University of Liège
Aerospace & Mechanical Engineering

Aircraft Structures
Structural & Loading Discontinuities

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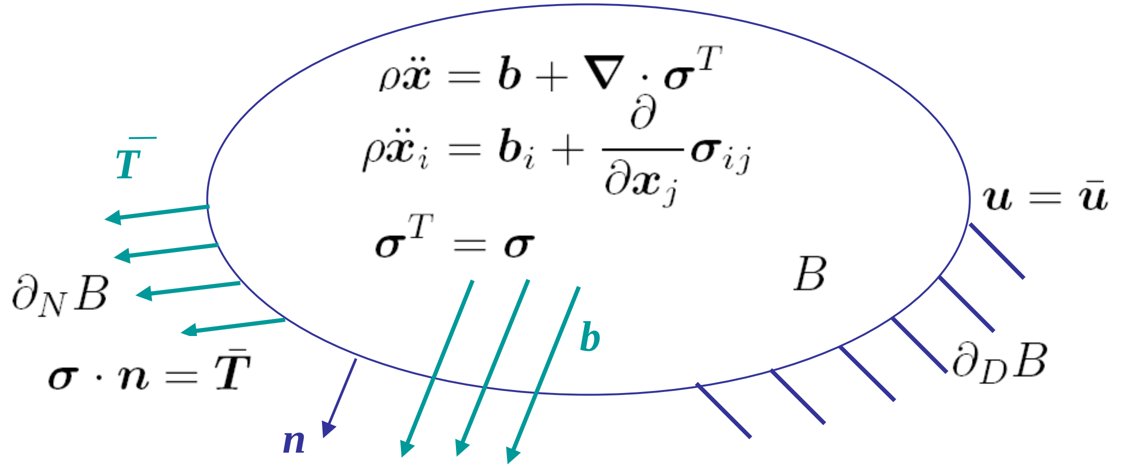
- Balance of body B
  - Momenta balance
    - Linear
    - Angular
  - Boundary conditions
    - Neumann
    - Dirichlet



- Small deformations with linear elastic, homogeneous & isotropic material
  - (Small) Strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla})$, or $\begin{cases} \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{x}_i}\boldsymbol{u}_j + \frac{\partial}{\partial \boldsymbol{x}_j}\boldsymbol{u}_i\right) \\ \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}(\boldsymbol{u}_{j,i} + \boldsymbol{u}_{i,j}) \end{cases}$
  - Hooke's law $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon}$, or $\boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl}\boldsymbol{\varepsilon}_{kl}$

    with $\mathcal{H}_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{1+\nu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$

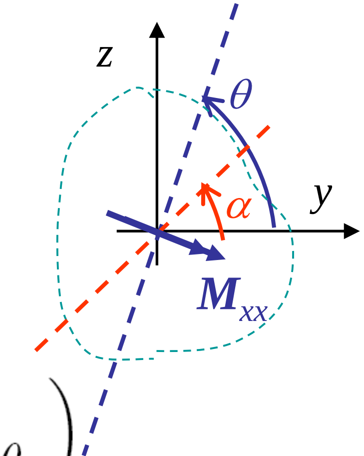
    $\lambda = K - 2\mu/3$ $\quad 2\mu$
  - Inverse law $\boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma}$

    with $\mathcal{G}_{ijkl} = \frac{1+\nu}{E}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right) - \frac{\nu}{E}\delta_{ij}\delta_{kl}$

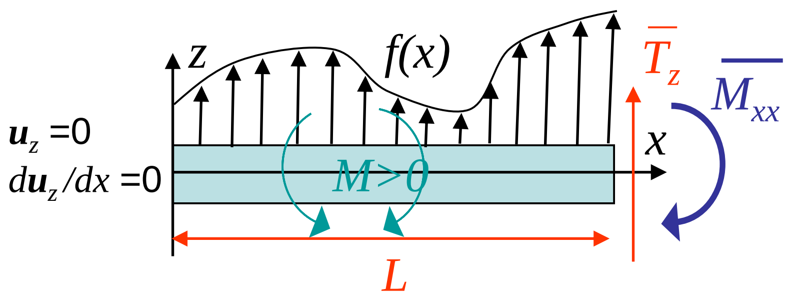
- General expression for unsymmetrical beams
  - Stress $\boldsymbol{\sigma}_{xx} = \kappa E z \cos\alpha - \kappa E y \sin\alpha$

    With $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{\kappa E} \begin{pmatrix} I_{yy} & -I_{yz} \\ -I_{yz} & I_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$
  - Curvature

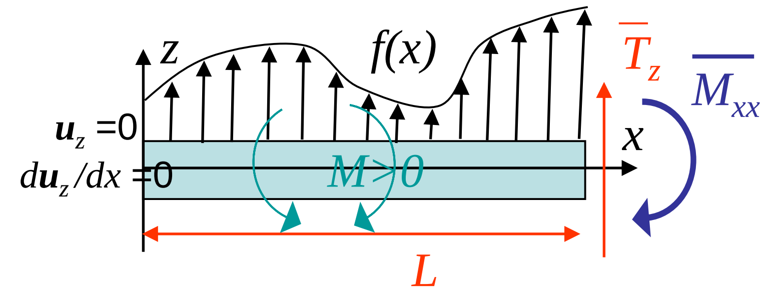
    $$\begin{pmatrix} -\boldsymbol{u}_{z,xx} \\ \boldsymbol{u}_{y,xx} \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{E\left(I_{yy}I_{zz} - I_{yz}I_{yz}\right)} \begin{pmatrix} I_{zz} & I_{yz} \\ I_{yz} & I_{yy} \end{pmatrix} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$
  - In the principal axes $I_{yz} = 0$



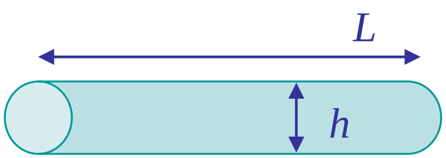
- Euler-Bernoulli equation in the principal axis
  - $\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f(x)$ for x in [0 L]
  - BCs $\begin{cases} -\frac{\partial}{\partial x}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\Big|_{0,\,L} = \bar{T}_z\big|_{0,\,L} \\ -EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\Big|_{0,\,L} = \bar{M}_{xx}\big|_{0,\,L} \end{cases}$
  - Similar equations for $\boldsymbol{u}_y$



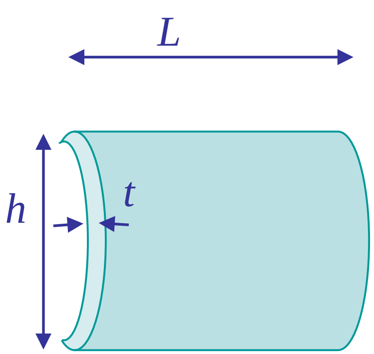
- General relationships
  - $$\begin{cases} f_z(x) = -\partial_x T_z = -\partial_{xx} M_y \\ f_y(x) = -\partial_x T_y = \partial_{xx} M_z \end{cases}$$



- Two problems considered
  - Thick symmetrical section



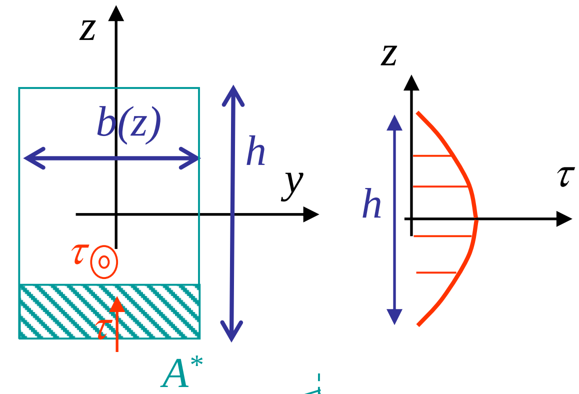
    - Shear stresses are small compared to bending stresses if $h/L << 1$
  - Thin-walled (unsymmetrical) sections



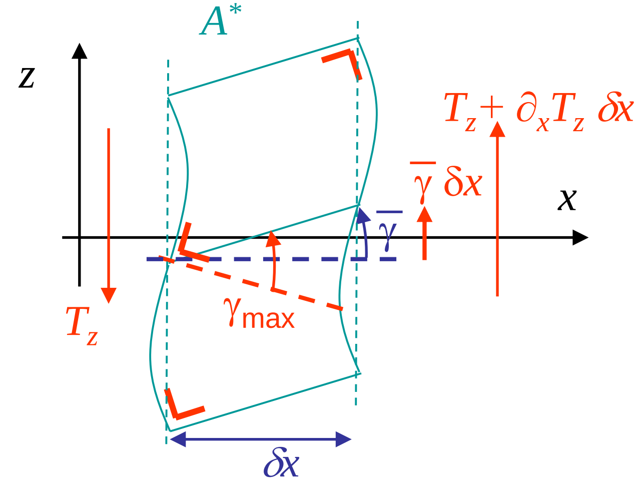
    - Shear stresses are not small compared to bending stresses
    - Deflection mainly results from bending stresses
    - 2 cases
      - Open thin-walled sections
        - » Shear = shearing through the shear center + torque
      - Closed thin-walled sections
        - » Twist due to shear has the same expression as torsion

Beam shearing: linear elasticity summary

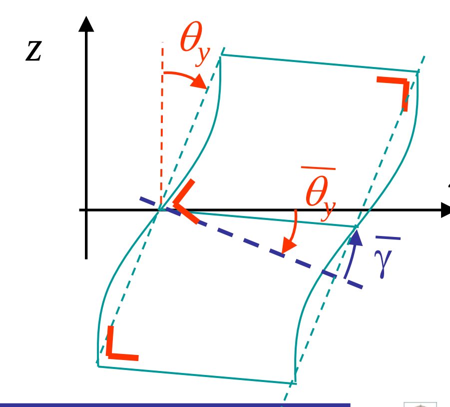
- Shearing of symmetrical thick-section beams
  - Stress $\boldsymbol{\sigma}_{zx} = -\dfrac{T_z S_n(z)}{I_{yy} b(z)}$
    - With $S_n(z) = \displaystyle\int_{A^*} z dA$
    - Accurate only if $h > b$



  - Energetically consistent averaged shear strain
    - $\bar{\gamma} = \dfrac{T_z}{A'\mu}$ with $A' = \dfrac{1}{\int_A \frac{S_n^2}{I_{yy}^2 b^2} dA}$
    - Shear center on symmetry axes

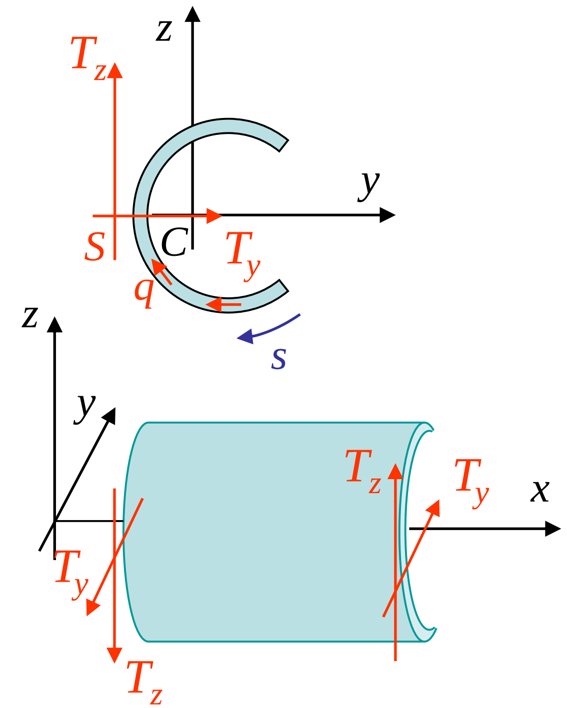


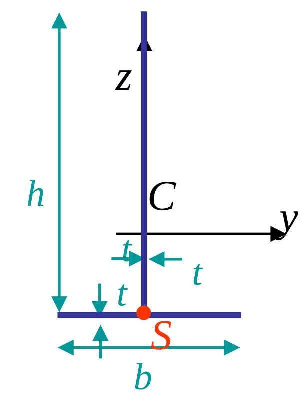
  - Timoshenko equations
    - $\bar{\gamma} = 2\bar{\boldsymbol{\varepsilon}}_{xz} = \dfrac{\partial \boldsymbol{u}_x}{\partial z} + \dfrac{\partial \boldsymbol{u}_z}{\partial x} = \theta_y + \partial_x \boldsymbol{u}_z$ & $\kappa = \dfrac{\partial \theta_y}{\partial x}$
    - On [0 L]: $\begin{cases} \dfrac{\partial}{\partial_x}\left(EI\dfrac{\partial \theta_y}{\partial x}\right) - \mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right) = 0 \\ \dfrac{\partial}{\partial x}\left(\mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right)\right) = -f \end{cases}$



Beam shearing: linear elasticity summary

- Shearing of open thin-walled section beams
  - Shear flow $q = t\tau$
    - $$q(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s tz ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s ty ds'$$
    - In the principal axes
      $$q(s) = -\frac{T_z}{I_{yy}}\int_0^s tz ds' - \frac{T_y}{I_{zz}}\int_0^s ty ds'$$
  - Shear center S
    - On symmetry axes
    - At walls intersection
    - Determined by momentum balance
  - Shear loads correspond to
    - Shear loads passing through the shear center &
    - Torque





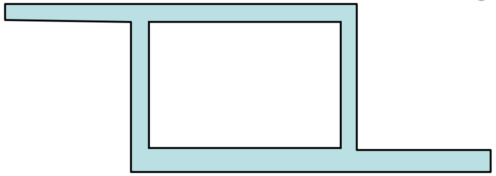
- Shearing of closed thin-walled section beams
  - Shear flow $q = t\tau$
    - $q(s) = q_o(s) + q(0)$
    - Open part (for anticlockwise of q, s)

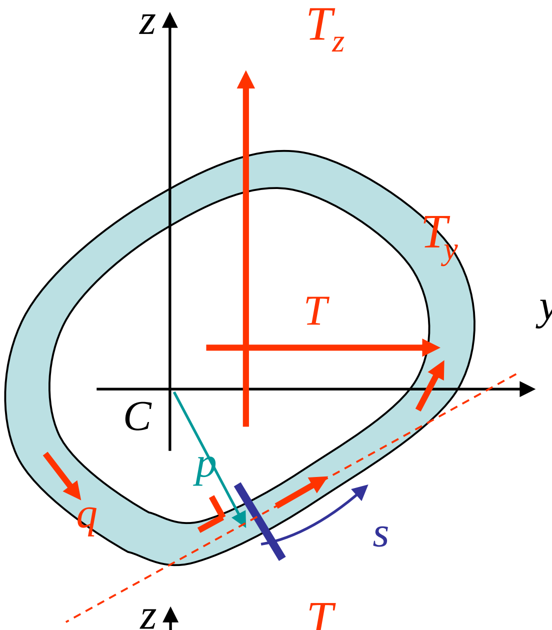
$$q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s t(s')\,z(s')\,ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s t(s')\,y(s')\,ds'$$

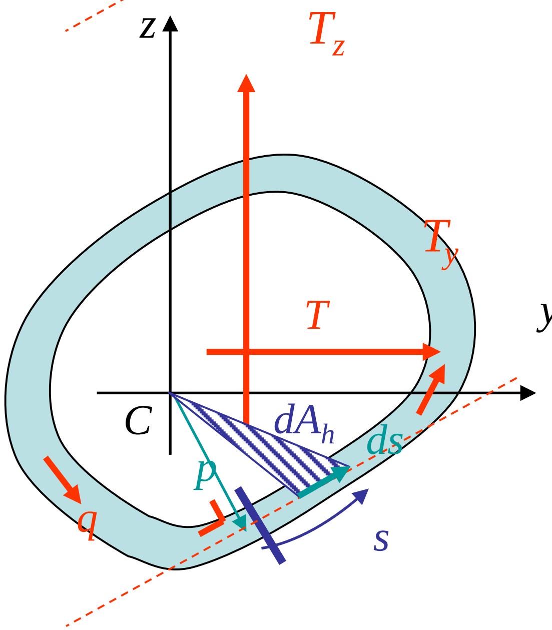
    - Constant twist part

$$q(s=0) = \frac{y_T T_z - z_T T_y - \oint p(s)\,q_o(s)\,ds}{2A_h}$$

    - The $q(0)$ is related to the closed part of the section, but there is a $q_o(s)$ in the open part which should be considered for the shear torque $\oint p(s)\,q_o(s)\,ds$



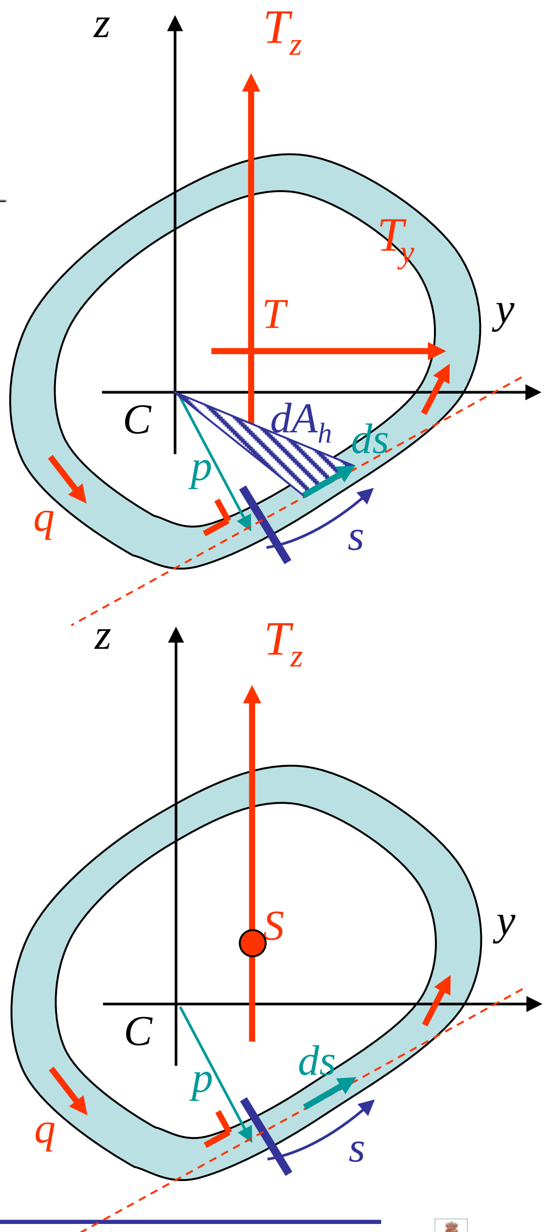




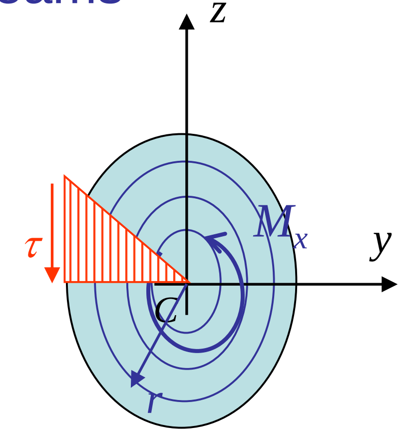
- Shearing of closed thin-walled section beams
  - Warping around twist center R
    - $$\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \int_0^s \frac{q}{\mu t} ds - \frac{1}{A_h} \oint \frac{q}{\mu t} ds \left\{ A_{Cp}(s) + \frac{z_R\left[y(s) - y(0)\right] - y_R\left[z(s) - z(0)\right]}{2} \right\}$$
    - With $\boldsymbol{u}_x(0) = \frac{\oint t \boldsymbol{u}_x(s)\, ds}{\oint t(s)\, ds}$
      - $\boldsymbol{u}_x(0)=0$ for symmetrical section if origin on the symmetry axis
  - Shear center S
    - Compute q for shear passing thought S
    - Use

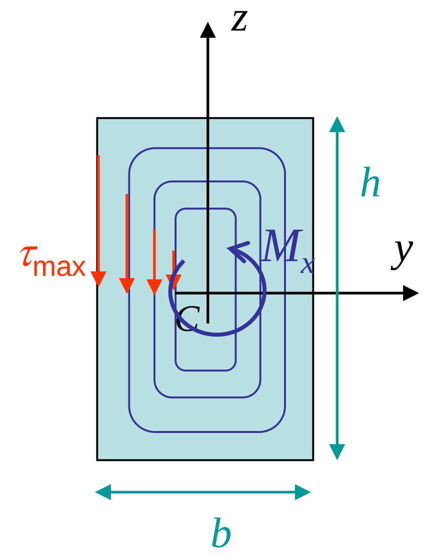
$$q(s=0) = \frac{y_T T_z - z_T T_y - \oint p(s)\, q_o(s)\, ds}{2A_h}$$

With point S=T



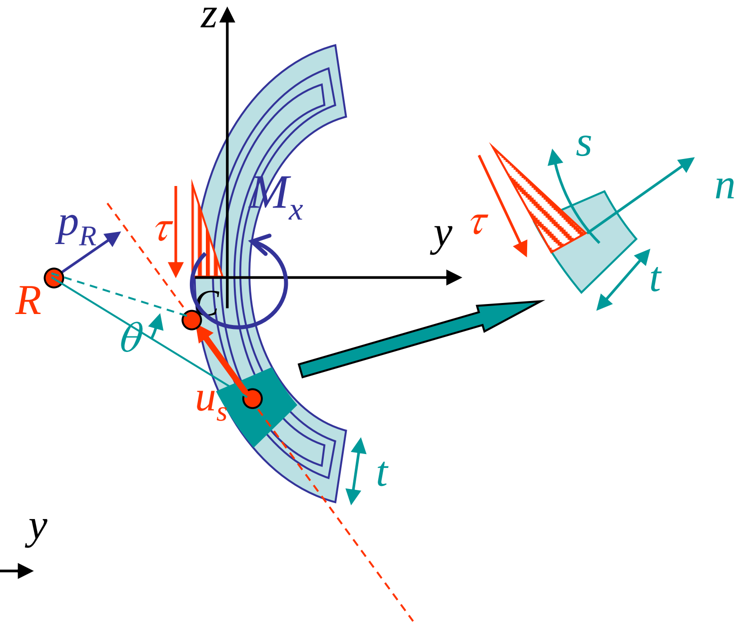
- Torsion of symmetrical thick-section beams
  - Circular section
    - $\tau = \mu\gamma = r\mu\theta_{,x}$
    - $C = \dfrac{M_x}{\theta_{,x}} = \int_A \mu r^2 dA$
  - Rectangular section
    - $\tau_{\max} = \dfrac{M_x}{\alpha h b^2}$
    - $C = \dfrac{M_x}{\theta_{,x}} = \beta h b^3 \mu$
    - If $h >> b$
      - $\tau_{xy} = 0$ & $\tau_{xz} = 2\mu y\theta_{,x}$
      - $\tau_{\max} = \dfrac{3M_x}{hb^2}$
      - $C = \dfrac{M_x}{\theta_{,x}} = \dfrac{hb^3\mu}{3}$



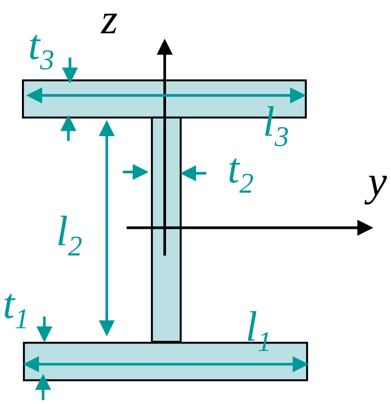


h/b	1	1.5	2	4	∞
α	0.208	0.231	0.246	0.282	1/3
β	0.141	0.196	0.229	0.281	1/3

- Torsion of open thin-walled section beams
  - Approximated solution for twist rate
    - Thin curved section
      - $\tau_{xs} = 2\mu n \theta_{,x}$
      - $C = \frac{M_x}{\theta_{,x}} = \frac{1}{3}\int \mu t^3 ds$



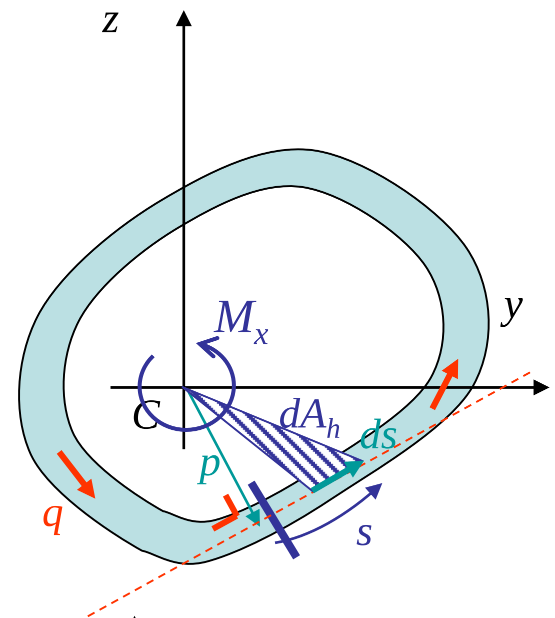
    - Rectangles
      - $\tau_{\max_i} = \mu t_i \theta_{,x}$
      - $\frac{M_x}{\theta_{,x}} = \sum_i \frac{l_i t_i^3 \mu}{3}$



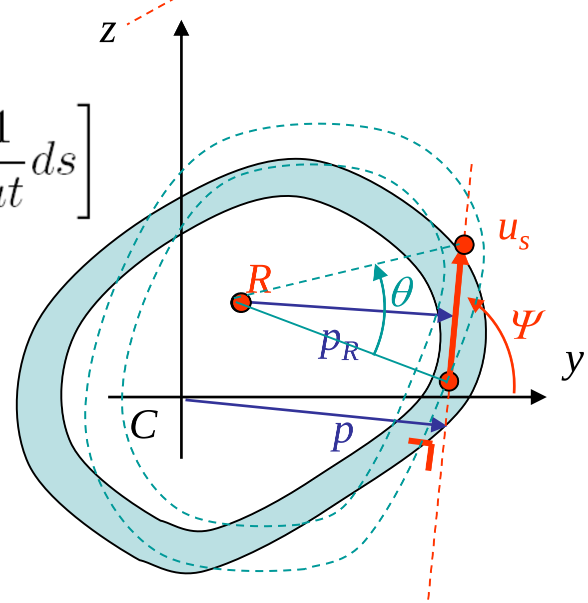
  - Warping of s-axis
    - $\boldsymbol{u}_x^s(s) = \boldsymbol{u}_x^s(0) - \theta_{,x}\int_0^s p_R ds' = \boldsymbol{u}_x^s(0) - 2A_{R_p}(s)\,\theta_{,x}$

- Torsion of closed thin-walled section beams
  - Shear flow due to torsion $M_x = 2A_h q$
  - Rate of twist
    - $\theta_{,x} = \frac{M_x}{4A_h^2} \oint \frac{1}{\mu t} ds$
    - Torsion rigidity for constant μ

$$I_T = \frac{4A_h^2}{\oint \frac{1}{t} ds} \leq I_p = \int_A r^2 dA$$

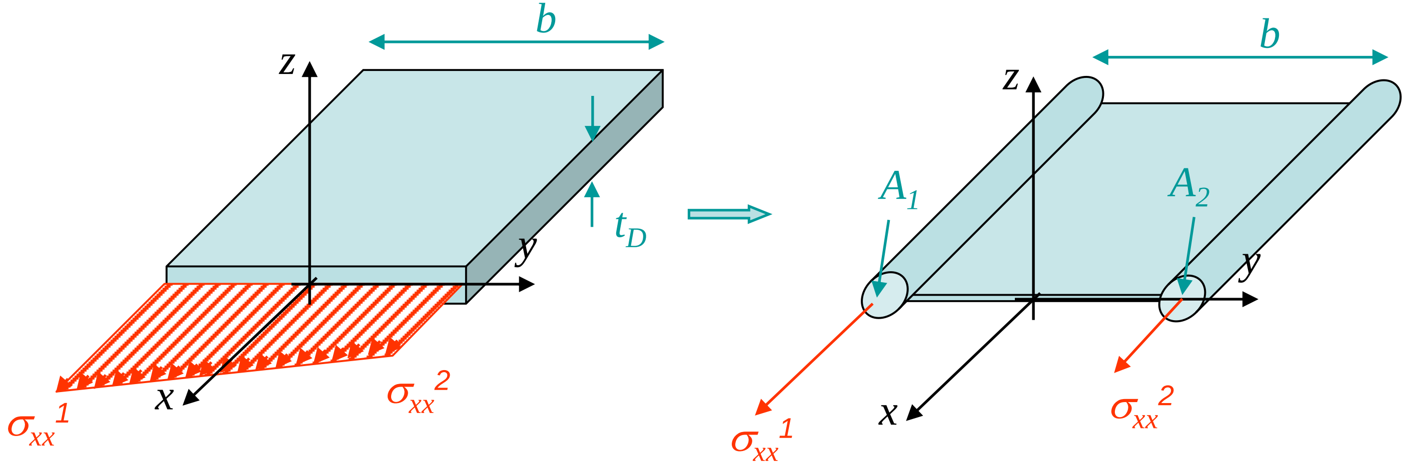


  - Warping due to torsion
    - $\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \frac{M_x}{2A_h}\left[\int_0^s \frac{1}{\mu t} ds - \frac{A_{R_p}(s)}{A_h} \oint \frac{1}{\mu t} ds\right]$
    - A_{Rp} from twist center



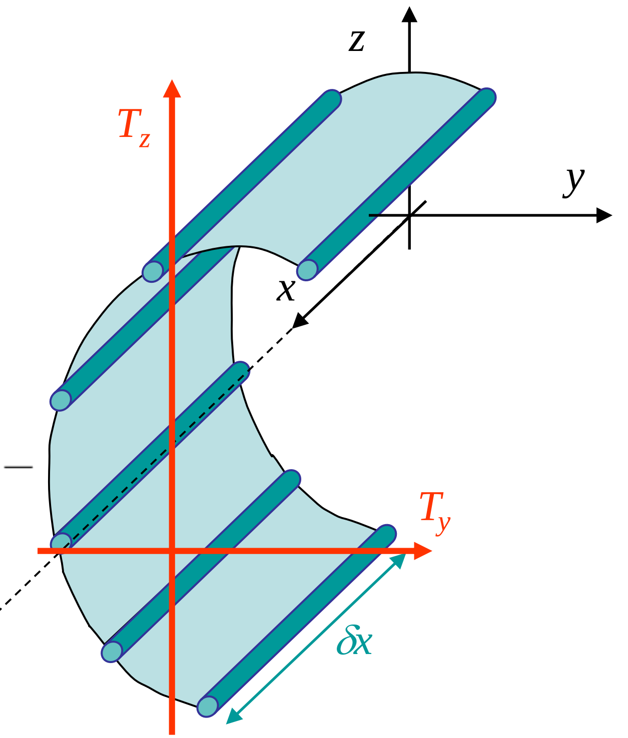
- Panel idealization
  - Booms' area **depending on loading**
    - For linear direct stress distribution

$$\begin{cases} A_1 = \frac{t_D b}{6}\left(2 + \frac{\boldsymbol{\sigma}_{xx}^2}{\boldsymbol{\sigma}_{xx}^1}\right) \\ A_2 = \frac{t_D b}{6}\left(2 + \frac{\boldsymbol{\sigma}_{xx}^1}{\boldsymbol{\sigma}_{xx}^2}\right) \end{cases}$$



- Consequence on bending
  - If Direct stress due to bending is carried by booms only
    - The position of the neutral axis, and thus the second moments of area
      - Refer to the direct stress carrying area only
      - Depend on the loading case only
- Consequence on shearing
  - Open part of the shear flux
    - Shear flux for open sections

$$q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\sigma} z ds + \sum_{i:\, s_i \leq s} z_i A_i\right] - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\sigma} y ds + \sum_{i:\, s_i \leq s} y_i A_i\right]$$



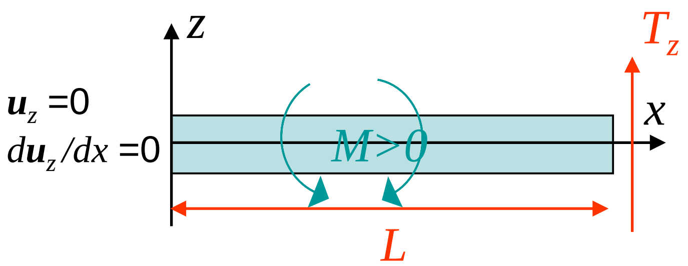
- Consequence on torsion
  - If no axial constraint
    - Torsion analysis does not involve axial stress
    - So torsion is unaffected by the structural idealization

- Virtual displacement
  - In linear elasticity the general formula of virtual displacement reads

$$\int_0^L \int_A \boldsymbol{\sigma}^{(1)} : \boldsymbol{\varepsilon} dA dx = P^{(1)} \Delta_P$$

- $\boldsymbol{\sigma}^{(1)}$ is the stress distribution corresponding to a (unit) load $P^{(1)}$
- Δ_P is the energetically conjugated displacement to P in the direction of $P^{(1)}$ that corresponds to the strain distribution $\boldsymbol{\varepsilon}$

  - Example bending of semi cantilever beam



- $$\int_0^L \int_A \boldsymbol{\sigma}_{xx}^{(1)} \boldsymbol{\varepsilon}_{xx} dA dx = \Delta_P u$$

    - In the principal axes

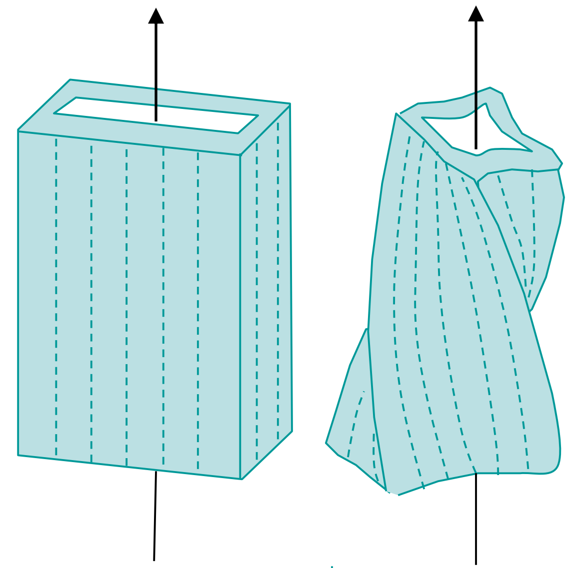
$$\Delta_P u \quad = \quad \frac{1}{EI_{yy}I_{zz}} \int_0^L \left\{ I_{zz} M_y^{(1)} M_y + I_{yy} M_z^{(1)} M_z \right\} dx$$

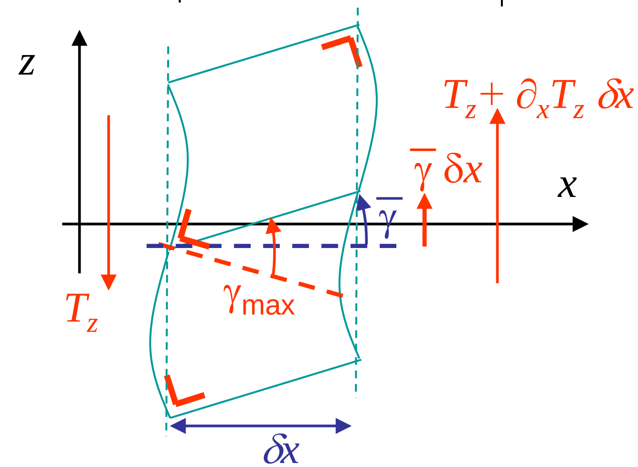
  - Example shearing of semi-cantilever beam

- $$\int_0^L \int_s q^{(1)} \frac{q}{\mu t} ds dx = \boldsymbol{T}^{(1)} \bar{\Delta} \boldsymbol{u} = \Delta_T u$$

Limitations of these theories

- Previously developed equations
  - Stresses & displacements produced by
    - Axial loads
    - Shear forces
    - Bending moments
    - Torsion
  - No allowance for constrained warping
    - Due to structural or loading discontinuities
    - Example torsion of a built-in beam
      - No warping allowed at clamping
  - Coupling shearing-bending neglected
    - Effect of shear strains on the direct stress
    - Shear strains prevent cross section to remain plane
    - Direct stress predicted by pure bending theory not correct anymore
    - For wing box, shear strains can be important



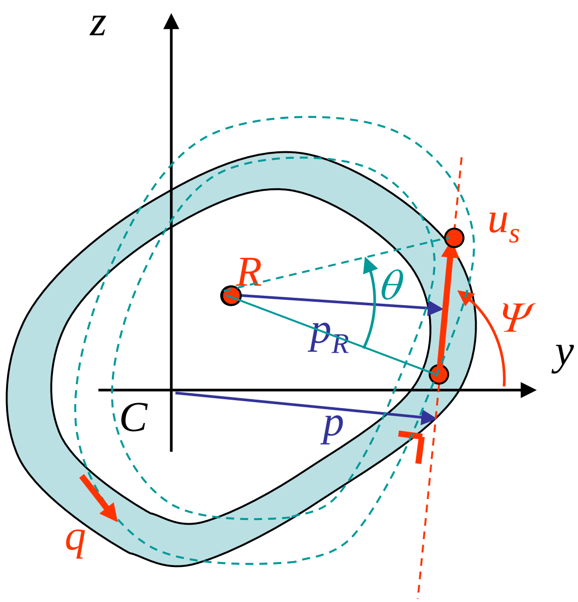


Limitations of these theories

- These effects can be analyzed on simple problems
  - Problem of axial constraint divided in two parts
    - Shear stress distribution calculated at the built-in section
    - Stress distribution calculated on the beam length for the separate loading cases of bending & torsion
  - Problem related to instabilities as buckling
    - See later
- For more complex problems
  - Finite element simulations required

- Shear stress distribution at a built-in end
  - Idealized or not cross-sections
  - Assume a beam with closed cross-section
    - Center of twist R
    - Undistorted section of the beam
    - Shear flow, displacements and rotation of the section were found to be

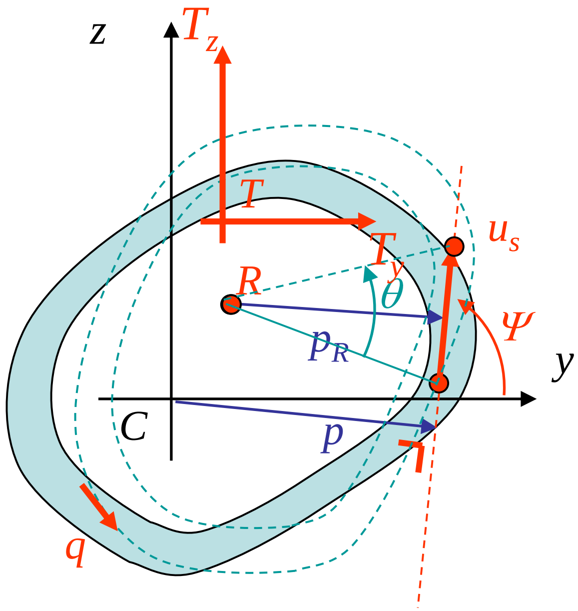
  - $$\frac{q}{\mu t} = \frac{\partial \boldsymbol{u}_x}{\partial s} + [p - y_R \sin\Psi + z_R \cos\Psi] \frac{\partial \theta}{\partial x}$$
  - With $\begin{cases} y_R = -\dfrac{\partial_x \boldsymbol{u}_z^C}{\partial_x \theta} \\ z_R = \dfrac{\partial_x \boldsymbol{u}_y^C}{\partial_x \theta} \end{cases}$



  - At built-in this relation simplifies into
    - $$\frac{q}{\mu t} = p\frac{\partial \theta}{\partial x} + \frac{\partial \boldsymbol{u}_z^C}{\partial x}\sin\Psi + \frac{\partial \boldsymbol{u}_y^C}{\partial x}\cos\Psi$$

Closed-section beam

- Shear stress distribution at a built-in end (2)
  - At built-in shear flux is written
    - $$\frac{q}{\mu t} = p\frac{\partial \theta}{\partial x} + \frac{\partial \boldsymbol{u}_z^C}{\partial x}\sin\Psi + \frac{\partial \boldsymbol{u}_y^C}{\partial x}\cos\Psi$$
    - By equilibrium
      - $$\boldsymbol{T}_y = \oint q\cos\Psi ds$$
      - $$\boldsymbol{T}_z = \oint q\sin\Psi ds$$
      - $$y_T\boldsymbol{T}_z - z_T\boldsymbol{T}_y = \oint pq ds$$



  - After substitution of shear flux

$$\left\{\begin{aligned}
\boldsymbol{T}_y &= \frac{\partial \theta}{\partial x}\oint \mu t p\cos\Psi ds + \frac{\partial \boldsymbol{u}_y^C}{\partial x}\oint \mu t\cos^2\Psi ds + \frac{\partial \boldsymbol{u}_z^C}{\partial x}\oint \mu t\cos\Psi\sin\Psi ds\\
\boldsymbol{T}_z &= \frac{\partial \theta}{\partial x}\oint \mu t p\sin\Psi ds + \frac{\partial \boldsymbol{u}_y^C}{\partial x}\oint \mu t\cos\Psi\sin\Psi ds + \frac{\partial \boldsymbol{u}_z^C}{\partial x}\oint \mu t\sin^2\Psi ds\\
y_T\boldsymbol{T}_z - z_T\boldsymbol{T}_y &= \frac{\partial \theta}{\partial x}\oint \mu t p^2 ds + \frac{\partial \boldsymbol{u}_y^C}{\partial x}\oint \mu t p\cos\Psi ds + \frac{\partial \boldsymbol{u}_z^C}{\partial x}\oint \mu t p\sin\Psi ds
\end{aligned}\right.$$

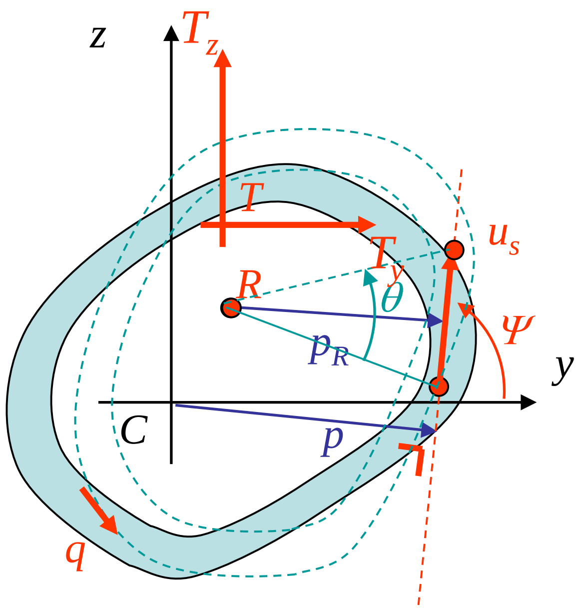
- Shear stress distribution at a built-in end (3)
  - New system of 3 equations and 3 unknowns

$$\begin{cases} \boldsymbol{T}_y = \frac{\partial \theta}{\partial x} \oint \mu t p \cos\Psi ds + \frac{\partial \boldsymbol{u}_y^C}{\partial x} \oint \mu t \cos^2\Psi ds + \frac{\partial \boldsymbol{u}_z^C}{\partial x} \oint \mu t \cos\Psi \sin\Psi ds \\ \boldsymbol{T}_z = \frac{\partial \theta}{\partial x} \oint \mu t p \sin\Psi ds + \frac{\partial \boldsymbol{u}_y^C}{\partial x} \oint \mu t \cos\Psi \sin\Psi ds + \frac{\partial \boldsymbol{u}_z^C}{\partial x} \oint \mu t \sin^2\Psi ds \\ y_T \boldsymbol{T}_z - z_T \boldsymbol{T}_y = \frac{\partial \theta}{\partial x} \oint \mu t p^2 ds + \frac{\partial \boldsymbol{u}_y^C}{\partial x} \oint \mu t p \cos\Psi ds + \frac{\partial \boldsymbol{u}_z^C}{\partial x} \oint \mu t p \sin\Psi ds \end{cases}$$

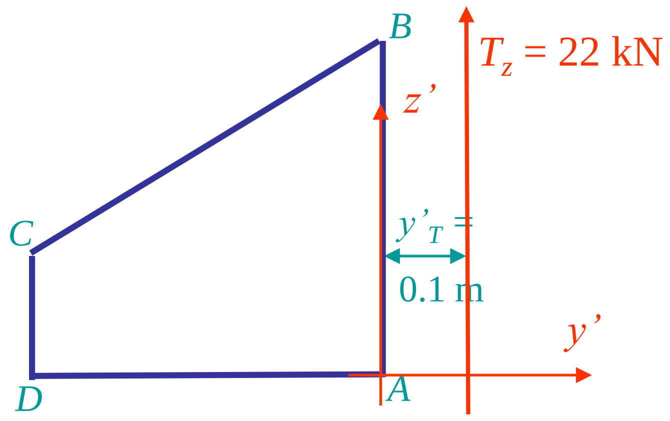
    - Solution of the system: $\frac{\partial \theta}{\partial x}$, $\frac{\partial \boldsymbol{u}_y^C}{\partial x}$ & $\frac{\partial \boldsymbol{u}_z^C}{\partial x}$
  - This solution is then substituted into

$$\frac{q}{\mu t} = p\frac{\partial \theta}{\partial x} + \frac{\partial \boldsymbol{u}_z^C}{\partial x} \sin\Psi + \frac{\partial \boldsymbol{u}_y^C}{\partial x} \cos\Psi$$

    - Shear flow and shear stress are then defined
    - Remains true for any choice of C as long as p is computed from there



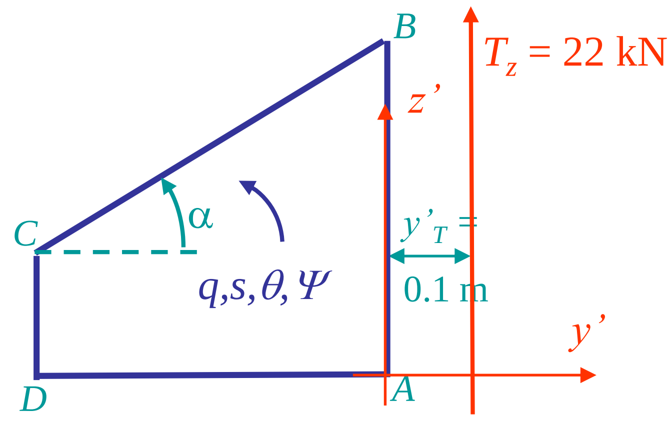
- Example
  - Built-in end
    - Section with constant shear modulus
  - Shear stress distribution?
  - Center of twist?



Wall	Length (m)	Thickness (mm)
AB	0.375	1.6
BC	0.500	1.0
CD	0.125	1.2
DA		1.0

- Deformation
  - Sign convention: >0 anticlockwise
  - Angle α: sin α = 0.25/0.5 ⟹ α = 30°
  - Coefficients

$$\oint tp\cos\Psi ds = \int_B^C p_A t^{BC}\cos\Psi^{BC} ds^{BC} + \int_C^D p_A t^{CD}\cos\Psi^{CD} ds^{CD}$$

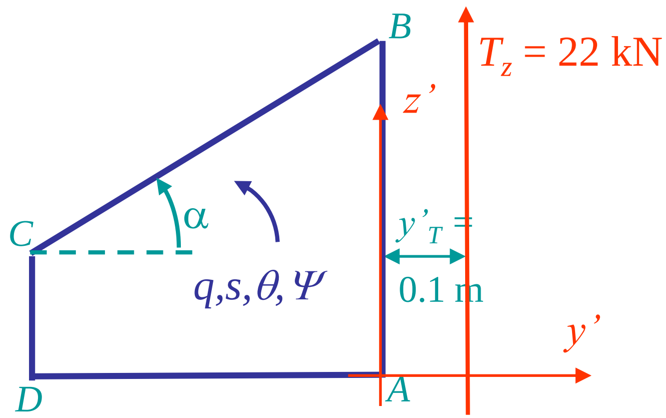


$$\Rightarrow \oint tp\cos\Psi ds = l^{BC}t^{BC}l^{AB}\cos\frac{\pi}{6}\cos\frac{7\pi}{6} + l^{CD}t^{CD}l^{AD}\cos\frac{3\pi}{2}$$

$$\Rightarrow \oint tp\cos\Psi ds = -0.5\ 10^{-3}\ 0.375\ \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} = -0.14\,10^{-3}\ \mathrm{m}^3$$

Wall	Length (m)	Thickness (mm)
AB	0.375	1.6
BC	0.500	1.0
CD	0.125	1.2
DA		1.0

- Deformation (2)
  - Coefficients (2)



$$\oint tp\sin\Psi ds = \int_B^C p_A t^{BC} \sin\Psi^{BC} ds^{BC} + \int_C^D p_A t^{CD} \sin\Psi^{CD} ds^{CD}$$

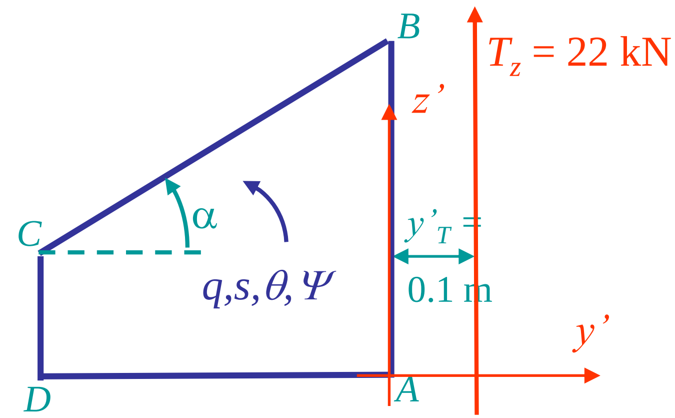
$$\Rightarrow \oint tp\sin\Psi ds = l^{BC} t^{BC} l^{AB} \cos\frac{\pi}{6} \sin\frac{7\pi}{6} + l^{CD} t^{CD} l^{BC} \cos\frac{\pi}{6} \sin\frac{3\pi}{2}$$

$$\Rightarrow \oint tp\sin\Psi ds = -0.5\ 10^{-3}\ 0.375\ \frac{\sqrt{3}}{2}\frac{1}{2} - 0.125\ 1.2\, 10^{-3}\ 0.5\frac{\sqrt{3}}{2} = -0.15\, 10^{-3}\ \mathrm{m}^3$$

Wall	Length (m)	Thickness (mm)
AB	0.375	1.6
BC	0.500	1.0
CD	0.125	1.2
DA		1.0

- Deformation (3)
  - Coefficients (3)

$$\oint t\cos^2\Psi ds = \int_A^B t^{AB}\cos^2\Psi^{AB} ds^{AB} + \int_B^C t^{BC}\cos^2\Psi^{BC} ds^{BC} + \int_C^D t^{CD}\cos^2\Psi^{CD} ds^{CD} + \int_D^A t^{DA}\cos^2\Psi^{DA} ds^{DA}$$



$$\Rightarrow \oint t\cos^2\Psi ds = l^{AB}t^{AB}\cos^2\frac{\pi}{2} + l^{BC}t^{BC}\cos^2\frac{7\pi}{6} + l^{CD}t^{CD}\cos^2\frac{3\pi}{2} + l^{DA}t^{DA}\cos^2 2\pi$$

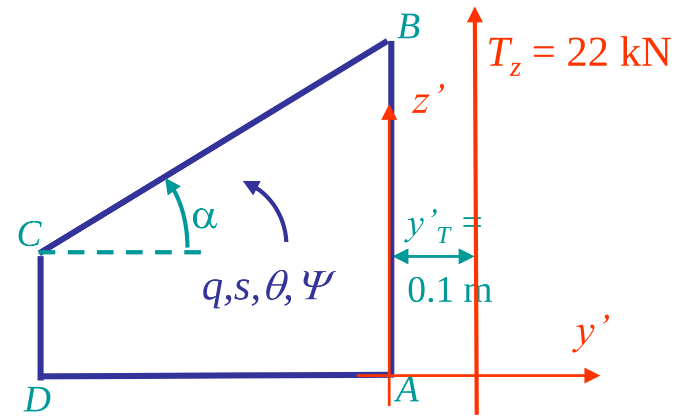
Wall	Length (m)	Thickness (mm)
AB	0.375	1.6
BC	0.500	1.0
CD	0.125	1.2
DA		1.0

$$\Rightarrow \oint t\cos^2\Psi ds = 0.5\ 10^{-3}\ \frac{3}{4} + 0.5\frac{\sqrt{3}}{2}\ 10^{-3} = 0.81\ 10^{-3}\ \text{m}^2$$

Closed-section beam

- Deformation (4)
  - Coefficients (4)

$$\oint t \sin^2 \Psi ds = \int_A^B t^{AB} \sin^2 \Psi^{AB} ds^{AB} + \int_B^C t^{BC} \sin^2 \Psi^{BC} ds^{BC} + \int_C^D t^{CD} \sin^2 \Psi^{CD} ds^{CD} + \int_D^A t^{DA} \sin^2 \Psi^{DA} ds^{DA}$$



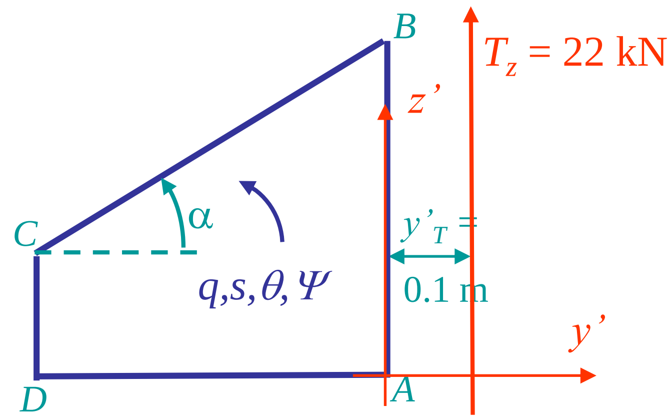
$$\Rightarrow \oint t \sin^2 \Psi ds = l^{AB} t^{AB} \sin^2 \frac{\pi}{2} + l^{BC} t^{BC} \sin^2 \frac{7\pi}{6} + -l^{CD} t^{CD} \sin^2 \frac{3\pi}{2} + l^{DA} t^{DA} \sin^2 2\pi$$

Wall	Length (m)	Thickness (mm)
AB	0.375	1.6
BC	0.500	1.0
CD	0.125	1.2
DA		1.0

$$\Rightarrow \oint t \sin^2 \Psi ds = 0.375\ 1.6\ 10^{-3} + 0.5\ 10^{-3}\ \frac{1}{4} + 0.125\ 1.2\ 10^{-3} = 0.88\ 10^{-3}\ \mathrm{m}^2$$

- Deformation (5)
  - Coefficients (5)

$$\oint t \sin\Psi \cos\Psi ds = \int_A^B t^{AB} \frac{\sin 2\Psi^{AB}}{2} ds^{AB} + \int_B^C t^{BC} \frac{\sin 2\Psi^{BC}}{2} ds^{BC} + \int_C^D t^{CD} \frac{\sin 2\Psi^{CD}}{2} ds^{CD} + \int_D^A t^{DA} \frac{\sin 2\Psi^{DA}}{2} ds^{DA}$$



$$\Rightarrow \oint t \sin\Psi \cos\Psi ds = l^{AB} t^{AB} \frac{\sin \pi}{2} + l^{BC} t^{BC} \frac{\sin \frac{7\pi}{3}}{2} + l^{CD} t^{CD} \frac{\sin 3\pi}{2} + l^{DA} t^{DA} \frac{\sin 4\pi}{2}$$

Wall	Length (m)	Thickness (mm)
AB	0.375	1.6
BC	0.500	1.0
CD	0.125	1.2
DA		1.0

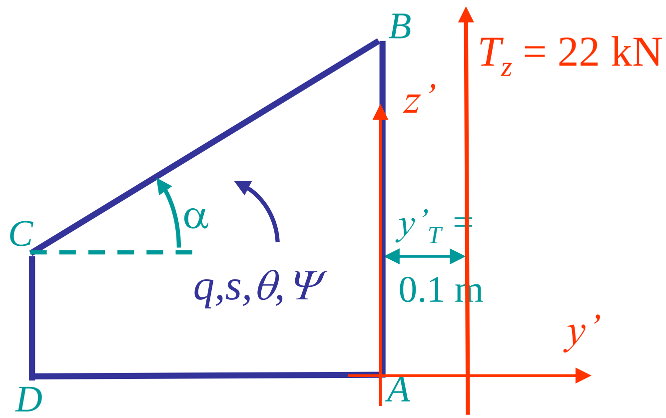
$$\Rightarrow \oint t \sin\Psi \cos\Psi ds = 0.5\ 10^{-3} \frac{\sqrt{3}}{4} = 0.22\,10^{-3}\ \text{m}^2$$

- Deformation (6)
  - Coefficients (6)

$$\oint tp^2 ds = \int_B^C p_A^2 t^{BC} ds^{BC} + \int_C^D p_A^2 t^{CD} ds^{CD}$$

$$\Rightarrow \oint tp^2 ds = l^{BC} t^{BC} \left(l^{AB} \cos\frac{\pi}{6} \right)^2 + l^{CD} t^{CD} \left(l^{BC} \cos\frac{\pi}{6} \right)^2$$

$$\Rightarrow \oint tp^2 ds = 0.5\ 10^{-3}\ 0.375^2\ \frac{3}{4} + 0.125\ 1.2\ 10^{-3}\ 0.5^2\ \frac{3}{4} = 0.081\ 10^{-3}\ \text{m}^4$$



Wall	Length (m)	Thickness (mm)
AB	0.375	1.6
BC	0.500	1.0
CD	0.125	1.2
DA		1.0

- Deformation (7)
  - System with origin of the axis at point A ($C \Longrightarrow A$)
    - $$\boldsymbol{T}_y = \frac{\partial \theta}{\partial x} \oint \mu t p \cos\Psi ds + \frac{\partial \boldsymbol{u}_y^C}{\partial x} \oint \mu t \cos^2\Psi ds + \frac{\partial \boldsymbol{u}_z^C}{\partial x} \oint \mu t \cos\Psi \sin\Psi ds$$

$$\Longrightarrow -0.14\,10^{-3}\ \mathrm{m}^3\ \mu\frac{\partial \theta}{\partial x} + 0.81\,10^{-3}\ \mathrm{m}^2\ \mu\frac{\partial \boldsymbol{u}_y^A}{\partial x} + 0.22\,10^{-3}\ \mathrm{m}^2\ \mu\frac{\partial \boldsymbol{u}_z^A}{\partial x} = 0$$

- $$\boldsymbol{T}_z = \frac{\partial \theta}{\partial x} \oint \mu t p \sin\Psi ds + \frac{\partial \boldsymbol{u}_y^C}{\partial x} \oint \mu t \cos\Psi \sin\Psi ds + \frac{\partial \boldsymbol{u}_z^C}{\partial x} \oint \mu t \sin^2\Psi ds$$

$$\Longrightarrow -0.15\,10^{-3}\ \mathrm{m}^3\ \mu\frac{\partial \theta}{\partial x} + 0.22\,10^{-3}\ \mathrm{m}^2\ \mu\frac{\partial \boldsymbol{u}_y^A}{\partial x} + 0.88\,10^{-3}\ \mathrm{m}^2\ \mu\frac{\partial \boldsymbol{u}_z^A}{\partial x} = 22\,10^3\ \mathrm{N}$$

- $$y_T \boldsymbol{T}_z - z_T \boldsymbol{T}_y = \frac{\partial \theta}{\partial x} \oint \mu t p^2 ds + \frac{\partial \boldsymbol{u}_y^C}{\partial x} \oint \mu t p \cos\Psi ds + \frac{\partial \boldsymbol{u}_z^C}{\partial x} \oint \mu t p \sin\Psi ds$$

$$\Longrightarrow 0.081\,10^{-3}\ \mathrm{m}^4\ \mu\frac{\partial \theta}{\partial x} - 0.14\,10^{-3}\ \mathrm{m}^3\ \mu\frac{\partial \boldsymbol{u}_y^A}{\partial x} - 0.15\,10^{-3}\ \mathrm{m}^3\ \mu\frac{\partial \boldsymbol{u}_z^A}{\partial x} = 2.2\,10^3 \mathrm{N}\cdot\mathrm{m}$$

- Deformation (8)
  - System (2)
    - $$-0.14\,10^{-3}\ \mathrm{m}^3\ \mu\frac{\partial\theta}{\partial x} + 0.81\,10^{-3}\ \mathrm{m}^2\ \mu\frac{\partial \boldsymbol{u}_y^A}{\partial x} + 0.22\,10^{-3}\ \mathrm{m}^2\ \mu\frac{\partial \boldsymbol{u}_z^A}{\partial x} = 0$$

      $$\Longrightarrow \mu\frac{\partial\theta}{\partial x} = 5.79\ \mathrm{m}^{-1}\ \mu\frac{\partial \boldsymbol{u}_y^A}{\partial x} + 1.57\ \mathrm{m}^{-1}\ \mu\frac{\partial \boldsymbol{u}_z^A}{\partial x}$$

    - $$-0.15\,10^{-3}\ \mathrm{m}^3\ \mu\frac{\partial\theta}{\partial x} + 0.22\,10^{-3}\ \mathrm{m}^2\ \mu\frac{\partial \boldsymbol{u}_y^A}{\partial x} + 0.88\,10^{-3}\ \mathrm{m}^2\ \mu\frac{\partial \boldsymbol{u}_z^A}{\partial x} = 22\,10^3\ \mathrm{N}$$

      $$\Longrightarrow -0.65\,10^{-3}\ \mathrm{m}^2\ \mu\frac{\partial \boldsymbol{u}_y^A}{\partial x} + 0.64\,10^{-3}\ \mathrm{m}^2\ \mu\frac{\partial \boldsymbol{u}_z^A}{\partial x} = 22\,10^3\ \mathrm{N}$$

      $$\Longrightarrow \mu\frac{\partial \boldsymbol{u}_y^A}{\partial x} = 0.98\mu\frac{\partial \boldsymbol{u}_z^A}{\partial x} - 33.85\,10^6\ \mathrm{N}\cdot\mathrm{m}^{-2}$$

      $$\Longrightarrow \mu\frac{\partial\theta}{\partial x} = 7.24\ \mathrm{m}^{-1}\ \mu\frac{\partial \boldsymbol{u}_z^A}{\partial x} - 196\,10^6\ \mathrm{N}\cdot\mathrm{m}^{-2}$$

    - $$0.081\,10^{-3}\ \mathrm{m}^4\ \mu\frac{\partial\theta}{\partial x} - 0.14\,10^{-3}\ \mathrm{m}^3\ \mu\frac{\partial \boldsymbol{u}_y^A}{\partial x} - 0.15\,10^{-3}\ \mathrm{m}^3\ \mu\frac{\partial \boldsymbol{u}_z^A}{\partial x} = 2.2\,10^3\mathrm{N}\cdot\mathrm{m}$$

      $$\Longrightarrow \mu\frac{\partial\theta}{\partial x} = 123\,10^6\ \mathrm{N}\cdot\mathrm{m}^{-3}$$

      $$\Longrightarrow \mu\frac{\partial \boldsymbol{u}_y^A}{\partial x} = 9.3\,10^6\ \mathrm{N}\cdot\mathrm{m}^{-2}$$

      $$\Longrightarrow \mu\frac{\partial \boldsymbol{u}_z^A}{\partial x} = 44\,10^6\ \mathrm{N}\cdot\mathrm{m}^{-2}$$

- Shear flux
  - $\frac{q}{\mu t} = p_A \frac{\partial \theta}{\partial x} + \frac{\partial \boldsymbol{u}_z^A}{\partial x} \sin \Psi + \frac{\partial \boldsymbol{u}_y^A}{\partial x} \cos \Psi$
  - Wall *AB*

$$q^{AB} = t^{AB} \mu \frac{\partial \boldsymbol{u}_z^A}{\partial x}$$

$$\Longrightarrow q^{AB} = 1.6\,10^{-3} 44\,10^6 = 70\,10^3 \text{ N} \cdot \text{m}^{-1}$$

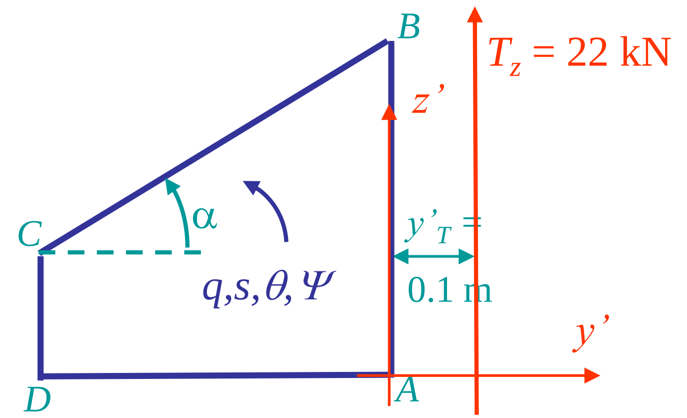
$$\Longrightarrow \tau^{AB} = \frac{q^{AB}}{t^{AB}} = 44 \text{ MPa}$$

  - Wall *DA*

$$q^{DA} = t^{DA} \mu \frac{\partial \boldsymbol{u}_y^A}{\partial x} \cos 2\pi$$

$$\Longrightarrow q^{DA} = 10^{-3}\ 9.3\,10^6 = 9.3\,10^3 \text{ N} \cdot \text{m}^{-1}$$

$$\Longrightarrow \tau^{DA} = \frac{q^{DA}}{t^{DA}} = \frac{9.3\,10^3}{10^{-3}} = 9.3 \text{ MPa}$$



Wall	Length (m)	Thickness (mm)
AB	0.375	1.6
BC	0.500	1.0
CD	0.125	1.2
DA		1.0

- Shear flux (2)

  – $$\frac{q}{\mu t} = p_A \frac{\partial \theta}{\partial x} + \frac{\partial \boldsymbol{u}_z^A}{\partial x} \sin \Psi + \frac{\partial \boldsymbol{u}_y^A}{\partial x} \cos \Psi$$

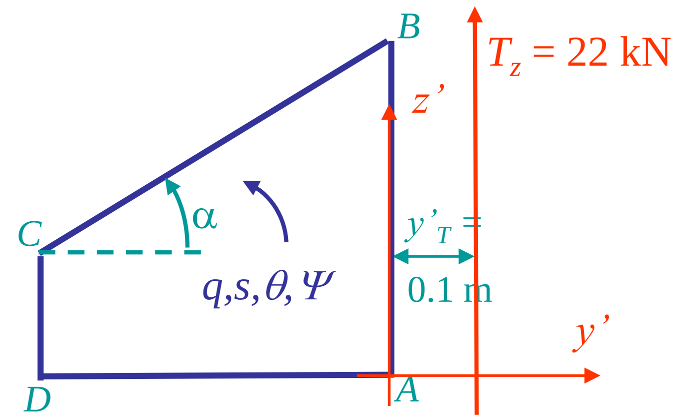
  – Wall *BC*

$$q^{BC} = p_A t^{BC} \mu \frac{\partial \theta}{\partial x} + t^{BC} \mu \frac{\partial \boldsymbol{u}_z^A}{\partial x} \sin \Psi^{BC} + t^{BC} \mu \frac{\partial \boldsymbol{u}_y^A}{\partial x} \cos \Psi^{BC}$$

$$\Longrightarrow q^{BC} = l^{AB} \cos \frac{\pi}{6} t^{BC} \mu \frac{\partial \theta}{\partial x} + t^{BC} \sin \frac{7\pi}{6} \mu \frac{\partial \boldsymbol{u}_z^A}{\partial x} + t^{BC} \cos \frac{7\pi}{6} \mu \frac{\partial \boldsymbol{u}_y^A}{\partial x}$$

$$\Longrightarrow q^{BC} = 0.375 \frac{\sqrt{3}}{2} 10^{-3}\, 123\, 10^6 - 10^{-3} \frac{1}{2}\, 44\, 10^6 - 10^{-3} \frac{\sqrt{3}}{2}\, 9.3\, 10^6 = 9.9\, 10^3 \text{ N}\cdot\text{m}^{-1}$$

$$\Longrightarrow \tau^{BC} = \frac{q^{BC}}{t^{BC}} = \frac{9.9\, 10^3}{10^{-3}} = 9.9 \text{ MPa}$$



Wall	Length (m)	Thickness (mm)
AB	0.375	1.6
BC	0.500	1.0
CD	0.125	1.2
DA		1.0

- Shear flux (3)

  - $$-\frac{q}{\mu t} = p_A \frac{\partial \theta}{\partial x} + \frac{\partial \boldsymbol{u}_z^A}{\partial x} \sin \Psi + \frac{\partial \boldsymbol{u}_y^A}{\partial x} \cos \Psi$$

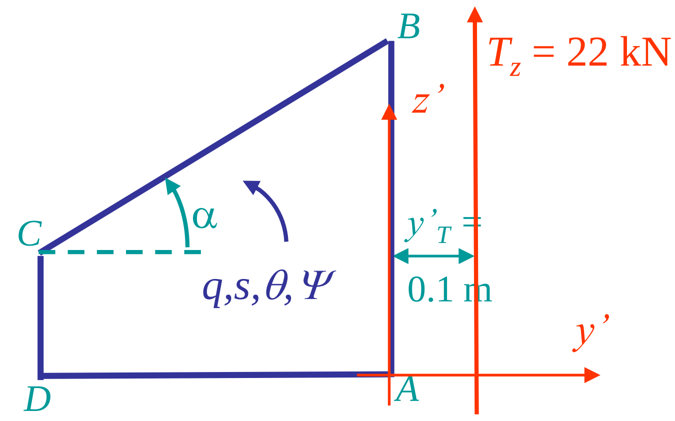
  - Wall *CD*

$$q^{CD} = p_A t^{CD} \mu \frac{\partial \theta}{\partial x} + t^{CD} \mu \frac{\partial \boldsymbol{u}_z^A}{\partial x} \sin \Psi^{CD} + t^{CD} \mu \frac{\partial \boldsymbol{u}_y^A}{\partial x} \cos \Psi^{CD}$$

$$\Rightarrow q^{CD} = l^{BC} \cos \frac{\pi}{6} t^{CD} \mu \frac{\partial \theta}{\partial x} + t^{CD} \sin \frac{3\pi}{2} \mu \frac{\partial \boldsymbol{u}_z^A}{\partial x}$$

$$\Rightarrow q^{CD} = 0.5 \frac{\sqrt{3}}{2}\, 1.2\, 10^{-3}\, 123\, 10^6 - 1.2\, 10^{-3}\, 44\, 10^6 = 11.1\, 10^3 \text{ N} \cdot \text{m}^{-1}$$

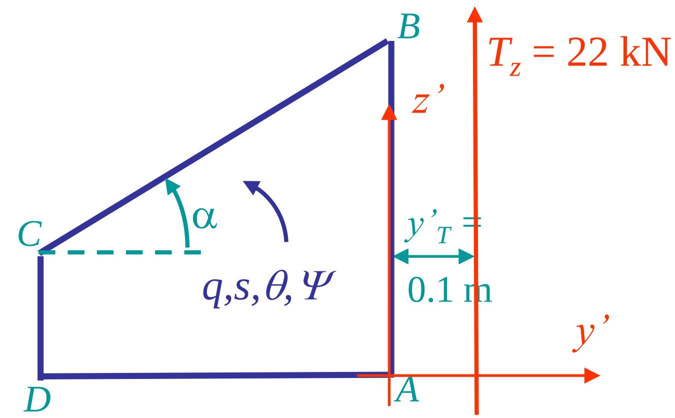
$$\Rightarrow \tau^{CD} = \frac{q^{CD}}{t^{CD}} = \frac{11.1\, 10^3}{1.2\, 10^{-3}} = 9.3 \text{ MPa}$$



Wall	Length (m)	Thickness (mm)
AB	0.375	1.6
BC	0.500	1.0
CD	0.125	1.2
DA		1.0

- Center of twist
  - System linked to point *A*

$$\Rightarrow \begin{cases} y_R = -\frac{\partial_x \boldsymbol{u}_z^A}{\partial_x \theta} = -\frac{44\,10^6}{123\,10^6} = -0.36 \text{ m} \\ z_R = \frac{\partial_x \boldsymbol{u}_y^A}{\partial_x \theta} = \frac{9.3\,10^6}{123\,10^6} = 0.076 \text{ m} \end{cases}$$



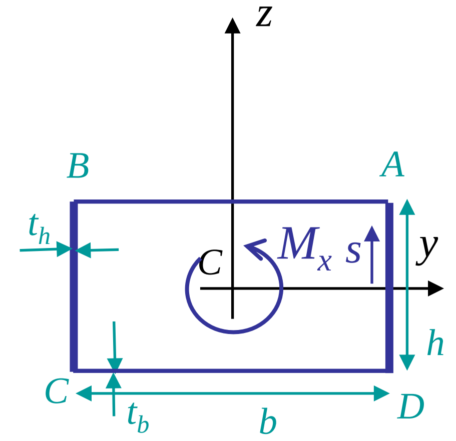
- Remarks
  - The center of twist
    - Depends on loading (y_T and T)
    - Does not correspond to the center of shear
    - Due to the warping constrain
  - Shear flux discontinuitiy at corners
    - Requires booms in order of avoiding stress concentrations

Wall	Length (m)	Thickness (mm)
AB	0.375	1.6
BC	0.500	1.0
CD	0.125	1.2
DA		1.0

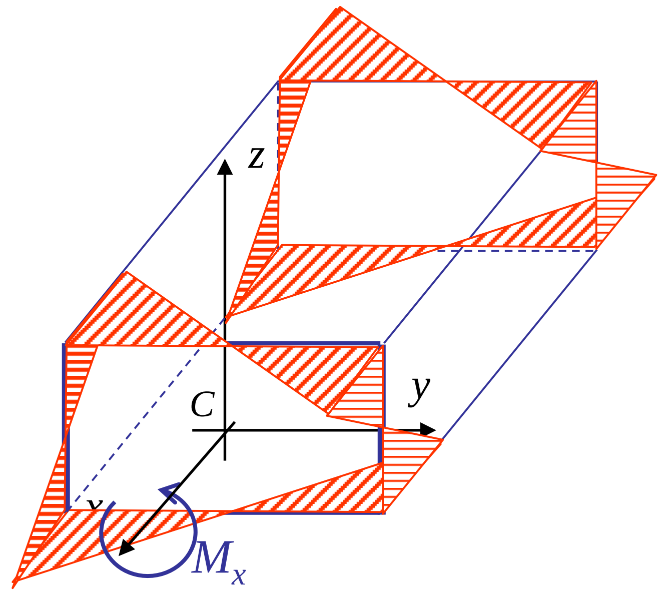
Closed-section beam

- Thin walled rectangular-section beam subjected to torsion
  – In the case of free warping, we found

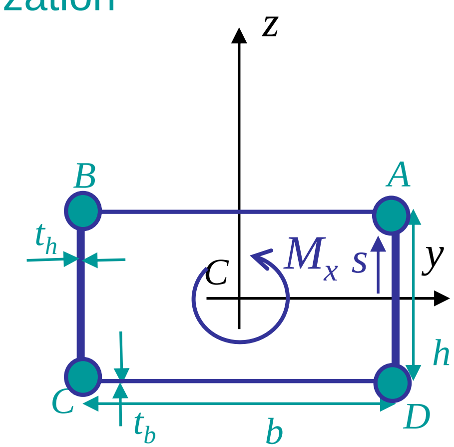
    - $$\begin{cases} \boldsymbol{u}_x^A = \boldsymbol{u}_x^C = \dfrac{M_x}{8\mu h b}\left(\dfrac{h}{t_h} - \dfrac{b}{t_b}\right) \\ \boldsymbol{u}_x^B = \boldsymbol{u}_x^D = \dfrac{M_x}{8\mu h b}\left(\dfrac{b}{t_b} - \dfrac{h}{t_h}\right) \end{cases}$$
    - $$\theta_{,x} = \frac{M_x}{2\mu h^2 b^2}\left(\frac{h}{t_h} + \frac{b}{t_b}\right)$$

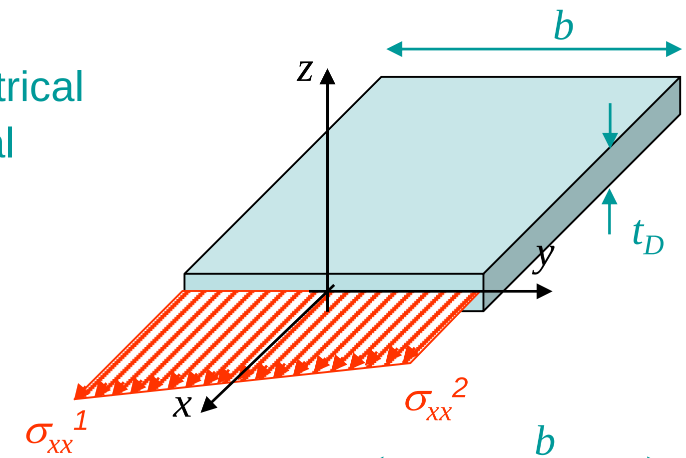


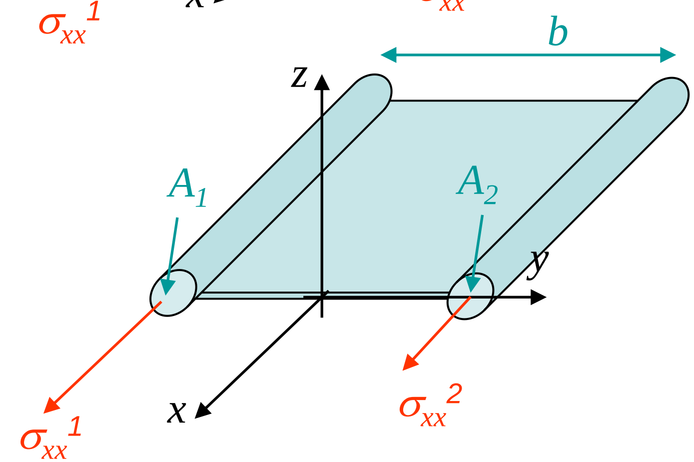
  – If warping is constrained (built-in end)
    - Direct stress are introduced
    - Different shear stress distribution



- Thin walled rectangular-section beam subjected to torsion (2)
  - Idealization
    - Warping to be suppressed is linear & symmetrical
      ⇒ Direct stress also linear & symmetrical
    - Idealization







- Four identical booms carrying direct stress only

$$A = \frac{bt_b}{6}(2-1) + \frac{ht_h}{6}(2-1)$$

$$\Rightarrow A = \frac{bt_b + ht_h}{6}$$

- Panels carry shear flux only

$$A_1 = \frac{t_D b}{6}\left(2 + \frac{\boldsymbol{\sigma}_{xx}^2}{\boldsymbol{\sigma}_{xx}^1}\right)$$

$$A_2 = \frac{t_D b}{6}\left(2 + \frac{\boldsymbol{\sigma}_{xx}^1}{\boldsymbol{\sigma}_{xx}^2}\right)$$

- Thin walled rectangular-section beam subjected to torsion (3)
  - Warping at a given section
    - Shearing (see beam lecture)

$$\begin{cases} q = \tau t = \mu t \gamma \\ \gamma = 2\boldsymbol{\varepsilon}_{xs} = \dfrac{\partial \boldsymbol{u}_s}{\partial x} + \dfrac{\partial \boldsymbol{u}_x}{\partial s} \end{cases}$$

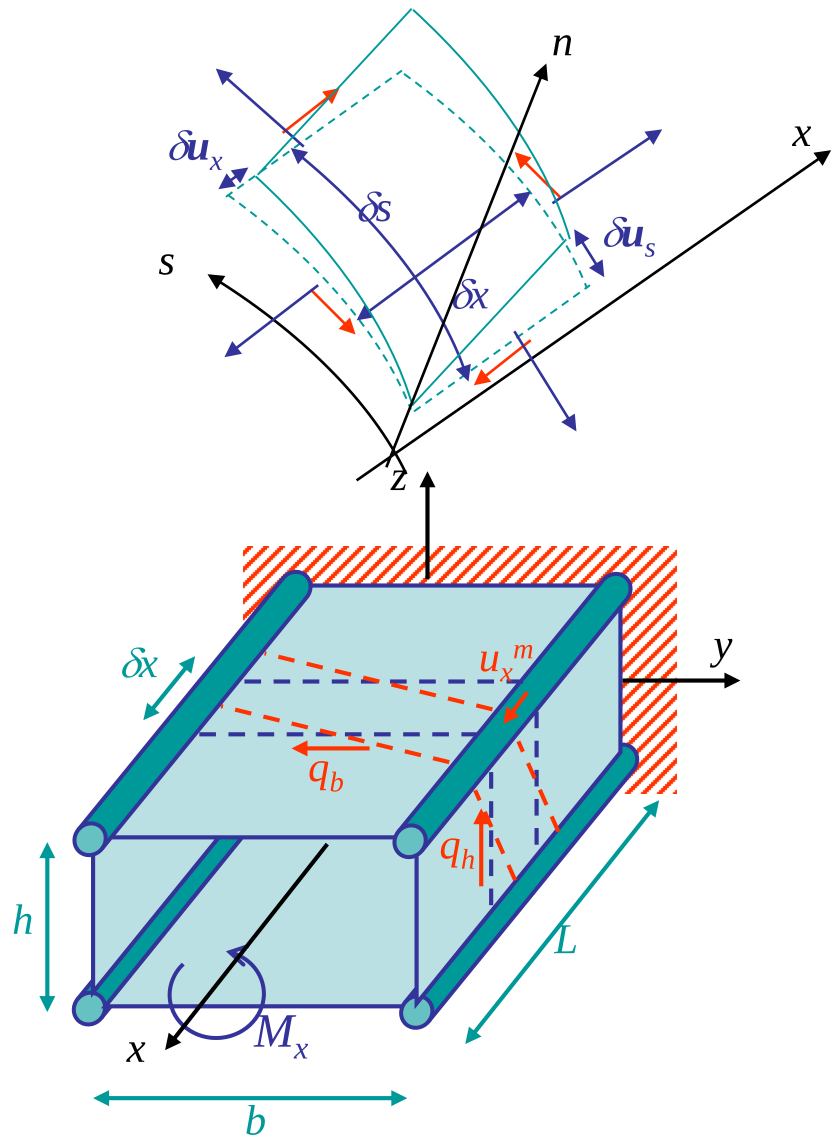
$$\Rightarrow \quad q = \mu t \left(\boldsymbol{u}_{s,x} + \boldsymbol{u}_{x,s} \right)$$

    - Warping
      - If u_x^m is the maximum warping
      - On webs

$$\boldsymbol{u}_{x,s} = \boldsymbol{u}_{x,z} = \frac{\boldsymbol{u}_x^m}{h/2}$$

      - On covers

$$\boldsymbol{u}_{x,s} = -\boldsymbol{u}_{x,y} = -\frac{\boldsymbol{u}_x^m}{b/2}$$

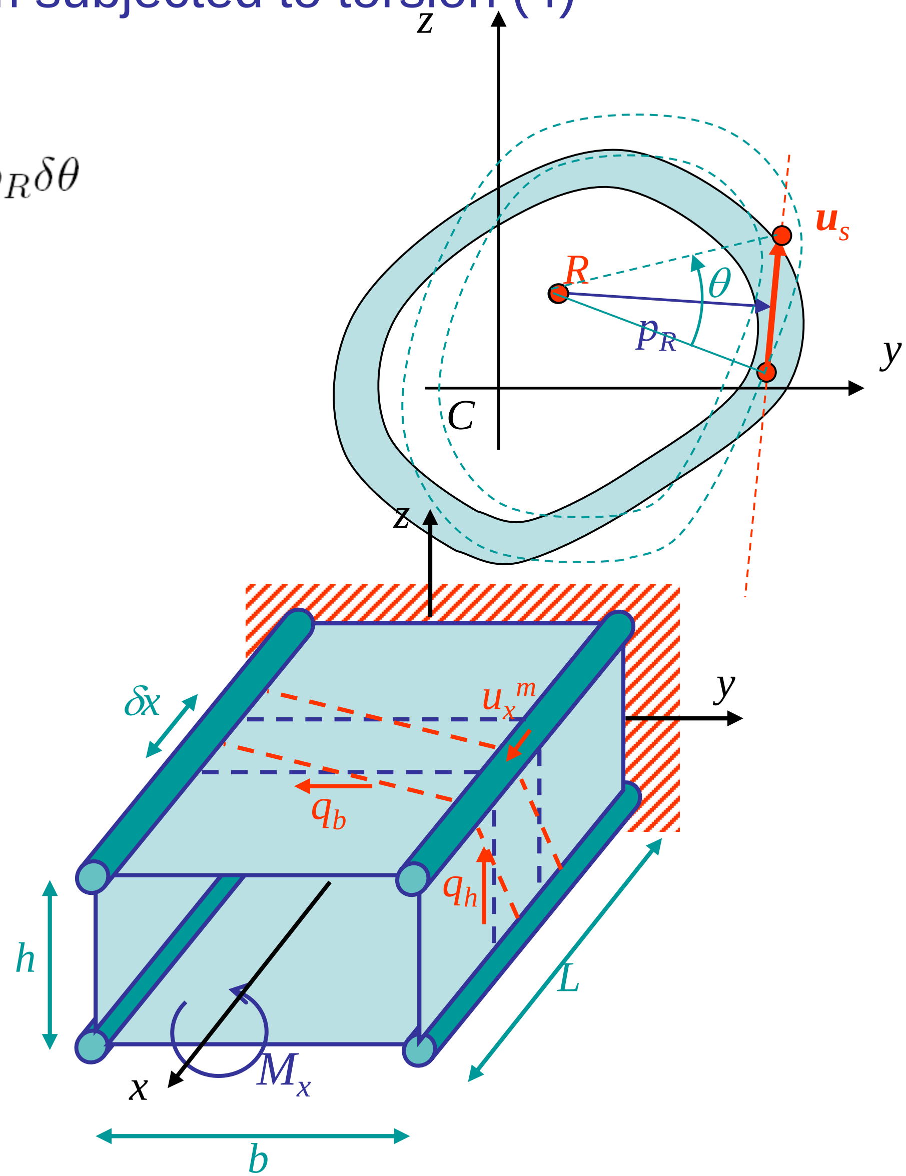


- Thin walled rectangular-section beam subjected to torsion (4)
  - Warping of a given section (2)
    - Kinematics
      - See lecture on beams $\delta \boldsymbol{u}_s = p_R \delta\theta$
    - As twist center is at section center (by symmetry)
      - On webs $\boldsymbol{u}_{s,x} = \frac{b}{2}\theta_{,x}$
      - On covers $\boldsymbol{u}_{s,x} = \frac{h}{2}\theta_{,x}$
  - Combining results
    - On webs

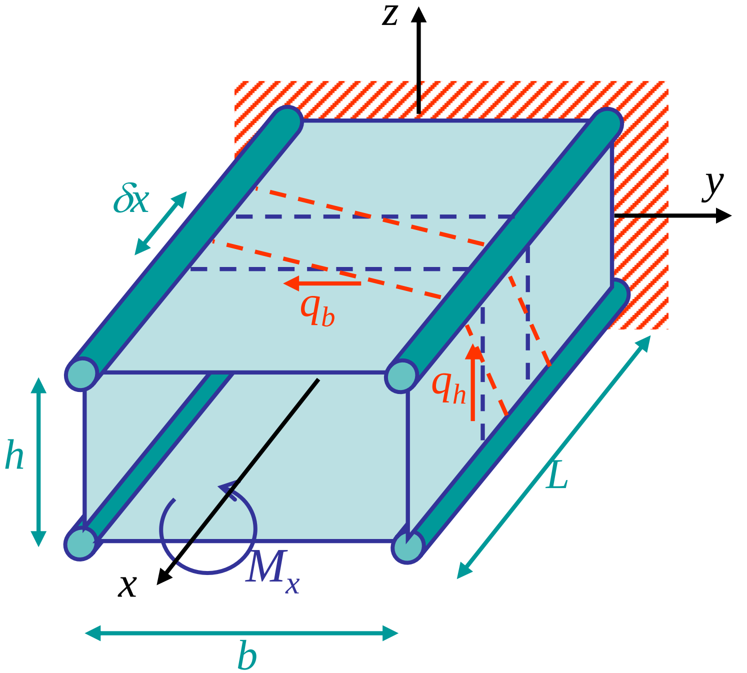
$$q_h = \mu t_h \left(\frac{b}{2}\theta_{,x} + \frac{2}{h}\boldsymbol{u}_x^m\right)$$

    - On covers

$$q_b = \mu t_b \left(\frac{h}{2}\theta_{,x} - \frac{2}{b}\boldsymbol{u}_x^m\right)$$



- Thin walled rectangular-section beam subjected to torsion (5)
  - Torque
    - From shear flow q_h & q_b



$$M_x = \oint q p_C \, ds = 2\frac{h}{2} q_b b + 2\frac{b}{2} q_h h$$

$$\Longrightarrow M_x = bh\left(q_b + q_h\right)$$

    - Using $\begin{cases} q_h = \mu t_h \left(\frac{b}{2}\theta_{,x} + \frac{2}{h}\boldsymbol{u}_x^m\right) \\ q_b = \mu t_b \left(\frac{h}{2}\theta_{,x} - \frac{2}{b}\boldsymbol{u}_x^m\right) \end{cases}$

$$\Longrightarrow M_x = \mu t_h \left(\frac{b^2 h}{2}\theta_{,x} + 2b\boldsymbol{u}_x^m\right) + \mu t_b \left(\frac{bh^2}{2}\theta_{,x} - 2h\boldsymbol{u}_x^m\right)$$

  - Twist rate is directly obtained

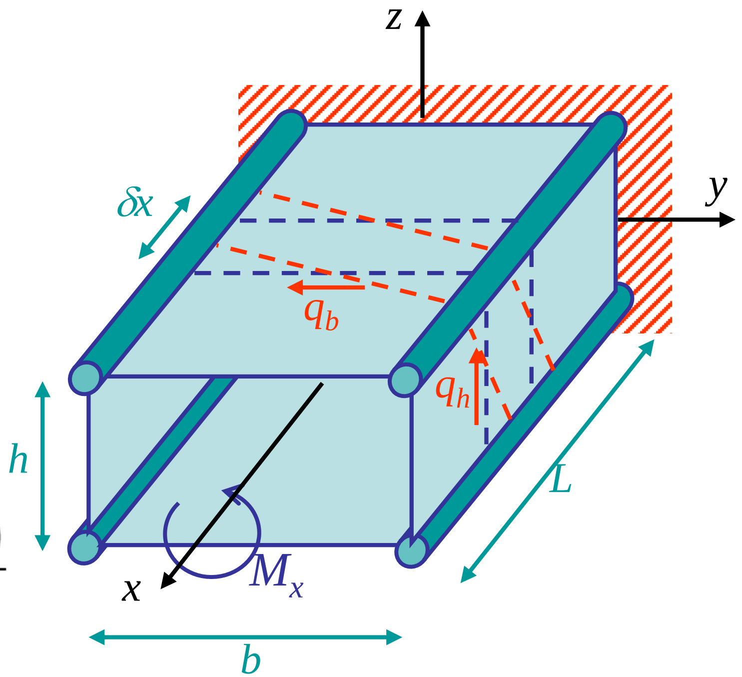
$$\theta_{,x} = \frac{2M_x}{\mu t_h b^2 h + \mu t_b h^2 b} + \frac{4\boldsymbol{u}_x^m\left(t_b h - t_h b\right)}{t_h b^2 h + t_b h^2 b}$$

- Thin walled rectangular-section beam subjected to torsion (6)
  - Shear flows
    - From shear flow q_h & q_b

$$\begin{cases} q_h = \mu t_h \left(\frac{b}{2}\theta_{,x} + \frac{2}{h}\boldsymbol{u}_x^m \right) \\ q_b = \mu t_b \left(\frac{h}{2}\theta_{,x} - \frac{2}{b}\boldsymbol{u}_x^m \right) \end{cases}$$

    - Using

$$\theta_{,x} = \frac{2M_x}{\mu t_h b^2 h + \mu t_b h^2 b} + \frac{4\boldsymbol{u}_x^m \left(t_b h - t_h b\right)}{t_h b^2 h + t_b h^2 b}$$



$$\Rightarrow \begin{cases} q_h = \frac{M_x t_h}{t_h b h + t_b h^2} + \boldsymbol{u}_x^m \frac{4\mu t_h t_b}{t_h b + t_b h} \\ q_b = \frac{M_x t_b}{t_h b^2 + t_b h b} - \boldsymbol{u}_x^m \frac{4\mu t_b t_h}{t_b h + t_h b} \end{cases}$$

  - Missing balance equation is obtained from boom balance

- Thin walled rectangular-section beam subjected to torsion (7)
  - Boom (of section A) balance equation
    - $(\boldsymbol{\sigma}_{xx} + \partial_x \boldsymbol{\sigma}_{xx} \delta x)\, A - \boldsymbol{\sigma}_{xx} A + q_b \delta x - q_h \delta x = 0$

$$\Longrightarrow \quad A \partial_x \boldsymbol{\sigma}_{xx} + q_b - q_h = 0$$

    - As boom carries direct stress only

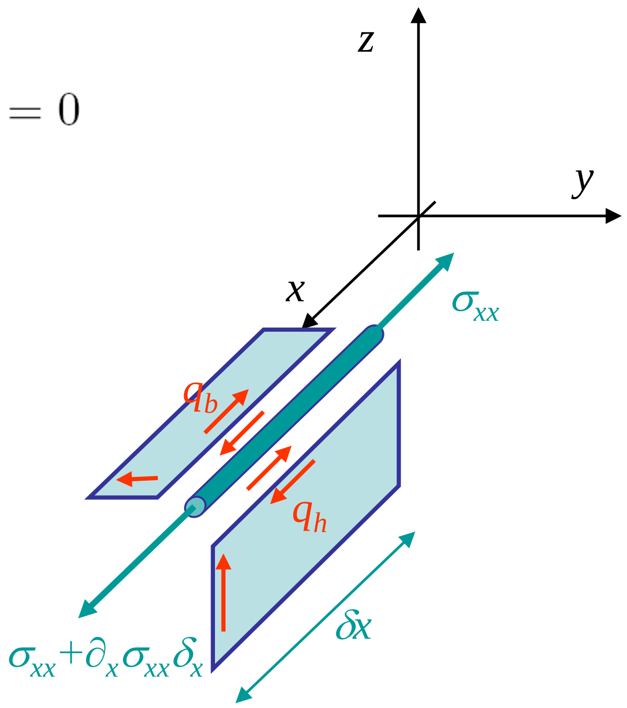
$$\boldsymbol{\sigma}_{xx} = E \partial_x \boldsymbol{u}_x^m$$

$$\Longrightarrow \quad EA \frac{\partial^2 \boldsymbol{u}_x^m}{\partial x^2} + q_b - q_h = 0$$

    - With

$$\begin{cases} q_h = \dfrac{M_x t_h}{t_h b h + t_b h^2} + \boldsymbol{u}_x^m \dfrac{4\mu t_h t_b}{t_h b + t_b h} \\ q_b = \dfrac{M_x t_b}{t_h b^2 + t_b h b} - \boldsymbol{u}_x^m \dfrac{4\mu t_b t_h}{t_b h + t_h b} \end{cases}$$

$$\Longrightarrow \quad EA \frac{\partial^2 \boldsymbol{u}_x^m}{\partial x^2} + \frac{M_x}{hb} \frac{t_b h - t_h b}{t_h b + t_b h} - \frac{8\mu t_h t_b}{t_h b + t_b h} \boldsymbol{u}_x^m = 0$$



Closed-section beam

- Thin walled rectangular-section beam subjected to torsion (8)
  - Differential equation
    - $$\frac{\partial^2 \boldsymbol{u}_x^m}{\partial x^2} - w^2 \boldsymbol{u}_x^m = -\frac{M_x}{EAhb}\frac{t_b h - t_h b}{t_h b + t_b h} \quad \text{with} \quad w^2 = \frac{1}{EA}\frac{8\mu t_h t_b}{t_h b + t_b h}$$
  - Solution
    - General form $\boldsymbol{u}_x^m(x) = C_1 \cosh wx + C_2 \sinh wx + \frac{M_x}{8\mu hb}\frac{t_b h - t_h b}{t_h t_b}$
    - Boundary conditions at $x = 0$ (constraint warping)

$$\boldsymbol{u}_x^m(0) = C_1 + \frac{M_x}{8\mu hb}\frac{t_b h - t_h b}{t_h t_b} = 0 \Longrightarrow C_1 = -\frac{M_x}{8\mu hb}\frac{t_b h - t_h b}{t_h t_b}$$

    - Boundary conditions at $x = L$ (free edge)

$$\partial_x \boldsymbol{u}_x^m(L) = wC_1 \sinh wL + wC_2 \cosh wL = 0$$

$$\Longrightarrow C_2 = -C_1 \tanh wL = \frac{M_x}{8\mu hb}\frac{t_b h - t_h b}{t_h t_b}\tanh wL$$

    - Final form

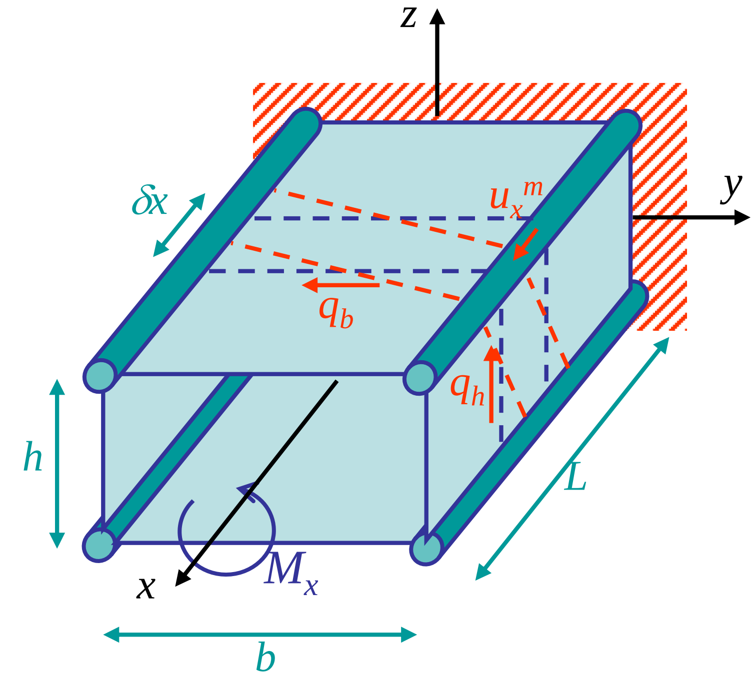
$$\boldsymbol{u}_x^m(x) = \frac{M_x}{8\mu hb}\frac{t_b h - t_h b}{t_h t_b}(1 + \tanh wL \sinh wx - \cosh wx)$$

$$\Longrightarrow \boldsymbol{u}_x^m(x) = \frac{M_x}{8\mu hb}\frac{t_b h - t_h b}{t_h t_b}\left(1 - \frac{\cosh(wL - wx)}{\cosh wL}\right)$$

- Thin walled rectangular-section beam subjected to torsion (9)
  - Warping

$$\boldsymbol{u}_x^m(x) = \frac{M_x}{8\mu hb}\frac{t_b h - t_h b}{t_h t_b}\left(1 - \frac{\cosh(wL - wx)}{\cosh wL}\right)$$

- At free end: $\boldsymbol{u}_x^L = \boldsymbol{u}_x^m(L) = \frac{M_x}{8\mu hb}\frac{t_b h - t_h b}{t_h t_b}\left(1 - \frac{1}{\cosh wL}\right)$
- To be compared with the warping of the free-free beam
  - $\boldsymbol{u}_x^A = \boldsymbol{u}_x^C = \frac{M_x}{8\mu hb}\left(\frac{h}{t_h} - \frac{b}{t_b}\right)$
  - Same for $L \to \infty$

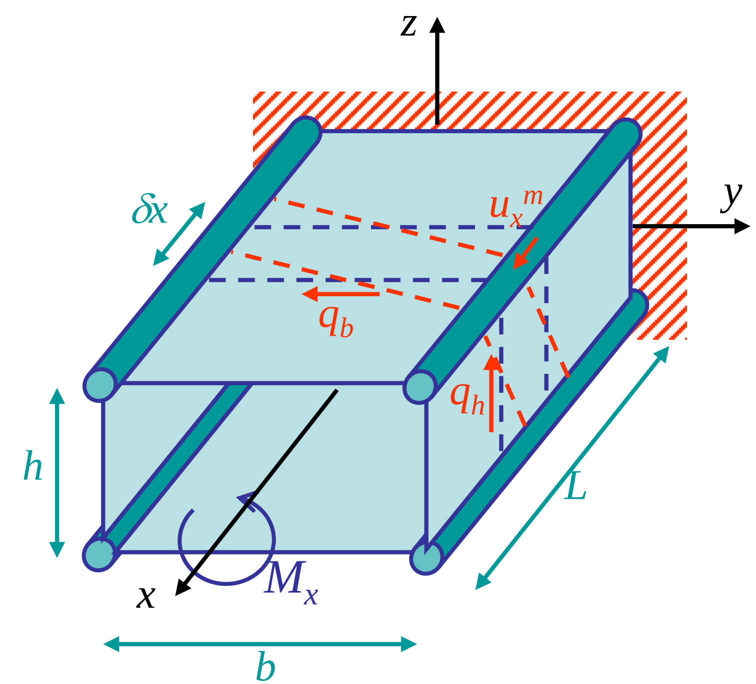


- Thin walled rectangular-section beam subjected to torsion (10)
  - Direct stress in booms

$$\boldsymbol{\sigma}_{xx} = E\partial_x \boldsymbol{u}_x^m = wE\frac{M_x}{8\mu hb}\frac{t_b h - t_h b}{t_h t_b}\frac{\sinh\left(wL - wx\right)}{\cosh wL}$$

  - Direct load in booms

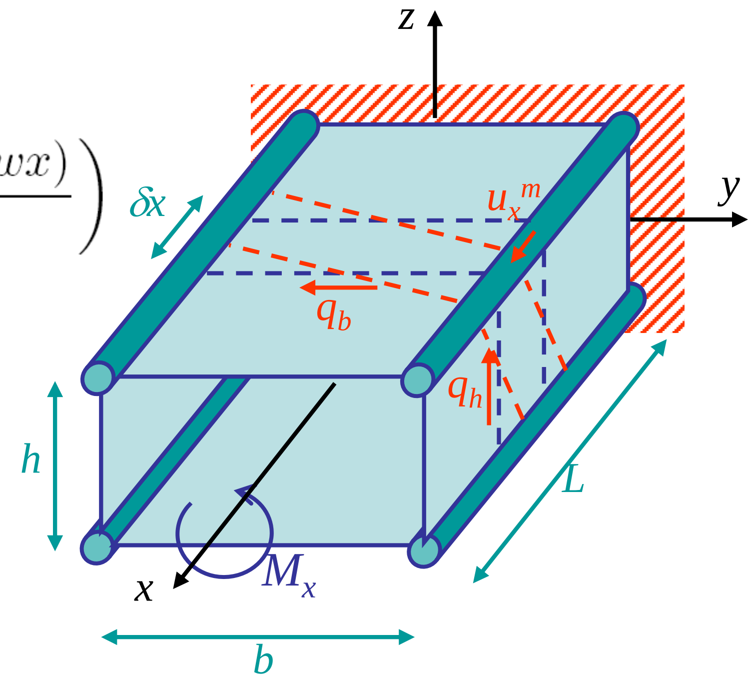
$$P_x = A\boldsymbol{\sigma}_{xx} = wEA\frac{M_x}{8\mu hb}\frac{t_b h - t_h b}{t_h t_b}\frac{\sinh\left(wL - wx\right)}{\cosh wL}$$



- Thin walled rectangular-section beam subjected to torsion (11)
  - Shear flow
    - Using

$$\boldsymbol{u}_x^m(x) = \frac{M_x}{8\mu hb}\frac{t_b h - t_h b}{t_h t_b}\left(1 - \frac{\cosh(wL - wx)}{\cosh wL}\right)$$

    - The shear flows becomes



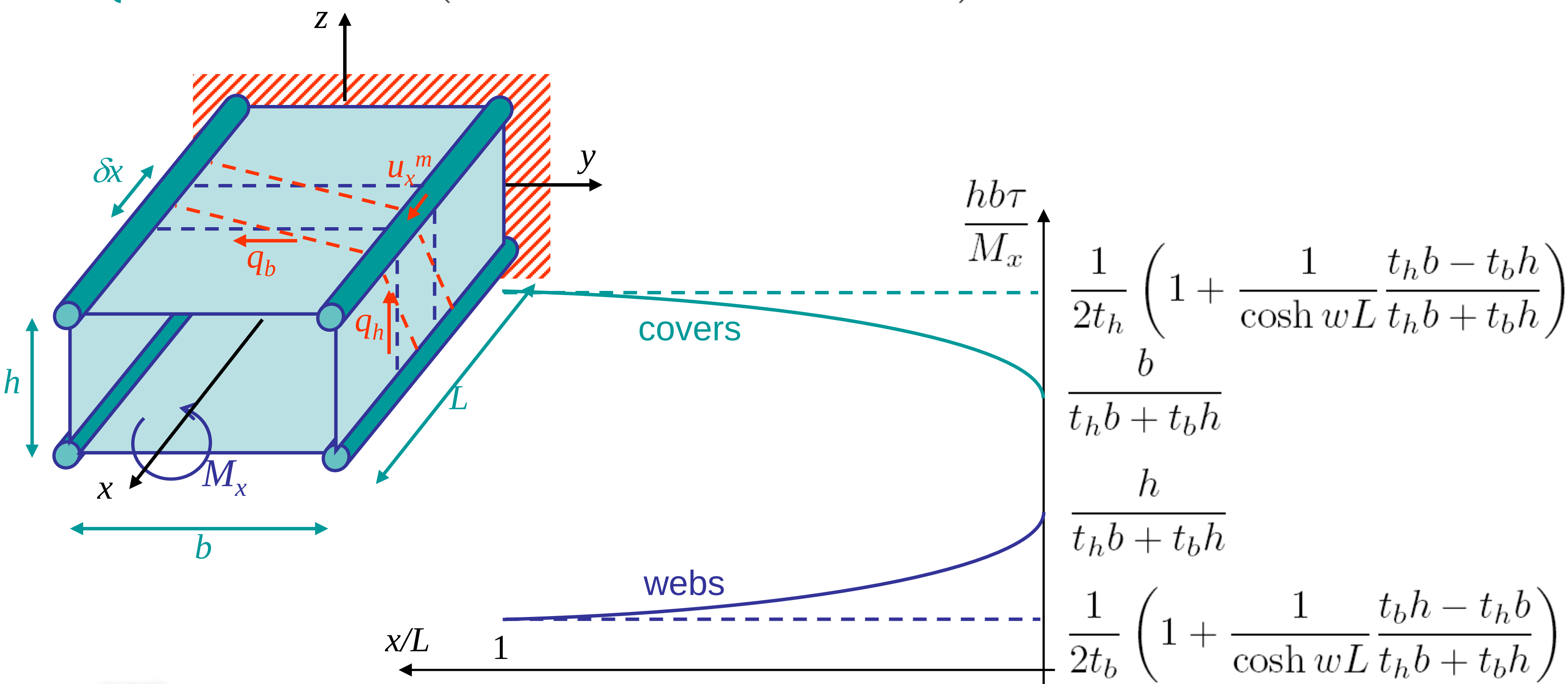
$$\begin{cases} q_h = \dfrac{M_x t_h}{t_h bh + t_b h^2} + \boldsymbol{u}_x^m \dfrac{4\mu t_h t_b}{t_h b + t_b h} \\ q_b = \dfrac{M_x t_b}{t_h b^2 + t_b hb} - \boldsymbol{u}_x^m \dfrac{4\mu t_b t_h}{t_b h + t_h b} \end{cases}$$

$$\Rightarrow \begin{cases} q_h = \dfrac{M_x}{2hb}\left(1 + \dfrac{t_h b - t_b h}{t_h b + t_b h}\dfrac{\cosh(wL - wx)}{\cosh wL}\right) \\ q_b = \dfrac{M_x}{2hb}\left(1 + \dfrac{t_b h - t_h b}{t_h b + t_b h}\dfrac{\cosh(wL - wx)}{\cosh wL}\right) \end{cases}$$

Closed-section beam

- Thin walled rectangular-section beam subjected to torsion (12)
  - Shear stress

$$\begin{cases} \tau_h = \dfrac{q_h}{t_h} = \dfrac{M_x}{2hbt_h}\left(1 + \dfrac{t_h b - t_b h}{t_h b + t_b h}\dfrac{\cosh\left(wL - wx\right)}{\cosh wL}\right) \\ \tau_b = \dfrac{q_b}{t_b} = \dfrac{M_x}{2hbt_b}\left(1 + \dfrac{t_b h - t_h b}{t_h b + t_b h}\dfrac{\cosh\left(wL - wx\right)}{\cosh wL}\right) \end{cases}$$



- Thin walled rectangular-section beam subjected to torsion (13)
  - Rate of twist
    - Using

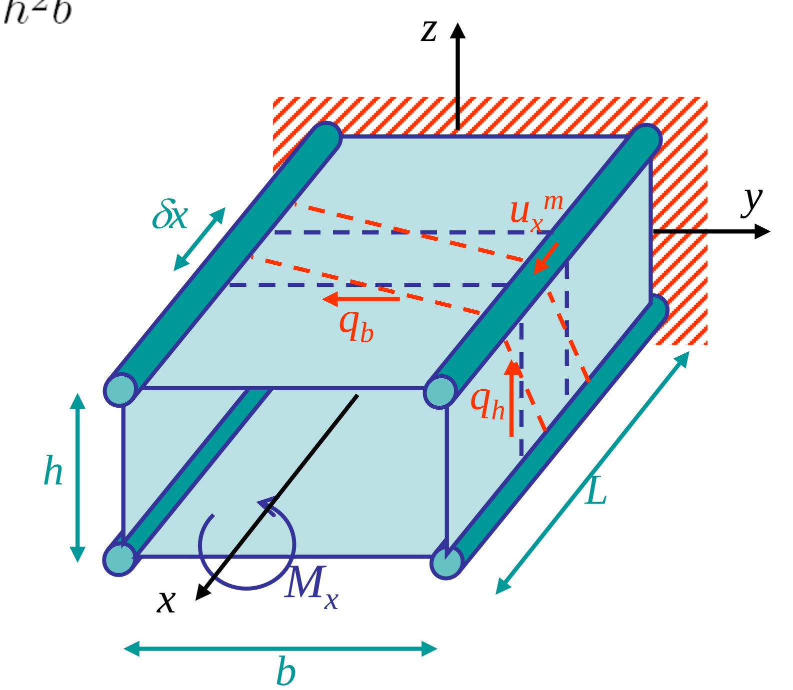
$$\boldsymbol{u}_x^m(x) = \frac{M_x}{8\mu h b}\frac{t_b h - t_h b}{t_h t_b}\left(1 - \frac{\cosh(wL - wx)}{\cosh wL}\right)$$

    - The rate of twist becomes

$$\theta_{,x} = \frac{2M_x}{\mu t_h b^2 h + \mu t_b h^2 b} + \frac{4\boldsymbol{u}_x^m (t_b h - t_h b)}{t_h b^2 h + t_b h^2 b}$$

$$\Rightarrow \theta_{,x} = \frac{M_x}{2\mu b^2 h^2 t_h t_b}\left[t_b h + t_h b - \frac{(t_b h - t_h b)^2}{t_h b + t_b h}\frac{\cosh(wL - wx)}{\cosh wL}\right]$$

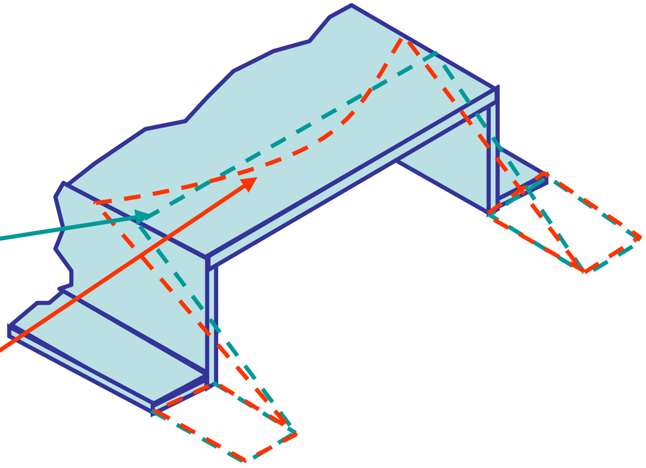
  - To be compared with the unconstraint theory
    - $$\theta_{,x} = \frac{M_x}{2\mu h^2 b^2}\left(\frac{h}{t_h} + \frac{b}{t_b}\right)$$
    - Constraint reduces the twist rate



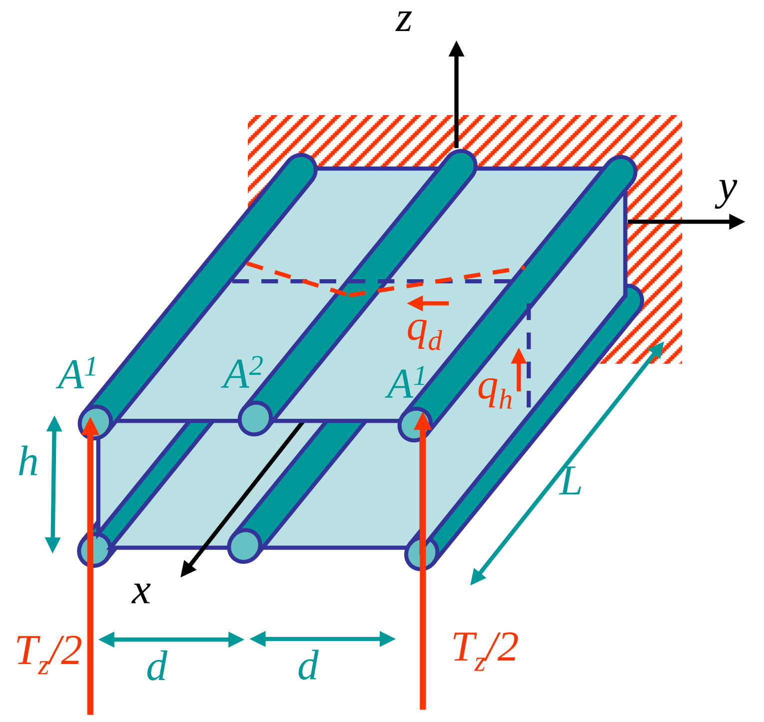
- Problem of axial constraint
  - In previous example the twist center was known by symmetry
  - In the general case
    - Twist center differs from shear center due to axial constraint
    - Proceed by increment of ΔL
      - Shear stress distribution calculated at the built-in section
        - » As in first example
        - » Allows determination of the twist center AT THAT SECTION
      - Use the previously developed theory on ΔL
      - New stress distribution on the new section
        - » New twist center
        - » …

Closed-section beam

- Shear lag
  - Beam shearing
    - Shear strain in cross-section
    - Deformation of cross-section
    - Elementary theory of bending
      - For pure bending
      - Not valid anymore due to cross section deformation
    - New distribution of direct stress
  - For wings
    - Wide & thin walled beam
    - Shear distortion of upper and lower skins causes redistribution of stress in the stringers



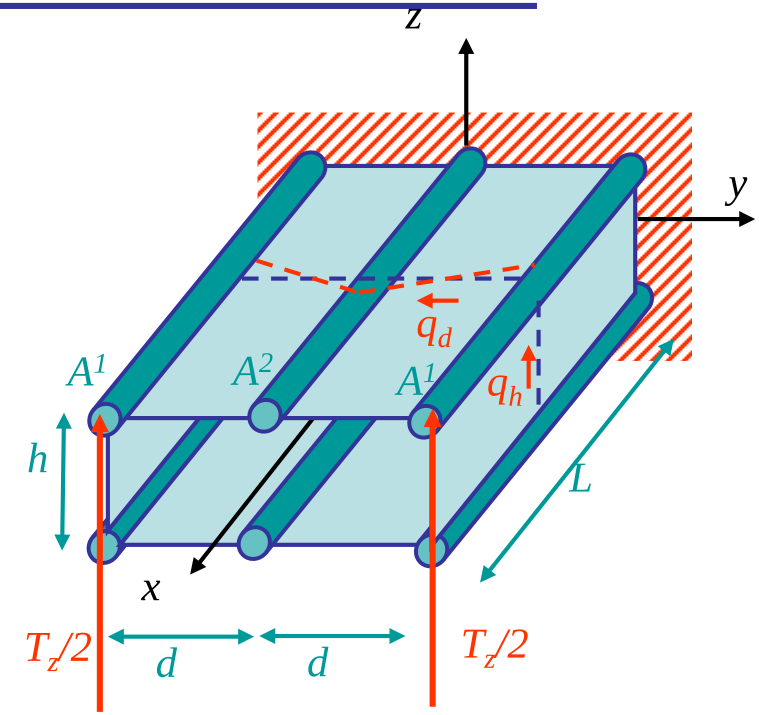
- Example
  - Assumptions
    - Doubly symmetrical 6-boom beam
    - Shear load trough shear center ⟹ No twist ⟹ No warping due to twist
    - Uniform panel thickness *t*
    - Shear loads applied at corner booms



- Shear lag (2)
  - For a given section
    - Uniform shear flow between booms
    - Shear flow in web should balance the shear load $\Longrightarrow q_h = \frac{T_z}{2h}$
    - Corner booms subjected to opposite loads P^1 , with, by equilibrium

$$P^1 + \partial_x P^1 \delta x - P^1 - q_h \delta x + q_d \delta x = 0$$

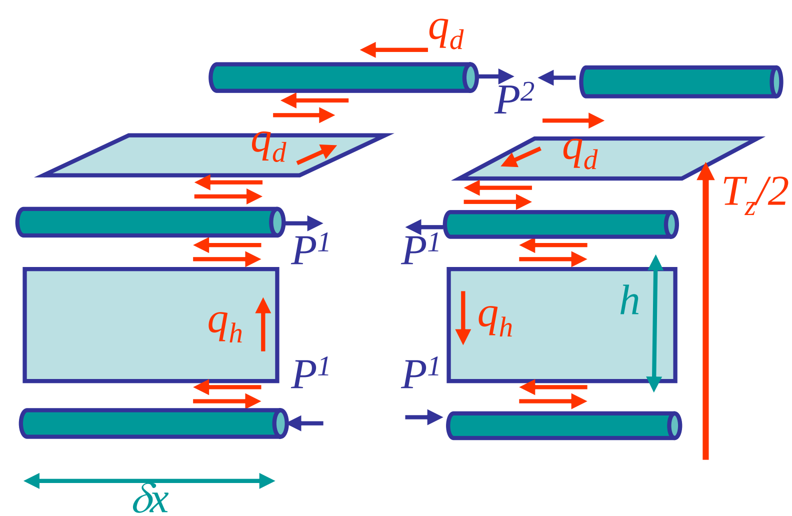
$$\Longrightarrow \partial_x P^1 = \frac{T_z}{2h} - q_d$$



    - Equilibrium of central boom
      - Due to symmetric distribution of q_d

$$P^2 + \partial_x P^2 \delta x - P^2 - 2 q_d \delta x = 0$$

$$\Longrightarrow \partial_x P^2 = 2 q_d$$

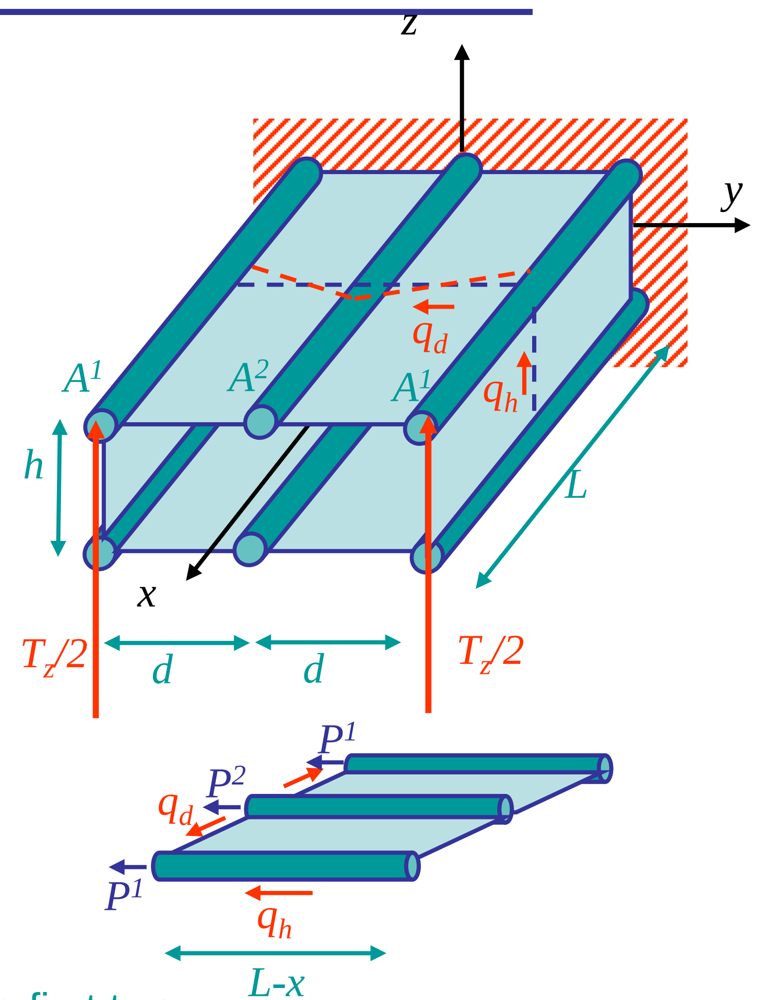


- Shear lag (3)
  - For a given section (2)
    - Equilibrium of the cover
      - At the free end

$$2P^1 + P^2 + 2q_h (L - x) = 0$$

$$\Longrightarrow 2P^1 + P^2 = \frac{T_z}{h}(x - L)$$

    - Summary
      - $\partial_x P^1 = \frac{T_z}{2h} - q_d$
      - $\partial_x P^2 = 2q_d$
      - $2P^1 + P^2 = \frac{T_z}{h}(x - L)$
      - Third equation is the integration of the first two
  - 3 unknowns so one equation is missing
    - Compatibility



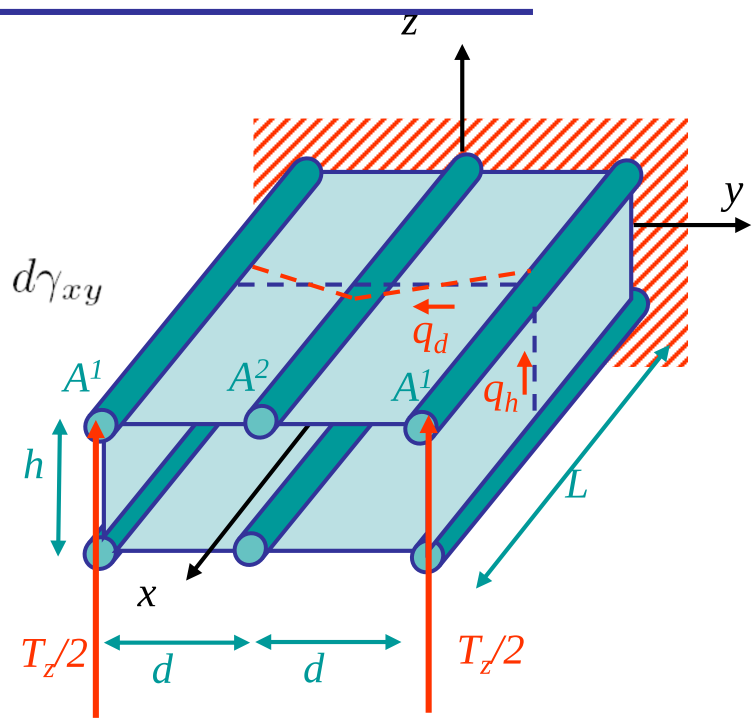
- Shear lag (4)
  - Deformations of top cover

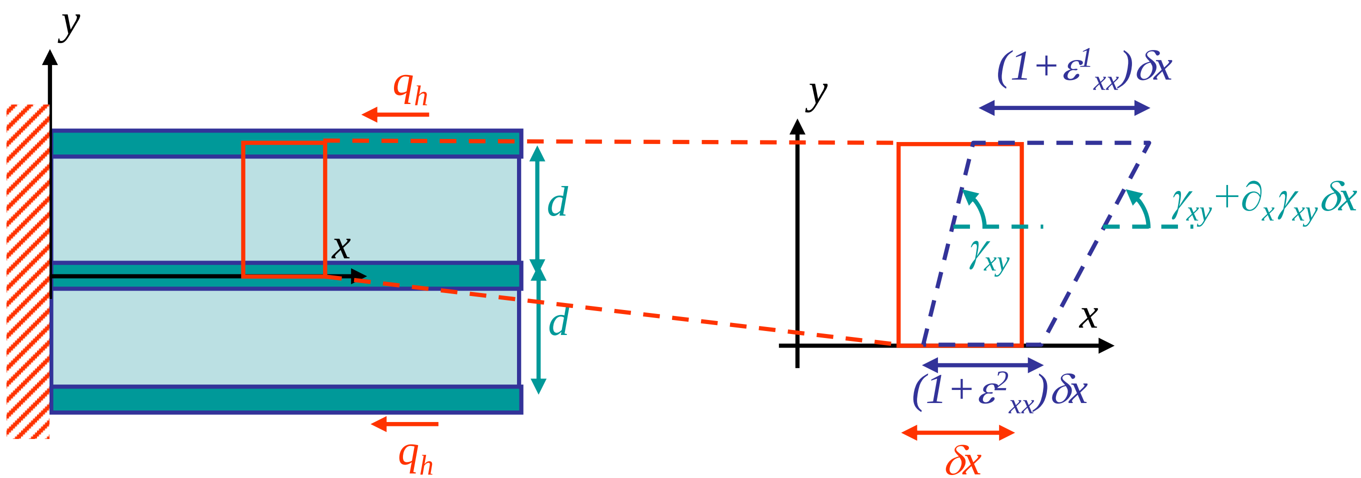
$$\left(1+\varepsilon_{xx}^{1}\right)\delta x=\left(1+\varepsilon_{xx}^{2}\right)\delta x+d\left(\gamma_{xy}+\partial_x\gamma_{xy}\delta x\right)-d\gamma_{xy}$$

$$\Rightarrow \partial_x\gamma_{xy}=\frac{\varepsilon_{xx}^{1}-\varepsilon_{xx}^{2}}{d}=\frac{P^1}{dEA^1}-\frac{P^2}{dEA^2}$$

- As $q_d=-\mu t\gamma_{xy}$

$$\Rightarrow -\frac{1}{\mu t}\partial_x q_d=\frac{P^1}{dEA^1}-\frac{P^2}{dEA^2}$$

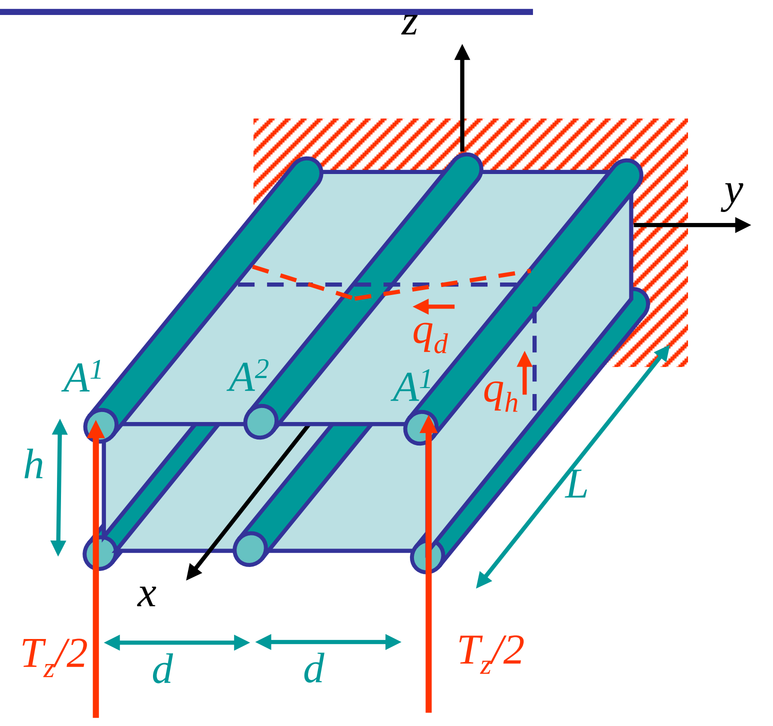




- Shear lag (5)
  - Equations
    - $\partial_x P^1 = \frac{T_z}{2h} - q_d$
    - $\partial_x P^2 = 2q_d$
    - $2P^1 + P^2 = \frac{T_z}{h}(x-L)$
    - $-\frac{1}{\mu t}\partial_x q_d = \frac{P^1}{dEA^1} - \frac{P^2}{dEA^2}$

$$\Rightarrow -\frac{1}{2\mu t}\partial^2_{xx}P^2 = \frac{\frac{T_z}{h}(x-L) - P^2}{2dEA^1} - \frac{P^2}{dEA^2}$$

$$\Rightarrow \partial^2_{xx}P^2 - \frac{2\mu t}{dE}\left(\frac{1}{A^2} + \frac{1}{2A^1}\right)P^2 = \frac{2\mu t T_z (L-x)}{2hdEA^1}$$



  - General solution
    - $P^2 = C_1 \cosh w(L-x) + C_2 \sinh w(L-x) - \frac{T_z (L-x)}{h\left(\frac{2A^1}{A^2}+1\right)}$

with $w^2 = \frac{2\mu t}{dE}\left(\frac{1}{A^2} + \frac{1}{2A^1}\right)$

- Shear lag (6)
  - General solution
    - $P^2 = C_1 \cosh w\,(L-x) + C_2 \sinh w\,(L-x) - \dfrac{T_z\,(L-x)}{h\left(\frac{2A^1}{A^2}+1\right)}$
  - Boundary conditions
    - Zero axial load at $x = L$ ⟹ $C_1 = 0$
    - Zero shear deformation at $x = 0$
      - As $\partial_x P^2 = 2q_d$ & $q_d = -\mu t \gamma_{xy}$

$$\partial_x P^2\,(x=0) = -C_2 w \cosh wL + \frac{T_z}{h\left(\frac{2A^1}{A^2}+1\right)} = 0$$

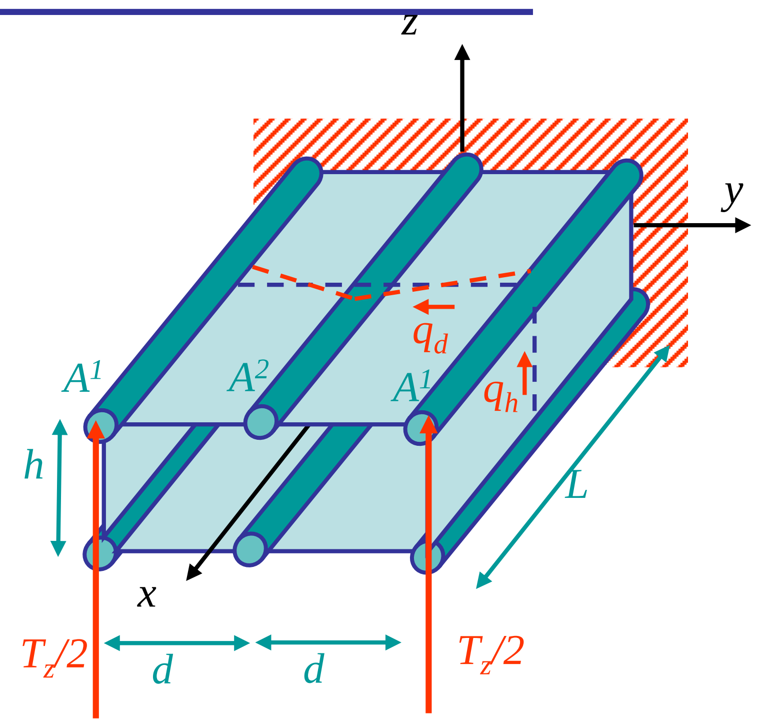
$$\Longrightarrow C_2 = \frac{T_z}{wh\cos wL\left(\frac{2A^1}{A^2}+1\right)}$$

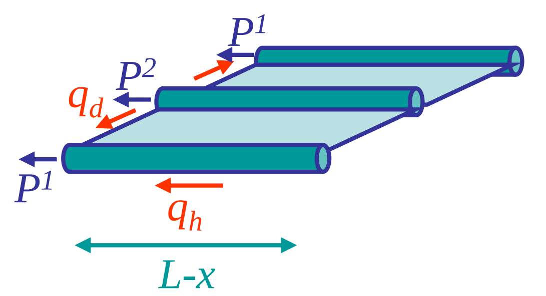
  - Booms direct loadings
    - $P^2 = -\dfrac{T_z}{h\left(\frac{2A^1}{A^2}+1\right)}\left(L - x - \dfrac{\sinh w\,(L-x)}{w\cosh wL}\right)$
    - $P^1 = \dfrac{T_z}{2h}\,(x-L) - \dfrac{P^2}{2} = \dfrac{T_z}{2h\left(\frac{2A^1}{A^2}+1\right)}\left(\dfrac{2A^1}{A^2}\,(x-L) - \dfrac{\sinh w\,(L-x)}{w\cosh wL}\right)$

- Shear lag (7)
  - Direct load in top cover
    - $$\boldsymbol{\sigma}^2 = \frac{P^2}{A^2} = -\frac{T_z}{h\left(2A^1 + A^2\right)} \left(L - x - \frac{\sinh w\left(L-x\right)}{w\cosh wL}\right)$$
    - $$\boldsymbol{\sigma}^1 = \frac{P^1}{A^1} = -\frac{T_z}{h\left(2A^1 + A^2\right)} \left(\left(L - x\right) + \frac{A^2}{2A^1}\frac{\sinh w\left(L-x\right)}{w\cosh wL}\right)$$
    - Pure bending theory leads to

$$\boldsymbol{\sigma}^1 = \boldsymbol{\sigma}^2 = -\frac{T_z}{h\left(2A^1 + A^2\right)}\left(L-x\right)$$

  - Compared to pure bending theory
    - Compression in central boom is lower
    - Compression in corner boom is higher





- Shear lag (8)
  - Shearing of top cover
    - As $\partial_x P^2 = 2q_d$

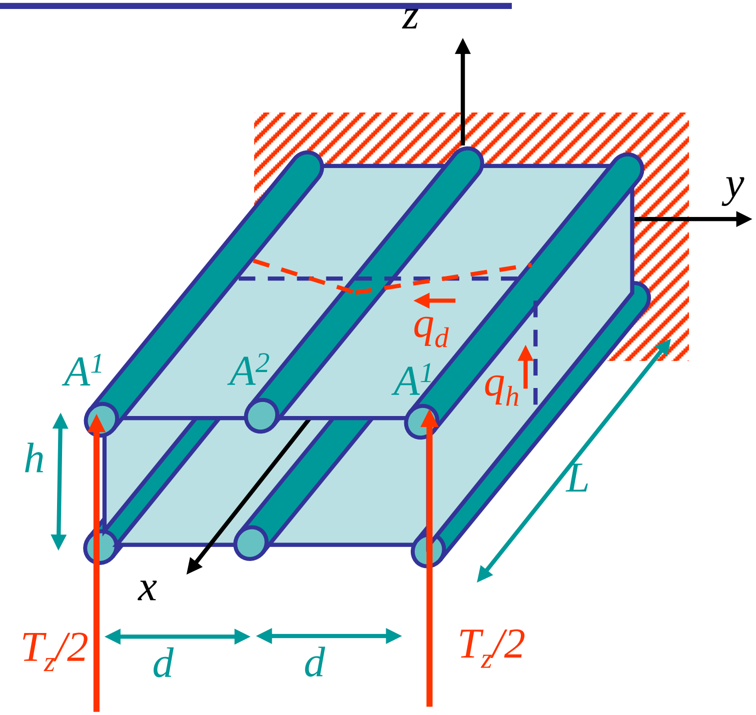
$$\Rightarrow \quad q_d = \frac{\partial_x P^2}{2}$$

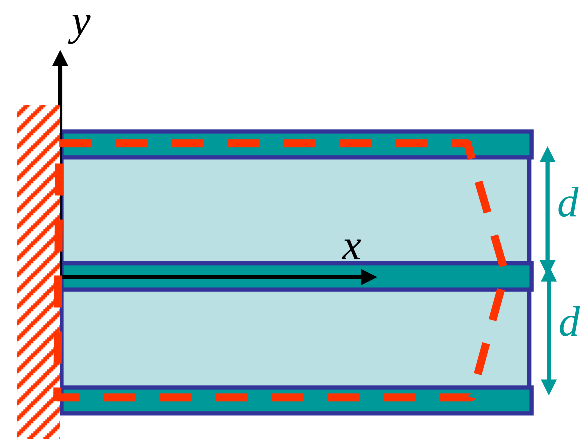
$$= \frac{T_z}{2h\left(\frac{2A^1}{A^2}+1\right)}\left(1 - \frac{\cosh w\,(L-x)}{\cosh wL}\right)$$

  - Deformation of top cover

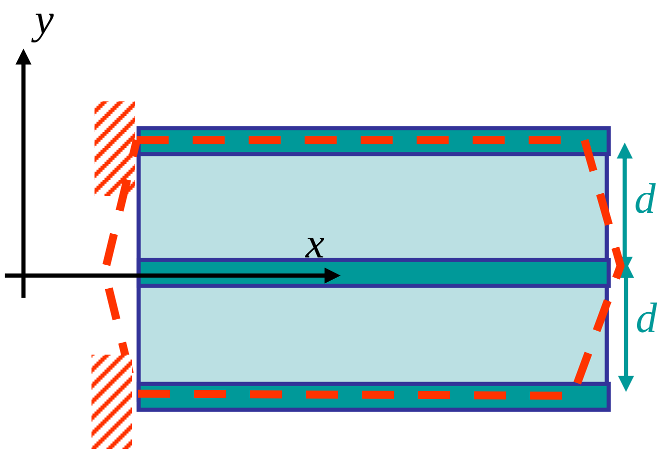
$$\gamma_{xy} = -\frac{q_d}{\mu t}$$

$$= -\frac{T_z}{2h\mu t\left(\frac{2A^1}{A^2}+1\right)}\left(1 - \frac{\cosh w\,(L-x)}{\cosh wL}\right)$$

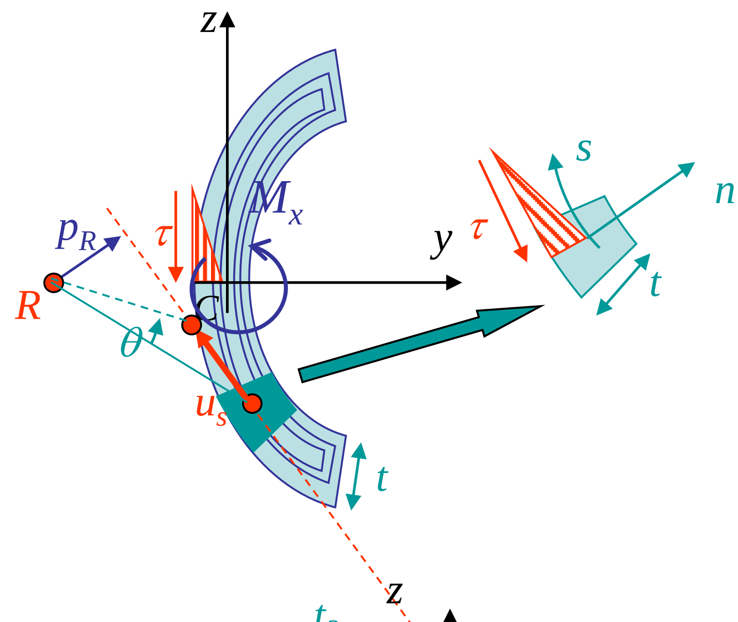




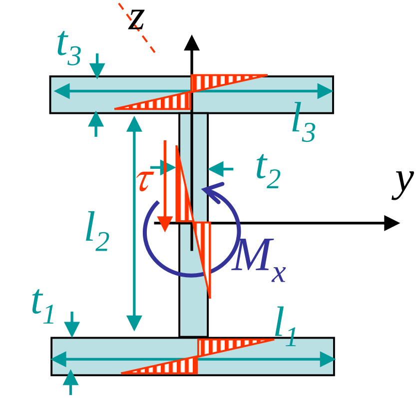
- Shear lag (9)
  - Remark
    - The solution depends on BCs
    - For a realistic wing structure, intermediate stringers have different BCs

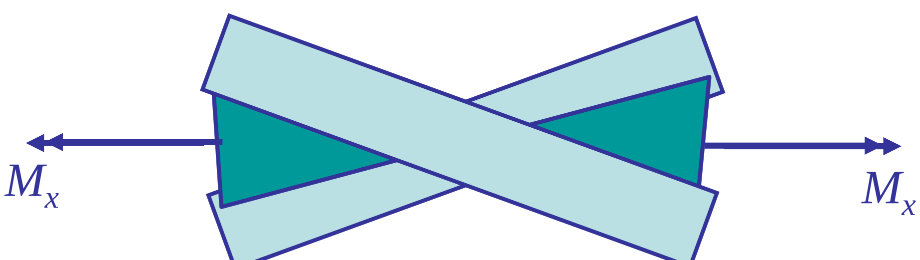


- I-section beam subjected to torsion without built-in end
  - Reminder
    - Shear
      - $\tau_{xs} = 2\mu n \theta_{,x}$
      - $C = \frac{M_x}{\theta_{,x}} = \frac{1}{3}\int \mu t^3 ds$
      - Or $\frac{M_x}{\theta_{,x}} = \sum_i \frac{l_i t_i^3 \mu}{3}$
    - Warping
      - $\boldsymbol{u}_x^s(s) = \boldsymbol{u}_x^s(0) - \theta_{,x}\int_0^s p_R ds'$

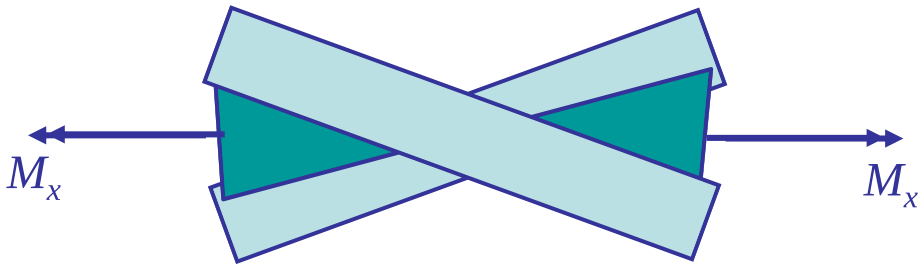


  - Particular case of the I-Section beam
    - There is no shear stress at mid plane of flanges
    - They remain rectangular after torsion

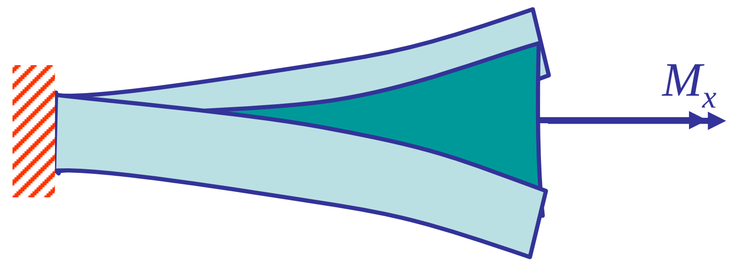




- I-section beam subjected to torsion with built-in end
  - Contrarily to the free/free beam



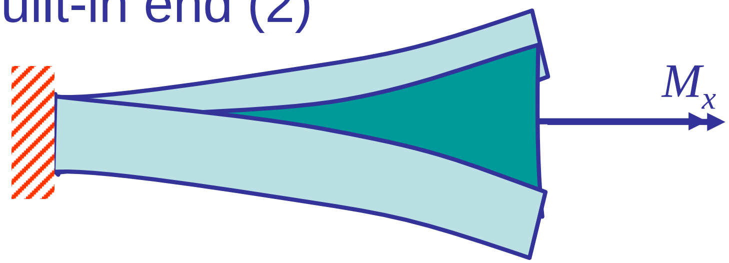
  - Presence of the built-end leads to deformation of the flanges



    - The beam still twists but with a non-constant twist rate
    - Method of solving: Combination of
      - Saint-Venant shear stress
      - Bending of flanges

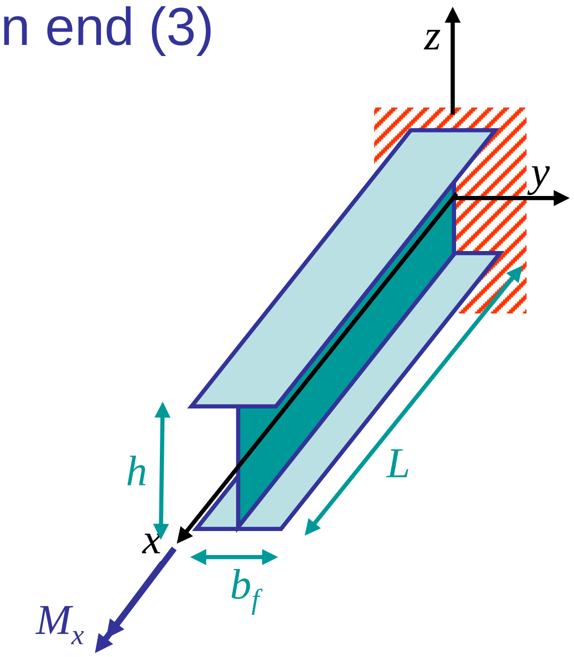
$$\Longrightarrow \quad M_x = M_x^t + M_x^b$$

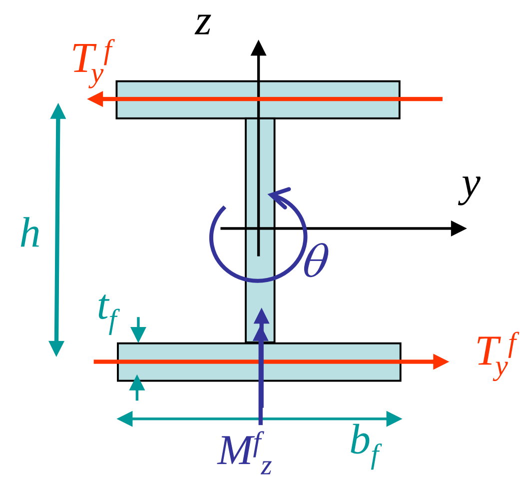
- I-section beam subjected to torsion with built-in end (2)
  - Saint-Venant shear stress
    - $M_x^t = C\theta_{,x}$
    - Where $\theta_{,x}$ is not constant



- I-section beam subjected to torsion with built-in end (3)
  - Bending of the flanges
    - For a given section
      - Angle of torsion θ
      - Lateral displacement of lower flange
        - » $\boldsymbol{u}_y = \frac{\theta h}{2}$
      - Bending moment in lower flange
        - » $M_z^f = EI_{zz}^f \boldsymbol{u}_{y,xx}$
        - » With $I_{zz}^f = \frac{t_f b_f^3}{12}$
        - » It has been assumed that displacement of the flange results from bending only
      - Shearing in the lower flange
        - » $T_y^f = -M_{z,x}^f = -EI_{zz}^f \boldsymbol{u}_{y,xxx}$

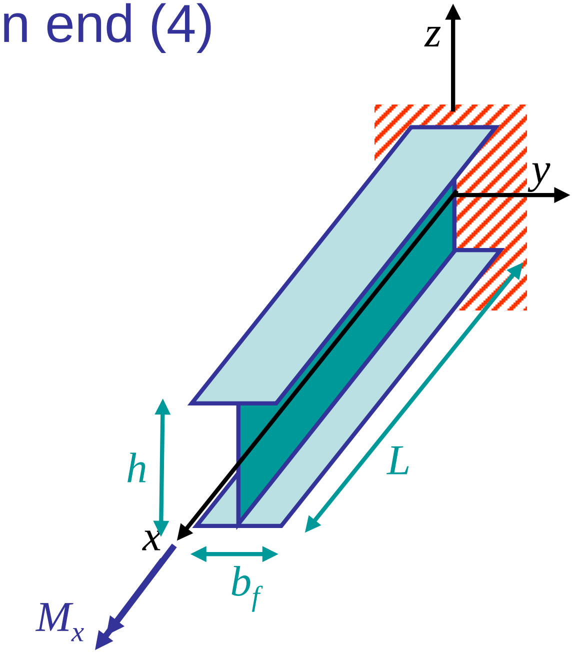
$$\Rightarrow T_y^f = -\frac{hEI_{zz}^f}{2}\theta_{,xxx}$$





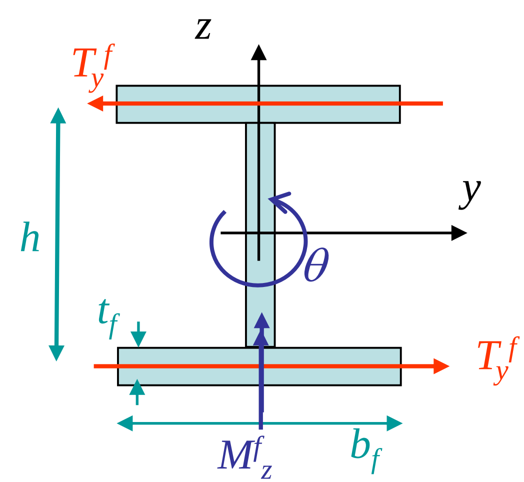
- I-section beam subjected to torsion with built-in end (4)
  - Bending of the flanges (2)
    - For a given section (2)
      - Shearing in the lower flange
        - » $T_y^f = -\frac{hEI_{zz}^f}{2}\theta_{,xxx}$
      - As shearing in top flange is in opposite direction, moment due to bending of the flange becomes

$$M_x^b = hT_y^f = -\frac{h^2EI_{zz}^f}{2}\theta_{,xxx}$$



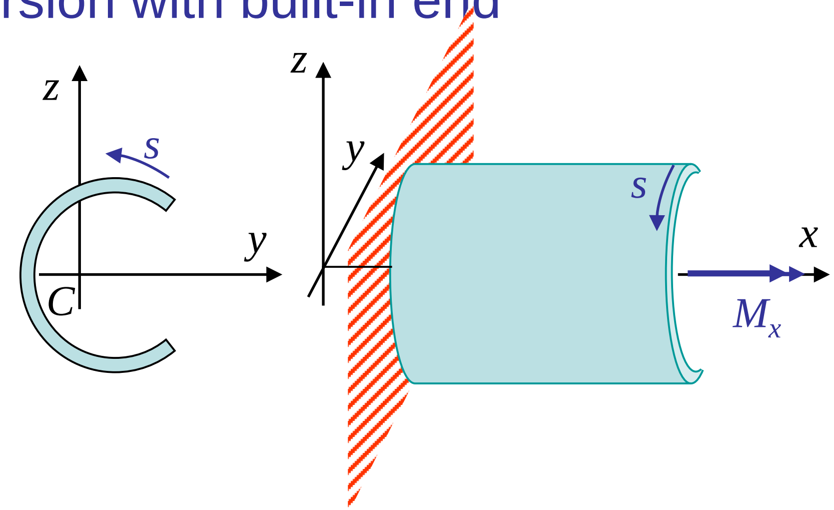
  - Total torque on the beam
    - $M_x = M_x^t + M_x^b$

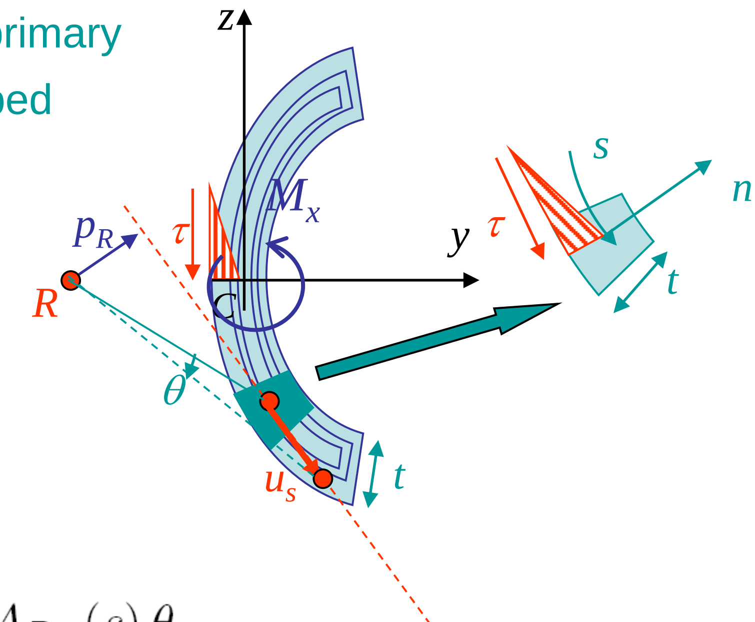
$$\Longrightarrow M_x = C\theta_{,x} - \frac{h^2EI_{zz}^f}{2}\theta_{,xxx}$$



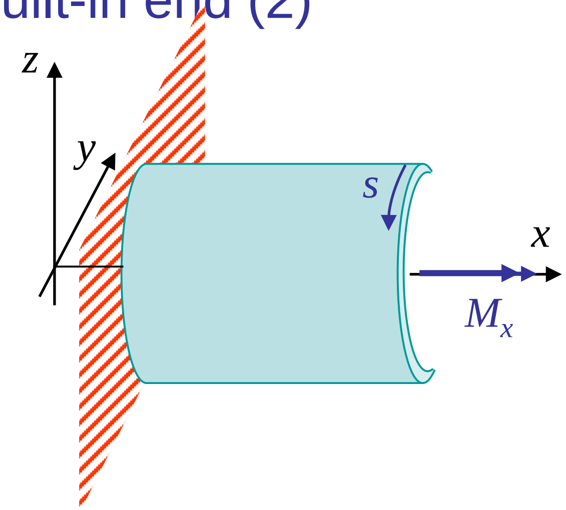
- Arbitrary-section beam subjected to torsion with built-in end
  - Wagner torsion theory
    - Assumptions
      - Length >> sectional dimensions
      - Undistorted cross-section
      - Shear stress at midsection negligible
        - » But shear load not negligible
    - Under these assumptions, we can use the primary warping (of mid section) expression developed for torsion of free/free open-section beams
      - $$\boldsymbol{u}_x^s(s) = \boldsymbol{u}_x^s(0) - \theta_{,x}\int_0^s p_R ds'$$
        $$= \boldsymbol{u}_x^s(0) - 2A_{R_p}(s)\,\theta_{,x}$$
    - As twist rate is not constant
      - There is a direct induced stress

$$\boldsymbol{\sigma}^{\Gamma}_{xx}(s) = E\boldsymbol{u}_{x,x} = E\boldsymbol{u}^s_{x,x}(0) - 2EA_{R_p}(s)\,\theta_{,xx}$$





- Arbitrary-section beam subjected to torsion with built-in end (2)
  - Wagner torsion theory (2)
    - Direct stress resulting from primary warping
      - $\boldsymbol{\sigma}^{\Gamma}_{xx}(s) = E\boldsymbol{u}^{s}_{x,x}(0) - 2EA_{R_p}(s)\,\theta_{,xx}$
    - As only a torsion couple is applied
      - Integrating on the whole section C x t should lead to 0

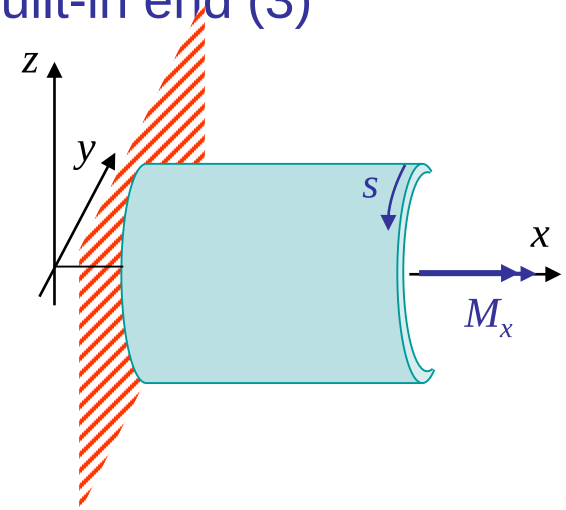


$$\Longrightarrow \int_C t\boldsymbol{\sigma}^{\Gamma} ds = 0$$

$$\Longrightarrow \boldsymbol{u}^{s}_{x,x}(0)\int_C Etds - \theta_{,xx}\int_C Et2A_{R_p}(s)\,ds = 0$$

$$\Longrightarrow \boldsymbol{u}^{s}_{x,x}(0) = \frac{\theta_{,xx}\int_C Et2A_{R_p}(s)\,ds}{\int_C Etds}$$

- Arbitrary-section beam subjected to torsion with built-in end (3)
  - Wagner torsion theory (3)
    - Direct stress resulting from primary warping (2)
      - $\boldsymbol{\sigma}^{\Gamma}_{xx}(s) = E\boldsymbol{u}^{s}_{x,x}(0) - 2EA_{R_p}(s)\,\theta_{,xx}$
    - As only a torsion couple is applied (2)
      - $\boldsymbol{u}^{s}_{x,x}(0) = \dfrac{\theta_{,xx}\int_C Et2A_{R_p}(s)\,ds}{\int_C Etds}$



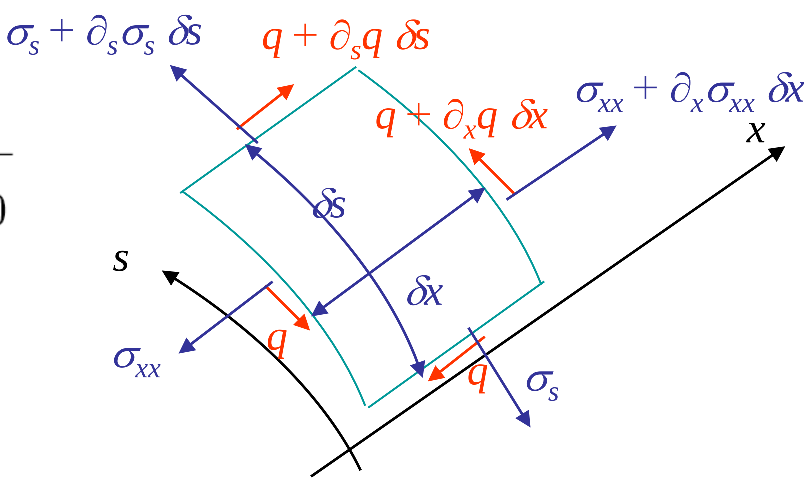
- Direct stress is equilibrated by shear flow
  - See lecture on beams

$$(\boldsymbol{\sigma}_{xx} + \partial_x\boldsymbol{\sigma}_{xx}\delta x)\,t\delta s - \boldsymbol{\sigma}_{xx}t\delta s + (q + \partial_s q\delta s)\,\delta_x - q\delta x = 0$$

$$\Longrightarrow t\partial_x\boldsymbol{\sigma}_{xx} + \partial_s q = 0$$

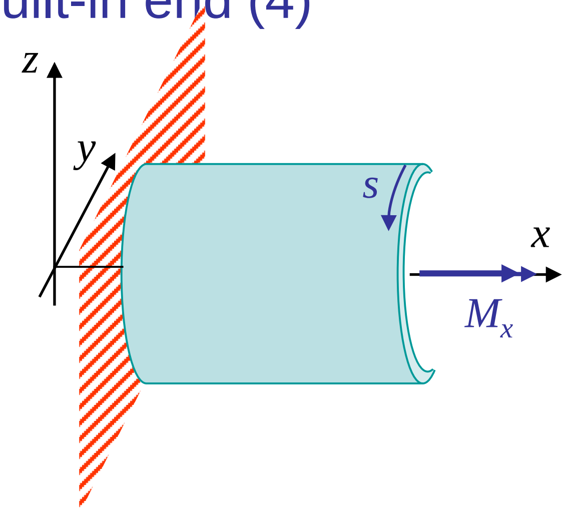
  - In this case

$$q^{\Gamma}_{,s} = -t\boldsymbol{\sigma}^{\Gamma}_{xx,x}$$



$$\Longrightarrow q^{\Gamma}_{,s}(s) = -Et\boldsymbol{u}^{s}_{x,xx}(0) + 2EtA_{R_p}(s)\,\theta_{,xxx}$$

- Arbitrary-section beam subjected to torsion with built-in end (4)
  - Wagner torsion theory (4)
    - Equations



$$-\ \boldsymbol{\sigma}^{\Gamma}_{xx}(s) = E\boldsymbol{u}^{s}_{x,x}(0) - 2EA_{R_p}(s)\,\theta_{,xx}$$

$$-\ \boldsymbol{u}^{s}_{x,x}(0) = \frac{\theta_{,xx}\int_C Et2A_{R_p}(s)\,ds}{\int_C Etds}$$

$$-\ q^{\Gamma}_{,s}(s) = -Et\boldsymbol{u}^{s}_{x,xx}(0) + 2EtA_{R_p}(s)\,\theta_{,xxx}$$

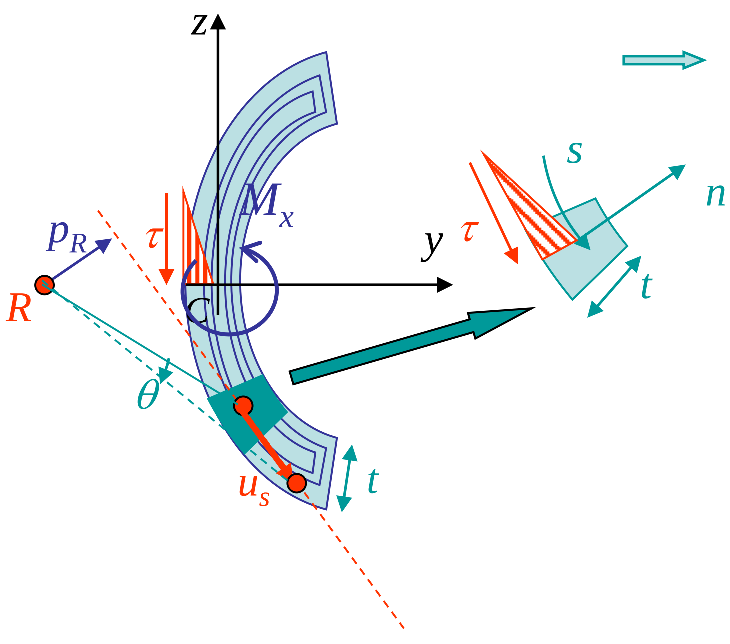
    - As for s = 0 (free edge) q(0) = 0

$$-\ q^{\Gamma}_{,s}(s) = \left(-\frac{\int_C Et2A_{R_p}(s)\,ds}{\int_C Etds} + 2A_{R_p}(s)\right)Et\theta_{,xxx}$$

$$\Longrightarrow q^{\Gamma}(s) = \left(-\frac{\int_C Et2A_{R_p}(s)\,ds}{\int_C Etds}Ets + \int_0^s 2EtA_{R_p}(s')\,ds'\right)\theta_{,xxx}$$

- Arbitrary-section beam subjected to torsion with built-in end (5)
  - Wagner torsion theory (5)
    - Torque
      - $M_x^b = \int_C p_R q^\Gamma(s)\, ds$
      - With $q^\Gamma(s) = \left(-\frac{\int_C Et2A_{R_p}(s)\, ds}{\int_C Etds} Ets + \int_0^s 2EtA_{R_p}(s')\, ds' \right) \theta_{,xxx}$

$$\Rightarrow M_x^b = \left(-\frac{\int_C Et2A_{R_p}(s)\, ds}{\int_C Etds} \int_C p_R Etsds + \int_C \left\{ p_R \int_0^s 2EtA_{R_p}(s')\, ds' \right\} ds \right) \theta_{,xxx}$$



- Arbitrary-section beam subjected to torsion with built-in end (6)
  - Wagner torsion theory (6)
    - Torque (2)

$$M_x^b = \left(-\frac{\int_C Et2A_{R_p}(s)\,ds}{\int_C Etds}\int_C p_R Etsds + \int_C \left\{p_R \int_0^s 2EtA_{R_p}(s')\,ds'\right\} ds\right)\theta_{,xxx}$$

    - Using $p_R = 2A_{R_p,s}$ the second term becomes

$$\int_C \left\{2A_{R_p,s}\int_0^s Et2A_{R_p}(s')\,ds'\right\} ds = 2A_{R_p}(s)\int_0^s Et2A_{R_p}(s')\,ds'\Big|_0^L - \int_C 4A_{R_p}^2 Etds$$

      - For $s = 0$, $A_{Rp} = 0$
      - For $s = L$, as the edge is free, there is no shear flux

$$\Longrightarrow 0 = q^\Gamma(L) = \left(-\frac{\int_C Et2A_{R_p}(s)\,ds}{\int_C Etds} EtL + \int_C Et2A_{R_p}(s')\,ds'\right)\theta_{,xxx}$$

      - Using these two boundary conditions, second term is rewritten

$$\int_C \left\{2A_{R_p,s}\int_0^s Et2A_{R_p}(s')\,ds'\right\} ds = \frac{\int_C Et2A_{R_p}(s)\,ds}{\int_C Etds} 2A_{R_p}(L)\,EtL - \int_C 4A_{R_p}^2 Etds$$

Open-section beam

- Arbitrary-section beam subjected to torsion with built-in end (7)
  - Wagner torsion theory (7)
    - Torque (3)

$$M_x^b = \left(- \frac{\int_C Et2A_{R_p}(s)\,ds}{\int_C Etds} \int_C p_R Etsds + \int_C \left\{ p_R \int_0^s 2EtA_{R_p}(s')\,ds' \right\} ds \right) \theta_{,xxx}$$

    - Using $p_R = 2A_{R_p,s}$ the integral of first term becomes

$$\int_C 2A_{R_p,s} Etsds = \cancel{2A_{R_p}(s)}\, Ets\big|_0^L - \int_C 2A_{R_p} Etds$$

    - As for s = 0, A_{Rp} = 0, and using

$$\int_C \left\{ 2A_{R_p,s} \int_0^s Et2A_{R_p}(s')\,ds' \right\} ds =$$

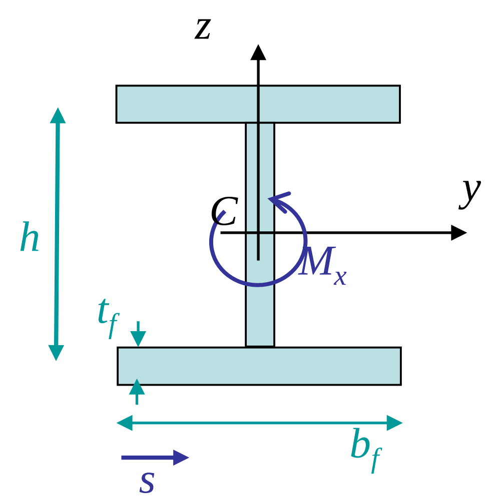
$$\frac{\int_C Et2A_{R_p}(s)\,ds}{\int_C Etds} \cancel{2A_{R_p}}(L)\, EtL - \int_C 4A_{R_p}^2 Etds$$

    - The final expression reads

$$M_x^b = \left(\frac{\left(\int_C Et2A_{R_p}(s)\,ds\right)^2}{\int_C Etds} - \int_C 4A_{R_p}^2 Etds \right) \theta_{,xxx}$$

- Arbitrary-section beam subjected to torsion with built-in end (8)
  - General expression for torque
    - $M_x = M_x^t + M_x^b \Longrightarrow M_x = C\theta_{,x} - C^\Gamma \theta_{,xxx}$
    - With $C^\Gamma = \int_C 4A_{R_p}^2 Etds - \frac{\left(\int_C Et2A_{R_p}(s)\,ds\right)^2}{\int_C Etds}$
  - Case of the I-section beam
    - Center of twist is the center of symmetry C
    - For the web: $A_{Rp}(s) = 0 \Longrightarrow$ no contribution to C^Γ
    - For lower flange

$$A_{R_p}(s) = \frac{hs}{4} \Longrightarrow \begin{cases} \int_C Et2A_{R_p}(s)\,ds = Et_f \frac{hb_f^2}{4} \\ \int_C Et4A_{R_p}^2(s)\,ds = Et_f \frac{h^2 b_f^3}{12} \end{cases}$$



- For the I-section

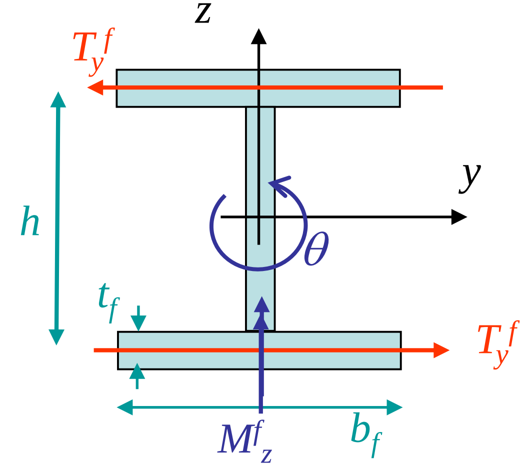
$$C^\Gamma = 2\left(Et_f \frac{h^2 b_f^3}{12} - Et_f \frac{h^2 b_f^3}{16}\right) = Et_f \frac{h^2 b_f^3}{24}$$

- Arbitrary-section beam subjected to torsion with built-in end (9)
  - Case of the I-section beam (2)
    - Expression $M_x = C\theta_{,x} - C^\Gamma \theta_{,xxx}$
      - With $C^\Gamma = \int_C 4A_{R_p}^2 Et ds - \frac{\left(\int_C Et2A_{R_p}(s)\, ds\right)^2}{\int_C Et ds}$

$$\Longrightarrow C^\Gamma = 2\left(Et_f \frac{h^2 b_f^3}{12} - Et_f \frac{h^2 b_f^3}{16}\right) = Et_f \frac{h^2 b_f^3}{24}$$

    - To be compared with

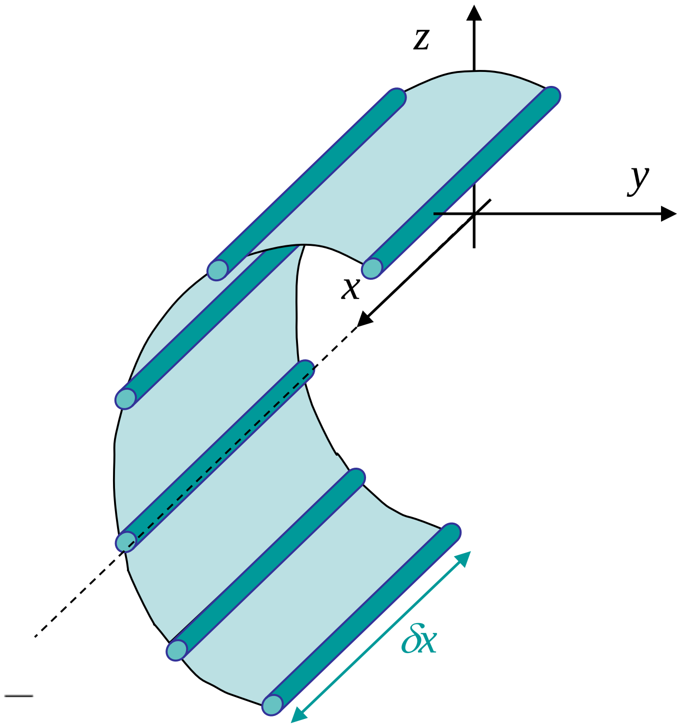
$$M_x = C\theta_{,x} - \frac{h^2 EI_{zz}^f}{2}\theta_{,xxx}$$



- Idealized beam subjected to torsion with built-in end
  - For idealized sections with booms
    - In expression

$$C^{\Gamma} = \int_C 4A_{R_p}^2 Etds - \frac{\left(\int_C Et2A_{R_p}(s)\,ds\right)^2}{\int_C Etds}$$

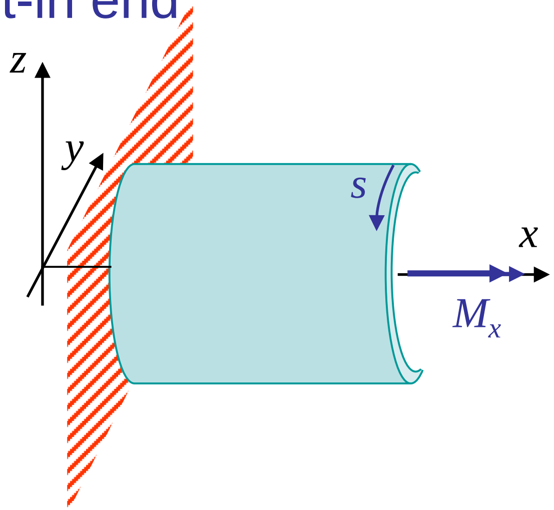
    - The direct stress is carried out by
      - t_{direct} &
      - Booms of section A_i



$$\Rightarrow C^{\Gamma} = \int_C 4A_{R_p}^2 Et_{\text{direct}} ds + \sum_i 4A_{R_p}^2\left(s^i\right) EA_i - \frac{\left(\int_C Et_{\text{direct}} 2A_{R_p}(s)\,ds + \sum_i 2A_{R_p}\left(s^i\right) EA_i\right)^2}{\int_C Et_{\text{direct}} ds + \sum_i EA_i}$$

- Applications of beam subjected to torsion with built-in end
  - Solution for pure torque
    - $M_x = C\theta_{,x} - C^{\Gamma}\theta_{,xxx}$

$$\Longrightarrow \theta_{,xxx} - w^2\theta_{,x} = -\frac{w^2}{C}M_x \quad \text{with} \quad w^2 = \frac{C}{C^{\Gamma}}$$



    - Solution

$$\theta_{,x} = C_1 \cosh wx + C_2 \sinh wx + \frac{M_x}{C}$$

    - Boundary conditions
      - At built-in end $x = 0$: No warping, and as $\boldsymbol{u}_x^s(s) = \boldsymbol{u}_x^s(0) - \theta_{,x}\int_0^s p_R ds'$

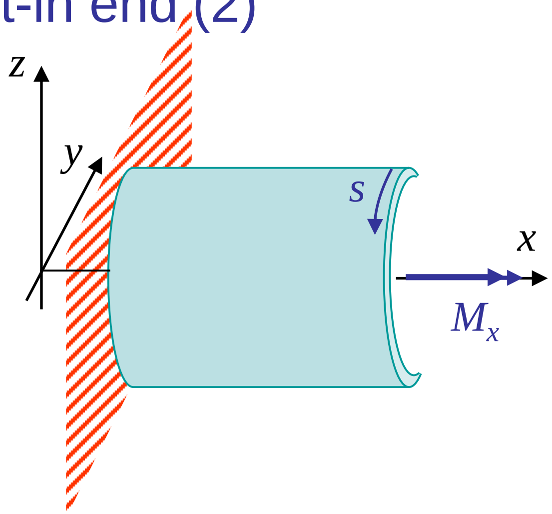
$$\Longrightarrow \theta_{,x}(0) = 0 \Longrightarrow C_1 = -\frac{M_x}{C}$$

      - At free end $x = L$: no direct load,

and as $\begin{cases} \boldsymbol{\sigma}_{xx}^{\Gamma}(s) = E\boldsymbol{u}_{x,x}^s(0) - 2EA_{R_p}(s)\theta_{,xx} \\ \boldsymbol{u}_{x,x}^s(0) = \dfrac{\theta_{,xx}\int_C Et2A_{R_p}(s)\,ds}{\int_C Etds} \end{cases}$

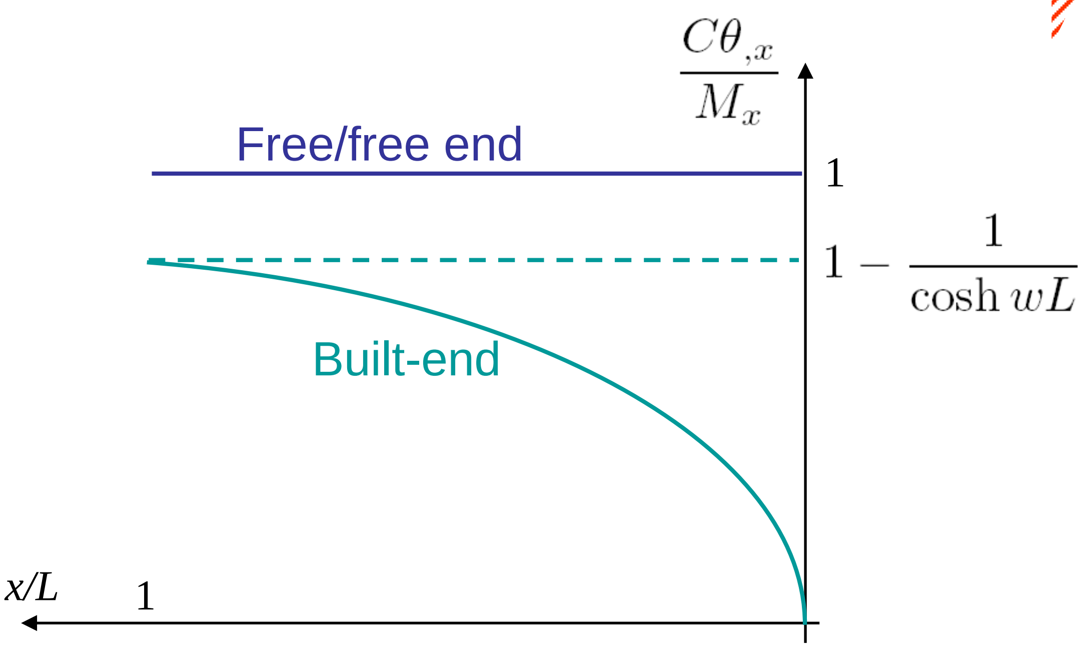
$$\Longrightarrow \theta_{,xx}(L) = 0 \Longrightarrow C_2 = \frac{M_x}{C}\tanh wL$$

- Applications of beam subjected to torsion with built-in end (2)
  - Solution for pure torque (2)
    - Twist rate

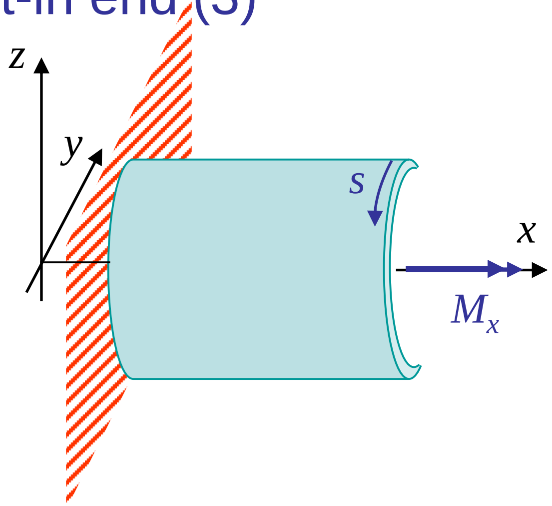


$$\theta_{,x} = \frac{M_x}{C}\left(1 - \cosh wx + \tanh wL \sinh wx\right)$$

$$\Rightarrow \theta_{,x} = \frac{M_x}{C}\left(1 - \frac{\cosh\left(wL - wx\right)}{\cosh wL}\right)$$



- Applications of beam subjected to torsion with built-in end (3)
  - Solution for pure torque (3)



- Angle of twist
  - As $\theta_{,x} = \frac{M_x}{C}\left(1 - \frac{\cosh\left(wL - wx\right)}{\cosh wL}\right)$

$$\Longrightarrow \theta\left(x\right) = \frac{M_x}{C}\left(x + \frac{\sinh\left(wL - wx\right)}{w \cosh wL} + C_3\right)$$

  - Boundary condition at built end x = 0: No twist

$$0 = \theta\left(0\right) = \frac{M_x}{C}\left(\frac{\sinh wL}{w \cosh wL} + C_3\right)$$

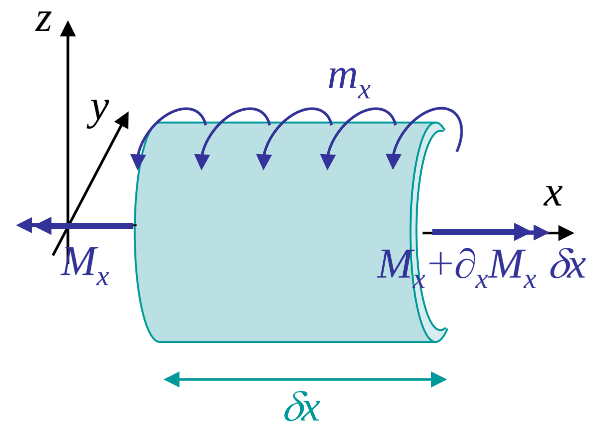
$$\Longrightarrow \theta\left(x\right) = \frac{M_x}{C}\left(x + \frac{\sinh\left(wL - wx\right)}{w \cosh wL} - \frac{\sinh wL}{w \cosh wL}\right)$$

- At free end

$$\theta\left(L\right) = \frac{M_x L}{C}\left(1 - \frac{\tanh wL}{wL}\right)$$

Reduction compared to free-free case

- Applications of beam subjected to torsion with built-in end (4)
  - Distributed torque loading m_x
    - Two contributions to torque
      - $M_x = M_x^t + M_x^b$
    - Balance equation



$$M_x^t + \partial_x M_x^t \delta x + M_x^b + \partial_x M_x^b \delta x + m_x \delta x = M_x^t + M_x^b$$

$$\Longrightarrow \partial_x M_x = \partial_x M_x^t + \partial_x M_x^b = -m_x$$

- As $\left\{ \begin{array}{l} M_x^t = C\theta_{,x} \\ M_x^b = -C^\Gamma \theta_{,xxx} \end{array} \right\} \Longrightarrow \partial_x \left(C^\Gamma \theta_{,xxx} - C\theta_{,x} \right) = m_x(x)$
- To be solved with adequate boundary conditions
  - Built-in end: $\theta = 0$ & $\theta_{,x} = 0$ (no warping)
  - Free end: $\theta_{,xx} = 0$ (no direct stress) & No torque at free end

- Remark
  - We have studied
    - Axial loading resulting from torsion
    - A similar theory can be derived to deduce torsion resulting from axial loading

References

- Lecture notes
  - Aircraft Structures for engineering students, T. H. G. Megson, Butterworth-Heinemann, An imprint of Elsevier Science, 2003, ISBN 0 340 70588 4
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    - Mécanique des matériaux, C. Massonet & S. Cescotto, De boek Université, 1994, ISBN 2-8041-2021-X