions
    - Built-in end: \( \theta = 0 \) & \( \theta, x = 0 \) (no warping)
    - Free end: \( \theta, xx = 0 \) (no direct stress) & No torque at free end
• Remark
  – We have studied
    • Axial loading resulting from torsion
    • A similar theory can be derived to deduce torsion resulting from axial loading
References

• Lecture notes

• Other references
  – Books