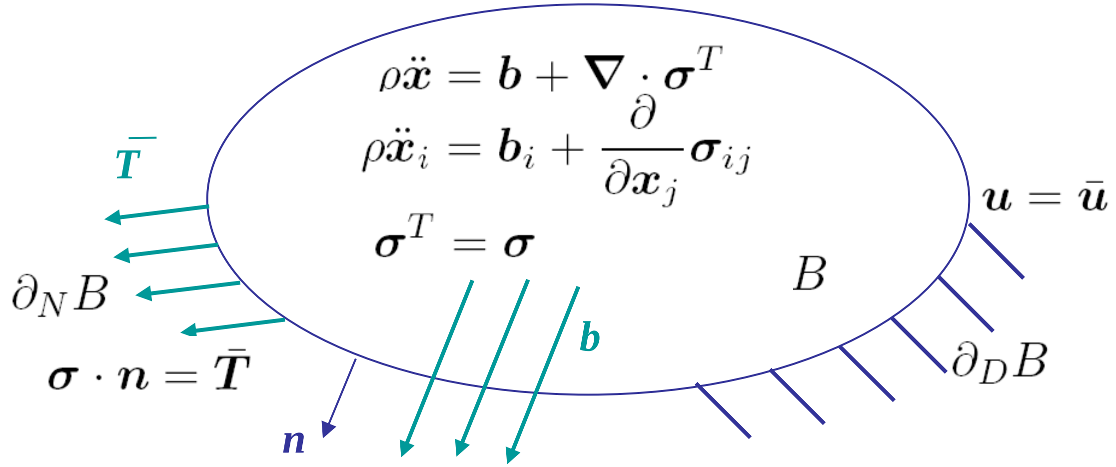
University of Liège
Aerospace & Mechanical Engineering

Aircraft Structures
Introduction to Shells

Ludovic Noels

Computational & Multiscale Mechanics of Materials – CM3
http://www.ltas-cm3.ulg.ac.be/
Chemin des Chevreuils 1, B4000 Liège
L.Noels@ulg.ac.be

- Balance of body B
  - Momenta balance
    - Linear
    - Angular
  - Boundary conditions
    - Neumann
    - Dirichlet



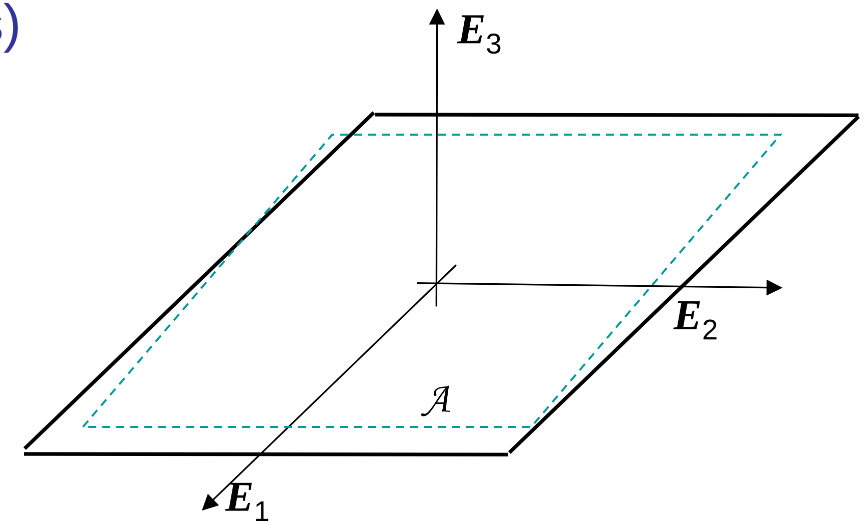
- Small deformations with linear elastic, homogeneous & isotropic material
  - (Small) Strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla}\right)$, or $\begin{cases} \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{x}_i}\boldsymbol{u}_j + \frac{\partial}{\partial \boldsymbol{x}_j}\boldsymbol{u}_i\right) \\ \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\boldsymbol{u}_{j,i} + \boldsymbol{u}_{i,j}\right) \end{cases}$
  - Hooke's law $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon}$, or $\boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl}\boldsymbol{\varepsilon}_{kl}$

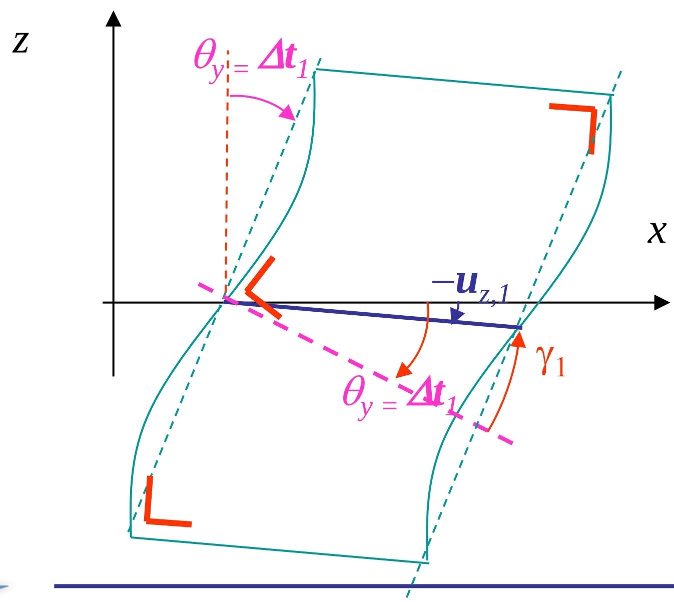
    with $$\mathcal{H}_{ijkl} = \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda = K - 2\mu/3}\delta_{ij}\delta_{kl} + \underbrace{\frac{E}{1+\nu}}_{2\mu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$$
  - Inverse law $\boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma}$

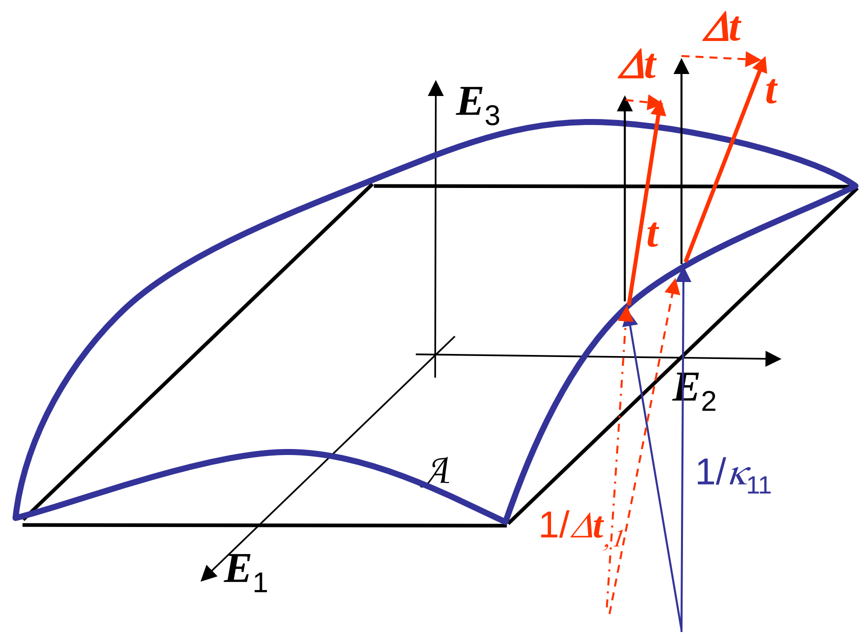
    with $$\mathcal{G}_{ijkl} = \frac{1+\nu}{E}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right) - \frac{\nu}{E}\delta_{ij}\delta_{kl}$$

Reissner-Mindlin plate summary

- Deformations (small transformations)
  - In plane membrane
    - $\boxed{\varepsilon_{\alpha\beta} = \dfrac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2}}$
  - Curvature
    - $\boxed{\kappa_{\alpha\beta} = \dfrac{\boldsymbol{\Delta t}_{\alpha,\beta} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2}}$
  - Out-of-plane sliding
    - $\boxed{\gamma_{\alpha} = \boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_{\alpha}}$

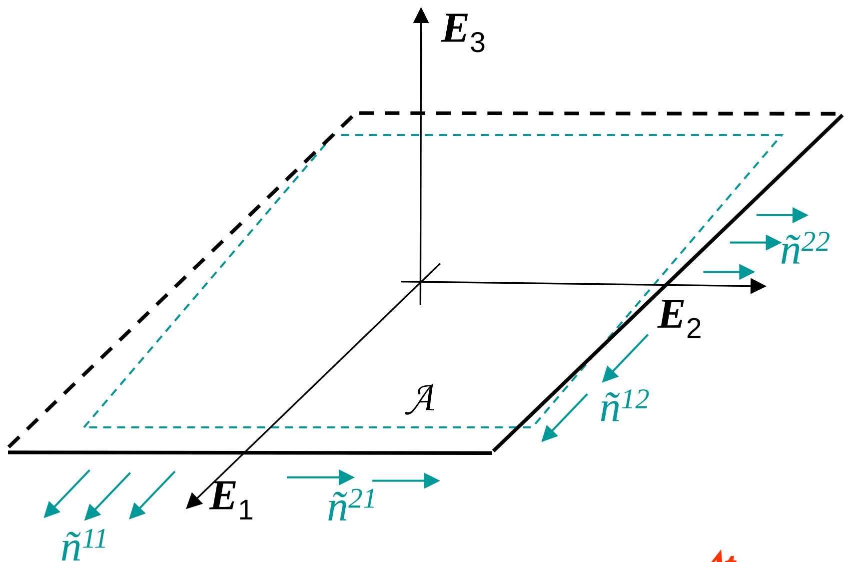


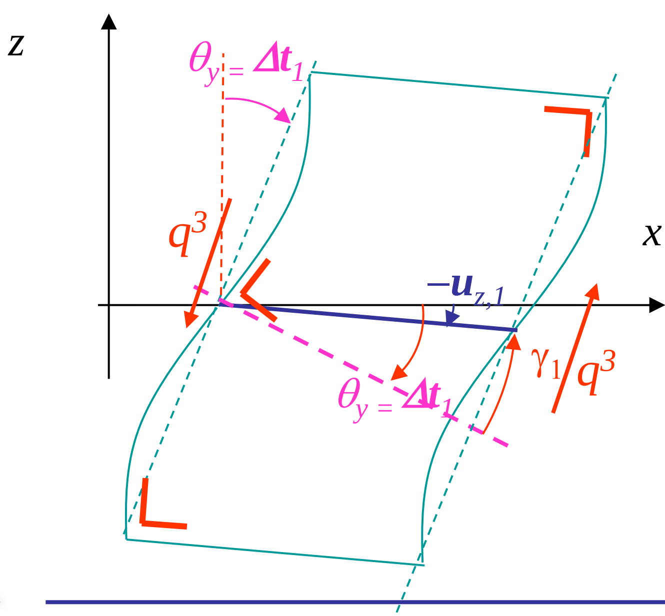


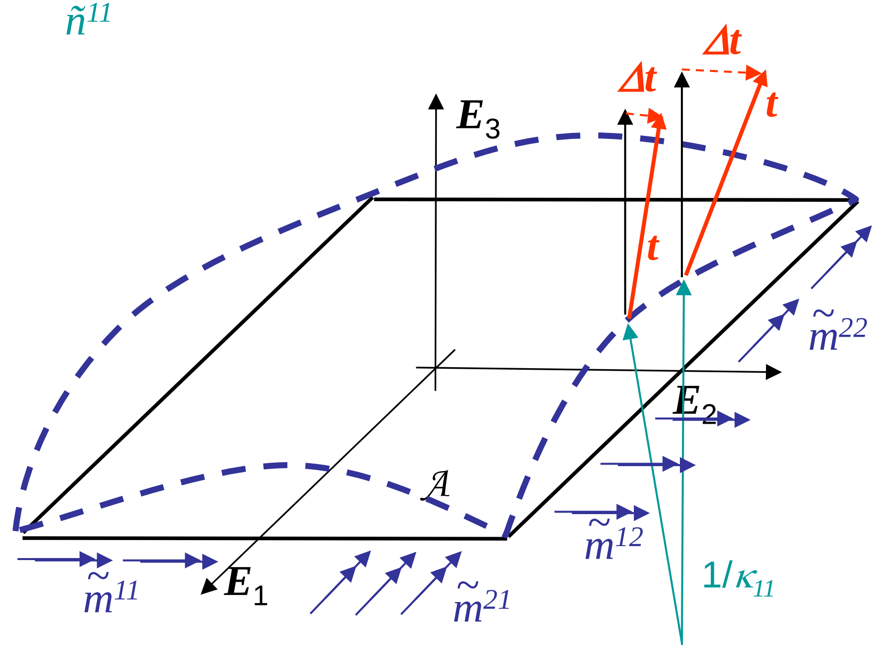


Reissner-Mindlin plate summary

- Resultant stresses in linear elasticity
  - Membrane stress
    - $\boxed{\tilde{n}^{\alpha\beta} = \mathcal{H}_n^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta}}$
  - Bending stress
    - $\boxed{\tilde{m}^{\alpha\beta} = \mathcal{H}_m^{\alpha\beta\gamma\delta} \kappa_{\gamma\delta}}$
  - Out-of-plane shear stress
    - $\boxed{\tilde{q}^{\alpha} = \frac{1}{2} \mathcal{H}_q^{\alpha\beta} \gamma_\beta}$



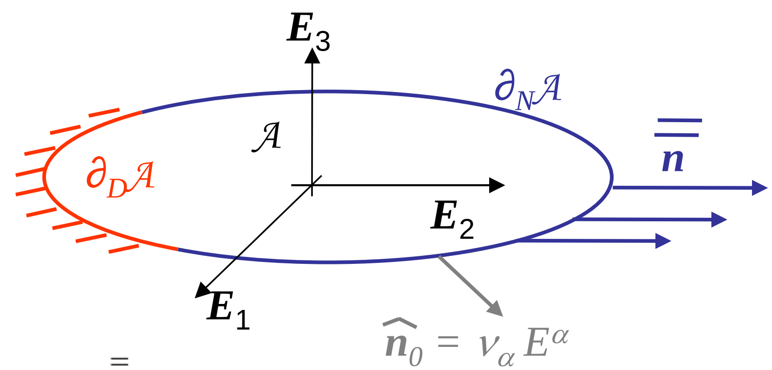




- Resultant Hooke tensor in linear elasticity
  - Membrane mode
    - $\mathcal{H}_n^{\alpha\beta\gamma\delta} = \frac{h_0 E}{1-\nu^2}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta} + \frac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$
  - Bending mode
    - $\mathcal{H}_m^{\alpha\beta\gamma\delta} = \frac{h_0^3 E}{12\left(1-\nu^2\right)}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta} + \frac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$
  - Shear mode
    - $\mathcal{H}_q^{\alpha\beta} = \frac{E h_0}{1+\nu}\frac{A'}{A}\delta^{\alpha\beta}$

- Resultant equations
  - Membrane mode
    - $\boxed{\left(\boldsymbol{n}^{\alpha}\right)_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}\ddot{\boldsymbol{u}}}$
    - $\boldsymbol{n}^{\alpha} = \tilde{n}^{\alpha\beta}\boldsymbol{E}_{\beta} + \tilde{q}^{\alpha}\boldsymbol{E}_{3}$
      - $\tilde{n}^{\alpha\beta} = \mathcal{H}_{n}^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}$ with $\varepsilon_{\alpha\beta} = \dfrac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2}$
      - $\tilde{q}^{\alpha} = \dfrac{1}{2}\mathcal{H}_{q}^{\alpha\beta}\gamma_{\beta}$ with $\gamma_{\alpha} = \boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_{\alpha}$
    - Clearly, the solution can be directly computed in plane Oxy (constant $\mathcal{H}_n$)
      - $\mathcal{H}_{n}^{\alpha\beta\gamma\delta}\dfrac{\boldsymbol{u}_{\gamma,\delta\alpha} + \boldsymbol{u}_{\delta,\gamma\alpha}}{2} + \bar{\boldsymbol{n}}_{\beta} = \bar{\rho}\ddot{\boldsymbol{u}}_{\beta}$
      - Boundary conditions
        - » Neumann $\boldsymbol{u}_{\alpha} = \bar{\boldsymbol{u}}_{\alpha}$
        - » Dirichlet

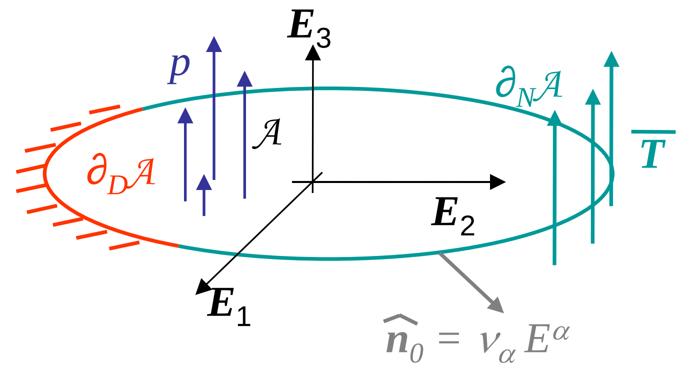
$$\boldsymbol{n}_{\beta}^{\alpha}\nu_{\alpha} = \mathcal{H}_{n}^{\alpha\beta\gamma\delta}\frac{\boldsymbol{u}_{\gamma,\delta\alpha} + \boldsymbol{u}_{\delta,\gamma\alpha}}{2}\nu_{\alpha} = \bar{\bar{\boldsymbol{n}}}_{\beta}$$



- Remaining equation along E^3: $\mathcal{H}_{q}^{\alpha\beta}\dfrac{\boldsymbol{u}_{3,\beta\alpha} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} + \bar{\boldsymbol{n}}_{3} = \bar{\rho}\ddot{\boldsymbol{u}}_{3}$

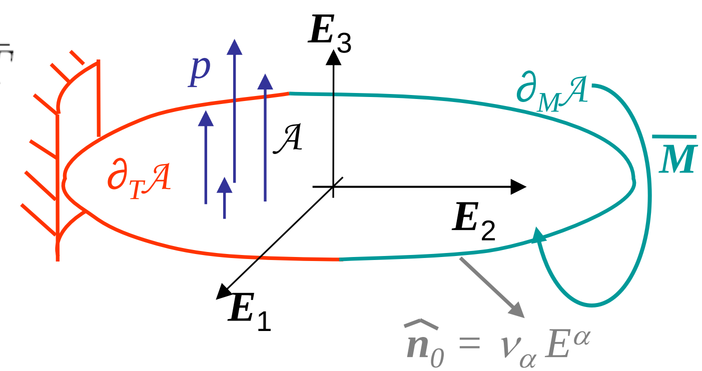
- Resultant equations (2)
  - Bending mode
    - $\boxed{\ddot{\boldsymbol{t}} I_p = \bar{\tilde{\boldsymbol{m}}} - \left(\boldsymbol{n}^3 - \lambda \boldsymbol{E}_3\right) + \left(\tilde{\boldsymbol{m}}^{\alpha}\right)_{,\alpha}}$
    - $\tilde{\boldsymbol{m}}^{\alpha} = \tilde{m}^{\alpha\beta} \boldsymbol{E}_{\beta}$ & $\boldsymbol{n}^3 = \tilde{q}^{\alpha} \boldsymbol{E}_{\alpha}$
      - $\tilde{m}^{\alpha\beta} = \mathcal{H}_m^{\alpha\beta\gamma\delta} \kappa_{\gamma\delta}$ with $\kappa_{\alpha\beta} = \dfrac{\boldsymbol{\Delta t}_{\alpha,\beta} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2}$
      - $\tilde{q}^{\alpha} = \dfrac{1}{2} \mathcal{H}_q^{\alpha\beta} \gamma_{\beta}$ with $\gamma_{\alpha} = \boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_{\alpha}$
    - Solution is obtained by projecting into the plane Oxy (constant $\mathcal{H}_q$, $\mathcal{H}_m$)
      - $I_p \ddot{\boldsymbol{\Delta t}}_{\alpha} = \bar{\tilde{\boldsymbol{m}}}_{\alpha} - \dfrac{1}{2} \mathcal{H}_q^{\alpha\beta} \left(\boldsymbol{u}_{3,\beta} + \boldsymbol{\Delta t}_{\beta}\right) + \mathcal{H}_m^{\alpha\beta\gamma\delta} \dfrac{\boldsymbol{\Delta t}_{\gamma,\delta\beta} + \boldsymbol{\Delta t}_{\delta,\gamma\beta}}{2}$
      - 2 equations (α=1, 2) with 3 unknowns ($\Delta \boldsymbol{t}_1$, $\Delta \boldsymbol{t}_2$, $\boldsymbol{u}_3$)
      - Use remaining equation $\mathcal{H}_q^{\alpha\beta} \dfrac{\boldsymbol{u}_{3,\beta\alpha} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} + \bar{\boldsymbol{n}}_3 = \bar{\rho} \ddot{\boldsymbol{u}}_3$

- Resultant equations (3)
  - Bending mode (2)
    - 3 equations with 3 unknowns
      - $I_p \ddot{\boldsymbol{\Delta t}}_\alpha = \bar{\tilde{\boldsymbol{m}}}_\alpha - \frac{1}{2}\mathcal{H}_q^{\alpha\beta}\left(\boldsymbol{u}_{3,\beta} + \boldsymbol{\Delta t}_\beta\right) + \mathcal{H}_m^{\alpha\beta\gamma\delta}\frac{\boldsymbol{\Delta t}_{\gamma,\delta\beta} + \boldsymbol{\Delta t}_{\delta,\gamma\beta}}{2}$
      - $\mathcal{H}_q^{\alpha\beta}\frac{\boldsymbol{u}_{3,\beta\alpha} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} + \bar{\boldsymbol{n}}_3 = \bar{\rho}\ddot{\boldsymbol{u}}_3$



    - To be completed by BCs
      - Low order constrains
        - » Displacement $\boldsymbol{u}_3 = \bar{\boldsymbol{u}}_3$ or
        - » Shearing

$$\boldsymbol{n}_3^\alpha \nu_\alpha = \mathcal{H}_q^{\alpha\beta}\frac{\boldsymbol{u}_{3,\beta\alpha} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} = \bar{T}$$



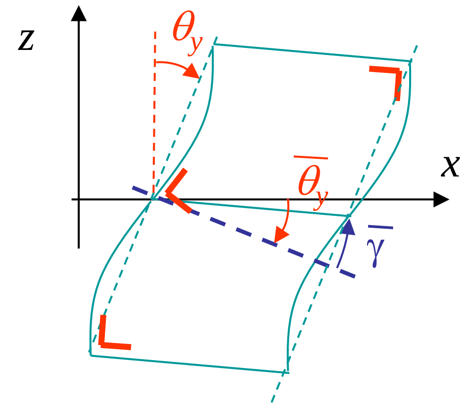
      - High order
        - » Rotation $\boldsymbol{\Delta t} = \bar{\boldsymbol{\Delta t}}$ or
        - » Bending

$$\tilde{\boldsymbol{m}}_\beta^\alpha \nu_\alpha = \mathcal{H}_m^{\alpha\beta\gamma\delta}\frac{\boldsymbol{\Delta t}_{\gamma,\delta} + \boldsymbol{\Delta t}_{\delta,\gamma}}{2}\nu_\alpha = \bar{\boldsymbol{M}}_\beta$$

- Resultant equations (4)
  - Remarks
    - Compare bending equations
      - $I_p \ddot{\boldsymbol{\Delta t}}_\alpha = \bar{\bar{\tilde{\boldsymbol{m}}}}_\alpha - \frac{1}{2}\mathcal{H}_q^{\alpha\beta}\left(\boldsymbol{u}_{3,\beta} + \boldsymbol{\Delta t}_\beta\right) + \mathcal{H}_m^{\alpha\beta\gamma\delta}\frac{\boldsymbol{\Delta t}_{\gamma,\delta\beta} + \boldsymbol{\Delta t}_{\delta,\gamma\beta}}{2}$
      - $\mathcal{H}_q^{\alpha\beta}\frac{\boldsymbol{u}_{3,\beta\alpha} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} + \bar{\boldsymbol{n}}_3 = \bar{\rho}\ddot{\boldsymbol{u}}_3$
      - With Timoshenko beam equations

$$\frac{\partial}{\partial_x}\left(EI\frac{\partial\theta_y}{\partial x}\right) - \mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right) = 0$$

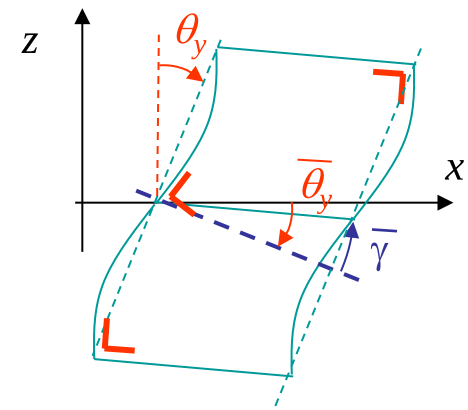
$$\frac{\partial}{\partial x}\left(\mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right)\right) = -f$$



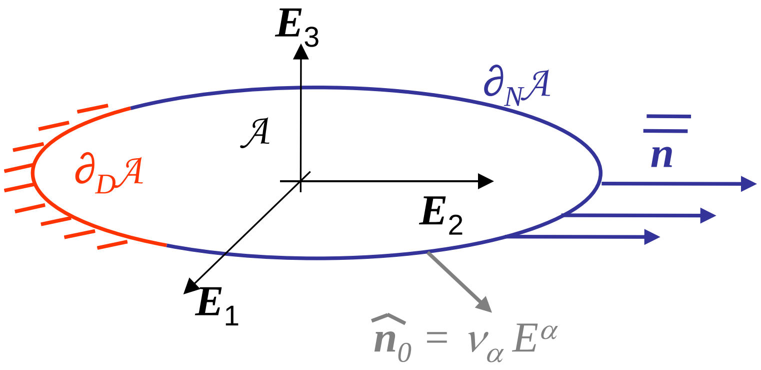
    - Membrane and bending equations are uncoupled
      - No initial curvature
      - Small deformations (equilibrium on non curved configuration)

Shear effect

- Kirchhoff assumption
  - As
    - $$\mathcal{H}_m^{\alpha\beta\gamma\delta} = \overbrace{\frac{h_0^3 E}{12\left(1-\nu^2\right)}}^{D}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta} + \frac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$$
    - $$\mathcal{H}_q^{\alpha\beta} = \frac{Eh_0}{1+\nu}\frac{A'}{A}\delta^{\alpha\beta}$$
  - Kirchhoff assumption requires
    - $$\frac{D(1+\nu)A}{L^2 E h_0 A'} = \frac{h_0^2 A}{12 L^2 (1-\nu) A'} \ll 1$$
    - Where L is a characteristic distance



- Membrane mode
  - On $\mathcal{A}$: $\mathcal{H}_n^{\alpha\beta\gamma\delta} \frac{\boldsymbol{u}_{\gamma,\delta\alpha} + \boldsymbol{u}_{\delta,\gamma\alpha}}{2} + \bar{\boldsymbol{n}}_\beta = \bar{\rho}\ddot{\boldsymbol{u}}_\beta$
    - With $\mathcal{H}_n^{\alpha\beta\gamma\delta} = \frac{h_0 E}{1-\nu^2}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta} + \frac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$

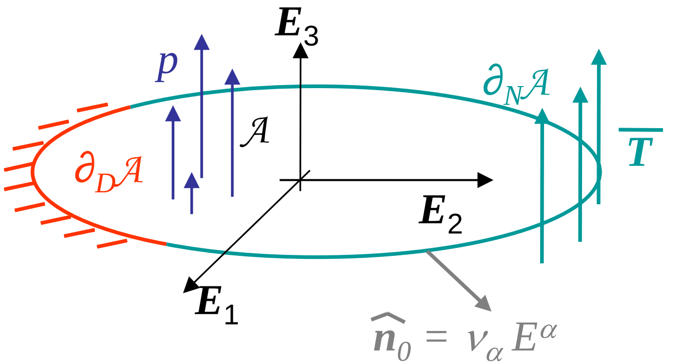


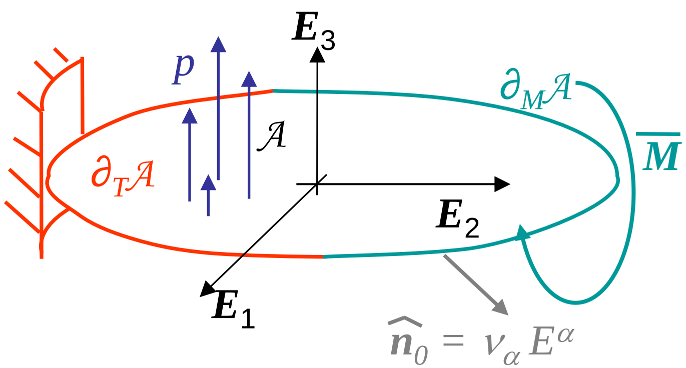
  - Completed by appropriate BCs
    - Dirichlet $\boldsymbol{u}_\alpha = \bar{\boldsymbol{u}}_\alpha$
    - Neumann $\boldsymbol{n}_\beta^\alpha \nu_\alpha = \mathcal{H}_n^{\alpha\beta\gamma\delta} \frac{\boldsymbol{u}_{\gamma,\delta\alpha} + \boldsymbol{u}_{\delta,\gamma\alpha}}{2} \nu_\alpha = \bar{\bar{\boldsymbol{n}}}_\beta$

Kirchhoff-Love plate summary

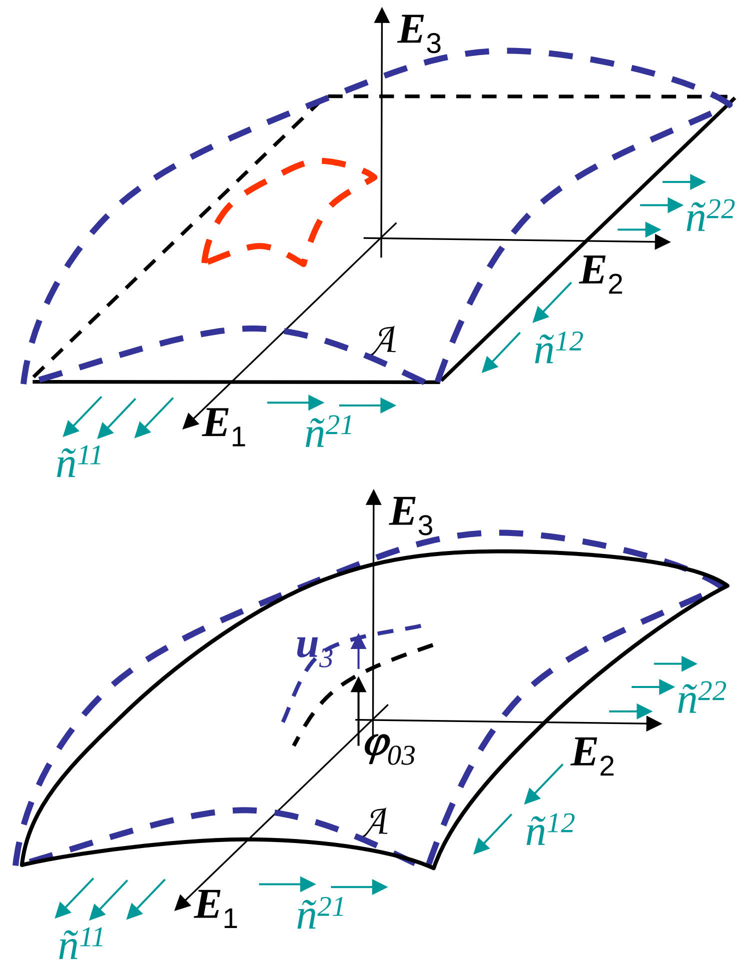
- Bending mode
  - On $\mathcal{A}$: $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\alpha\beta} = p$
    - With $\mathcal{H}_m^{\alpha\beta\gamma\delta} = \overbrace{\frac{h_0^3 E}{12\left(1-\nu^2\right)}}^{D}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta} + \frac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$
  - Completed by appropriate BCs
    - Low order
      - On $\partial_N\mathcal{A}$: $-\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\beta}\nu_\alpha = \bar{T}$
      - On $\partial_D\mathcal{A}$: $\boldsymbol{u}_3 = \bar{\boldsymbol{u}}_3$
    - High order
      - On $\partial_T\mathcal{A}$: $\boldsymbol{\Delta t} = \bar{\boldsymbol{\Delta t}}$

        with $\boldsymbol{\Delta t} = -\boldsymbol{u}_{3,\alpha}\boldsymbol{E}_\alpha$
      - On $\partial_M\mathcal{A}$: $-\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)\nu_\beta = \bar{M}\nu_\alpha$



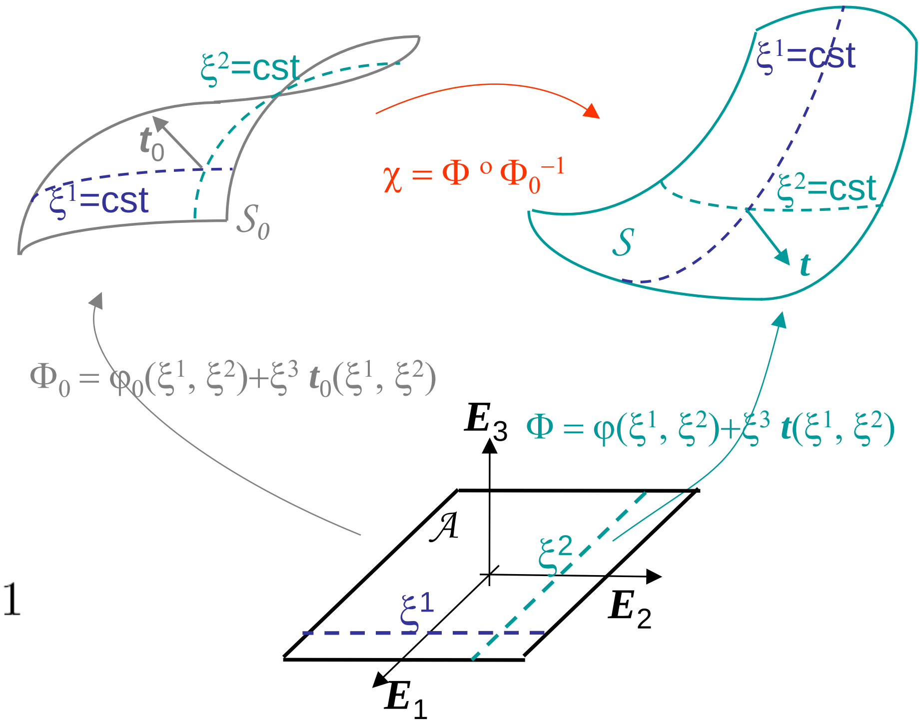


- Membrane-bending coupling
  - The first order theory is uncoupled
  - For second order theory
    - On $\mathcal{A}$: $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} - \tilde{n}^{\alpha\beta}\boldsymbol{u}_3\right)_{,\alpha\beta} = p$
    - Tension increases the bending stiffness of the plate
  - In case of small initial curvature (κ >>)
    - On $\mathcal{A}$: $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} - \tilde{n}^{\alpha\beta}\boldsymbol{u}_3 - \tilde{n}^{\alpha\beta}\boldsymbol{\varphi}_{03}\right)_{,\alpha\beta} = p$
    - Tension induces bending effect
- General theories
  - For not small initial curvature:
    - Linear shells
  - To fully account for tension effect
    - Non-Linear shells



- Shell kinematics
  - In the reference frame $\boldsymbol{E}_i$
    - The shell is described by

$$\boldsymbol{\xi} = \xi^I \boldsymbol{E}_I \quad \text{with} \begin{cases} (\xi^1, \xi^2) \in \mathcal{A} \\ \xi^3 \in [-\frac{h}{2}; \frac{h}{2}] \end{cases}$$



  - Initial configuration $\mathcal{S}_0$ mapping
    - Neutral plane $\boldsymbol{\varphi}_0\left(\xi^1, \xi^2\right)$
    - Cross section $\boldsymbol{t}_0(\xi^1, \xi^2)\,, \;\|\boldsymbol{t}\| = 1$
    - Thin body

$$\boldsymbol{x}_0 = \boldsymbol{\Phi}_0\left(\xi^I\right) = \boldsymbol{\varphi}_0\left(\xi^\alpha\right) + \xi^3 \boldsymbol{t}_0(\xi^1, \xi^2)$$

  - Deformed configuration $\mathcal{S}$ mapping
    - Thin body $\boldsymbol{x} = \boldsymbol{\Phi}\left(\xi^I\right) = \boldsymbol{\varphi}\left(\xi^\alpha\right) + \xi^3 \boldsymbol{t}(\xi^1, \xi^2)$
  - Two-point deformation mapping $\boldsymbol{\chi} = \boldsymbol{\Phi} \circ \boldsymbol{\Phi}_0^{-1}$

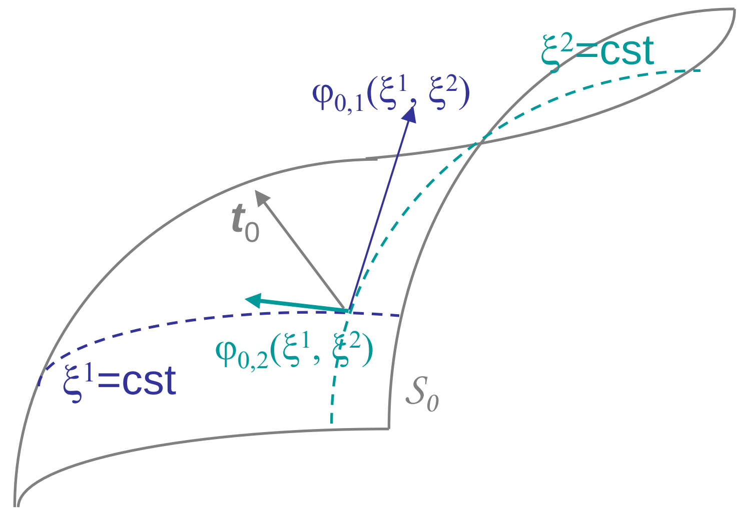
- Shell kinematics (2)
  - Deformation gradient
    - Two-point deformation mapping $\chi = \mathbf{\Phi} \circ \mathbf{\Phi}_0^{-1}$
    - Two-point deformation gradient $\mathbf{F} = \nabla\mathbf{\Phi} \circ \left[\nabla\mathbf{\Phi}_0\right]^{-1}$

      In frame E^I ⟹ with respect to ξ^I

      - Small strain deformation gradient $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^T\right) - \mathbf{I}$
  - It is more convenient to evaluate these tensors in a convected basis
    - Example $\boldsymbol{g}_{0I}$ basis convected to S_0
      - As S_0 is described by $\boldsymbol{x}_0 = \mathbf{\Phi}_0\left(\xi^I\right) = \boldsymbol{\varphi}_0\left(\xi^\alpha\right) + \xi^3 \boldsymbol{t}_0(\xi^1, \xi^2)$
      - One has $\nabla\mathbf{\Phi}_0 = \boldsymbol{g}_{0I} \otimes \boldsymbol{E}^I$ with the convected basis

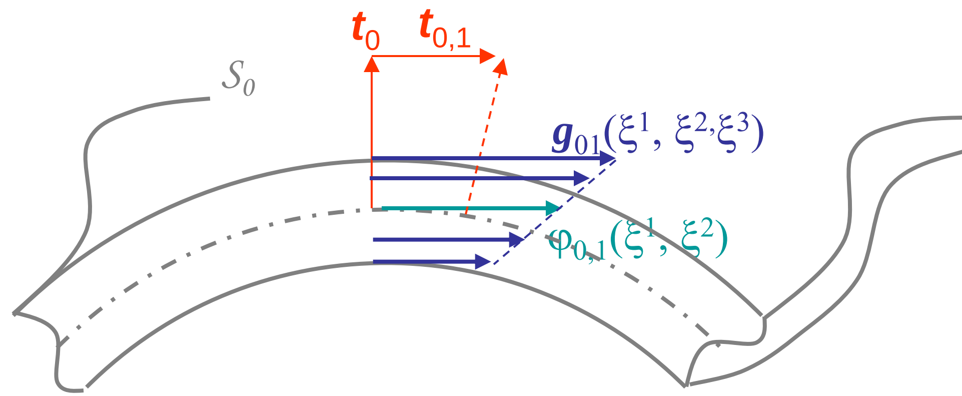
$$\begin{cases} \boldsymbol{g}_{0\alpha} = \dfrac{\partial \mathbf{\Phi}_0}{\partial \xi^\alpha} = \boldsymbol{\varphi}_{0,\alpha} + \xi^3 \boldsymbol{t}_{0,\alpha} \\ \boldsymbol{g}_{03} = \dfrac{\partial \mathbf{\Phi}_0}{\partial \xi^3} = \boldsymbol{t}_0 \end{cases}$$

      - The picture shows the basis for $\xi^3 = 0$
        » $\boldsymbol{g}_{0\alpha}\ (\xi^3=0) = \boldsymbol{\varphi}_{0,\alpha}$



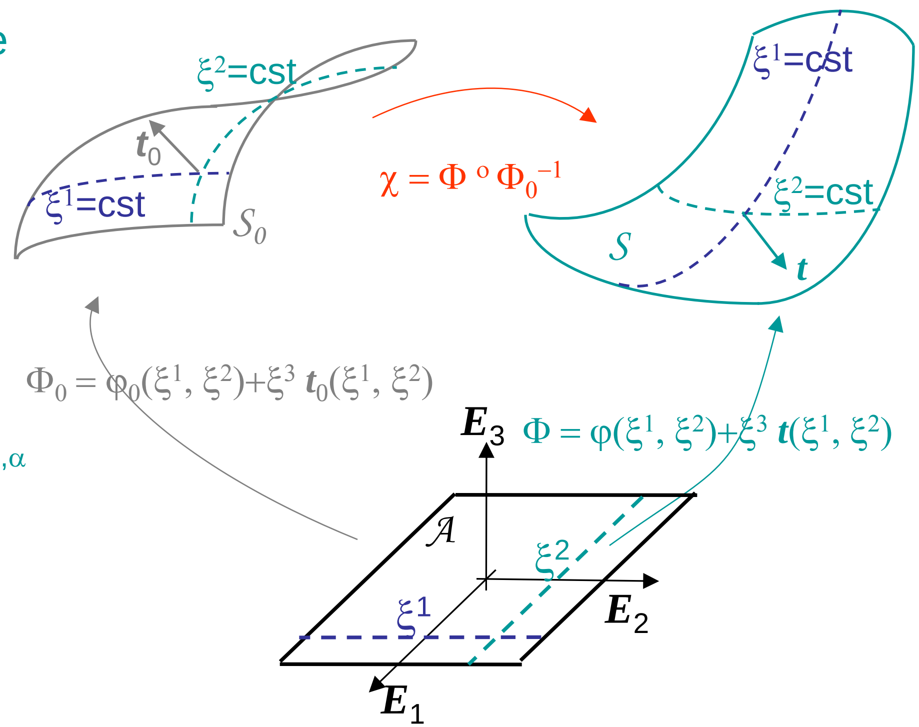
- Shell kinematics (3)
  - Convected basis $\boldsymbol{g}_{0I}$ to $\mathcal{S}_0$

$$\begin{cases} \boldsymbol{g}_{0\alpha} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^\alpha} = \boldsymbol{\varphi}_{0,\alpha} + \xi^3 \boldsymbol{t}_{0,\alpha} \\ \boldsymbol{g}_{03} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^3} = \boldsymbol{t}_0 \end{cases}$$

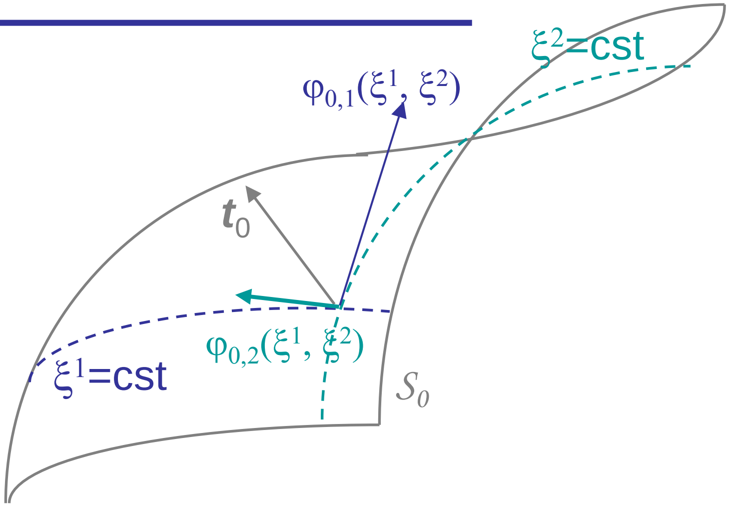


  - For $\xi^3 = 0$
    - $\boldsymbol{g}_{0\alpha}$ are tangent to the mid-surface
    - $\boldsymbol{g}_{0\alpha}$ (ξ^3=0) = $\varphi_{0,\alpha}$
  - For $\xi^3 \neq 0$
    - If there is an initial curvature

      $\boldsymbol{g}_{0\alpha}$ depends on ξ^3
    - Initial curvature is measured by $\boldsymbol{t}_{0,\alpha}$



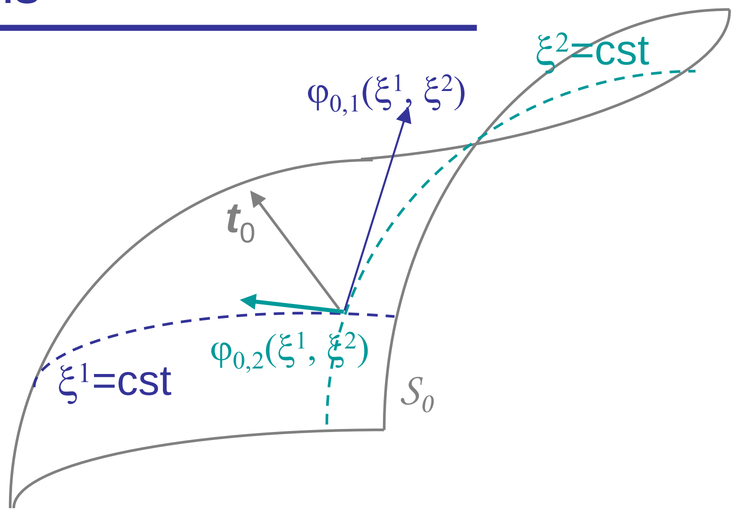
- Shell kinematics (4)
  - Convected basis $\boldsymbol{g}_{0I}$ to $\mathcal{S}_0$ (2)



$$\begin{cases} \boldsymbol{g}_{0\alpha} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^\alpha} = \boldsymbol{\varphi}_{0,\alpha} + \xi^3 \boldsymbol{t}_{0,\alpha} \\ \boldsymbol{g}_{03} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^3} = \boldsymbol{t}_0 \end{cases}$$

- The basis is not orthonormal
  - A vector component is still defined as $a_I = \boldsymbol{a} \cdot \boldsymbol{g}_{0I}$
  - So can $\boldsymbol{a} = a_I \boldsymbol{g}_{0I}$ be written?
    - » If so $\boldsymbol{a} \cdot \boldsymbol{g}_{0I} = a_J \underbrace{\boldsymbol{g}_{0J} \cdot \boldsymbol{g}_{0I}}_{\neq \delta_{IJ}} \neq a_I$ which is not consistent

- Shell kinematics (5)
  - Convected basis $\boldsymbol{g}_{0I}$ to S_0 (3)



$$\begin{cases} \boldsymbol{g}_{0\alpha} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^\alpha} = \boldsymbol{\varphi}_{0,\alpha} + \xi^3 \boldsymbol{t}_{0,\alpha} \\ \boldsymbol{g}_{03} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^3} = \boldsymbol{t}_0 \end{cases}$$

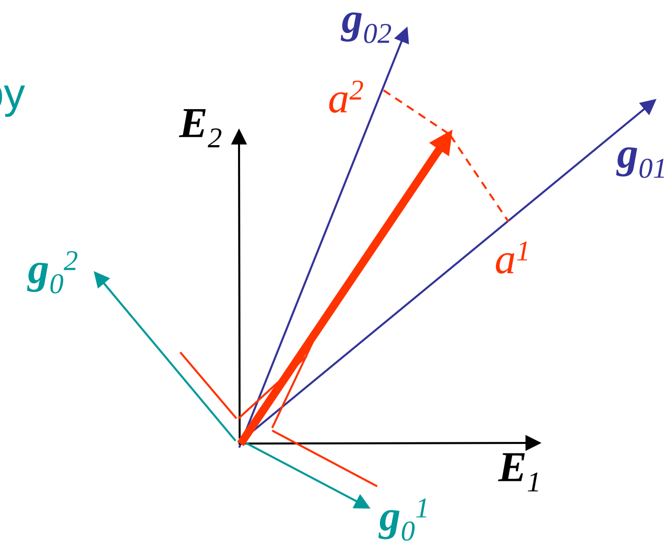
- The basis is not orthonormal (2)
  - A conjugate basis $\boldsymbol{g}_0^I$ has to be defined
    - » Such that $\boldsymbol{g}_{0I} \cdot \boldsymbol{g}_0^J = \delta_{IJ}$
    - » So vector components are defined by

$$a_I = \boldsymbol{a} \cdot \boldsymbol{g}_{0I} \text{ \& } a^I = \boldsymbol{a} \cdot \boldsymbol{g}_0^I$$

- » The vector can be represented by

$$\boldsymbol{a} = a^J \boldsymbol{g}_{0J} = a_I \boldsymbol{g}_0^I$$

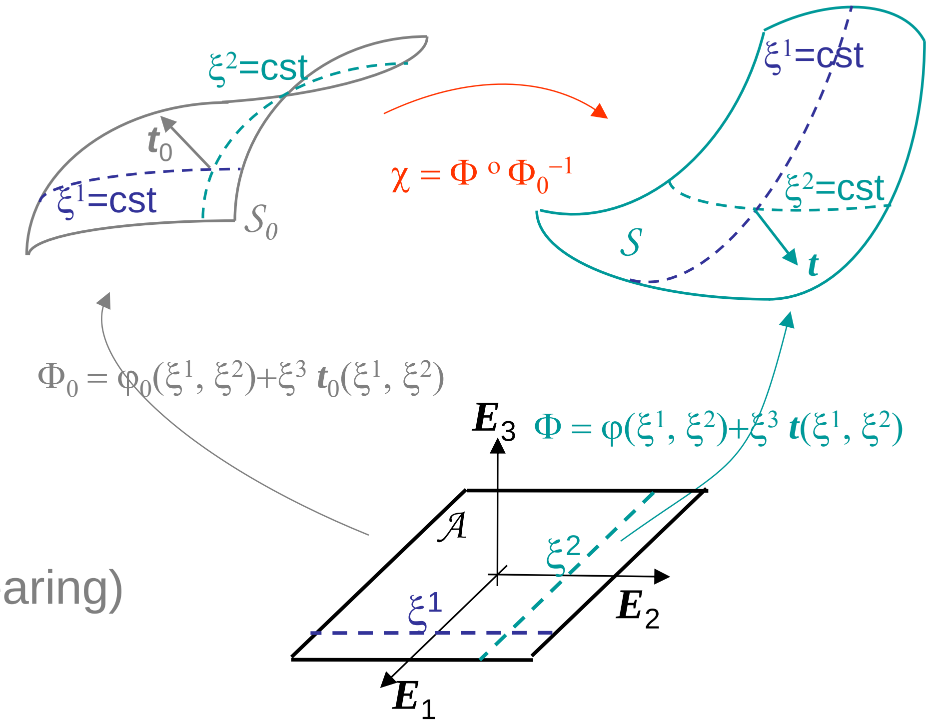
as $\begin{cases} \boldsymbol{a} \cdot \boldsymbol{g}_{0I} = a_J \boldsymbol{g}_0^J \cdot \boldsymbol{g}_{0I} = a_I \\ \boldsymbol{a} \cdot \boldsymbol{g}_0^I = a^J \boldsymbol{g}_{0J} \cdot \boldsymbol{g}_0^I = a^I \end{cases}$



- For plates $\begin{cases} \boldsymbol{g}_{0\alpha} = \boldsymbol{\varphi}_{0,\alpha} = \boldsymbol{E}_\alpha = \boldsymbol{E}^\alpha \\ \boldsymbol{g}_{03} = \boldsymbol{t}_0 = \boldsymbol{E}_3 = \boldsymbol{E}^3 \end{cases}$

- Assumptions
  - Small deformations/displacements
    - $\mathcal{S} \simeq \mathcal{S}_0$
    - $\int_{\mathcal{S}} \simeq \int_{\mathcal{S}_0}$
    - $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 + \boldsymbol{u}$ with

$$\|\boldsymbol{u}\| << \sqrt[3]{|\mathcal{S}_0|}$$



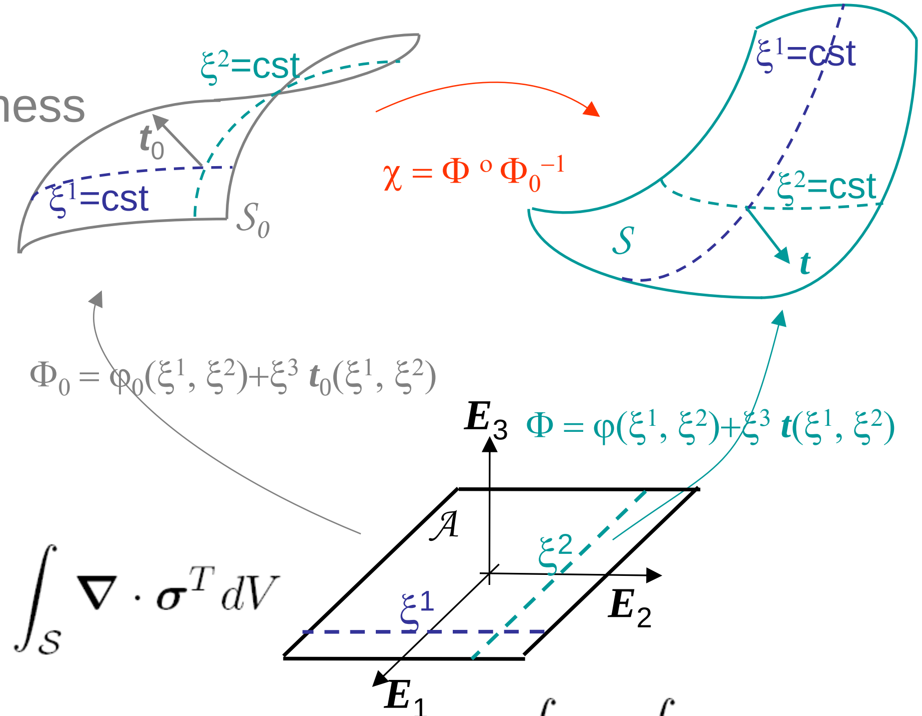
  - Kirchhoff-Love assumption (no shearing)
    - Normal is assumed to remain
      - Planar
      - Perpendicular to the neutral plane

$$\Rightarrow \begin{cases} \boldsymbol{t}_0 = \boldsymbol{E}_3 \\ \boldsymbol{t} = \dfrac{\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}}{\|\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}\|} \end{cases}$$

  - Reissner-Mindlin (shearing is allowed)
    - Normal is assumed to remain planar
    - But not perpendicular to neutral plane

Shell equations

- Same idea as for plates
  - Avoiding discretization on the thickness
    - $\boldsymbol{u}$ and $\boldsymbol{t}$ constant on the thickness
  - Equations are integrated on the thickness



- Linear momentum equation
  - $\rho \ddot{\boldsymbol{x}} = \boldsymbol{b} + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T$

$$\Longrightarrow \int_{\mathcal{S}} \left\{ \rho \left(\ddot{\boldsymbol{\varphi}} + \xi^3 \ddot{\boldsymbol{t}} \right) \right\} dV = \int_{\mathcal{S}} \boldsymbol{b} dV + \int_{\mathcal{S}} \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T dV$$

  - Small transformations assumptions ($\mathcal{S} \simeq \mathcal{S}_0$, $\boldsymbol{\nabla} \simeq \boldsymbol{\nabla}_0$, $\int_{\mathcal{S}} \simeq \int_{\mathcal{S}_0}$)

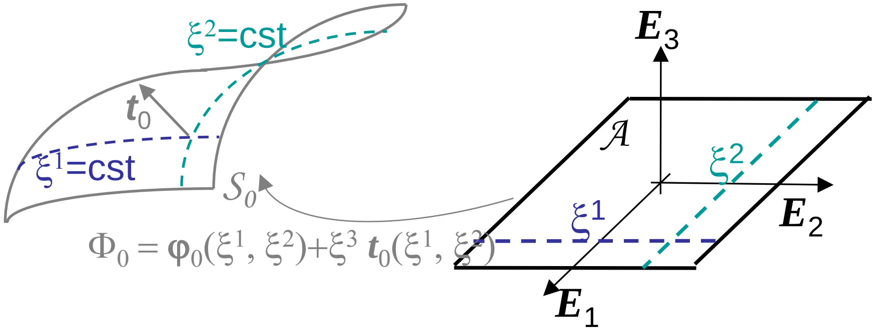
$$\Longrightarrow \int_{\mathcal{S}_0} \left\{ \rho_0 \left(\ddot{\boldsymbol{\varphi}} + \xi^3 \ddot{\boldsymbol{t}} \right) \right\} dV = \int_{\mathcal{S}_0} \boldsymbol{b} dV + \int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV$$

    - Using $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 + \boldsymbol{u}$

$$\Longrightarrow \int_{\mathcal{S}_0} \left\{ \rho_0 \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}} \right) \right\} dV = \int_{\mathcal{S}_0} \boldsymbol{b} dV + \int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV$$

- Linear momentum equation (2)
  - Inertial term (2)

$$\int_{\mathcal{S}_0} \left\{\rho_0 \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right)\right\} dV$$



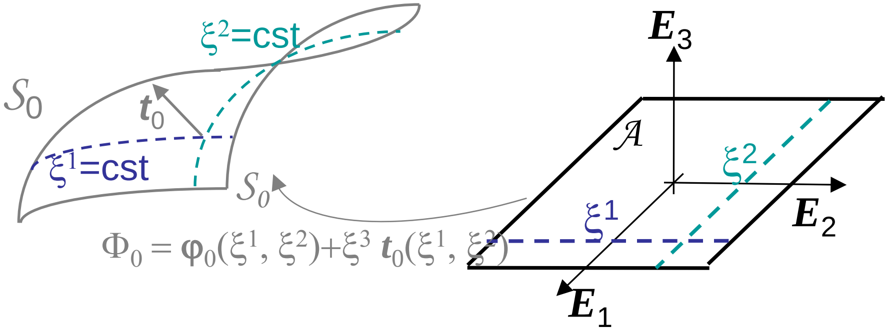
- Main idea in shells
  - $\boldsymbol{u}$ and $\boldsymbol{t}$ constant on the thickness
  - $\boldsymbol{u}$ and $\boldsymbol{t}$ defined in terms of ξ^1 & ξ^2 $\Longrightarrow$ integrate equations in $\mathcal{A}$ x [-h_0/2 h_0/2]
- To transform an integral on $\mathcal{S}_0$ to an integral on $\mathcal{A}$ x [-h_0/2 h_0/2]
  - Consider the Jacobian of the mapping: $j_0 = \left|\boldsymbol{\nabla}\boldsymbol{\Phi}_0\right|$

$$\Longrightarrow \int_{\mathcal{S}_0} \left\{\rho_0 \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right)\right\} dV = \int_{\mathcal{A}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \rho_0 j_0 \ddot{\boldsymbol{u}} d\xi^3 d\mathcal{A} + \int_{\mathcal{A}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} j_0 \rho_0 \xi^3 \ddot{\boldsymbol{t}} d\xi^3 d\mathcal{A}$$

  - Expression of the Jacobian?

Resultant equilibrium equations

- Jacobian
  - Mapping Φ_0 from $\mathcal{A}$ x [-h_0/2 h_0/2] to $\mathcal{S}_0$
    - Jacobian: $j_0 = |\nabla \Phi_0|$
      - Gradient of mapping



$$\nabla \Phi_0 = \boldsymbol{g}_{0I} \otimes \boldsymbol{E}^I$$

With $\boldsymbol{g}_{0\alpha} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^\alpha} = \boldsymbol{\varphi}_{0,\alpha} + \xi^3 \boldsymbol{t}_{0,\alpha}$ & $\boldsymbol{g}_{03} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^3} = \boldsymbol{t}_0$

$$\Longrightarrow j_0 = \begin{vmatrix} \boldsymbol{g}_{01} \cdot \boldsymbol{E}^1 & \boldsymbol{g}_{02} \cdot \boldsymbol{E}^1 & \boldsymbol{g}_{03} \cdot \boldsymbol{E}^1 \\ \boldsymbol{g}_{01} \cdot \boldsymbol{E}^2 & \boldsymbol{g}_{02} \cdot \boldsymbol{E}^2 & \boldsymbol{g}_{03} \cdot \boldsymbol{E}^2 \\ \boldsymbol{g}_{01} \cdot \boldsymbol{E}^3 & \boldsymbol{g}_{02} \cdot \boldsymbol{E}^3 & \boldsymbol{g}_{03} \cdot \boldsymbol{E}^3 \end{vmatrix} = (\boldsymbol{g}_{01} \wedge \boldsymbol{g}_{02}) \cdot \boldsymbol{t}_0$$

Laplace formula

$$\Longrightarrow j_0 = \underbrace{(\boldsymbol{\varphi}_{0,1} \wedge \boldsymbol{\varphi}_{0,2})\cdot \boldsymbol{t}_0}_{\bar{j}_0} + \xi^3 (\boldsymbol{\varphi}_{0,1} \wedge \boldsymbol{t}_{0,2} + \boldsymbol{t}_{0,1} \wedge \boldsymbol{\varphi}_{0,2}) \cdot \boldsymbol{t}_0 + {\xi^3}^2 (\boldsymbol{t}_{0,1} \wedge \boldsymbol{t}_{0,2}) \cdot \boldsymbol{t}_0$$

$\bar{j}_0 = (\boldsymbol{\varphi}_{0,1} \wedge \boldsymbol{\varphi}_{0,2}) \cdot \boldsymbol{t}_0 = \|\boldsymbol{\varphi}_{0,1} \wedge \boldsymbol{\varphi}_{0,2}\|$ as $\boldsymbol{t}_0$ is initially perpendicular to mid-surface

- Mid-surface Jacobian $\bar{j}_0$ corresponds to the change of mid-plane surface between the curvilinear frame (ξ^1, ξ^2) and the initial configuration $\mathcal{S}_0$

- Linear momentum equation (3)
  - Inertial term (3)

$$\int_{\mathcal{S}_0} \left\{\rho_0 \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right)\right\} dV = \int_{\mathcal{A}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \rho_0 j_0 \ddot{\boldsymbol{u}} d\xi^3 d\mathcal{A} + \int_{\mathcal{A}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} j_0 \rho_0 \xi^3 \ddot{\boldsymbol{t}} d\xi^3 d\mathcal{A}$$

$$\Longrightarrow \int_{\mathcal{S}_0} \left\{\rho_0 \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right)\right\} dV = \int_{\mathcal{A}} \ddot{\boldsymbol{u}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} j_0 \rho_0 d\xi^3 d\mathcal{A} + \int_{\mathcal{A}} \ddot{\boldsymbol{t}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} j_0 \rho_0 \xi^3 d\xi^3 d\mathcal{A}$$

- Position of the mid-surface is redefined such that
  - $\int_{h_0} j_0 \rho_0 \xi^3 d\xi^3 = 0$
  - Rigorously, integration is not always from $-h_0/2$ to $h_0/2$ as the mid-surface does not always correspond to the geometrical center
  - We will consider ρ_0 constant on the thikcness (but not j_0)
  - We define the surface density $\bar{\rho}_0 = \frac{1}{\bar{j}_0} \int_{h_0} j_0 \rho_0 d\xi^3$

$$\Longrightarrow \int_{\mathcal{S}_0} \left\{\rho_0 \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right)\right\} dV = \int_{\mathcal{A}} \bar{j}_0 \bar{\rho}_0 \ddot{\boldsymbol{u}} d\mathcal{A}$$

- Linear momentum equation (4)
  - Volume loading

$$\int_{\mathcal{S}_0} \boldsymbol{b} dV = \int_{\mathcal{A}} \int_{h_0} j_0 \boldsymbol{b} d\xi^3 d\mathcal{A} \Longrightarrow \int_{\mathcal{S}_0} \boldsymbol{b} dV = \int_{\mathcal{A}} \bar{j}_0 \bar{\boldsymbol{b}} d\mathcal{A}$$

    - With the surface force $\bar{\boldsymbol{b}} = \frac{1}{\bar{j}_0} \int_{h_0} j_0 \boldsymbol{b} d\xi^3$

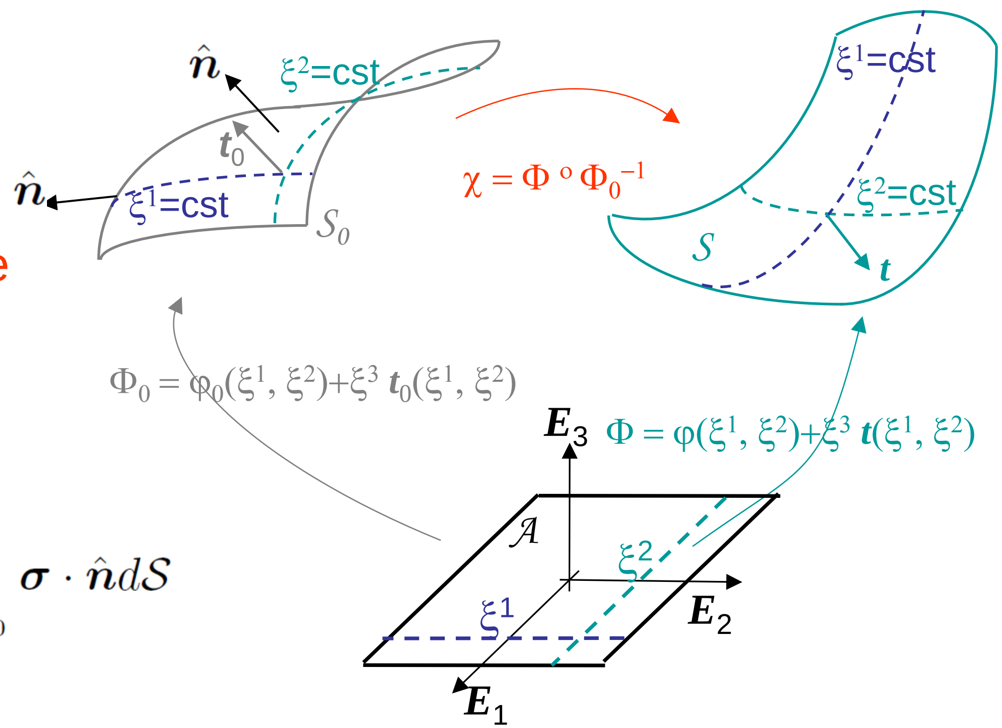
  - Stress term

$$\int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV$$

Divergence in the frame linked to $\mathcal{S}_0$ (not in E^I)

$\Longrightarrow$ Normal $\hat{\boldsymbol{n}}$ to $\mathcal{S}_0$

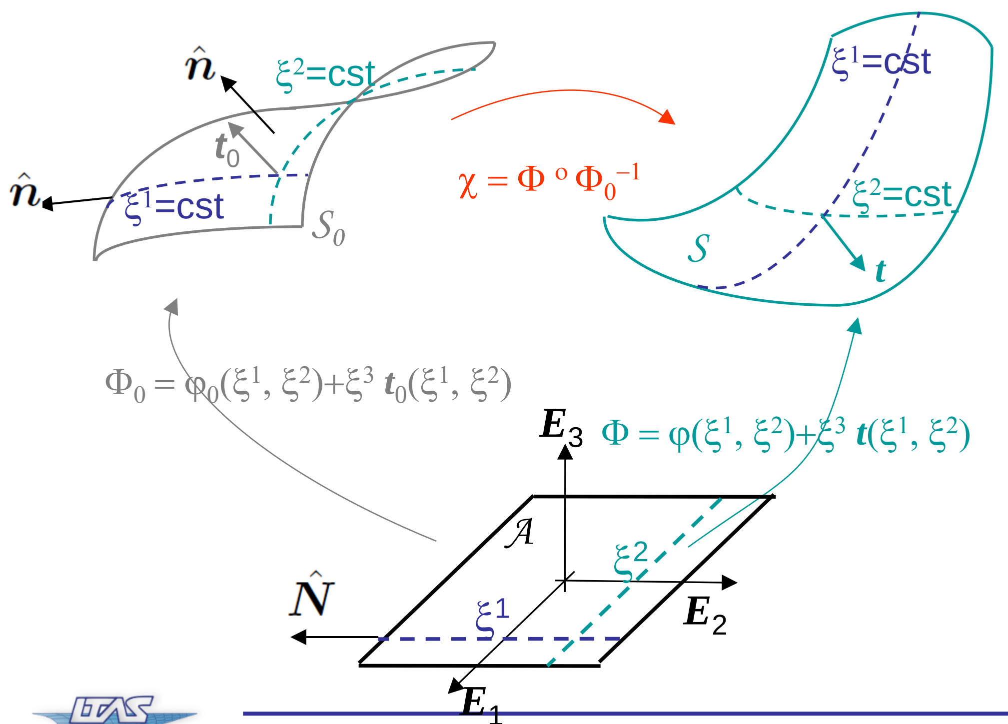
$$\int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV = \int_{\partial \mathcal{S}_0} \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}} d\mathcal{S}$$



- Linear momentum equation (5)
  - Stress term (2) $\int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV = \int_{\partial \mathcal{S}_0} \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}} d\mathcal{S}$
    - How to transform surface integral $d\mathcal{S}$ in $\mathcal{S}_0$ to in surface integral dS in $\mathcal{A}$ x h_0 ?
    - Nanson's Formula, for a mapping Φ_0: $d\mathcal{S}\hat{\boldsymbol{n}} = |\boldsymbol{\Phi}_0|\, dS \boldsymbol{\Phi}_0^{-T} \cdot \hat{\boldsymbol{N}}$

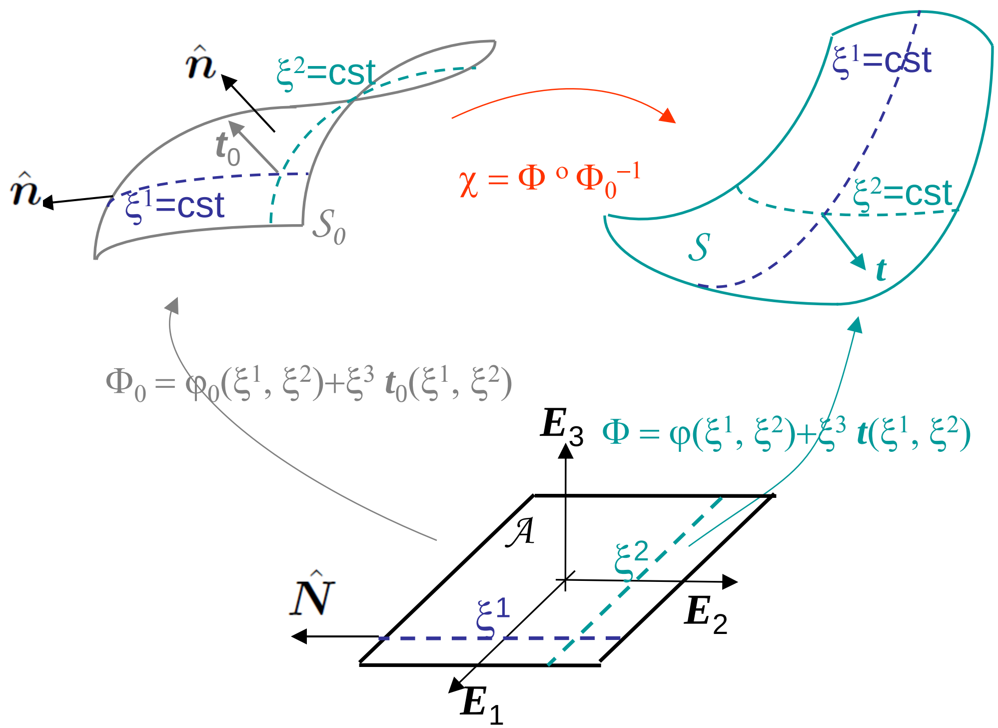
$\neq$

$$\Longrightarrow \int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV = \int_{\partial(\mathcal{A} \times h_0)} j_0 \boldsymbol{\sigma} \cdot \boldsymbol{\Phi}_0^{-T} \cdot \hat{\boldsymbol{N}} dS$$



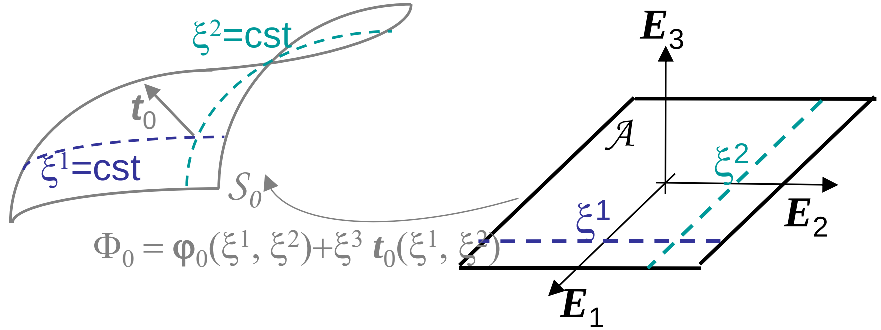
- Linear momentum equation (6)
  - Stress term (3) $\int_{\mathcal{S}_0} \nabla_0 \cdot \boldsymbol{\sigma}^T dV = \int_{\partial(\mathcal{A}\times h_0)} j_0 \boldsymbol{\sigma} \cdot \boldsymbol{\Phi}_0^{-T} \cdot \hat{\mathbf{N}} dS$
    - As $\nabla \boldsymbol{\Phi}_0 = \boldsymbol{g}_{0I} \otimes \boldsymbol{E}^I$ & $\left(\boldsymbol{E}_J \otimes \boldsymbol{g}_0^J\right)\left(\boldsymbol{g}_{0I} \otimes \boldsymbol{E}^I\right) = \boldsymbol{E}_J \delta_{IJ} \otimes \boldsymbol{E}^I = \mathbf{I}$

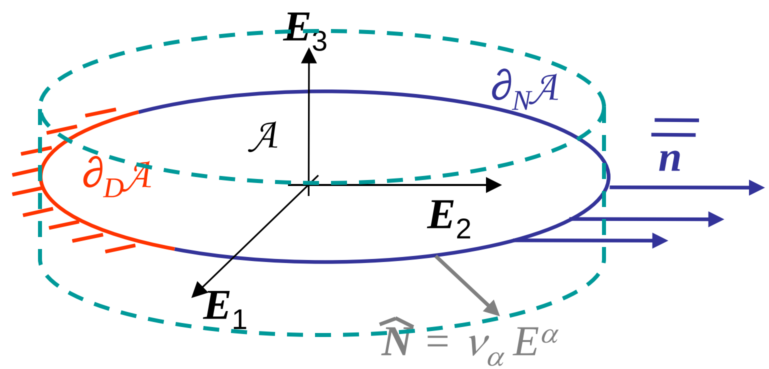
$$\Rightarrow \boldsymbol{\Phi}_0^{-1} = \left(\boldsymbol{E}_J \otimes \boldsymbol{g}_0^J\right) \quad \Rightarrow \int_{\mathcal{S}_0} \nabla_0 \cdot \boldsymbol{\sigma}^T dV = \int_{\partial(\mathcal{A}\times h_0)} j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^I \boldsymbol{E}_I \cdot \hat{\mathbf{N}} dS$$



- Linear momentum equation (7)
  - Stress term (4)
    - $$\int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV = \int_{\partial(\mathcal{A}\times h_0)} j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^I \boldsymbol{E}_I \cdot \hat{\mathbf{N}} dS$$
    - Where $\widehat{\boldsymbol{N}}$ is the normal

      in the reference frame
    - On top/bottom faces $\widehat{\boldsymbol{N}} = \pm \boldsymbol{E}^3$
    - On lateral surface: $\widehat{\boldsymbol{N}} = \nu_\alpha \boldsymbol{E}^\alpha$



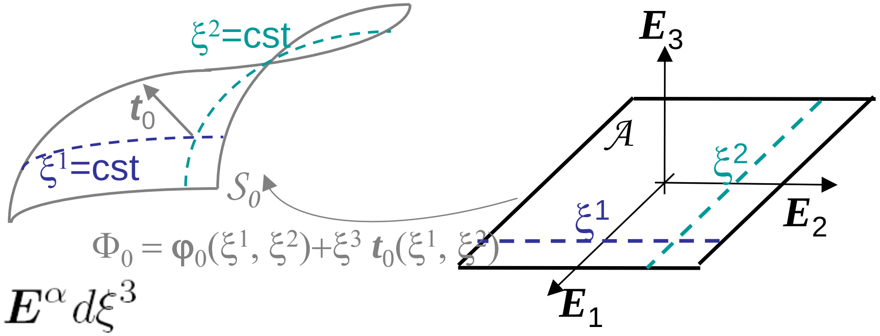


$$\int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV = \int_{\mathcal{A}} j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3 \big|_{h_0} d\mathcal{A} + \int_{\partial\mathcal{A}} \int_{h_0} j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha d\xi^3 \nu_\alpha dl$$

  - Let us define the resultant stresses: $\boldsymbol{n}^\alpha = \frac{1}{\bar{j}_0} \int_{h_0} j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha d\xi^3$

- Linear momentum equation (8)
  - Resultant stresses

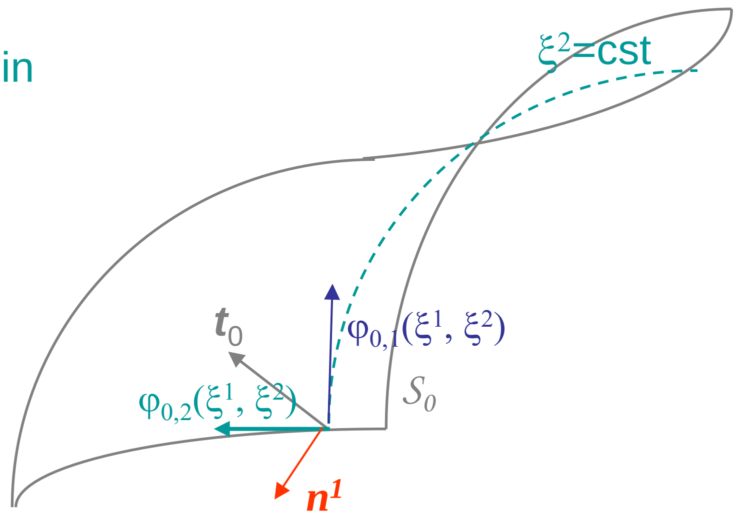
$$\boldsymbol{n}^{\alpha} = \frac{1}{\bar{j}_0}\int_{h_0} j_0 \boldsymbol{\sigma}\cdot\boldsymbol{g}_0^{\alpha} d\xi^3$$



for plates we found $\boldsymbol{n}^{\alpha} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}\cdot\boldsymbol{E}^{\alpha} d\xi^3$

- Compared to plates
  - There is the Jacobian mapping (1 for plates)
  - Cauchy stresses have to be projected in the body basis
    - » $\boldsymbol{g}^{\alpha}$ for shells
    - » E^I for plates
- We can write it in terms of the components in the mid-surface basis
  - $\boldsymbol{n}^{\alpha} = n^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + q^{\alpha}\boldsymbol{t}_0$

$$\begin{cases} n^{\alpha\beta} = \boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{n}^{\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0} j_0 \boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{g}_0^{\alpha} d\xi^3 \\ q^{\alpha} = \boldsymbol{t}_0\cdot\boldsymbol{n}^{\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0} j_0 \boldsymbol{t}_0\cdot\boldsymbol{\sigma}\cdot\boldsymbol{g}_0^{\alpha} d\xi^3 \end{cases}$$



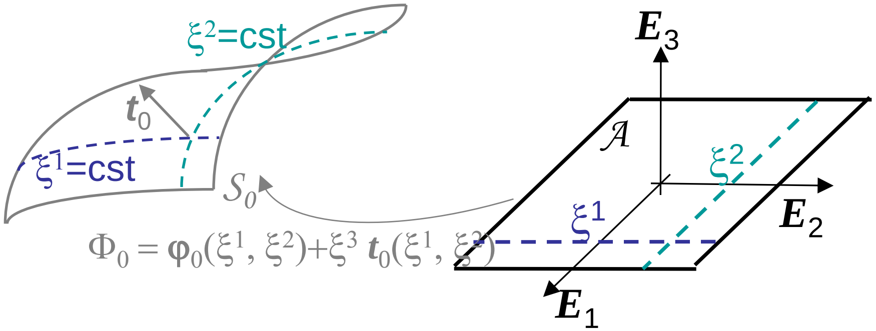
- Linear momentum equation (9)
  - From
    - $\int_{\mathcal{S}_0} \boldsymbol{b} dV = \int_{\mathcal{A}} \bar{j}_0 \bar{\boldsymbol{b}} d\mathcal{A}$
    - $\int_{\mathcal{S}_0} \left\{\rho_0 \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right)\right\} dV = \int_{\mathcal{A}} \bar{j}_0 \bar{\rho}_0 \ddot{\boldsymbol{u}} d\mathcal{A}$
    - $\int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV = \int_{\mathcal{A}} j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3 \big|_{h_0} d\mathcal{A} + \int_{\partial\mathcal{A}} \bar{j}_0 \boldsymbol{n}^\alpha \nu_\alpha dl$
    - Applying Gauss theorem on last term leads to

$$\int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV = \int_{\mathcal{A}} j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3 \big|_{h_0} d\mathcal{A} + \int_{\mathcal{A}} \left(\bar{j}_0 \boldsymbol{n}^\alpha\right)_{,\alpha} d\mathcal{A}$$

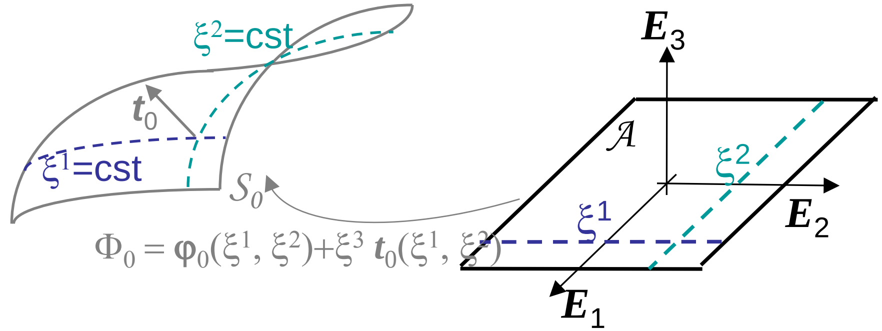
  - Resultant linear momentum equation

$$\int_{\mathcal{S}_0} \left\{\rho_0 \left(\ddot{\boldsymbol{\varphi}} + \xi^3 \ddot{\boldsymbol{t}}\right)\right\} dV = \int_{\mathcal{S}_0} \boldsymbol{b} dV + \int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV$$

$$\Longrightarrow \int_{\mathcal{A}} \bar{j}_0 \bar{\rho}_0 \ddot{\boldsymbol{u}} d\mathcal{A} = \int_{\mathcal{A}} \left\{\bar{j}_0 \bar{\boldsymbol{b}} + j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3 \big|_{h_0} + \left(\bar{j}_0 \boldsymbol{n}^\alpha\right)_{,\alpha}\right\} d\mathcal{A}$$



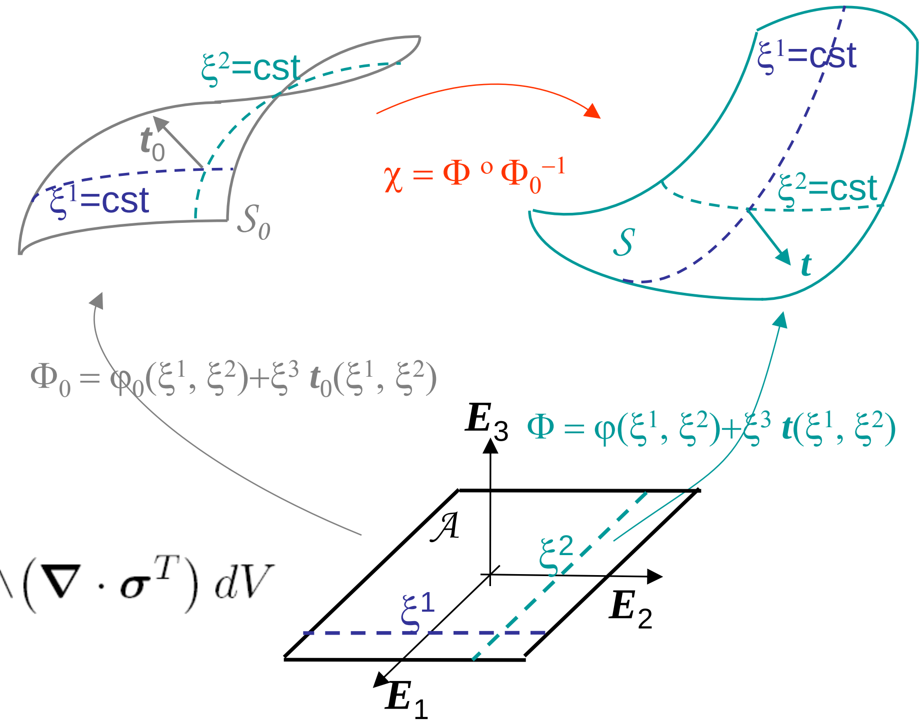
- Linear momentum equation (10)
  - Defining
    - Surface density $\bar{\rho}_0 = \frac{1}{\bar{j}_0} \int_{h_0} j_0 \rho_0 d\xi^3$
    - Surface loading $\bar{\boldsymbol{b}} = \frac{1}{\bar{j}_0} \int_{h_0} j_0 \boldsymbol{b} d\xi^3$
    - Resultant loading $\bar{\boldsymbol{n}} = \frac{1}{\bar{j}_0} \left[\bar{j}_0 \bar{\boldsymbol{b}} + \left(j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3 \right)\Big|_{h_0} \right]$



  - The linear momentum equation $\rho \ddot{\boldsymbol{x}} = \boldsymbol{b} + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T$ becomes after being integrated on the volume: $\int_{\mathcal{A}} \bar{j}_0 \bar{\rho}_0 \ddot{\boldsymbol{u}} d\mathcal{A} = \int_{\mathcal{A}} \left\{ \bar{j}_0 \bar{\boldsymbol{b}} + j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3 \Big|_{h_0} + \left(\bar{j}_0 \boldsymbol{n}^\alpha \right)_{,\alpha} \right\} d\mathcal{A}$
    - Is rewritten in the Cosserat plane A as $\boxed{\frac{1}{\bar{j}_0} \left(\bar{j}_0 \boldsymbol{n}^\alpha \right)_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}_0 \ddot{\boldsymbol{u}}}$
    - With the resultant stresses $\boldsymbol{n}^\alpha = \frac{1}{\bar{j}_0} \int_{h_0} j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha d\xi^3$
    - Remark, for plates we found $\left(\boldsymbol{n}^\alpha \right)_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho} \ddot{\boldsymbol{u}}$

- Angular momentum equation
  - $\rho\ddot{\boldsymbol{x}} = \boldsymbol{b} + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^T$

$\Rightarrow$ $\rho\Phi\wedge\ddot{\boldsymbol{x}} = \Phi\wedge\boldsymbol{b} + \Phi\wedge\left(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^T\right)$



$\Rightarrow$ $$\int_{\mathcal{S}}\rho\left(\boldsymbol{\varphi}+\xi^3\boldsymbol{t}\right)\wedge\left(\ddot{\boldsymbol{\varphi}}+\xi^3\ddot{\boldsymbol{t}}\right)dV = \int_{\mathcal{S}}\left(\boldsymbol{\varphi}+\xi^3\boldsymbol{t}\right)\wedge\boldsymbol{b}dV + \int_{\mathcal{S}}\left(\boldsymbol{\varphi}+\xi^3\boldsymbol{t}\right)\wedge\left(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^T\right)dV$$

  - Small transformations assumptions

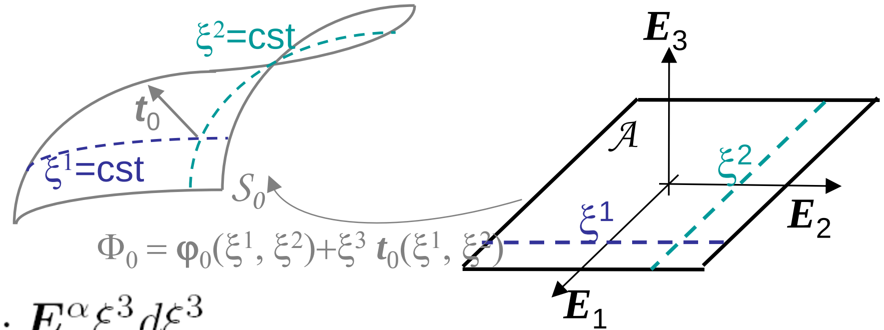
$\Rightarrow$ $$\int_{\mathcal{S}_0}\rho_0\left(\boldsymbol{\varphi}+\xi^3\boldsymbol{t}\right)\wedge\left(\ddot{\boldsymbol{\varphi}}+\xi^3\ddot{\boldsymbol{t}}\right)dV = \int_{\mathcal{S}_0}\left(\boldsymbol{\varphi}+\xi^3\boldsymbol{t}\right)\wedge\boldsymbol{b}dV + \int_{\mathcal{S}_0}\left(\boldsymbol{\varphi}+\xi^3\boldsymbol{t}\right)\wedge\left(\boldsymbol{\nabla}_0\cdot\boldsymbol{\sigma}^T\right)dV$$

- Using $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 + \boldsymbol{u}$

$\Rightarrow$ $$\int_{\mathcal{S}_0}\rho_0\left(\boldsymbol{\varphi}_0+\boldsymbol{u}+\xi^3\boldsymbol{t}\right)\wedge\left(\ddot{\boldsymbol{u}}+\xi^3\ddot{\boldsymbol{t}}\right)dV = \int_{\mathcal{S}_0}\left(\boldsymbol{\varphi}_0+\boldsymbol{u}+\xi^3\boldsymbol{t}\right)\wedge\boldsymbol{b}dV + \int_{\mathcal{S}_0}\left(\boldsymbol{\varphi}_0+\boldsymbol{u}+\xi^3\boldsymbol{t}\right)\wedge\left(\boldsymbol{\nabla}_0\cdot\boldsymbol{\sigma}^T\right)dV$$

- Angular momentum equation (2)
  - Proceeding as before (see annex 1), leads to
    - $\boldsymbol{t}_0 \wedge \ddot{\boldsymbol{t}} I_p \bar{j}_0 = \boldsymbol{t}_0 \wedge \bar{\tilde{\boldsymbol{m}}} \bar{j}_0 - \bar{j}_0 \boldsymbol{t}_0 \wedge \boldsymbol{n}^3 + \boldsymbol{t}_0 \wedge \left(\tilde{\boldsymbol{m}}^\alpha \bar{j}_0\right)_{,\alpha}$
    - If λ is an undefined pressure applied through the thickness, the resultant angular momentum equation reads $\boxed{\ddot{\boldsymbol{t}} I_p = \bar{\tilde{\boldsymbol{m}}} - \boldsymbol{n}^3 + \frac{1}{\bar{j}_0}\left(\tilde{\boldsymbol{m}}^\alpha \bar{j}_0\right)_{,\alpha} + \lambda \boldsymbol{t}_0}$
    - With
      - $\tilde{\boldsymbol{m}}^\alpha = \frac{1}{\bar{j}_0} \int_{h_0} \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha j_0 \xi^3 d\xi^3$
      - $\bar{\tilde{\boldsymbol{m}}} = \frac{1}{\bar{j}_0} \left(j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}^3 \xi^3 \right)\Big|_{h_0} + \frac{1}{\bar{j}_0} \int_{h_0} j_0 \boldsymbol{b} {\xi^3}^2 d\xi^3$
      - $I_p = \frac{1}{\bar{j}_0} \int_{h_0} \rho_0 j_0 {\xi^3}^2 d\xi^3$
    - Remark, for plates: $\ddot{\boldsymbol{t}} I_p = \bar{\tilde{\boldsymbol{m}}} - \left(\boldsymbol{n}^3 - \lambda \boldsymbol{E}_3\right) + \left(\tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha}$

- Angular momentum equation (3)
  - Resultant bending stresses

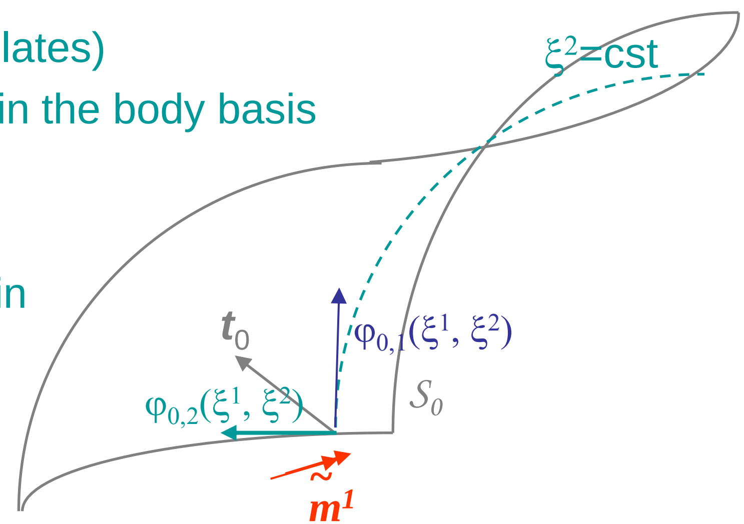


- $\tilde{\boldsymbol{m}}^{\alpha} = \frac{1}{\bar{j}_0}\int_{h_0} \boldsymbol{\sigma}\cdot\boldsymbol{g}_0^{\alpha} j_0 \xi^3 d\xi^3$

  for plates we found $\tilde{\boldsymbol{m}}^{\alpha} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}\cdot\boldsymbol{E}^{\alpha}\xi^3 d\xi^3$

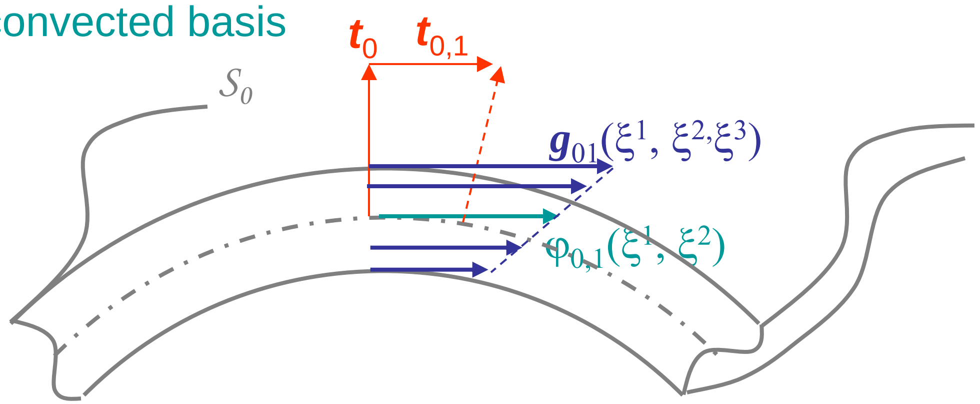
- Compared to plates
  - There is the Jacobian mapping (1 for plates)
  - Cauchy stresses have to be projected in the body basis
    - » $\boldsymbol{g}^{\alpha}$ for shells
    - » E^I for plates

- We can write it in terms of the components in the mid-surface basis
  - $\tilde{\boldsymbol{m}}^{\alpha} = \tilde{m}^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + \tilde{m}^{3\alpha}\boldsymbol{t}_0$



$$\Rightarrow \begin{cases} \tilde{m}^{\beta\alpha} = \boldsymbol{\varphi}_0^{,\beta}\cdot\tilde{\boldsymbol{m}}^{\alpha} = \frac{1}{\bar{j}_0}\int_{h_0} j_0\boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{g}_0^{\alpha}\xi^3 d\xi^3 \\ \tilde{m}^{3\alpha} = \boldsymbol{t}_0\cdot\tilde{\boldsymbol{m}}^{\alpha} = \frac{1}{\bar{j}_0}\int_{h_0} j_0\boldsymbol{t}_0\cdot\boldsymbol{\sigma}\cdot\boldsymbol{g}_0^{\alpha}\xi^3 d\xi^3 \end{cases}$$

- Resultant equations summary
  - Linear momentum
    - $\boxed{\frac{1}{\bar{j}_0}\left(\bar{j}_0 \boldsymbol{n}^{\alpha}\right)_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}_0 \ddot{\boldsymbol{u}}}$
    - Resultant stresses $\boldsymbol{n}^{\alpha} = \frac{1}{\bar{j}_0}\int_{h_0} j_0 \boldsymbol{\sigma}\cdot \boldsymbol{g}_0^{\alpha} d\xi^3$
  - Angular momentum
    - $\boxed{\ddot{\boldsymbol{t}} I_p = \bar{\tilde{\boldsymbol{m}}} - \boldsymbol{n}^3 + \frac{1}{\bar{j}_0}\left(\tilde{\boldsymbol{m}}^{\alpha}\bar{j}_0\right)_{,\alpha} + \lambda \boldsymbol{t}_0}$
    - Resultant bending stresses $\tilde{\boldsymbol{m}}^{\alpha} = \frac{1}{\bar{j}_0}\int_{h_0} \boldsymbol{\sigma}\cdot \boldsymbol{g}_0^{\alpha} j_0 \xi^3 d\xi^3$
  - Interpretation
    - Everything is projected on the convected basis
    - We have the ncoupling traction/bending as for plate only if no curvature ($\boldsymbol{t}_{0,\alpha}$=0)
    - What happens if initial curvature?



Material law

- What is missing is the link between the deformations and the stresses
  - Idea: as the discretization does not involve the thickness, the deformations should be evaluated at neutral plane too
  - Works only in linear elasticity
  - For non-linear materials, Cauchy tensor has to be evaluated through the thickness

- Deformations
  - Deformation gradient
    - Two-point deformation mapping $\boldsymbol{\chi} = \boldsymbol{\Phi} \circ \boldsymbol{\Phi}_0^{-1}$
    - Two-point deformation gradient $\mathbf{F} = \boldsymbol{\nabla}\boldsymbol{\Phi} \circ \left[\boldsymbol{\nabla}\boldsymbol{\Phi}_0\right]^{-1}$

In frame E^I ⇒ with respect to ξ^I

      - Small strain deformation gradient $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^T\right) - \mathbf{I}$
  - In terms of convected bases
    - We have the reference convected basis

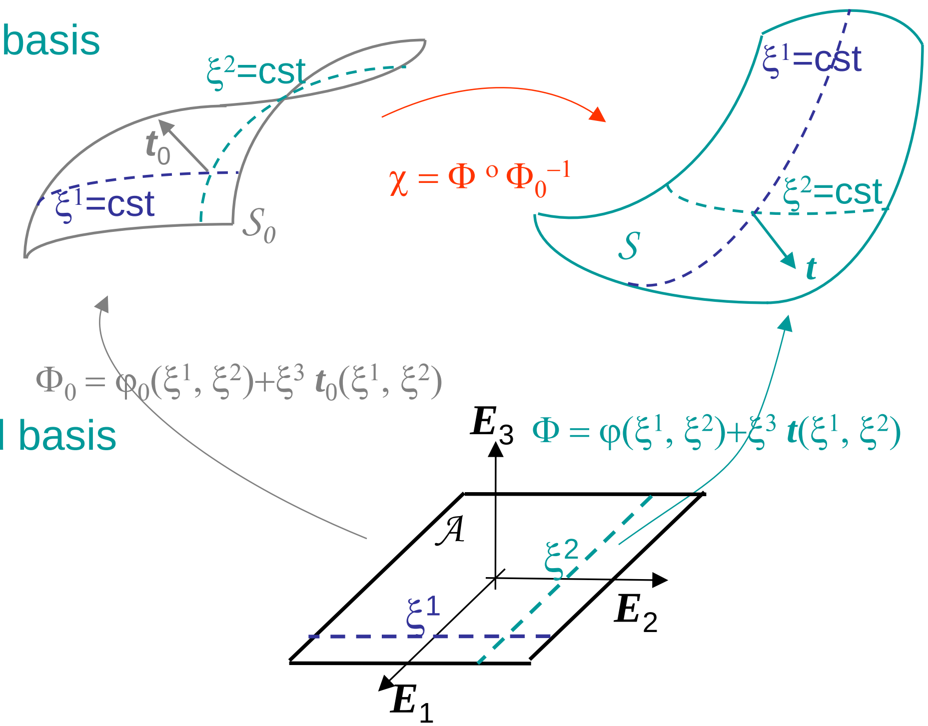
$$\boldsymbol{\nabla}\boldsymbol{\Phi}_0 = \boldsymbol{g}_{0I} \otimes \boldsymbol{E}^I$$

$$\begin{cases} \boldsymbol{g}_{0\alpha} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^\alpha} = \boldsymbol{\varphi}_{0,\alpha} + \xi^3 \boldsymbol{t}_{0,\alpha} \\ \boldsymbol{g}_{03} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^3} = \boldsymbol{t}_0 \end{cases}$$

    - Similarly, the deformed convected basis

$$\boldsymbol{\nabla}\boldsymbol{\Phi} = \boldsymbol{g}_I \otimes \boldsymbol{E}^I$$

$$\begin{cases} \boldsymbol{g}_{\alpha} = \dfrac{\partial \boldsymbol{\Phi}}{\partial \xi^\alpha} = \boldsymbol{\varphi}_{,\alpha} + \xi^3 \boldsymbol{t}_{,\alpha} \\ \boldsymbol{g}_{3} = \dfrac{\partial \boldsymbol{\Phi}}{\partial \xi^3} = \boldsymbol{t}_{,\alpha} \end{cases}$$



- Deformations (2)
  - Deformation gradient in terms of convected bases
    - $\mathbf{F} = \boldsymbol{\nabla}\boldsymbol{\Phi} \circ [\boldsymbol{\nabla}\boldsymbol{\Phi}_0]^{-1}$
    - With $\boldsymbol{\nabla}\boldsymbol{\Phi}_0 = \boldsymbol{g}_{0I} \otimes \boldsymbol{E}^I$, $\boldsymbol{\Phi}_0^{-1} = \left(\boldsymbol{E}_J \otimes \boldsymbol{g}_0^J\right)$ & $\boldsymbol{\nabla}\boldsymbol{\Phi} = \boldsymbol{g}_I \otimes \boldsymbol{E}^I$
    - As $\boldsymbol{E}^I \cdot \boldsymbol{E}_J = \delta_{IJ}$ $\Longrightarrow$ $\mathbf{F} = \boldsymbol{g}_I \otimes \boldsymbol{g}_0^I = \boldsymbol{g}_\alpha \otimes \boldsymbol{g}_0^\alpha + \boldsymbol{t} \otimes \boldsymbol{g}_0^3$
      - Remarks:
        - » As the basis is not orthonormal $\boldsymbol{g}^3 \neq \boldsymbol{g}_3 = t$
        - » Identity matrix should also be defined in the convected basis
        - » When $\boldsymbol{g} = \boldsymbol{g}_0$, there is no deformation $\Longrightarrow$ $\boldsymbol{F} = \boldsymbol{I}$

$$\Longrightarrow \mathbf{I} = \boldsymbol{g}_{0I} \otimes \boldsymbol{g}_0^I = \boldsymbol{g}_{0\alpha} \otimes \boldsymbol{g}_0^\alpha + \boldsymbol{t}_0 \otimes \boldsymbol{g}_0^3$$

    - Small deformations

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^T\right) - \mathbf{I}$$

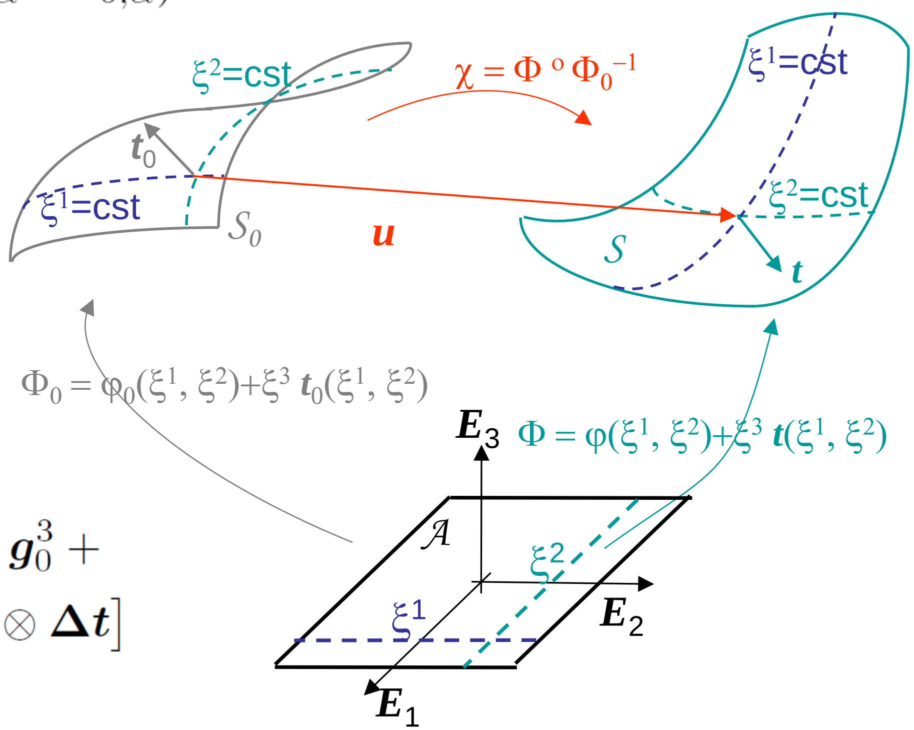
$$\Longrightarrow \boldsymbol{\varepsilon} = \frac{1}{2}\left[(\boldsymbol{g}_\alpha - \boldsymbol{g}_{0\alpha}) \otimes \boldsymbol{g}_0^\alpha + (\boldsymbol{t} - \boldsymbol{t}_0) \otimes \boldsymbol{g}_0^3 + \boldsymbol{g}_0^\alpha \otimes (\boldsymbol{g}_\alpha - \boldsymbol{g}_{0\alpha}) + \boldsymbol{g}_0^3 \otimes (\boldsymbol{t} - \boldsymbol{t}_0)\right]$$

- Deformations (3)
  - Small deformation tensor
    - $$\boldsymbol{\varepsilon} = \frac{1}{2}\left[(\boldsymbol{g}_\alpha - \boldsymbol{g}_{0\alpha}) \otimes \boldsymbol{g}_0^\alpha + (\boldsymbol{t} - \boldsymbol{t}_0) \otimes \boldsymbol{g}_0^3 + \boldsymbol{g}_0^\alpha \otimes (\boldsymbol{g}_\alpha - \boldsymbol{g}_{0\alpha}) + \boldsymbol{g}_0^3 \otimes (\boldsymbol{t} - \boldsymbol{t}_0)\right]$$
    - If $\boldsymbol{u} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_0$ is the displacement of the mid-surface

$$\boldsymbol{g}_\alpha - \boldsymbol{g}_{0\alpha} = \boldsymbol{\varphi}_{,\alpha} - \boldsymbol{\varphi}_{0,\alpha} + \xi^3 \left(\boldsymbol{t}_{,\alpha} - \boldsymbol{t}_{0,\alpha}\right)$$

$$\Longrightarrow \boldsymbol{g}_\alpha - \boldsymbol{g}_{0\alpha} = \boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}$$

    - With $\boldsymbol{t} = \boldsymbol{t}_0 + \boldsymbol{\Delta t}$



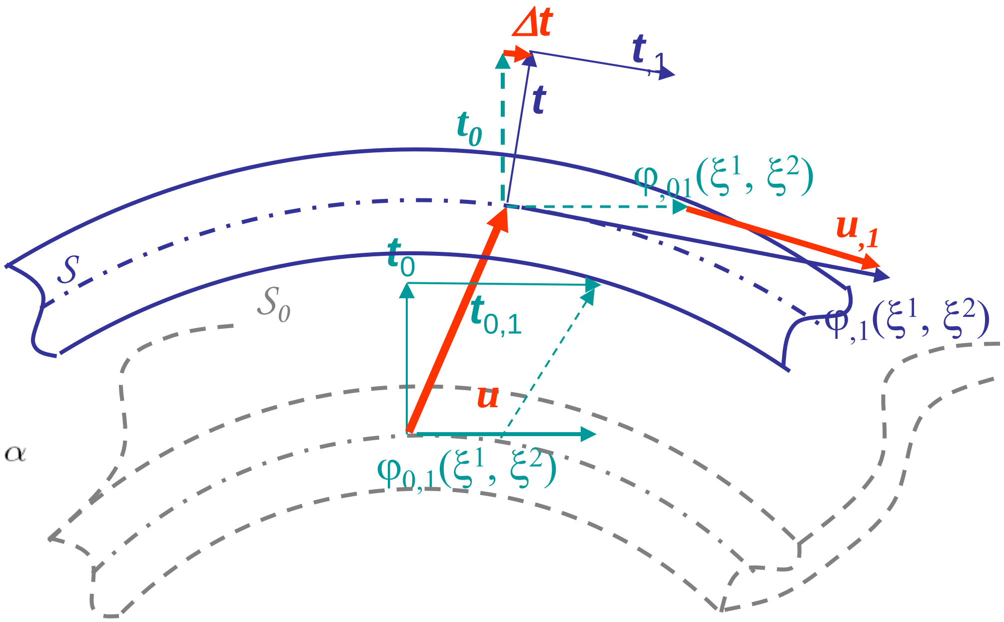
$$\Longrightarrow \boldsymbol{\varepsilon} = \frac{1}{2}\left[\left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) \otimes \boldsymbol{g}_0^\alpha + \boldsymbol{\Delta t} \otimes \boldsymbol{g}_0^3 + \boldsymbol{g}_0^\alpha \otimes \left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) + \boldsymbol{g}_0^3 \otimes \boldsymbol{\Delta t}\right]$$

- Deformations (4)
  - Small deformation tensor (2)
    - $\boldsymbol{\varepsilon} = \frac{1}{2}\left[\left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) \otimes \boldsymbol{g}_0^{\alpha} + \boldsymbol{\Delta t} \otimes \boldsymbol{g}_0^3 + \boldsymbol{g}_0^{\alpha} \otimes \left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) + \boldsymbol{g}_0^3 \otimes \boldsymbol{\Delta t}\right]$
  - Remarks: As $||\boldsymbol{t}|| = ||\boldsymbol{t}_0|| = 1$
    - $1 = \boldsymbol{t} \cdot \boldsymbol{t} = \boldsymbol{t}_0 \cdot \boldsymbol{t}_0 + 2\boldsymbol{\Delta t} \cdot \boldsymbol{t}_0 + \cancel{\boldsymbol{\Delta t} \cdot \boldsymbol{\Delta t}}$

      Second order

      $\Longrightarrow \boldsymbol{\Delta t} \cdot \boldsymbol{t}_0 = 0$

      $\Longrightarrow \boldsymbol{\Delta t}_{,\alpha} \cdot \boldsymbol{t}_0 = -\boldsymbol{\Delta t} \cdot \boldsymbol{t}_{0,\alpha}$
    - $0 = \partial_{\alpha}\left(\boldsymbol{t} \cdot \boldsymbol{t}\right) = 2\boldsymbol{t}_{,\alpha} \cdot \boldsymbol{t}$
    - $0 = \partial_{\alpha}\left(\boldsymbol{t}_0 \cdot \boldsymbol{t}_0\right) = 2\boldsymbol{t}_{0,\alpha} \cdot \boldsymbol{t}_0$

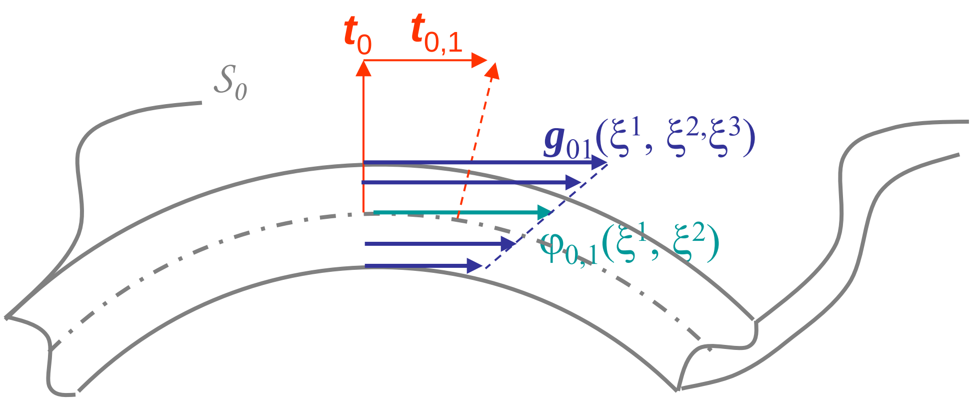


- Deformations (5)
  - Components of the deformation tensor
    - $\boldsymbol{\varepsilon} = \frac{1}{2}\left[\left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) \otimes \boldsymbol{g}_0^\alpha + \boldsymbol{\Delta t} \otimes \boldsymbol{g}_0^3 + \boldsymbol{g}_0^\alpha \otimes \left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) + \boldsymbol{g}_0^3 \otimes \boldsymbol{\Delta t}\right]$
    - But a vector can always be expressed in a basis

$$\boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{g}_{0I}\left(\xi^3 = 0\right) \boldsymbol{g}_0^I\left(\xi^3 = 0\right)$$

$$\begin{aligned} \boldsymbol{\varepsilon} \;=\; \frac{1}{2}\Big[& \left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) \cdot \boldsymbol{\varphi}_{0,\beta} \boldsymbol{\varphi}_0^{,\beta} \otimes \boldsymbol{g}_0^\alpha + \left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) \cdot \boldsymbol{t}_0 \boldsymbol{t}_0 \otimes \boldsymbol{g}_0^\alpha + \\ & \boldsymbol{\Delta t} \cdot \boldsymbol{\varphi}_{0,\beta} \boldsymbol{\varphi}_0^{,\beta} \otimes \boldsymbol{g}_0^3 + \boldsymbol{\Delta t} \cdot \boldsymbol{t}_0 \boldsymbol{t}_0 \otimes \boldsymbol{g}_0^3 + \\ & \boldsymbol{g}_0^\alpha \otimes \boldsymbol{\varphi}_0^{,\beta} \boldsymbol{\varphi}_{0,\beta} \cdot \left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) + \boldsymbol{g}_0^\alpha \otimes \boldsymbol{t}_0 \boldsymbol{t}_0 \cdot \left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) + \\ & \boldsymbol{g}_0^3 \otimes \boldsymbol{\varphi}_0^{,\beta} \boldsymbol{\varphi}_{0,\beta} \cdot \boldsymbol{\Delta t} + \boldsymbol{g}_0^3 \otimes \boldsymbol{t}_0 \boldsymbol{t}_0 \cdot \boldsymbol{\Delta t} \Big] \end{aligned}$$

- Deformations (6)
  - Components of the deformation tensor (2)
    - $$\boldsymbol{\varepsilon} = \frac{1}{2}\Big[\left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) \cdot \boldsymbol{\varphi}_{0,\beta}\boldsymbol{\varphi}_0^{,\beta} \otimes \boldsymbol{g}_0^{\alpha} + \left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) \cdot \boldsymbol{t}_0\boldsymbol{t}_0 \otimes \boldsymbol{g}_0^{\alpha} + \boldsymbol{\Delta t} \cdot \boldsymbol{\varphi}_{0,\beta}\boldsymbol{\varphi}_0^{,\beta} \otimes \boldsymbol{g}_0^{3} + \cancel{\boldsymbol{\Delta t} \cdot \boldsymbol{t}_0\boldsymbol{t}_0 \otimes \boldsymbol{g}_0^{3}} + \boldsymbol{g}_0^{\alpha} \otimes \boldsymbol{\varphi}_0^{,\beta}\boldsymbol{\varphi}_{0,\beta} \cdot \left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) + \boldsymbol{g}_0^{\alpha} \otimes \boldsymbol{t}_0\boldsymbol{t}_0 \cdot \left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) + \boldsymbol{g}_0^{3} \otimes \boldsymbol{\varphi}_0^{,\beta}\boldsymbol{\varphi}_{0,\beta} \cdot \boldsymbol{\Delta t} + \boldsymbol{g}_0^{3} \otimes \cancel{\boldsymbol{t}_0\boldsymbol{t}_0 \cdot \boldsymbol{\Delta t}}\Big]$$
    - If thickness of the shell is reduced compared to the curvature
      - Then $\xi^3 << \kappa$ & $\xi^3 \boldsymbol{t}_{0,\alpha} \sim \xi^3/\kappa << \boldsymbol{\varphi}_{0,\alpha}$
      - An approximation consist in considering the mid-surface basis in the dyadic products
      - $\boldsymbol{g}_0^{\alpha} \rightarrow \boldsymbol{\varphi}_0^{,\alpha}$
      - $\boldsymbol{g}_0^{3} \rightarrow \boldsymbol{t}_0$ as initially $\boldsymbol{t}_0$ is perpendicular to the mid-surface



- Deformations (7)
  - Components of the deformation tensor (3)
    - $$\begin{aligned}\boldsymbol{\varepsilon} \;=\; \frac{1}{2}\Big[&\left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) \cdot \boldsymbol{\varphi}_{0,\beta}\boldsymbol{\varphi}_0^{,\beta} \otimes \boldsymbol{g}_0^{\alpha} + \left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) \cdot \boldsymbol{t}_0\boldsymbol{t}_0 \otimes \boldsymbol{g}_0^{\alpha} + \\ &\boldsymbol{\Delta t} \cdot \boldsymbol{\varphi}_{0,\beta}\boldsymbol{\varphi}_0^{,\beta} \otimes \boldsymbol{g}_0^{3} + \cancel{\boldsymbol{\Delta t} \cdot \boldsymbol{t}_0\boldsymbol{t}_0 \otimes \boldsymbol{g}_0^{3}} + \\ &\boldsymbol{g}_0^{\alpha} \otimes \boldsymbol{\varphi}_0^{,\beta}\boldsymbol{\varphi}_{0,\beta} \cdot \left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) + \boldsymbol{g}_0^{\alpha} \otimes \boldsymbol{t}_0\boldsymbol{t}_0 \cdot \left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) + \\ &\boldsymbol{g}_0^{3} \otimes \boldsymbol{\varphi}_0^{,\beta}\boldsymbol{\varphi}_{0,\beta} \cdot \boldsymbol{\Delta t} + \cancel{\boldsymbol{g}_0^{3} \otimes \boldsymbol{t}_0\boldsymbol{t}_0 \cdot \boldsymbol{\Delta t}}\Big]\end{aligned}$$
    - Thickness reduced compared to the curvature: $\boldsymbol{g}_0^{\alpha} \rightarrow \boldsymbol{\varphi}_0^{,\alpha}$ & $\boldsymbol{g}_0^{3} \rightarrow \boldsymbol{t}_0$

$$\Longrightarrow \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left[\left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) \cdot \boldsymbol{\varphi}_{0,\beta} + \left(\boldsymbol{u}_{,\beta} + \xi^3 \boldsymbol{\Delta t}_{,\beta}\right) \cdot \boldsymbol{\varphi}_{0,\alpha}\right]\boldsymbol{\varphi}_0^{,\beta} \otimes \boldsymbol{\varphi}_0^{,\alpha} +$$

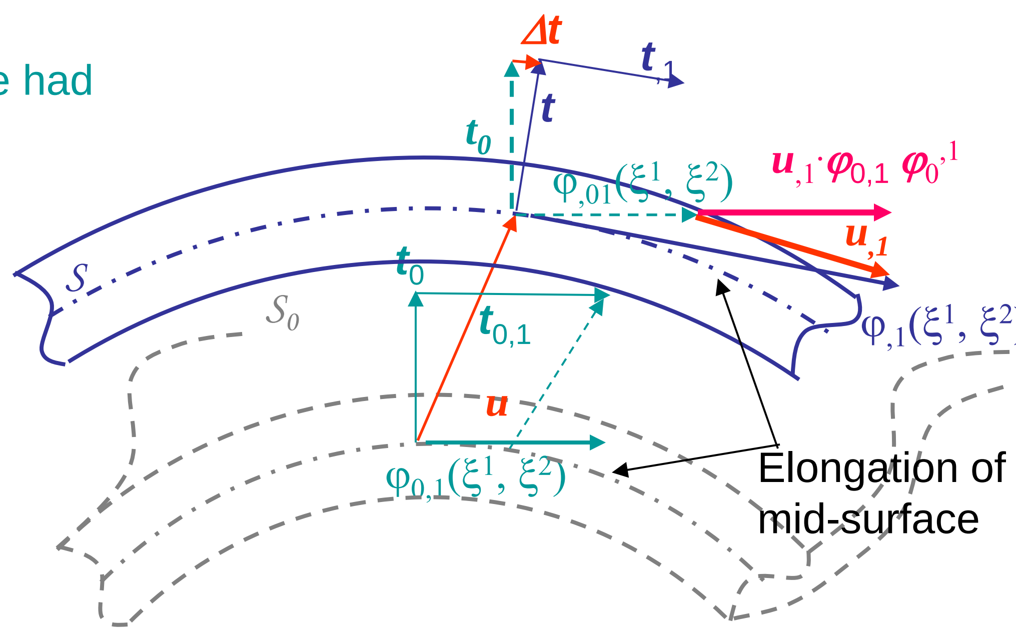
$$\frac{1}{2}\left[\left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right) \cdot \boldsymbol{t}_0 + \boldsymbol{\Delta t} \cdot \boldsymbol{\varphi}_{0,\alpha}\right]\left[\boldsymbol{t}_0 \otimes \boldsymbol{\varphi}_0^{,\alpha} + \boldsymbol{\varphi}_0^{,\alpha} \otimes \boldsymbol{t}_0\right]$$

- Deformations (8)
  - Components of the deformation tensor (4)
    - $$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right)\cdot\boldsymbol{\varphi}_{0,\beta} + \left(\boldsymbol{u}_{,\beta} + \xi^3 \boldsymbol{\Delta t}_{,\beta}\right)\cdot\boldsymbol{\varphi}_{0,\alpha}\right]\boldsymbol{\varphi}_0^{,\beta}\otimes\boldsymbol{\varphi}_0^{,\alpha} + \frac{1}{2}\left[\left(\boldsymbol{u}_{,\alpha} + \xi^3 \boldsymbol{\Delta t}_{,\alpha}\right)\cdot\boldsymbol{t}_0 + \boldsymbol{\Delta t}\cdot\boldsymbol{\varphi}_{0,\alpha}\right]\left[\boldsymbol{t}_0\otimes\boldsymbol{\varphi}_0^{,\alpha} + \boldsymbol{\varphi}_0^{,\alpha}\otimes\boldsymbol{t}_0\right]$$
    - Defining
      - $\varepsilon_{\alpha\beta} = \dfrac{\boldsymbol{u}_{,\alpha}\cdot\boldsymbol{\varphi}_{0,\beta} + \boldsymbol{u}_{,\beta}\cdot\boldsymbol{\varphi}_{0,\alpha}}{2}$
      - $\kappa_{\alpha\beta} = \dfrac{\boldsymbol{\Delta t}_{,\alpha}\cdot\boldsymbol{\varphi}_{0,\beta} + \boldsymbol{\Delta t}_{,\beta}\cdot\boldsymbol{\varphi}_{0,\alpha}}{2}$
      - $2\delta_\alpha = \gamma_\alpha = \boldsymbol{u}_{,\alpha}\cdot\boldsymbol{t}_0 + \boldsymbol{\Delta t}\cdot\boldsymbol{\varphi}_{0,\alpha}$

$$\Longrightarrow \boldsymbol{\varepsilon} = \left(\varepsilon_{\alpha\beta} + \xi^3\kappa_{\alpha\beta}\right)\boldsymbol{\varphi}_0^{,\alpha}\otimes\boldsymbol{\varphi}_0^{,\beta} + \frac{1}{2}\left(\gamma_\alpha + \xi^3\boldsymbol{\Delta t}_{,\alpha}\cdot\boldsymbol{t}_0\right)\left(\boldsymbol{t}_0\otimes\boldsymbol{\varphi}_0^{,\alpha} + \boldsymbol{\varphi}_0^{,\alpha}\otimes\boldsymbol{t}_0\right)$$

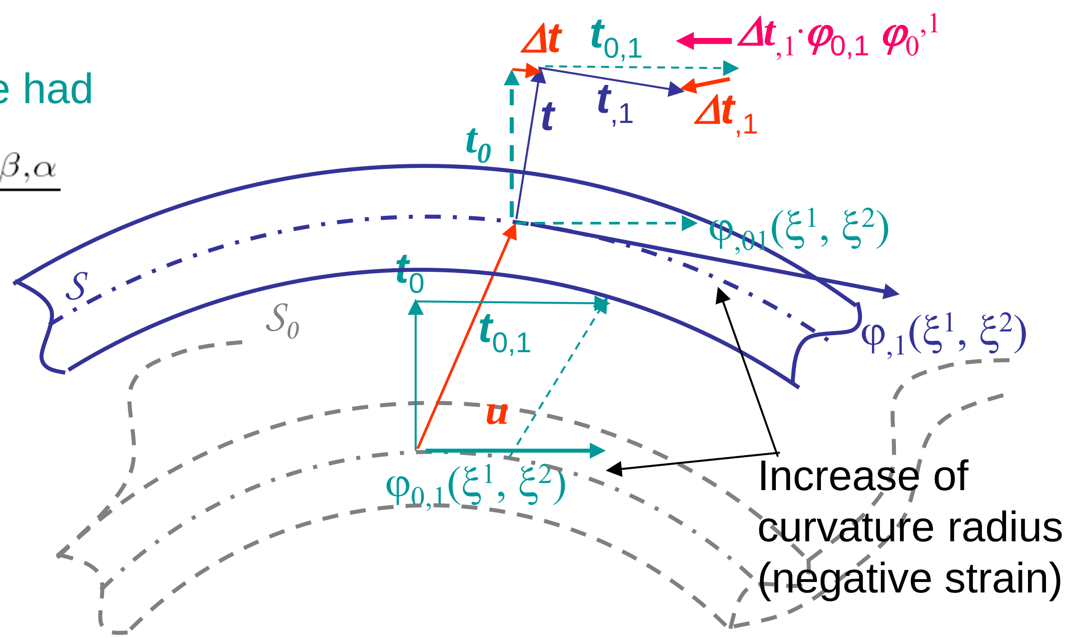
- Deformations (9)
  - Deformation mode
    - $\boldsymbol{\varepsilon} = \left(\varepsilon_{\alpha\beta} + \xi^3 \kappa_{\alpha\beta}\right) \boldsymbol{\varphi}_0^{;\alpha} \otimes \boldsymbol{\varphi}_0^{;\beta} + \frac{1}{2}\left(\gamma_\alpha + \xi^3 \boldsymbol{\Delta t}_{,\alpha} \cdot \boldsymbol{t}_0\right)\left(\boldsymbol{t}_0 \otimes \boldsymbol{\varphi}_0^{;\alpha} + \boldsymbol{\varphi}_0^{;\alpha} \otimes \boldsymbol{t}_0\right)$
    - Term in $\boxed{\varepsilon_{\alpha\beta} = \frac{\boldsymbol{u}_{,\alpha} \cdot \boldsymbol{\varphi}_{0,\beta} + \boldsymbol{u}_{,\beta} \cdot \boldsymbol{\varphi}_{0,\alpha}}{2}}$
      - Corresponds to the relative “in-plane” deformation of the mid-surface plane
      - Relative because expressed in the metric of the convected basis at mid-surface
      - Remark, for plates we had

$$\varepsilon_{\alpha\beta} = \frac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2}$$



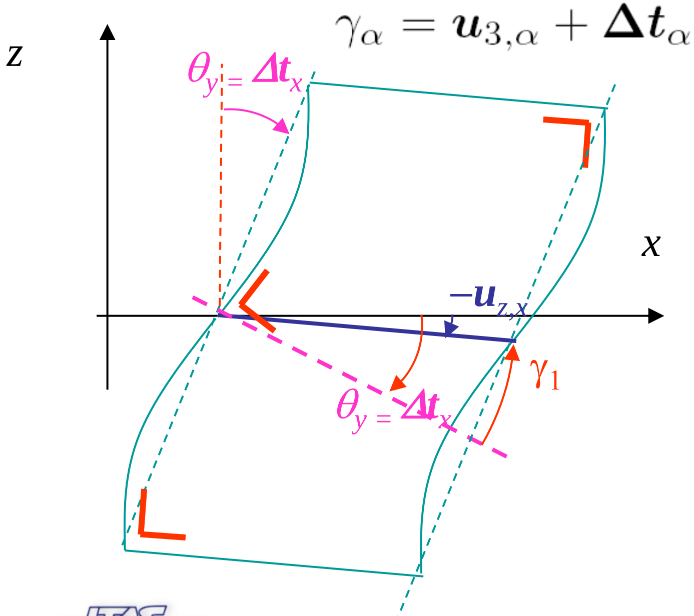
- Deformations (10)
  - Deformation mode (2)
    - $\boldsymbol{\varepsilon} = \left(\varepsilon_{\alpha\beta} + \xi^3 \kappa_{\alpha\beta}\right) \boldsymbol{\varphi}_0^{;\alpha} \otimes \boldsymbol{\varphi}_0^{;\beta} + \frac{1}{2}\left(\gamma_\alpha + \xi^3 \boldsymbol{\Delta t}_{,\alpha} \cdot \boldsymbol{t}_0\right)\left(\boldsymbol{t}_0 \otimes \boldsymbol{\varphi}_0^{;\alpha} + \boldsymbol{\varphi}_0^{;\alpha} \otimes \boldsymbol{t}_0\right)$
    - Term in $\boxed{\kappa_{\alpha\beta} = \frac{\boldsymbol{\Delta t}_{,\alpha} \cdot \boldsymbol{\varphi}_{0,\beta} + \boldsymbol{\Delta t}_{,\beta} \cdot \boldsymbol{\varphi}_{0,\alpha}}{2}}$
      - Corresponds to the relative change in curvature of the mid-surface plane
      - Relative because expressed in the metric of the convected basis at mid-surface
      - Remark, for plates we had

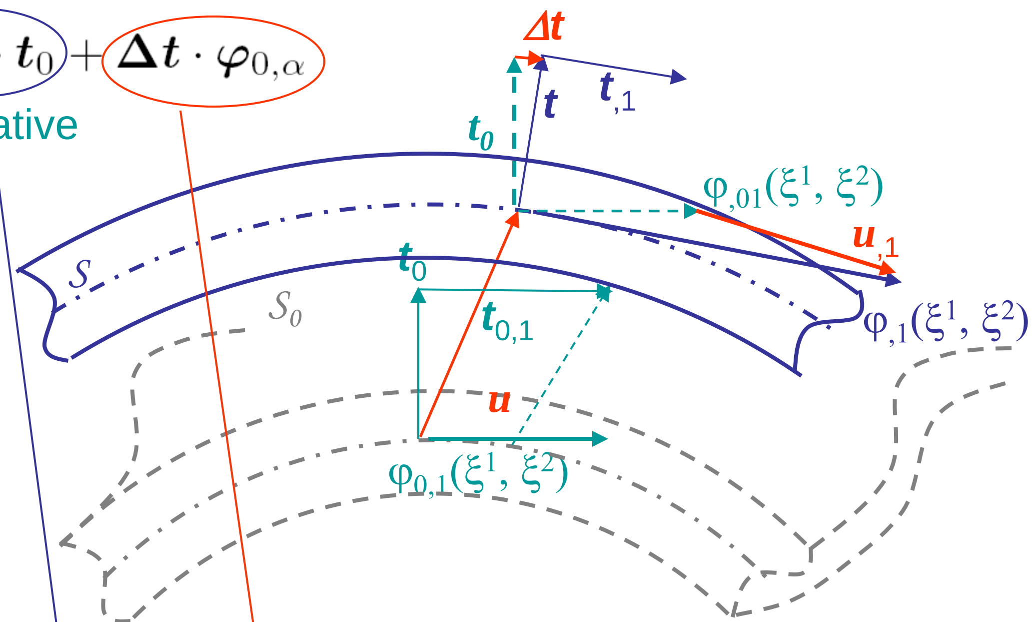
$$\kappa_{\alpha\beta} = \frac{\boldsymbol{\Delta t}_{\alpha,\beta} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2}$$



- Deformations (11)
  - Deformation mode (3)
    - $\boldsymbol{\varepsilon} = \left(\varepsilon_{\alpha\beta} + \xi^3 \kappa_{\alpha\beta}\right) \boldsymbol{\varphi}_0^{;\alpha} \otimes \boldsymbol{\varphi}_0^{;\beta} + \frac{1}{2}\left(\gamma_\alpha + \xi^3 \boldsymbol{\Delta t}_{,\alpha} \cdot \boldsymbol{t}_0\right)\left(\boldsymbol{t}_0 \otimes \boldsymbol{\varphi}_0^{;\alpha} + \boldsymbol{\varphi}_0^{;\alpha} \otimes \boldsymbol{t}_0\right)$
    - Term in $2\delta_\alpha = \gamma_\alpha = \boldsymbol{u}_{,\alpha} \cdot \boldsymbol{t}_0 + \boldsymbol{\Delta t} \cdot \boldsymbol{\varphi}_{0,\alpha}$
      - Corresponds to the relative average change in neutral plane direction
      - Plate analogy

$$\gamma_\alpha = \boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_\alpha$$

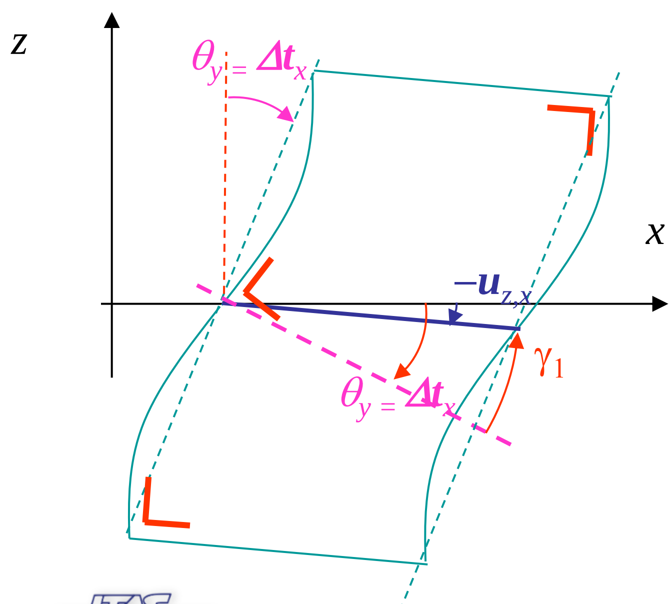


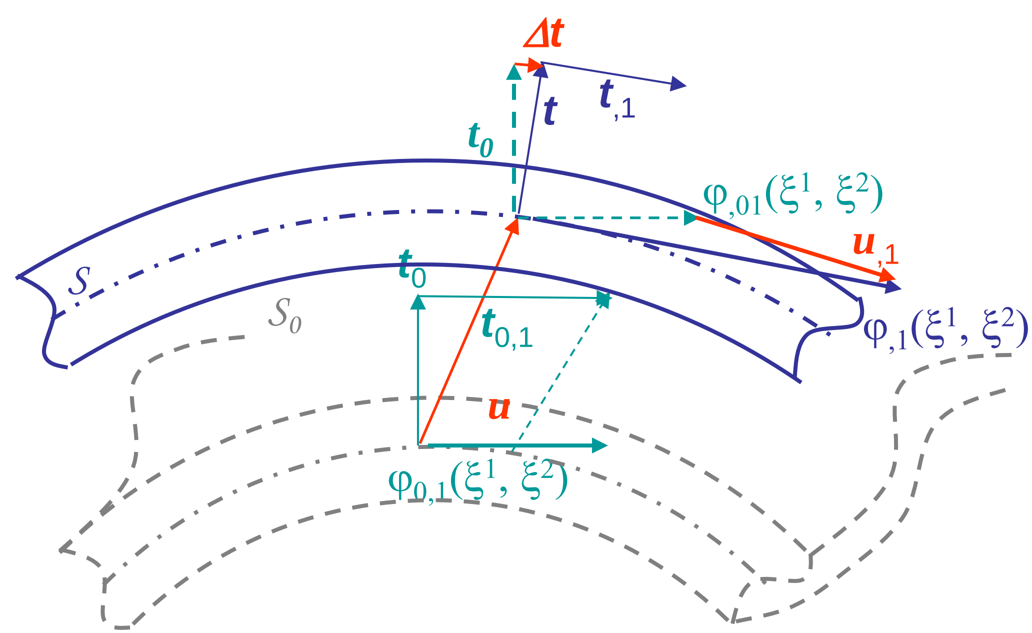


Change of the neutral plane direction resulting from
1) Bending
2) Out-of-plane shearing

- Deformations (12)
  - Deformation mode (4)
    - $\boldsymbol{\varepsilon} = \left(\varepsilon_{\alpha\beta} + \xi^3 \kappa_{\alpha\beta}\right) \boldsymbol{\varphi}_0^{,\alpha} \otimes \boldsymbol{\varphi}_0^{,\beta} + \frac{1}{2}\left(\gamma_\alpha + \xi^3 \boldsymbol{\Delta t}_{,\alpha} \cdot \boldsymbol{t}_0\right)\left(\boldsymbol{t}_0 \otimes \boldsymbol{\varphi}_0^{,\alpha} + \boldsymbol{\varphi}_0^{,\alpha} \otimes \boldsymbol{t}_0\right)$
    - Term in $\xi^3 \boldsymbol{\Delta t}_{,\alpha} \cdot \boldsymbol{t}_0$
      - Thickness dependence of the out-of-plane shearing
      - For thin structure we consider the average one

        ⇒ neglected





- Deformations (13)
  - Deformation modes (5)
    - $\boldsymbol{\varepsilon} = \left(\varepsilon_{\alpha\beta} + \xi^3 \kappa_{\alpha\beta}\right) \boldsymbol{\varphi}_0^{;\alpha} \otimes \boldsymbol{\varphi}_0^{;\beta} + \frac{1}{2}\left(\gamma_\alpha + \xi^3 \boldsymbol{\Delta t}_{,\alpha} \cdot \boldsymbol{t}_0\right)\left(\boldsymbol{t}_0 \otimes \boldsymbol{\varphi}_0^{;\alpha} + \boldsymbol{\varphi}_0^{;\alpha} \otimes \boldsymbol{t}_0\right)$
    - With $\xi^3 \boldsymbol{\Delta t}_{,\alpha} \cdot \boldsymbol{t}_0$ neglected
    - Through-the-thickness elongation
      - In this model there is no through-the-thickness elongation
      - Actually the plate is in plane-σ state, meaning there is such a deformation
        - » To be introduced: ξ^3 $\boldsymbol{t}$ should be substituted by $\lambda_h(\xi^3)$ $\boldsymbol{t}$ in the shell kinematics
        - » We have to introduce it to get the plane-σ effect
        - » In small deformations this term would lead to second order effects on other components

$$\Longrightarrow \boldsymbol{\varepsilon} = \left(\varepsilon_{\alpha\beta} + \xi^3 \kappa_{\alpha\beta}\right) \boldsymbol{\varphi}_0^{;\alpha} \otimes \boldsymbol{\varphi}_0^{;\beta} + \frac{1}{2}\gamma_\alpha\left(\boldsymbol{t}_0 \otimes \boldsymbol{\varphi}_0^{;\alpha} + \boldsymbol{\varphi}_0^{;\alpha} \otimes \boldsymbol{t}_0\right) + \lambda_h \boldsymbol{t}_0 \otimes \boldsymbol{t}_0$$

- Hooke's law
  - Hooke's law in the convected basis metric
    - In reference frame: $\boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl}\boldsymbol{\varepsilon}_{kl}$

      with $\mathcal{H}_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{1+\nu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$
    - But here deformations are relative because in the metric of the convected basis

      $\Longrightarrow$ stress components should also be in this metric
    - Let us consider a basis $\boldsymbol{a}^I$,with $\boldsymbol{\varepsilon}$ components known in this basis

      $$\boldsymbol{\sigma}_{ij} = \mathcal{H}^{ijkl}\boldsymbol{\varepsilon}_{IJ}\left(\boldsymbol{a}^I \otimes \boldsymbol{a}^J\right)_{kl}$$
    - Component of $\boldsymbol{\sigma}$ in the conjugate basis can be deduced

      $$\boldsymbol{\sigma}_{ij} = \boldsymbol{\sigma}^{PQ}\left(\boldsymbol{a}_P \otimes \boldsymbol{a}_Q\right)_{ij} = \mathcal{H}^{ijkl}\boldsymbol{\varepsilon}_{IJ}\left(\boldsymbol{a}^I \otimes \boldsymbol{a}^J\right)_{kl}$$

      $$\Longrightarrow \boldsymbol{\sigma}^{PQ}\left(\boldsymbol{a}_P \otimes \boldsymbol{a}_Q\right)_{ij}\left(\boldsymbol{a}^M \otimes \boldsymbol{a}^N\right)_{ij} = \left(\boldsymbol{a}^M \otimes \boldsymbol{a}^N\right)_{ij}\mathcal{H}^{ijkl}\boldsymbol{\varepsilon}_{IJ}\left(\boldsymbol{a}^I \otimes \boldsymbol{a}^J\right)_{kl}$$

      $$\Longrightarrow \boldsymbol{\sigma}^{MN} = \left(\boldsymbol{a}^M \otimes \boldsymbol{a}^N\right)_{ij}\mathcal{H}^{ijkl}\boldsymbol{\varepsilon}_{IJ}\left(\boldsymbol{a}^I \otimes \boldsymbol{a}^J\right)_{kl}$$
    - Eventually $\boldsymbol{\sigma}^{MN} = \mathcal{H}^{IJKL}\boldsymbol{\varepsilon}_{IJ}$ with

$$\mathcal{H}^{IJKL} = \frac{E\nu}{(1+\nu)(1-2\nu)}\boldsymbol{a}^I\cdot\boldsymbol{a}^J\boldsymbol{a}^K\cdot\boldsymbol{a}^L + \frac{E}{2(1+\nu)}\left(\boldsymbol{a}^I\cdot\boldsymbol{a}^K\boldsymbol{a}^J\cdot\boldsymbol{a}^L + \boldsymbol{a}^I\cdot\boldsymbol{a}^L\boldsymbol{a}^J\cdot\boldsymbol{a}^K\right)$$

- Hooke's law (2)
  - From $\boldsymbol{\varepsilon} = \left(\varepsilon_{\alpha\beta} + \xi^3 \kappa_{\alpha\beta}\right) \boldsymbol{\varphi}_0{}_{,}{}^{\alpha} \otimes \boldsymbol{\varphi}_0{}_{,}{}^{\beta} + \frac{1}{2}\gamma_\alpha \left(\boldsymbol{t}_0 \otimes \boldsymbol{\varphi}_0{}_{,}{}^{\alpha} + \boldsymbol{\varphi}_0{}_{,}{}^{\alpha} \otimes \boldsymbol{t}_0\right) + \lambda_h \boldsymbol{t}_0 \otimes \boldsymbol{t}_0$
  - There are 4 contributions
    - Membrane mode $\boldsymbol{\sigma}_\varepsilon^{IJ} = \mathcal{H}^{IJ\alpha\beta}\varepsilon_{\alpha\beta}$
      - With $\mathcal{H}^{IJKL} = \frac{E\nu}{(1+\nu)(1-2\nu)} \boldsymbol{a}^I \cdot \boldsymbol{a}^J \boldsymbol{a}^K \cdot \boldsymbol{a}^L + \frac{E}{2(1+\nu)} \left(\boldsymbol{a}^I \cdot \boldsymbol{a}^K \boldsymbol{a}^J \cdot \boldsymbol{a}^L + \boldsymbol{a}^I \cdot \boldsymbol{a}^L \boldsymbol{a}^J \cdot \boldsymbol{a}^K\right)$
    - In the conjugate convected basis $\boldsymbol{\varphi}_0{}_{,}{}^{\alpha}$, $\boldsymbol{t}_0$ (abuse of notation):
      - As $\boldsymbol{t}_0$ initially perpendicular to $\boldsymbol{\varphi}_0{}_{,}{}^{\alpha}$, and $\|\boldsymbol{t}_0\| = 1$
      - The non-zero components are

$$
\left\{
\begin{aligned}
\boldsymbol{\sigma}_\varepsilon^{\alpha\beta} &= \frac{E\nu}{(1+\nu)(1-2\nu)} \boldsymbol{\varphi}_0{}_{,}{}^{\alpha} \cdot \boldsymbol{\varphi}_0{}_{,}{}^{\beta} \boldsymbol{\varphi}_0{}_{,}{}^{\gamma} \cdot \boldsymbol{\varphi}_0{}_{,}{}^{\delta} \varepsilon_{\gamma\delta} + \frac{E}{1+\nu} \varepsilon_{\gamma\delta} \boldsymbol{\varphi}_0{}_{,}{}^{\alpha} \cdot \boldsymbol{\varphi}_0{}_{,}{}^{\gamma} \boldsymbol{\varphi}_0{}_{,}{}^{\beta} \cdot \boldsymbol{\varphi}_0{}_{,}{}^{\delta} \\
\boldsymbol{\sigma}_\varepsilon^{33} &= \frac{E\nu}{(1+\nu)(1-2\nu)} \boldsymbol{\varphi}_0{}_{,}{}^{\gamma} \cdot \boldsymbol{\varphi}_0{}_{,}{}^{\delta} \varepsilon_{\gamma\delta}
\end{aligned}
\right.
$$

- Hooke's law (3)
  - From $\boldsymbol{\varepsilon} = \left(\varepsilon_{\alpha\beta} + \xi^3 \kappa_{\alpha\beta}\right) \boldsymbol{\varphi}_{0}^{,\alpha} \otimes \boldsymbol{\varphi}_{0}^{,\beta} + \frac{1}{2}\gamma_\alpha \left(\boldsymbol{t}_0 \otimes \boldsymbol{\varphi}_{0}^{,\alpha} + \boldsymbol{\varphi}_{0}^{,\alpha} \otimes \boldsymbol{t}_0\right) + \lambda_h \boldsymbol{t}_0 \otimes \boldsymbol{t}_0$
  - There are 4 contributions (2)
    - Bending mode $\boldsymbol{\sigma}_\kappa^{IJ} = \mathcal{H}^{IJ\alpha\beta}\xi^3\kappa_{\alpha\beta}$
      - With $\mathcal{H}^{IJKL} = \frac{E\nu}{(1+\nu)(1-2\nu)}\boldsymbol{a}^I \cdot \boldsymbol{a}^J \boldsymbol{a}^K \cdot \boldsymbol{a}^L +$
        $\frac{E}{2(1+\nu)}\left(\boldsymbol{a}^I \cdot \boldsymbol{a}^K \boldsymbol{a}^J \cdot \boldsymbol{a}^L + \boldsymbol{a}^I \cdot \boldsymbol{a}^L \boldsymbol{a}^J \cdot \boldsymbol{a}^K\right)$
    - In the conjugate convected basis $\boldsymbol{\varphi}_0{}^{,\alpha}$, $\boldsymbol{t}_0$:
      - As $\boldsymbol{t}_0$ initially perpendicular to $\boldsymbol{\varphi}_0{}^{,\alpha}$, and $||\boldsymbol{t}_0|| = 1$
      - The non-zero components are

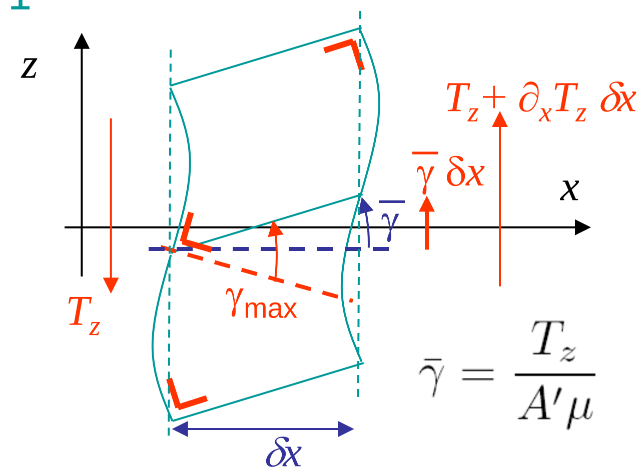
$$\begin{cases} \boldsymbol{\sigma}_\kappa^{\alpha\beta} = \frac{E\nu}{(1+\nu)(1-2\nu)}\boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\beta} \boldsymbol{\varphi}_{0}^{,\gamma} \cdot \boldsymbol{\varphi}_{0}^{,\delta} \xi^3 \kappa_{\gamma\delta} + \frac{E}{1+\nu}\boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\gamma} \boldsymbol{\varphi}_{0}^{,\beta} \cdot \boldsymbol{\varphi}_{0}^{,\delta} \xi^3 \kappa_{\gamma\delta} \\ \boldsymbol{\sigma}_\kappa^{33} = \frac{E\nu}{(1+\nu)(1-2\nu)}\boldsymbol{\varphi}_{0}^{,\gamma} \cdot \boldsymbol{\varphi}_{0}^{,\delta} \xi^3 \kappa_{\gamma\delta} \end{cases}$$

- Hooke's law (4)
  - From $\boldsymbol{\varepsilon} = \left(\varepsilon_{\alpha\beta} + \xi^3 \kappa_{\alpha\beta}\right) \boldsymbol{\varphi}_0^{,\alpha} \otimes \boldsymbol{\varphi}_0^{,\beta} + \underbrace{\frac{1}{2}\gamma_\alpha}_{\delta_\alpha} \left(\boldsymbol{t}_0 \otimes \boldsymbol{\varphi}_0^{,\alpha} + \boldsymbol{\varphi}_0^{,\alpha} \otimes \boldsymbol{t}_0\right) + \lambda_h \boldsymbol{t}_0 \otimes \boldsymbol{t}_0$
  - There are 4 contributions (3)
    - Shearing mode $\boldsymbol{\sigma}_\delta^{IJ} = \mathcal{H}^{IJ3\alpha}\delta_\alpha + \mathcal{H}^{IJ\alpha 3}\delta_\alpha$
      - With $\mathcal{H}^{IJKL} = \frac{E\nu}{(1+\nu)(1-2\nu)} \boldsymbol{a}^I \cdot \boldsymbol{a}^J \boldsymbol{a}^K \cdot \boldsymbol{a}^L +$
        $\frac{E}{2(1+\nu)} \left(\boldsymbol{a}^I \cdot \boldsymbol{a}^K \boldsymbol{a}^J \cdot \boldsymbol{a}^L + \boldsymbol{a}^I \cdot \boldsymbol{a}^L \boldsymbol{a}^J \cdot \boldsymbol{a}^K\right)$
    - In the conjugate convected basis $\boldsymbol{\varphi}_0^{,\alpha}$, $\boldsymbol{t}_0$:
      - As $\boldsymbol{t}_0$ initially perpendicular to $\boldsymbol{\varphi}_0^{,\alpha}$, and $\|\boldsymbol{t}_0\| = 1$
      - The non-zero components are

$$\boldsymbol{\sigma}_\delta^{\alpha 3} = \boldsymbol{\sigma}_\delta^{3\alpha} = \frac{E}{(1+\nu)} \boldsymbol{\varphi}_0^{,\alpha} \cdot \boldsymbol{\varphi}_0^{,\gamma} \delta_\gamma$$

      - To account for non uniformity of shearing

$$\boldsymbol{\sigma}_\delta^{\alpha 3} = \boldsymbol{\sigma}_\delta^{3\alpha} = \frac{E}{2(1+\nu)} \frac{A'}{A} \boldsymbol{\varphi}_0^{,\alpha} \cdot \boldsymbol{\varphi}_0^{,\gamma} \gamma_\gamma$$



$$\bar{\gamma} = \frac{T_z}{A'\mu}$$

- Hooke's law (5)
  - From $\boldsymbol{\varepsilon} = \left(\varepsilon_{\alpha\beta} + \xi^3 \kappa_{\alpha\beta}\right) \boldsymbol{\varphi}_0^{,\alpha} \otimes \boldsymbol{\varphi}_0^{,\beta} + \frac{1}{2}\gamma_\alpha \left(\boldsymbol{t}_0 \otimes \boldsymbol{\varphi}_0^{,\alpha} + \boldsymbol{\varphi}_0^{,\alpha} \otimes \boldsymbol{t}_0\right) + \lambda_h \boldsymbol{t}_0 \otimes \boldsymbol{t}_0$
  - There are 4 contributions (4)
    - Through-the thickness elongation $\boldsymbol{\sigma}_\lambda^{IJ} = \mathcal{H}^{IJ33}\lambda_h$
      - With $\mathcal{H}^{IJKL} = \frac{E\nu}{(1+\nu)(1-2\nu)} \boldsymbol{a}^I \cdot \boldsymbol{a}^J \boldsymbol{a}^K \cdot \boldsymbol{a}^L + \frac{E}{2(1+\nu)} \left(\boldsymbol{a}^I \cdot \boldsymbol{a}^K \boldsymbol{a}^J \cdot \boldsymbol{a}^L + \boldsymbol{a}^I \cdot \boldsymbol{a}^L \boldsymbol{a}^J \cdot \boldsymbol{a}^K\right)$
    - In the conjugate convected basis $\boldsymbol{\varphi}_0{}^{,\alpha}$, $\boldsymbol{t}_0$:
      - As $\boldsymbol{t}_0$ initially perpendicular to $\boldsymbol{\varphi}_0{}^{,\alpha}$, and $||\boldsymbol{t}_0|| = 1$
      - The non-zero components are

$$\left\{\begin{aligned} \boldsymbol{\sigma}_\lambda^{\alpha\beta} &= \frac{E\nu}{(1+\nu)(1-2\nu)} \boldsymbol{\varphi}_0^{,\alpha} \cdot \boldsymbol{\varphi}_0^{,\beta} \lambda_h \\ \boldsymbol{\sigma}_\lambda^{33} &= \left(\frac{E\nu}{(1+\nu)(1-2\nu)} + \frac{E}{1+\nu}\right)\lambda_h \end{aligned}\right.$$

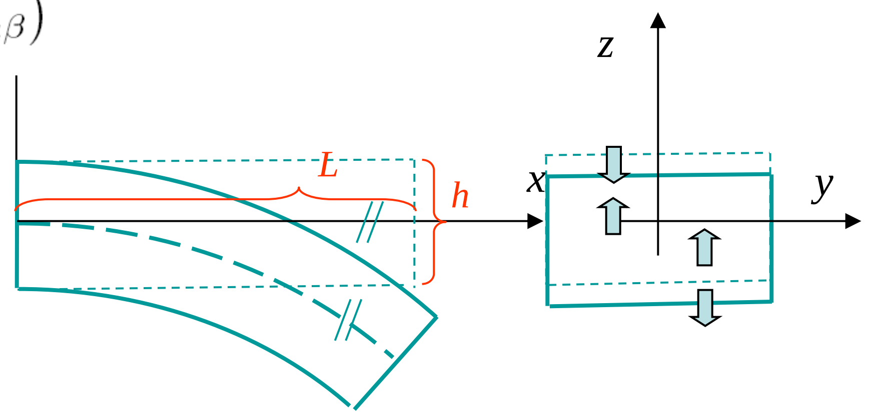
$$\Longrightarrow \quad \boldsymbol{\sigma}_\lambda^{33} = E\frac{1-\nu}{(1+\nu)(1-2\nu)}\lambda_h$$

- Resultant stresses
  - Plane-σ state
    - Contributions

$$\begin{cases} \boldsymbol{\sigma}_{\varepsilon}^{33} = \dfrac{E\nu}{(1+\nu)(1-2\nu)} \boldsymbol{\varphi}_{0}^{,\gamma} \cdot \boldsymbol{\varphi}_{0}^{,\delta} \varepsilon_{\gamma\delta} \\ \boldsymbol{\sigma}_{\kappa}^{33} = \dfrac{E\nu}{(1+\nu)(1-2\nu)} \boldsymbol{\varphi}_{0}^{,\gamma} \cdot \boldsymbol{\varphi}_{0}^{,\delta} \xi^{3} \kappa_{\gamma\delta} \\ \boldsymbol{\sigma}_{\lambda}^{33} = E \dfrac{1-\nu}{(1+\nu)(1-2\nu)} \lambda_{h} \end{cases}$$

$$\Rightarrow \lambda_{h} = \frac{\nu}{\nu - 1} \boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\beta} \left(\varepsilon_{\alpha\beta} + \xi^{3} \kappa_{\alpha\beta}\right)$$

    - Elongation depends on ξ^3
      - Part is stretched and part is compressed
      - For pure bending the change of sign is on the neutral axis



    - Average trough the thickness elongation

$$\frac{1}{\bar{j}_0 h_0} \int_{h_0} \lambda_h j_0 d\xi^3 = \frac{1}{\bar{j}_0 h_0} \int_{h_0} \frac{\nu}{\nu - 1} j_0 \boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\beta} \varepsilon_{\alpha\beta} d\xi^3$$

      - Depends on the membrane mode only

- Resultant stresses (2)
  - Plane σ-stated (2)
    - Values αβ can now be deduced

$$\begin{cases} \boldsymbol{\sigma}_{\varepsilon}^{\alpha\beta} = \dfrac{E\nu}{(1+\nu)(1-2\nu)} \boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\beta} \boldsymbol{\varphi}_{0}^{,\gamma} \cdot \boldsymbol{\varphi}_{0}^{,\delta} \varepsilon_{\gamma\delta} + \dfrac{E}{1+\nu} \varepsilon_{\gamma\delta} \boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\gamma} \boldsymbol{\varphi}_{0}^{,\beta} \cdot \boldsymbol{\varphi}_{0}^{,\delta} \\ \boldsymbol{\sigma}_{\kappa}^{\alpha\beta} = \dfrac{E\nu}{(1+\nu)(1-2\nu)} \boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\beta} \boldsymbol{\varphi}_{0}^{,\gamma} \cdot \boldsymbol{\varphi}_{0}^{,\delta} \xi^{3} \kappa_{\gamma\delta} + \dfrac{E}{1+\nu} \boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\gamma} \boldsymbol{\varphi}_{0}^{,\beta} \cdot \boldsymbol{\varphi}_{0}^{,\delta} \xi^{3} \kappa_{\gamma\delta} \\ \boldsymbol{\sigma}_{\lambda}^{\alpha\beta} = \dfrac{E\nu}{(1+\nu)(1-2\nu)} \boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\beta} \lambda_{h} \qquad \text{with } \lambda_{h} = \dfrac{\nu}{\nu - 1} \boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\beta} \left(\varepsilon_{\alpha\beta} + \xi^{3} \kappa_{\alpha\beta}\right) \end{cases}$$

    - Can be rewritten as a through-the-thickness constant term and a linear term
      - $\boldsymbol{\sigma}^{\alpha\beta} = \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} + \xi^{3} \boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta}$
      - With $\begin{cases} \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} = \dfrac{E\nu}{(1-\nu^{2})} \boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\beta} \boldsymbol{\varphi}_{0}^{,\gamma} \cdot \boldsymbol{\varphi}_{0}^{,\delta} \varepsilon_{\gamma\delta} + \dfrac{E}{1+\nu} \varepsilon_{\gamma\delta} \boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\gamma} \boldsymbol{\varphi}_{0}^{,\beta} \cdot \boldsymbol{\varphi}_{0}^{,\delta} \\ \boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta} = \dfrac{E\nu}{(1-\nu^{2})} \boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\beta} \boldsymbol{\varphi}_{0}^{,\gamma} \cdot \boldsymbol{\varphi}_{0}^{,\delta} \kappa_{\gamma\delta} + \boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\gamma} \boldsymbol{\varphi}_{0}^{,\beta} \cdot \boldsymbol{\varphi}_{0}^{,\delta} \kappa_{\gamma\delta} \end{cases}$
    - Out-of-plane shearing remains the same
      - $\boldsymbol{\sigma}_{\delta}^{\alpha 3} = \boldsymbol{\sigma}_{\delta}^{3\alpha} = \dfrac{E}{2(1+\nu)} \dfrac{A'}{A} \boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\gamma} \gamma_{\gamma}$

- Resultant stresses (3)
  - From stress fields
    - $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\alpha\beta}\boldsymbol{g}_{0\alpha} \otimes \boldsymbol{g}_{0\beta} + \boldsymbol{\sigma}^{\alpha 3}\left(\boldsymbol{g}_{0\alpha} \otimes \boldsymbol{t}_0 + \boldsymbol{t}_0 \otimes \boldsymbol{g}_{0\alpha}\right)$

$$\Longrightarrow \quad \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^{\gamma} = \boldsymbol{\sigma}^{\alpha\gamma}\boldsymbol{g}_{0\alpha} + \boldsymbol{\sigma}^{3\gamma}\boldsymbol{t}_0$$

With $\boldsymbol{\sigma}^{\alpha\beta} = \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} + \xi^3\boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta}$ & $\boldsymbol{\sigma}_{\delta}^{\alpha 3} = \boldsymbol{\sigma}_{\delta}^{3\alpha} = \frac{E}{2(1+\nu)}\frac{A'}{A}\boldsymbol{\varphi}_{0}^{,\alpha} \cdot \boldsymbol{\varphi}_{0}^{,\gamma}\gamma_{\gamma}$

  - Membrane resultant stress
    - $\boldsymbol{n}^{\alpha} = \frac{1}{\bar{j}_0}\int_{h_0} j_0\boldsymbol{\sigma} \cdot \boldsymbol{g}_0^{\alpha} d\xi^3$

$$\Longrightarrow \boldsymbol{n}^{\alpha} = \frac{1}{\bar{j}_0}\int_{h_0}\left[\boldsymbol{\sigma}^{\beta\alpha}\boldsymbol{g}_{0\beta} + \boldsymbol{\sigma}_{\delta}^{\alpha 3}\boldsymbol{t}_0\right] j_0 d\xi^3$$

    - As $\boldsymbol{g}_{0\alpha} = \frac{\partial \boldsymbol{\Phi}_0}{\partial \xi^{\alpha}} = \boldsymbol{\varphi}_{0,\alpha} + \xi^3\boldsymbol{t}_{0,\alpha}$

$$\Longrightarrow \boldsymbol{n}^{\alpha} = \frac{1}{\bar{j}_0}\int_{h_0}\left[\boldsymbol{\sigma}^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + \boldsymbol{\sigma}_{\delta}^{\alpha 3}\boldsymbol{t}_0\right] j_0 d\xi^3 + \frac{1}{\bar{j}_0}\int_{h_0}\boldsymbol{\sigma}^{\beta\alpha}\boldsymbol{t}_{0,\beta} j_0 \xi^3 d\xi^3$$

- Resultant stresses (4)
  - Membrane resultant stress (2)
    - $$\boldsymbol{n}^{\alpha} = \frac{1}{\bar{j}_0}\int_{h_0}\left[\boldsymbol{\sigma}^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + \boldsymbol{\sigma}_{\delta}^{\alpha 3}\boldsymbol{t}_0\right] j_0 d\xi^3 + \frac{1}{\bar{j}_0}\int_{h_0}\boldsymbol{\sigma}^{\beta\alpha}\boldsymbol{t}_{0,\beta} j_0 \xi^3 d\xi^3$$

      With $\boldsymbol{\sigma}^{\alpha\beta} = \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} + \xi^3\boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta}$ & $\boldsymbol{\sigma}_{\delta}^{\alpha 3} = \boldsymbol{\sigma}_{\delta}^{3\alpha} = \frac{E}{2(1+\nu)}\frac{A'}{A}\boldsymbol{\varphi}_0^{;\alpha}\cdot\boldsymbol{\varphi}_0^{;\gamma}\gamma_{\gamma}$
  - Membrane resultant stress components
    - We had previously defined $\boldsymbol{n}^{\alpha} = n^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + q^{\alpha}\boldsymbol{t}_0$

$$\Longrightarrow \begin{cases} n^{\beta\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0}\left[\boldsymbol{\sigma}^{\alpha\beta} + \xi^3\boldsymbol{\sigma}^{\gamma\alpha}\boldsymbol{t}_{0,\gamma}\cdot\boldsymbol{\varphi}_0^{;\beta}\right] j_0 d\xi^3 \\ q^{\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0}\boldsymbol{\sigma}_{\delta}^{\alpha 3} j_0 d\xi^3 \end{cases}$$

New term compared to plates

    - Similarly

$$\boldsymbol{n}^3 = \frac{1}{\bar{j}_0}\int_{h_0}\boldsymbol{\sigma}_{\delta}^{\alpha 3}\left[\boldsymbol{\varphi}_{0,\alpha} + \xi^3\boldsymbol{t}_{0,\alpha}\right] j_0 d\xi^3 = \frac{1}{\bar{j}_0}\int_{h_0}\boldsymbol{\sigma}_{\delta}^{\alpha 3}\boldsymbol{\varphi}_{0,\alpha} j_0 d\xi^3$$

$$\Longrightarrow \boldsymbol{n}^3 = q^{\alpha}\boldsymbol{\varphi}_{0,\alpha}$$

- Resultant stresses (5)
  - Membrane resultant stress components (2)
    - $\boldsymbol{n}^{\alpha} = n^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + q^{\alpha}\boldsymbol{t}_0$

$$\begin{cases} n^{\beta\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0}\left[\boldsymbol{\sigma}^{\alpha\beta} + \xi^3\boldsymbol{\sigma}^{\gamma\alpha}\boldsymbol{t}_{0,\gamma}\cdot\boldsymbol{\varphi}_0^{,\beta}\right] j_0 d\xi^3 \\ q^{\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0}\boldsymbol{\sigma}_{\delta}^{\alpha 3} j_0 d\xi^3 \end{cases}$$

?

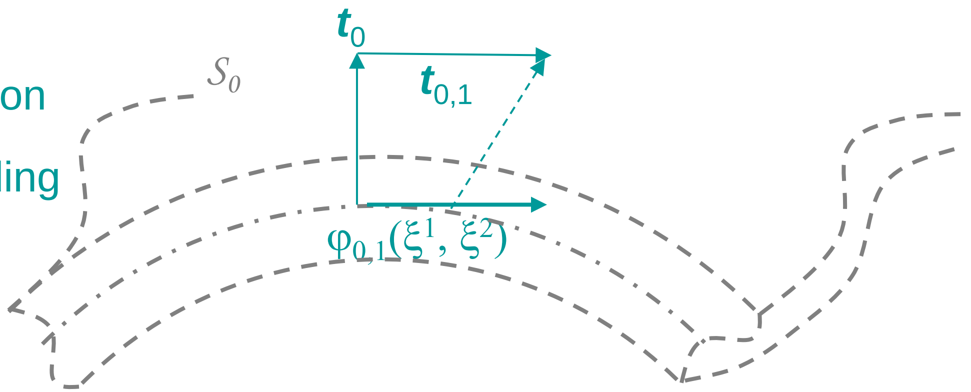
    - Due to the curvature $\boldsymbol{t}_{0,\alpha}$ has components in the basis $\boldsymbol{\varphi}_0{}^{,\beta}$

$\Longrightarrow$ $\boldsymbol{t}_{0,\alpha} = \lambda_{\alpha}^{\beta}\boldsymbol{\varphi}_{0,\beta}$ no component along $\boldsymbol{t}_0$ as $0 = \partial_{\alpha}\left(\boldsymbol{t}_0\cdot\boldsymbol{t}_0\right) = 2\boldsymbol{t}_{0,\alpha}\cdot\boldsymbol{t}_0$

    - As integration of
      - $\boldsymbol{\sigma}^{\alpha\beta}$ corresponds to tension
      - $\xi^3\boldsymbol{\sigma}^{\beta\gamma}$ corresponds to bending

Due to the initial curvature

there is a coupling



$\Longrightarrow$ $$n^{\beta\alpha} = \frac{1}{\bar{j}_0}\int_{h_0}\left[\boldsymbol{\sigma}^{\alpha\beta} + \xi^3\boldsymbol{\sigma}^{\gamma\alpha}\boldsymbol{t}_{0,\gamma}\cdot\boldsymbol{\varphi}_0^{,\beta}\right] j_0 d\xi^3$$

- Resultant stresses (6)
  - Membrane resultant stress components (3)
    - Using fields

$$\begin{cases} \boldsymbol{\sigma}^{\alpha\beta} = \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} + \xi^3 \boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta} \\ \boldsymbol{\sigma}_{\delta}^{\alpha 3} = \boldsymbol{\sigma}_{\delta}^{3\alpha} = \frac{E}{2(1+\nu)} \frac{A'}{A} \boldsymbol{\varphi}_{0,\alpha} \cdot \boldsymbol{\varphi}_{0,\gamma} \gamma_{\gamma} \end{cases}$$

    - Leads to (see annex 2)

$$\begin{cases} n^{\beta\alpha} = \frac{1}{\bar{j}_0} \int_{h_0} \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} j_0 d\xi^3 + \lambda_{\gamma}^{\beta} \frac{1}{\bar{j}_0} \int_{h_0} {\xi^3}^2 \boldsymbol{\sigma}_{\tilde{m}}^{\gamma\alpha} j_0 d\xi^3 = \tilde{n}^{\beta\alpha} + \lambda_{\gamma}^{\beta} \tilde{m}'^{\alpha\gamma} \\ q^{\alpha} = \frac{1}{\bar{j}_0} \int_{h_0} \boldsymbol{\sigma}_{\delta}^{\alpha 3} j_0 d\xi^3 = \tilde{q}^{\alpha} \end{cases}$$

    - With

$$\begin{cases} \tilde{n}^{\beta\alpha} = \mathcal{H}_n^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta} \\ \tilde{q}^{\alpha} = q^{\alpha} = \mathcal{H}_q^{\alpha\beta} \delta_{\beta} = \frac{1}{2} \mathcal{H}_q^{\alpha\beta} \gamma_{\beta} \\ \tilde{m}'^{\beta\alpha} = \mathcal{H}_m^{\alpha\beta\gamma\delta} \kappa_{\gamma\delta} \end{cases}$$

See later for interpretation

- Resultant stresses (7)
  - The resultant Hooke tensors in the shell metric reads (see annex 2)

$$\left\{\begin{array}{l}
\mathcal{H}_n^{\alpha\beta\gamma\delta} = \frac{h_0 E}{1-\nu^2}\left[\nu \boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\beta}\boldsymbol{\varphi}_0^{,\gamma}\cdot\boldsymbol{\varphi}_0^{,\delta} + \right.\\
\qquad\left.\frac{1-\nu}{2}\left(\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\gamma}\boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\varphi}_0^{,\delta} + \boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\delta}\boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\varphi}_0^{,\gamma}\right)\right]\\
\mathcal{H}_m^{\alpha\beta\gamma\delta} = \frac{h_0^3 E}{12\left(1-\nu^2\right)}\left[\nu \boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\beta}\boldsymbol{\varphi}_0^{,\gamma}\cdot\boldsymbol{\varphi}_0^{,\delta} + \right.\\
\qquad\left.\frac{1-\nu}{2}\left(\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\gamma}\boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\varphi}_0^{,\delta} + \boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\delta}\boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\varphi}_0^{,\gamma}\right)\right]\\
\mathcal{H}_q^{\alpha\beta} = \frac{E h_0}{1+\nu}\frac{A'}{A}\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\beta}
\end{array}\right.$$

  - Doing the same developments for the bending mode leads to (see annex 2)
    - $\tilde{\boldsymbol{m}}^{\alpha} = \tilde{m}^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + \tilde{m}^{3\alpha}\boldsymbol{t}_0$
    - With

New term compared to plates

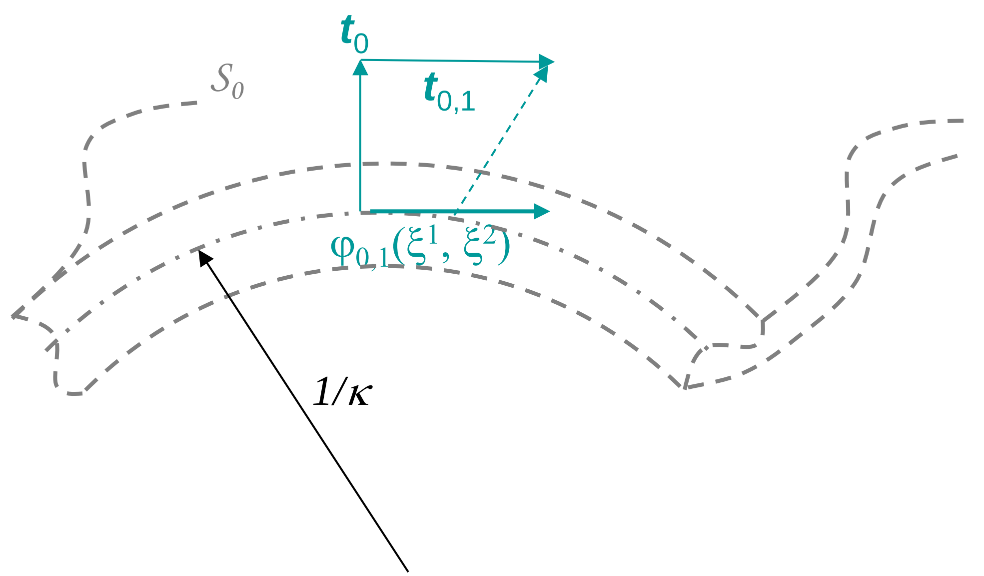
$$\left\{\begin{array}{l}
\tilde{m}^{\beta\alpha} = \frac{1}{\bar{j}_0}\int_{h_0}\boldsymbol{\sigma}_{\tilde{m}}^{\beta\alpha} j_0 {\xi^3}^2 d\xi^3 + \lambda_\gamma^\beta\frac{1}{\bar{j}_0}\int_{h_0}\boldsymbol{\sigma}_{\tilde{n}}^{\gamma\alpha} j_0 {\xi^3}^2 d\xi^3 = \tilde{m}'^{\beta\alpha} + \lambda_\gamma^\beta\frac{h_0^2}{12}\tilde{n}^{\gamma\alpha}\\
\tilde{m}^{3\alpha} = 0
\end{array}\right.$$

- Resultant equations
  - Equations
$$\begin{cases} \dfrac{1}{\bar{j}_0}\left(\bar{j}_0 \boldsymbol{n}^\alpha\right)_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}_0 \ddot{\boldsymbol{u}} \\ \ddot{\boldsymbol{t}} I_p = \bar{\tilde{\boldsymbol{m}}} - \boldsymbol{n}^3 + \dfrac{1}{\bar{j}_0}\left(\tilde{\boldsymbol{m}}^\alpha \bar{j}_0\right)_{,\alpha} + \lambda \boldsymbol{t}_0 \end{cases}$$
  - Resultant stresses
    - $$\begin{cases} \boldsymbol{n}^\alpha = n^{\beta\alpha} \boldsymbol{\varphi}_{0,\beta} + q^\alpha \boldsymbol{t}_0 \\ \tilde{\boldsymbol{m}}^\alpha = \tilde{m}^{\beta\alpha} \boldsymbol{\varphi}_{0,\beta} + \tilde{m}^{3\alpha} \boldsymbol{t}_0 \end{cases}, \quad \boldsymbol{n}^3 = q^\alpha \boldsymbol{\varphi}_{0,\alpha}$$
  - Coupling
    - $$\begin{cases} n^{\beta\alpha} = \tilde{n}^{\beta\alpha} + \lambda^\beta_\gamma \tilde{m}'^{\alpha\gamma} \\ q^\alpha = \tilde{q}^\alpha \\ \tilde{m}^{\beta\alpha} = \tilde{m}'^{\beta\alpha} + \lambda^\beta_\gamma \dfrac{h_0^2}{12} \tilde{n}^{\gamma\alpha} \\ \tilde{m}^{3\alpha} = 0 \end{cases}$$

Initial curvature of the shell $\quad \boldsymbol{t}_{0,\alpha} = \lambda^\beta_\alpha \boldsymbol{\varphi}_{0,\beta}$

  - Hookes'law
    - $$\begin{cases} \tilde{n}^{\beta\alpha} = \mathcal{H}_n^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta} \\ \tilde{m}'^{\beta\alpha} = \mathcal{H}_m^{\alpha\beta\gamma\delta} \kappa_{\gamma\delta} \\ \tilde{q}^\alpha = q^\alpha = \mathcal{H}_q^{\alpha\beta} \delta_\beta = \dfrac{1}{2} \mathcal{H}_q^{\alpha\beta} \gamma_\beta \end{cases} \quad \text{with} \quad \begin{cases} \varepsilon_{\alpha\beta} = \dfrac{\boldsymbol{u}_{,\alpha} \cdot \boldsymbol{\varphi}_{0,\beta} + \boldsymbol{u}_{,\beta} \cdot \boldsymbol{\varphi}_{0,\alpha}}{2} \\ \kappa_{\alpha\beta} = \dfrac{\boldsymbol{\Delta t}_{,\alpha} \cdot \boldsymbol{\varphi}_{0,\beta} + \boldsymbol{\Delta t}_{,\beta} \cdot \boldsymbol{\varphi}_{0,\alpha}}{2} \\ 2\delta_\alpha = \gamma_\alpha = \boldsymbol{u}_{,\alpha} \cdot \boldsymbol{t}_0 + \boldsymbol{\Delta t} \cdot \boldsymbol{\varphi}_{0,\alpha} \end{cases}$$

- Resultant equations (2)
  - Analysis of coupling
    - Force F, mid-plane length L, curvature radius $1/\kappa$, section A



    - First term of coupling

$$\underbrace{n^{\beta\alpha}}_{F/A} = \underbrace{\tilde{n}^{\beta\alpha}}_{F/A} + \underbrace{\underbrace{\lambda^{\beta}_{\gamma}}_{\kappa}\underbrace{\tilde{m}'^{\alpha\gamma}}_{FL/A}}_{FL\kappa/A}$$

      - In general $L \sim 1/\kappa$ for shells (for plates $L \ll 1/\kappa$)

    - Second term of coupling

$$\underbrace{\tilde{m}^{\beta\alpha}}_{FL/A} = \underbrace{\tilde{m}'^{\beta\alpha}}_{FL/A} + \underbrace{\underbrace{\lambda^{\beta}_{\gamma}}_{\kappa}\frac{h_0^2}{12}\underbrace{\tilde{n}^{\gamma\alpha}}_{F/A}}_{Fh_0^2\kappa/A}$$

      - If $h_0^2 \ll L/\kappa \sim 1/\kappa^2$, which was already assumed, this term can be neclected

- Resultant equations summary
  - Equations
$$\begin{cases} \frac{1}{\bar{j}_0}\left(\bar{j}_0 \boldsymbol{n}^\alpha\right)_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}_0 \ddot{\boldsymbol{u}} \\ \ddot{\boldsymbol{t}} I_p = \bar{\tilde{\boldsymbol{m}}} - \boldsymbol{n}^3 + \frac{1}{\bar{j}_0}\left(\tilde{\boldsymbol{m}}^\alpha \bar{j}_0\right)_{,\alpha} + \lambda \boldsymbol{t}_0 \end{cases}$$
  - Resultant stresses
    - $\begin{cases} \boldsymbol{n}^\alpha = n^{\beta\alpha} \boldsymbol{\varphi}_{0,\beta} + q^\alpha \boldsymbol{t}_0 \\ \tilde{\boldsymbol{m}}^\alpha = \tilde{m}^{\beta\alpha} \boldsymbol{\varphi}_{0,\beta} \\ \boldsymbol{n}^3 = q^\alpha \boldsymbol{\varphi}_{0,\alpha} \end{cases}$, $n^{\beta\alpha} = \tilde{n}^{\beta\alpha} + \lambda^\beta_\gamma \tilde{m}^{\alpha\gamma}$, $\boldsymbol{t}_{0,\alpha} = \lambda^\beta_\alpha \boldsymbol{\varphi}_{0,\beta}$
  - Hookes'law
    - $\begin{cases} \tilde{n}^{\beta\alpha} = \mathcal{H}_n^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta} \\ \tilde{m}^{\beta\alpha} = \mathcal{H}_m^{\alpha\beta\gamma\delta} \kappa_{\gamma\delta} \\ \tilde{q}^\alpha = q^\alpha = \mathcal{H}_q^{\alpha\beta} \delta_\beta = \frac{1}{2} \mathcal{H}_q^{\alpha\beta} \gamma_\beta \end{cases}$

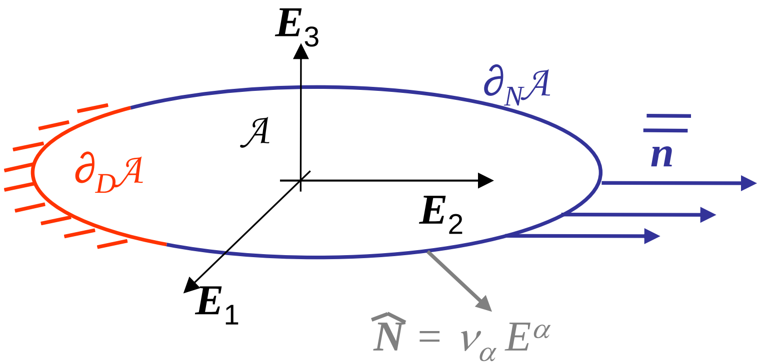
with $\begin{cases} \varepsilon_{\alpha\beta} = \dfrac{\boldsymbol{u}_{,\alpha} \cdot \boldsymbol{\varphi}_{0,\beta} + \boldsymbol{u}_{,\beta} \cdot \boldsymbol{\varphi}_{0,\alpha}}{2} \\ \kappa_{\alpha\beta} = \dfrac{\boldsymbol{\Delta t}_{,\alpha} \cdot \boldsymbol{\varphi}_{0,\beta} + \boldsymbol{\Delta t}_{,\beta} \cdot \boldsymbol{\varphi}_{0,\alpha}}{2} \\ 2\delta_\alpha = \gamma_\alpha = \boldsymbol{u}_{,\alpha} \cdot \boldsymbol{t}_0 + \boldsymbol{\Delta t} \cdot \boldsymbol{\varphi}_{0,\alpha} \end{cases}$

- Membrane equations
  - Assuming no external loading, $\frac{1}{\bar{j}_0}\left(\bar{j}_0 \boldsymbol{n}^\alpha\right)_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}_0 \ddot{\boldsymbol{u}}$ becomes

$$\left(\bar{j}_0 \mathcal{H}_n^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta} \boldsymbol{\varphi}_{0,\beta} + \bar{j}_0 \mathcal{H}_m^{\alpha\mu\gamma\delta} \lambda_\mu^\beta \kappa_{\gamma\delta} \boldsymbol{\varphi}_{0,\beta} + \bar{j}_0 \mathcal{H}_q^{\alpha\beta} \frac{\gamma_\beta}{2} \boldsymbol{t}_0\right)_{,\alpha} = \bar{j}_0 \bar{\rho}_0 \ddot{\boldsymbol{u}}$$

coupling

  - This equation can be projected into ($\boldsymbol{\varphi}_{0,\alpha}$) plane, and is completed by BCs
    - Neumann $\boldsymbol{u}_\alpha = \bar{\boldsymbol{u}}_\alpha$
    - Dirichlet $\boldsymbol{n}^\alpha \nu_\alpha = \bar{\bar{\boldsymbol{n}}}$
    - For plate we had



$$\mathcal{H}_n^{\alpha\beta\gamma\delta} \frac{\boldsymbol{u}_{\gamma,\delta\alpha} + \boldsymbol{u}_{\delta,\gamma\alpha}}{2} + \bar{\boldsymbol{n}}_\beta = \bar{\rho} \ddot{\boldsymbol{u}}_\beta$$

    - Remark, for shells $\boldsymbol{\varphi}_{0,\alpha\beta}$ is not equal to zero
  - This equation projected on $\boldsymbol{t}_0$ leads to the shearing equation
    - For plate we had

$$\mathcal{H}_q^{\alpha\beta} \frac{\boldsymbol{u}_{3,\beta\alpha} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} + \bar{\boldsymbol{n}}_3 = \bar{\rho} \ddot{\boldsymbol{u}}_3$$

- Bending equations
  - Assuming no external loading, $\ddot{\boldsymbol{t}} I_p = \tilde{\bar{\boldsymbol{m}}} - \boldsymbol{n}^3 + \frac{1}{\bar{j}_0}\left(\tilde{\boldsymbol{m}}^\alpha \bar{j}_0\right)_{,\alpha} + \lambda \boldsymbol{t}_0$ becomes

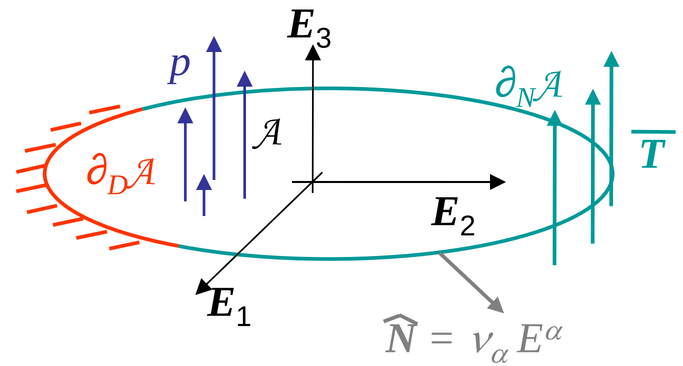
$$\bar{j}_0 I_p \ddot{\boldsymbol{\Delta t}} = \left(\bar{j}_0 \mathcal{H}_m^{\alpha\beta\gamma\delta} \kappa_{\gamma\delta} \boldsymbol{\varphi}_{0,\beta}\right)_{,\alpha} - \bar{j}_0 \mathcal{H}_q^{\alpha\beta} \frac{\gamma_\alpha}{2} \boldsymbol{\varphi}_{0,\beta}$$

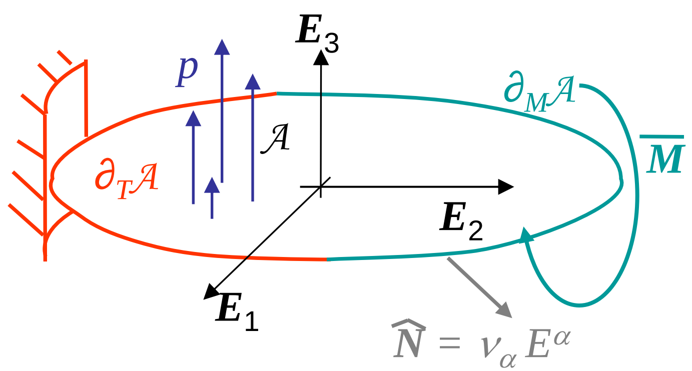
  - This equation can be projected into ($\boldsymbol{\varphi}_{0,\alpha}$) plane, and is completed by BCs
    - Low order $\boldsymbol{u}_3 = \bar{\boldsymbol{u}}_3$ or $\boldsymbol{t}_0 \cdot \boldsymbol{n}^\alpha \nu_\alpha = \bar{T}$
    - High order $\boldsymbol{\Delta t} = \bar{\boldsymbol{\Delta t}}$ or $\tilde{\boldsymbol{m}}_\beta^\alpha \nu_\alpha = \bar{M}_\beta$
    - For plate we had

$$I_p \ddot{\boldsymbol{\Delta t}}_\alpha = \tilde{\bar{\boldsymbol{m}}}_\alpha - \frac{1}{2} \mathcal{H}_q^{\alpha\beta} \left(\boldsymbol{u}_{3,\beta} + \boldsymbol{\Delta t}_\beta\right) + \mathcal{H}_m^{\alpha\beta\gamma\delta} \frac{\boldsymbol{\Delta t}_{\gamma,\delta\beta} + \boldsymbol{\Delta t}_{\delta,\gamma\beta}}{2}$$

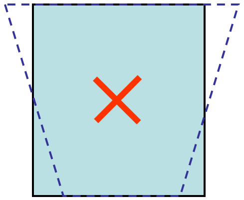
  - With the shearing equation

$$\mathcal{H}_q^{\alpha\beta} \frac{\boldsymbol{u}_{3,\beta\alpha} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} + \bar{\boldsymbol{n}}_3 = \bar{\rho} \ddot{\boldsymbol{u}}_3$$





- Membrane-bending coupling
  - If $\boldsymbol{t}_{0,\alpha} \neq 0$ (there is a curvature)
    - Tension and bending are coupled
    - This leads to locking with a FE integration: stiffness of the FE $\rightarrow \infty$
    - Locking can be avoided by
      - Adding internal degrees of freedom (EAS) $\Longrightarrow$ more expensive
      - Using only one Gauss point (reduced integration) $\Longrightarrow$ hourglass modes (zero energy spurious deformation modes)
      - Both methods lead to complex computational schemes

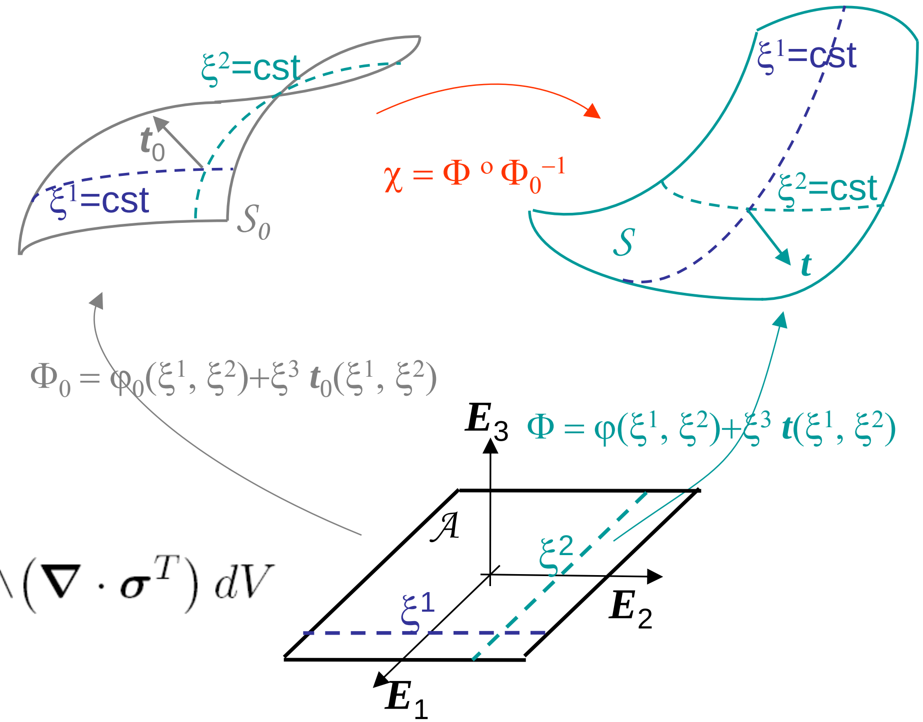


References

- Lecture notes
  - Aircraft Structures for engineering students, T. H. G. Megson, Butterworth-Heinemann, An imprint of Elsevier Science, 2003, ISBN 0 340 70588 4
- Other references
  - Papers

    - Simo JC, Fox DD. On a stress resultant geometrically exact shell model. Part I: formulation and optimal parametrization. *Computer Methods in Applied Mechanics and Engineering* 1989; **72**:267–304.
    - Simo JC, Fox DD, Rifai MS. On a stress resultant geometrically exact shell model. Part II: the linear theory, computational aspects. *Computer Methods in Applied Mechanics and Engineering* 1989; **73**:53–92.

- Angular momentum equation
  - $\rho \ddot{\boldsymbol{x}} = \boldsymbol{b} + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T$



$$\Rightarrow \rho \Phi \wedge \ddot{\boldsymbol{x}} = \Phi \wedge \boldsymbol{b} + \Phi \wedge \left(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T\right)$$

$$\Rightarrow \int_{\mathcal{S}} \rho \left(\boldsymbol{\varphi} + \xi^3 \boldsymbol{t}\right) \wedge \left(\ddot{\boldsymbol{\varphi}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV = \int_{\mathcal{S}} \left(\boldsymbol{\varphi} + \xi^3 \boldsymbol{t}\right) \wedge \boldsymbol{b} dV + \int_{\mathcal{S}} \left(\boldsymbol{\varphi} + \xi^3 \boldsymbol{t}\right) \wedge \left(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T\right) dV$$

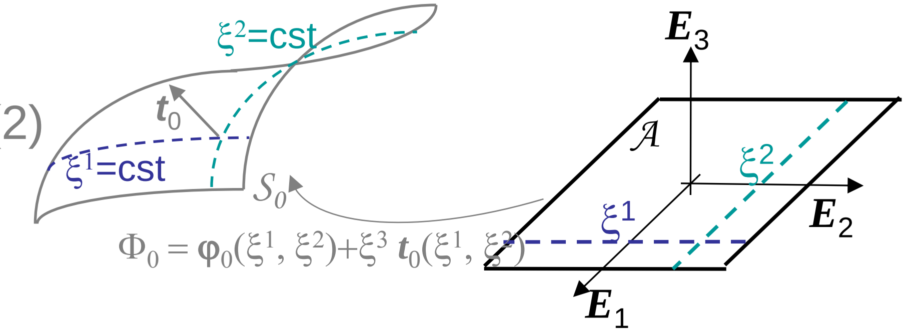
  - Small transformations assumptions

$$\Rightarrow \int_{\mathcal{S}_0} \rho_0 \left(\boldsymbol{\varphi} + \xi^3 \boldsymbol{t}\right) \wedge \left(\ddot{\boldsymbol{\varphi}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV = \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi} + \xi^3 \boldsymbol{t}\right) \wedge \boldsymbol{b} dV + \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi} + \xi^3 \boldsymbol{t}\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV$$

- Using $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 + \boldsymbol{u}$

$$\Rightarrow \int_{\mathcal{S}_0} \rho_0 \left(\boldsymbol{\varphi}_0 + \boldsymbol{u} + \xi^3 \boldsymbol{t}\right) \wedge \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV = \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \boldsymbol{u} + \xi^3 \boldsymbol{t}\right) \wedge \boldsymbol{b} dV + \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \boldsymbol{u} + \xi^3 \boldsymbol{t}\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV$$

- Angular momentum equation (2)
  - Small transformations assumptions (2)



$$\int_{\mathcal{S}_0} \rho_0 \left(\boldsymbol{\varphi}_0 + \boldsymbol{u} + \xi^3 \boldsymbol{t}\right) \wedge \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV = \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \boldsymbol{u} + \xi^3 \boldsymbol{t}\right) \wedge \boldsymbol{b} dV + \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \boldsymbol{u} + \xi^3 \boldsymbol{t}\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV$$

- As second order terms can be neglected

$$\Longrightarrow \int_{\mathcal{S}_0} \rho_0 \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0\right) \wedge \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV = \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0\right) \wedge \boldsymbol{b} dV + \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV$$

- Angular momentum equation (3)
  - Inertial term
    - $$\int_{\mathcal{S}_0} \rho_0 \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0\right) \wedge \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV =$$
      $$\int_{\mathcal{A}} \int_{h_0} \rho_0 j_0 \left\{ \boldsymbol{\varphi}_0 \wedge \ddot{\boldsymbol{u}} + \xi^3 \left(\boldsymbol{t}_0 \wedge \ddot{\boldsymbol{u}} + \boldsymbol{\varphi}_0 \wedge \ddot{\boldsymbol{t}}\right) + {\xi^3}^2 \boldsymbol{t}_0 \wedge \ddot{\boldsymbol{t}} \right\} d\xi^3 d\mathcal{A}$$
    - As
      - Main idea in plates is to consider $\boldsymbol{u}$ and $\boldsymbol{t}$ constant on the thickness
      - $\int_{h_0} j_0 \rho_0 \xi^3 d\xi^3 = 0$
      - $\bar{\rho}_0 = \frac{1}{\bar{j}_0} \int_{h_0} j_0 \rho_0 d\xi^3$

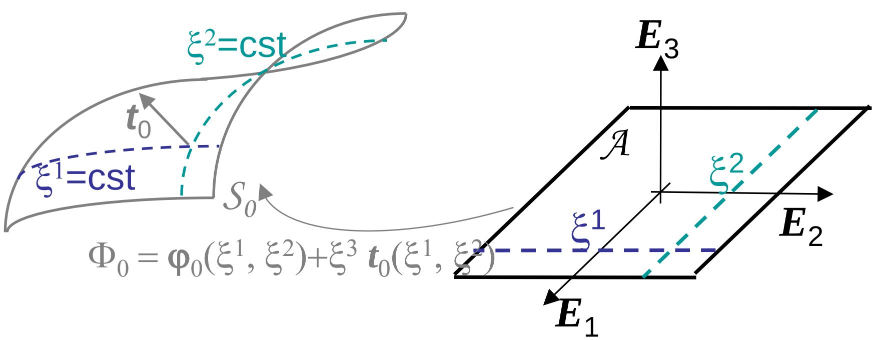
$$\Longrightarrow \int_{\mathcal{S}_0} \rho_0 \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0\right) \wedge \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV = \int_{\mathcal{A}} \bar{j}_0 \bar{\rho}_0 \boldsymbol{\varphi}_0 \wedge \ddot{\boldsymbol{u}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{t}_0 \wedge \ddot{\boldsymbol{t}} \bar{j}_0 I_p d\mathcal{A}$$

  - With $I_p = \frac{1}{\bar{j}_0} \int_{h_0} \rho_0 j_0 {\xi^3}^2 d\xi^3$

- Angular momentum equation (4)
  - Loading term
    - $$\int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0) \wedge \boldsymbol{b} dV = \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \int_{h_0} j_0 \boldsymbol{b} d\xi^3 d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{t}_0 \wedge \int_{h_0} j_0 \boldsymbol{b} \xi^3 d\xi^3 d\mathcal{A}$$

$$\Longrightarrow \int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0) \wedge \boldsymbol{b} dV = \int_{\mathcal{A}} \bar{j}_0 \boldsymbol{\varphi}_0 \wedge \bar{\boldsymbol{b}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{t}_0 \wedge \int_{h_0} j_0 \boldsymbol{b} \xi^3 d\xi^3 d\mathcal{A}$$

  - With the surface loading $\bar{\boldsymbol{b}} = \frac{1}{\bar{j}_0} \int_{h_0} j_0 \boldsymbol{b} d\xi^3$

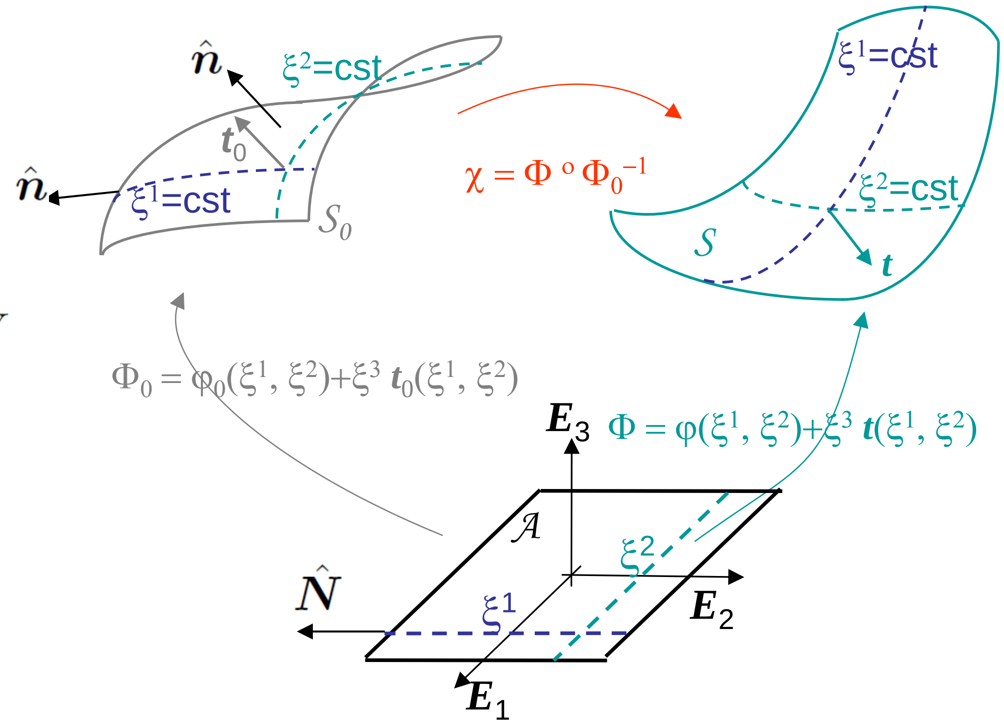


- Angular momentum equation (5)
  - Stress term
    - $$\int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0)\wedge(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T)\, dV = \int_{\mathcal{S}_0} \Phi_0 \wedge (\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T)\, dV$$

$$\Longrightarrow \int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0) \wedge (\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T)\, dV = \int_{\mathcal{S}_0} e_{ijk} \Phi_{0\,j} \boldsymbol{\nabla}_{0\,l} \boldsymbol{\sigma}_{kl} dV$$

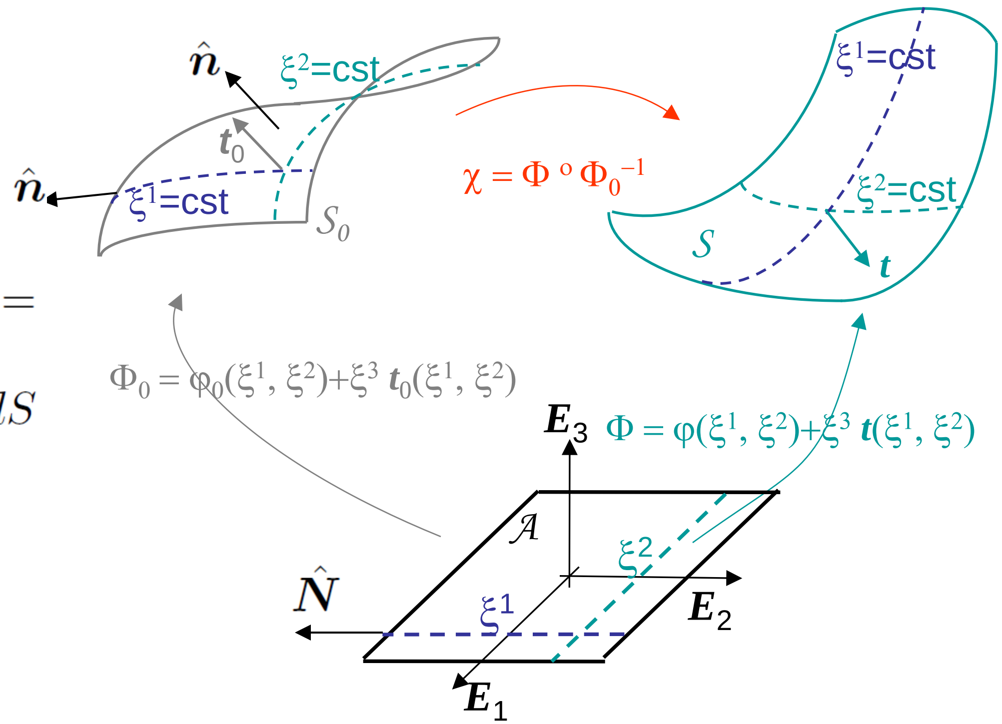
    - Integration by parts

$$\int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0)\wedge(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T)\, dV = \int_{\mathcal{S}_0} e_{ijk} \boldsymbol{\nabla}_{0\,l} \left(\Phi_{0\,j} \boldsymbol{\sigma}_{kl}\right) dV - \int_{\mathcal{S}_0} e_{ijk} \left(\boldsymbol{\nabla}_{0\,l} \Phi_{0\,j}\right) \boldsymbol{\sigma}_{kl} dV$$



- Angular momentum equation (6)
  - Stress term (2)
    - $$\int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0) \wedge (\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T)\, dV =$$
      $$\int_{\mathcal{S}_0} e_{ijk} \boldsymbol{\nabla}_{0\,l} (\Phi_{0\,j} \boldsymbol{\sigma}_{kl})\, dV - \int_{\mathcal{S}_0} e_{ijk} (\boldsymbol{\nabla}_{0\,l} \Phi_{0\,j})\, \boldsymbol{\sigma}_{kl} dV$$
      δ_{jl} as here grad is in $\mathcal{S}_0$ frame
      as $\boldsymbol{\sigma}$ symmetric
      $$\Longrightarrow \int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0) \wedge (\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T)\, dV = \int_{\partial \mathcal{S}_0} e_{ijk} \Phi_{0\,j} \boldsymbol{\sigma}_{kl} \hat{\boldsymbol{n}}_l dS - \int_{\mathcal{S}_0} \cancel{e_{ijk} \boldsymbol{\sigma}_{kj}} dV$$
    - Where $\hat{\boldsymbol{n}}$ is the normal to the shell $\mathcal{S}_0$

$$\int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0) \wedge (\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T)\, dV =$$
$$\int_{\partial \mathcal{S}_0} \Phi_0 \wedge (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}})\, dS$$



- Angular momentum equation (7)
  - Stress term (2)
    - $\int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0) \wedge (\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T)\, dV = \int_{\partial \mathcal{S}_0} \Phi_0 \wedge (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}})\, dS$
    - Transform surface integral $d\mathcal{S}$ in $\mathcal{S}_0$ to in surface integral dS in $\mathcal{A}$ x h_0
    - Nanson's Formula, for a mapping Φ_0: $d\mathcal{S}\hat{\boldsymbol{n}} = |\boldsymbol{\Phi}_0|\, dS \boldsymbol{\Phi}_0^{-T} \cdot \hat{\boldsymbol{N}}$

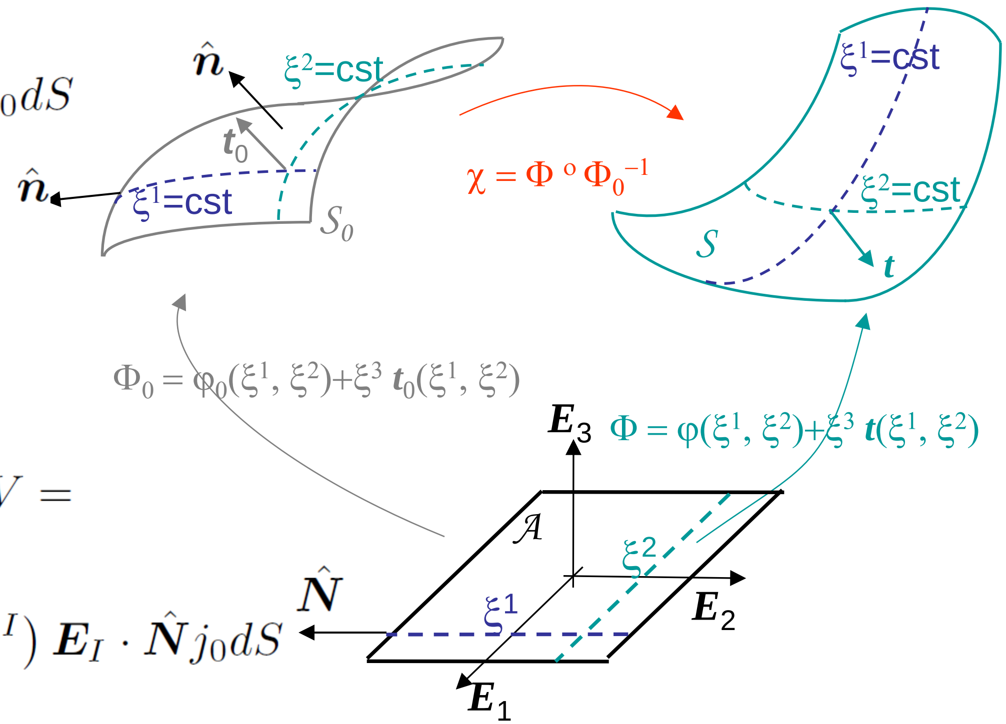
$$\Rightarrow \int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0) \wedge (\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T)\, dV =$$

$$\int_{\partial[\mathcal{A}\times h_0]} e_{ijk} \Phi_{0\,j} \boldsymbol{\sigma}_{kl} {\boldsymbol{\Phi}_0^{-1}}_{pl} \hat{\boldsymbol{N}}_p j_0 dS$$

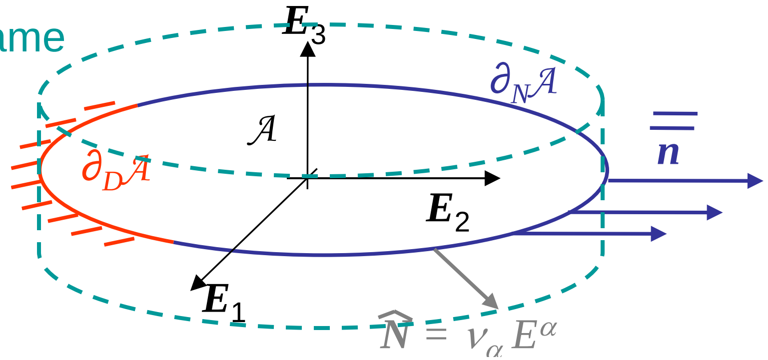
- As $\boldsymbol{\Phi}_0^{-1} = (\boldsymbol{E}_J \otimes \boldsymbol{g}_0^J)$

$$\Rightarrow \int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0) \wedge (\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T)\, dV =$$

$$\int_{\partial[\mathcal{A}\times h_0]} \Phi_0 \wedge (\boldsymbol{\sigma} \cdot \boldsymbol{g}^I)\, \boldsymbol{E}_I \cdot \hat{\boldsymbol{N}} j_0 dS$$



- Angular momentum equation (8)
  - Stress term (3)
    - $$\int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV = \int_{\partial[\mathcal{A}\times h_0]} \Phi_0 \wedge \left(\boldsymbol{\sigma} \cdot \boldsymbol{g}^I\right) \boldsymbol{E}_I \cdot \hat{\mathbf{N}} j_0 dS$$
    - Where $\widehat{\boldsymbol{N}}$ is the normal in the reference frame
      - On top/bottom faces: $\widehat{\boldsymbol{N}} = \pm \boldsymbol{E}^3$
      - On lateral surface: $\widehat{\boldsymbol{N}} = \nu_\alpha \boldsymbol{E}^\alpha$



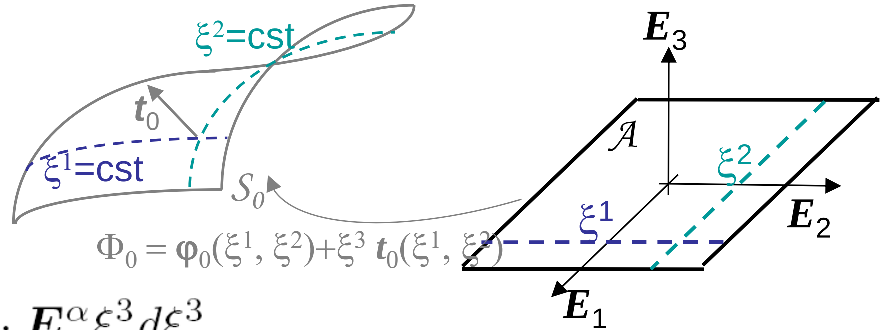
    - $$\int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV = \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3\right)\Big|_{h_0} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \int_{h_0} \left(j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha\right) d\xi^3 \nu_\alpha dl + \int_{\mathcal{A}} \boldsymbol{t}_0 \wedge \left(j_0 \xi^3 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3\right)\Big|_{h_0} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{t}_0 \wedge \int_{h_0} \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha j_0 \xi^3 d\xi^3 \nu_\alpha dl$$

- Angular momentum equation (9)
  - Stress term (4)
    - $$\int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0) \wedge (\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T)\, dV = \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge (j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3)\big|_{h_0} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \int_{h_0} (j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha)\, d\xi^3 \nu_\alpha dl + \int_{\mathcal{A}} \boldsymbol{t}_0 \wedge (j_0 \xi^3 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3)\big|_{h_0} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{t}_0 \wedge \int_{h_0} \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha j_0 \xi^3 d\xi^3 \nu_\alpha dl$$
    - Resultant bending stresses & resultant stresses
      - $\tilde{\boldsymbol{m}}^\alpha = \frac{1}{\bar{j}_0} \int_{h_0} \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha j_0 \xi^3 d\xi^3$, $\boldsymbol{n}^\alpha = \frac{1}{\bar{j}_0} \int_{h_0} j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha d\xi^3$

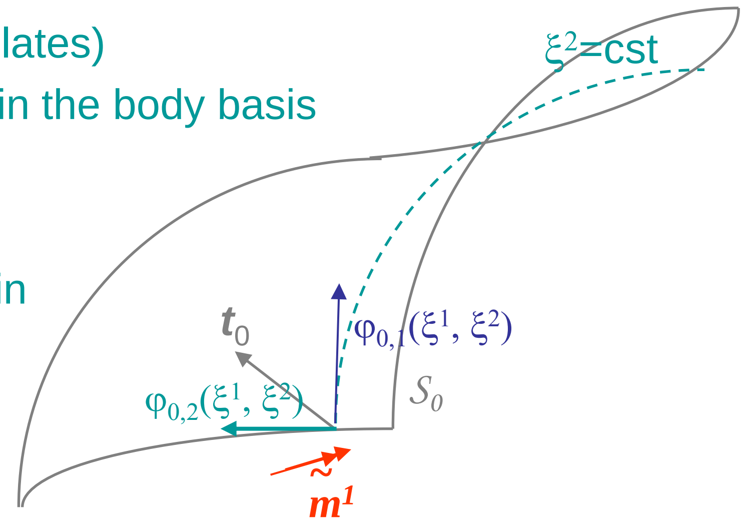
$$\Longrightarrow \int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0) \wedge (\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T)\, dV = \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge (j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3)\big|_{h_0} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \boldsymbol{n}^\alpha \bar{j}_0 \nu_\alpha dl + \int_{\mathcal{A}} \boldsymbol{t}_0 \wedge (j_0 \xi^3 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3)\big|_{h_0} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{t}_0 \wedge \tilde{\boldsymbol{m}}^\alpha \bar{j}_0 \nu_\alpha dl$$

- Angular momentum equation (10)
  - Resultant bending stresses
    - $\tilde{\boldsymbol{m}}^{\alpha} = \frac{1}{\bar{j}_0}\int_{h_0} \boldsymbol{\sigma}\cdot\boldsymbol{g}_0^{\alpha} j_0 \xi^3 d\xi^3$

      for plates we found $\tilde{\boldsymbol{m}}^{\alpha} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}\cdot\boldsymbol{E}^{\alpha}\xi^3 d\xi^3$



    - Compared to plates
      - There is the Jacobian mapping (1 for plates)
      - Cauchy stresses have to be projected in the body basis
        - » $\boldsymbol{g}^{\alpha}$ for shells
        - » E^I for plates
    - We can write it in terms of the components in the mid-surface basis
      - $\tilde{\boldsymbol{m}}^{\alpha} = \tilde{m}^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + \tilde{m}^{3\alpha}\boldsymbol{t}_0$



$$\Rightarrow \begin{cases} \tilde{m}^{\beta\alpha} = \boldsymbol{\varphi}_0^{,\beta}\cdot\tilde{\boldsymbol{m}}^{\alpha} = \frac{1}{\bar{j}_0}\int_{h_0} j_0 \boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{g}_0^{\alpha}\xi^3 d\xi^3 \\ \tilde{m}^{3\alpha} = \boldsymbol{t}_0\cdot\tilde{\boldsymbol{m}}^{\alpha} = \frac{1}{\bar{j}_0}\int_{h_0} j_0 \boldsymbol{t}_0\cdot\boldsymbol{\sigma}\cdot\boldsymbol{g}_0^{\alpha}\xi^3 d\xi^3 \end{cases}$$

- Angular momentum equation (11)
  - Stress term (5)
    - $$\int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV \quad = \quad \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3\right)\Big|_{h_0} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \boldsymbol{n}^\alpha \bar{j}_0 \nu_\alpha dl + \int_{\mathcal{A}} \boldsymbol{t}_0 \wedge \left(j_0 \xi^3 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3\right)\Big|_{h_0} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{t}_0 \wedge \tilde{\boldsymbol{m}}^\alpha \bar{j}_0 \nu_\alpha dl$$
    - Applying Gauss theorem

$$\int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV = \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3\right)\Big|_{h_0} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha \bar{j}_0 d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(\boldsymbol{n}^\alpha \bar{j}_0\right)_{,\alpha} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{t}_0 \wedge \left(j_0 \xi^3 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3\right)\Big|_{h_0} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{t}_{0,\alpha} \wedge \tilde{\boldsymbol{m}}^\alpha \bar{j}_0 d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{t}_0 \wedge \left(\bar{j}_0 \tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha} d\mathcal{A}$$

- Angular momentum equation (12)
  - From
    - $$\int_{\mathcal{S}_0} \rho_0 \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0\right) \wedge \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV = \int_{\mathcal{A}} \bar{j}_0 \bar{\rho}_0 \boldsymbol{\varphi}_0 \wedge \ddot{\boldsymbol{u}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{t}_0 \wedge \ddot{\boldsymbol{t}} \bar{j}_0 I_p d\mathcal{A}$$
    - $$\int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0\right) \wedge \boldsymbol{b} dV = \int_{\mathcal{A}} \bar{j}_0 \boldsymbol{\varphi}_0 \wedge \bar{\boldsymbol{b}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{t}_0 \wedge \int_{h_0} j_0 \boldsymbol{b} \xi^3 d\xi^3 d\mathcal{A}$$
    - $$\int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV =$$
$$\int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3\right)\big|_{h_0} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha \bar{j}_0 d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(\boldsymbol{n}^\alpha \bar{j}_0\right)_{,\alpha} d\mathcal{A} +$$
$$\int_{\mathcal{A}} \boldsymbol{t}_0 \wedge \left(j_0 \xi^3 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3\right)\big|_{h_0} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{t}_{0,\alpha} \wedge \tilde{\boldsymbol{m}}^\alpha \bar{j}_0 d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{t}_0 \wedge \left(\bar{j}_0 \tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha} d\mathcal{A}$$

  It comes

$$\bar{j}_0 \bar{\rho}_0 \boldsymbol{\varphi}_0 \wedge \ddot{\boldsymbol{u}} + \boldsymbol{t}_0 \wedge \ddot{\boldsymbol{t}} \bar{j}_0 I_p = \bar{j}_0 \boldsymbol{\varphi}_0 \wedge \bar{\boldsymbol{b}} + \boldsymbol{t}_0 \wedge \int_{h_0} j_0 \boldsymbol{b} \xi^3 d\xi^3 +$$
$$\boldsymbol{\varphi}_0 \wedge \left(\bar{j}_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha\right)\big|_{h_0} + \bar{j}_0 \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha + \boldsymbol{\varphi}_0 \wedge \left(\boldsymbol{n}^\alpha \bar{j}_0\right)_{,\alpha} +$$
$$\boldsymbol{t}_0 \wedge \left(\bar{j}_0 \xi^3 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha\right)\big|_{h_0} + \bar{j}_0 \boldsymbol{t}_{0,\alpha} \wedge \tilde{\boldsymbol{m}}^\alpha + \boldsymbol{t}_0 \wedge \left(\tilde{\boldsymbol{m}}^\alpha \bar{j}_0\right)_{,\alpha}$$

- Angular momentum equation (13)
  - Resultant form
    - $$\bar{j}_0 \bar{\rho}_0 \boldsymbol{\varphi}_0 \wedge \ddot{\boldsymbol{u}} + \boldsymbol{t}_0 \wedge \ddot{\boldsymbol{t}} \bar{j}_0 I_p = \bar{j}_0 \boldsymbol{\varphi}_0 \wedge \bar{\boldsymbol{b}} + \boldsymbol{t}_0 \wedge \int_{h_0} j_0 \boldsymbol{b} \xi^3 d\xi^3 + \boldsymbol{\varphi}_0 \wedge \left(\bar{j}_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha\right)\big|_{h_0} + \bar{j}_0 \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha + \boldsymbol{\varphi}_0 \wedge \left(\boldsymbol{n}^\alpha \bar{j}_0\right)_{,\alpha} + \boldsymbol{t}_0 \wedge \left(\bar{j}_0 \xi^3 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha\right)\Big|_{h_0} + \bar{j}_0 \boldsymbol{t}_{0,\alpha} \wedge \tilde{\boldsymbol{m}}^\alpha + \boldsymbol{t}_0 \wedge \left(\tilde{\boldsymbol{m}}^\alpha \bar{j}_0\right)_{,\alpha}$$
    - But the resultant linear momentum equation reads

$$\begin{cases} \frac{1}{\bar{j}_0} \left(\bar{j}_0 \boldsymbol{n}^\alpha\right)_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}_0 \ddot{\boldsymbol{u}} \\ \boldsymbol{n}^\alpha = \frac{1}{\bar{j}_0} \int_{h_0} j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha d\xi^3 \\ \bar{\boldsymbol{n}} = \frac{1}{\bar{j}_0} \left[\bar{j}_0 \bar{\boldsymbol{b}} + \left(j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3\right)\big|_{h_0}\right] \end{cases}$$

    - So the angular momentum equation reads

$$\boldsymbol{t}_0 \wedge \ddot{\boldsymbol{t}} \bar{j}_0 I_p = \boldsymbol{t}_0 \wedge \int_{h_0} j_0 \boldsymbol{b} \xi^3 d\xi^3 + \bar{j}_0 \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha + \boldsymbol{t}_0 \wedge \left(\bar{j}_0 \xi^3 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha\right)\big|_{h_0} + \bar{j}_0 \boldsymbol{t}_{0,\alpha} \wedge \tilde{\boldsymbol{m}}^\alpha + \boldsymbol{t}_0 \wedge \left(\tilde{\boldsymbol{m}}^\alpha \bar{j}_0\right)_{,\alpha}$$

- Angular momentum equation (14)
  - Resultant form (2)
    - $$\boldsymbol{t}_0 \wedge \ddot{\boldsymbol{t}}\bar{j}_0 I_p = \boldsymbol{t}_0 \wedge \int_{h_0} j_0 \boldsymbol{b} \xi^3 d\xi^3 + \bar{j}_0 \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha + \boldsymbol{t}_0 \wedge \left(\bar{j}_0 \xi^3 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha\right)\Big|_{h_0} + \bar{j}_0 \boldsymbol{t}_{0,\alpha} \wedge \tilde{\boldsymbol{m}}^\alpha + \boldsymbol{t}_0 \wedge \left(\tilde{\boldsymbol{m}}^\alpha \bar{j}_0\right)_{,\alpha}$$
    - Defining the applied torque $\bar{\tilde{\boldsymbol{m}}} = \frac{1}{\bar{j}_0} \left(j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}^3 \xi^3\right)\Big|_{h_0} + \frac{1}{\bar{j}_0} \int_{h_0} j_0 \boldsymbol{b} {\xi^3}^2 d\xi^3$

$$\Longrightarrow \boldsymbol{t}_0 \wedge \ddot{\boldsymbol{t}} I_p \bar{j}_0 = \boldsymbol{t}_0 \wedge \bar{\tilde{\boldsymbol{m}}} \bar{j}_0 + \bar{j}_0 \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha + \bar{j}_0 \boldsymbol{t}_{0,\alpha} \wedge \tilde{\boldsymbol{m}}^\alpha + \boldsymbol{t}_0 \wedge \left(\tilde{\boldsymbol{m}}^\alpha \bar{j}_0\right)_{,\alpha}$$

- Terms which are preventing from uncoupling the equations are

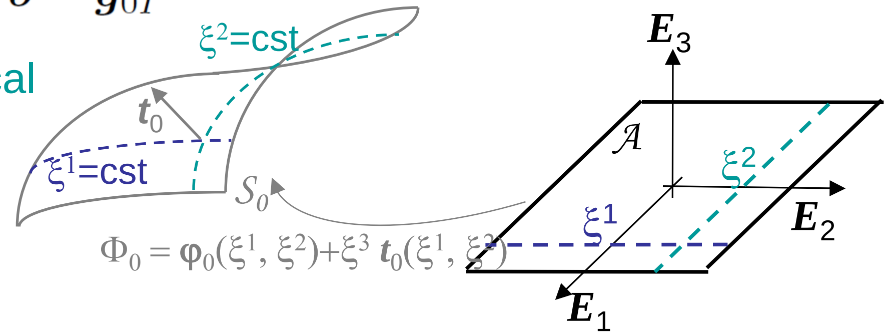
$$\bar{j}_0 \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha + \bar{j}_0 \boldsymbol{t}_{0,\alpha} \wedge \tilde{\boldsymbol{m}}^\alpha$$

- Angular momentum equation (15)
  - Terms $\bar{j}_0 \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha + \bar{j}_0 \boldsymbol{t}_{0,\alpha} \wedge \tilde{\boldsymbol{m}}^\alpha$
    - Let us rewrite the Cauchy stress tensor in terms of its components in the convected basis: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{IJ} \boldsymbol{g}_{0I} \otimes \boldsymbol{g}_{0J}$

$$\Longrightarrow \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^K = \boldsymbol{\sigma}^{IJ} \boldsymbol{g}_{0I} \otimes \boldsymbol{g}_{0J} \cdot \boldsymbol{g}_0^K = \boldsymbol{\sigma}^{IK} \boldsymbol{g}_{0I}$$

    - As Cauchy stress tensor is symmetrical

$$\boldsymbol{g}_{0K} \wedge \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^K = \boldsymbol{\sigma}^{IK} \boldsymbol{g}_{0K} \wedge \boldsymbol{g}_{0I} = 0$$



    - As $\begin{cases} \boldsymbol{g}_{0\alpha} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^\alpha} = \boldsymbol{\varphi}_{0,\alpha} + \xi^3 \boldsymbol{t}_{0,\alpha} \\ \boldsymbol{g}_{03} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^3} = \boldsymbol{t}_0 \end{cases}$

$$\Longrightarrow \left(\boldsymbol{\varphi}_{0,\alpha} + \xi^3 \boldsymbol{t}_{0,\alpha}\right) \wedge \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha + \boldsymbol{t}_0 \wedge \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3 = 0$$

    - After integration on the thickness:

$$\int_{h_0} \bar{j}_0 \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha d\xi^3 + \int_{h_0} \boldsymbol{t}_{0,\alpha} \wedge \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha \bar{j}_0 \xi^3 d\xi^3 + \int_{h_0} \boldsymbol{t}_0 \wedge \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3 \bar{j}_0 d\xi^3 = 0$$

- Angular momentum equation (16)
  - Terms $\bar{j}_0 \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha + \bar{j}_0 \boldsymbol{t}_{0,\alpha} \wedge \tilde{\boldsymbol{m}}^\alpha$ (2)
    - Using symmetry of Cauchy stress tensor

$$\int_{h_0} \bar{j}_0 \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha d\xi^3 + \int_{h_0} \boldsymbol{t}_{0,\alpha} \wedge \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha \bar{j}_0 \xi^3 d\xi^3 + \int_{h_0} \boldsymbol{t}_0 \wedge \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3 \bar{j}_0 d\xi^3 = 0$$

    - Defining the out-of-plane resultant stress $\boldsymbol{n}^3 = \frac{1}{\bar{j}_0} \int_{h_0} \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^3 j_0 d\xi^3$
    - As $\begin{cases} \boldsymbol{n}^\alpha = \frac{1}{\bar{j}_0} \int_{h_0} j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha d\xi^3 \\ \tilde{\boldsymbol{m}}^\alpha = \frac{1}{\bar{j}_0} \int_{h_0} \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha j_0 \xi^3 d\xi^3 \end{cases}$

$$\Longrightarrow \bar{j}_0 \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha + \bar{j}_0 \boldsymbol{t}_{0,\alpha} \wedge \tilde{\boldsymbol{m}}^\alpha = -\bar{j}_0 \boldsymbol{t}_0 \wedge \boldsymbol{n}^3$$

  - Equation

$$\boldsymbol{t}_0 \wedge \ddot{\boldsymbol{t}} I_p \bar{j}_0 = \boldsymbol{t}_0 \wedge \bar{\tilde{\boldsymbol{m}}} \bar{j}_0 + \bar{j}_0 \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha + \bar{j}_0 \boldsymbol{t}_{0,\alpha} \wedge \tilde{\boldsymbol{m}}^\alpha + \boldsymbol{t}_0 \wedge \left(\tilde{\boldsymbol{m}}^\alpha \bar{j}_0\right)_{,\alpha}$$

$$\Longrightarrow \boldsymbol{t}_0 \wedge \ddot{\boldsymbol{t}} I_p \bar{j}_0 = \boldsymbol{t}_0 \wedge \bar{\tilde{\boldsymbol{m}}} \bar{j}_0 - \bar{j}_0 \boldsymbol{t}_0 \wedge \boldsymbol{n}^3 + \boldsymbol{t}_0 \wedge \left(\tilde{\boldsymbol{m}}^\alpha \bar{j}_0\right)_{,\alpha}$$

- Angular momentum equation (17)
  - Resultant form (3)
    - $\boldsymbol{t}_0 \wedge \ddot{\boldsymbol{t}} I_p \bar{j}_0 = \boldsymbol{t}_0 \wedge \bar{\tilde{\boldsymbol{m}}} \bar{j}_0 - \bar{j}_0 \boldsymbol{t}_0 \wedge \boldsymbol{n}^3 + \boldsymbol{t}_0 \wedge \left(\tilde{\boldsymbol{m}}^\alpha \bar{j}_0\right)_{,\alpha}$
    - If λ is an undefined pressure applied through the thickness, the resultant angular momentum equation reads $\boxed{\ddot{\boldsymbol{t}} I_p = \bar{\tilde{\boldsymbol{m}}} - \boldsymbol{n}^3 + \frac{1}{\bar{j}_0}\left(\tilde{\boldsymbol{m}}^\alpha \bar{j}_0\right)_{,\alpha} + \lambda \boldsymbol{t}_0}$
    - With
      - $\tilde{\boldsymbol{m}}^\alpha = \frac{1}{\bar{j}_0} \int_{h_0} \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^\alpha j_0 \xi^3 d\xi^3$
      - $\bar{\tilde{\boldsymbol{m}}} = \frac{1}{\bar{j}_0} \left(j_0 \boldsymbol{\sigma} \cdot \boldsymbol{g}^3 \xi^3\right)\big|_{h_0} + \frac{1}{\bar{j}_0} \int_{h_0} j_0 \boldsymbol{b} {\xi^3}^2 d\xi^3$
      - $I_p = \frac{1}{\bar{j}_0} \int_{h_0} \rho_0 j_0 {\xi^3}^2 d\xi^3$
    - Remark, for plates: $\ddot{\boldsymbol{t}} I_p = \bar{\tilde{\boldsymbol{m}}} - \left(\boldsymbol{n}^3 - \lambda \boldsymbol{E}_3\right) + \left(\tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha}$

- Resultant stresses closed form
  - Membrane resultant stress components
    - $\boldsymbol{n}^{\alpha} = n^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + q^{\alpha}\boldsymbol{t}_0$, $\boldsymbol{t}_{0,\alpha} = \lambda_{\alpha}^{\beta}\boldsymbol{\varphi}_{0,\beta}$

$$\begin{cases} n^{\beta\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0}\left[\boldsymbol{\sigma}^{\alpha\beta} + \xi^3\lambda_{\gamma}^{\beta}\boldsymbol{\sigma}^{\gamma\alpha}\right] j_0 d\xi^3 \\ q^{\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0}\boldsymbol{\sigma}_{\delta}^{\alpha 3} j_0 d\xi^3 \end{cases}$$

    - As $\boldsymbol{\sigma}^{\alpha\beta} = \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} + \xi^3\boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta}$, $\boldsymbol{\sigma}_{\delta}^{\alpha 3} = \boldsymbol{\sigma}_{\delta}^{3\alpha} = \dfrac{E}{2(1+\nu)}\dfrac{A'}{A}\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\gamma}\gamma_{\gamma}$

      & $\displaystyle\int_{h_0} j_0\rho_0\xi^3 d\xi^3 = 0$, for constant density it leads to

$$\Rightarrow \begin{cases} n^{\beta\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0}\boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} j_0 d\xi^3 + \lambda_{\gamma}^{\beta}\dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0}{\xi^3}^2\boldsymbol{\sigma}_{\tilde{m}}^{\gamma\alpha} j_0 d\xi^3 = \tilde{n}^{\beta\alpha} + \lambda_{\gamma}^{\beta}\tilde{m}'^{\alpha\gamma} \\ q^{\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0}\boldsymbol{\sigma}_{\delta}^{\alpha 3} j_0 d\xi^3 = \tilde{q}^{\alpha} \end{cases}$$

With

$$\begin{cases} \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} = \dfrac{E\nu}{(1-\nu^2)}\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\beta}\boldsymbol{\varphi}_0^{,\gamma}\cdot\boldsymbol{\varphi}_0^{,\delta}\varepsilon_{\gamma\delta} + \dfrac{E}{1+\nu}\varepsilon_{\gamma\delta}\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\gamma}\boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\varphi}_0^{,\delta} \\ \boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta} = \dfrac{E\nu}{(1-\nu^2)}\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\beta}\boldsymbol{\varphi}_0^{,\gamma}\cdot\boldsymbol{\varphi}_0^{,\delta}\kappa_{\gamma\delta} + \boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\gamma}\boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\varphi}_0^{,\delta}\kappa_{\gamma\delta} \end{cases}$$

- Resultant stresses closed form (2)
  - Membrane resultant stress components (2)
    - $\boldsymbol{n}^{\alpha} = n^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + q^{\alpha}\boldsymbol{t}_0$, $\boldsymbol{t}_{0,\alpha} = \lambda_{\alpha}^{\beta}\boldsymbol{\varphi}_{0,\beta}$

with $$\left\{\begin{aligned} n^{\beta\alpha} &= \frac{1}{\bar{j}_0}\int_{h_0}\boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} j_0 d\xi^3 + \lambda_{\gamma}^{\beta}\frac{1}{\bar{j}_0}\int_{h_0}{\xi^3}^2\boldsymbol{\sigma}_{\tilde{m}}^{\gamma\alpha} j_0 d\xi^3 = \tilde{n}^{\beta\alpha} + \lambda_{\gamma}^{\beta}\tilde{m}'^{\alpha\gamma} \\ \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} &= \frac{E\nu}{(1-\nu^2)}\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\beta}\boldsymbol{\varphi}_0^{,\gamma}\cdot\boldsymbol{\varphi}_0^{,\delta}\varepsilon_{\gamma\delta} + \frac{E}{1+\nu}\varepsilon_{\gamma\delta}\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\gamma}\boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\varphi}_0^{,\delta} \\ \boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta} &= \frac{E\nu}{(1-\nu^2)}\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\beta}\boldsymbol{\varphi}_0^{,\gamma}\cdot\boldsymbol{\varphi}_0^{,\delta}\kappa_{\gamma\delta} + \boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\gamma}\boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\varphi}_0^{,\delta}\kappa_{\gamma\delta} \end{aligned}\right.$$

    - As $\int_{h_0} j_0 d\xi^3 \simeq \bar{j}_0 h_0$ & $\int_{h_0} j_0 {\xi^3}^2 d\xi^3 \simeq \bar{j}_0\frac{h_0^3}{12}$

$$\Rightarrow \left\{\begin{aligned} &\tilde{n}^{\beta\alpha} = \mathcal{H}_n^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} \text{ with } \mathcal{H}_n^{\alpha\beta\gamma\delta} = \frac{h_0 E}{1-\nu^2}\left[\nu\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\beta}\boldsymbol{\varphi}_0^{,\gamma}\cdot\boldsymbol{\varphi}_0^{,\delta} + \frac{1-\nu}{2}\left(\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\gamma}\boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\varphi}_0^{,\delta} + \boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\delta}\boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\varphi}_0^{,\gamma}\right)\right] \\ &\tilde{m}'^{\beta\alpha} = \mathcal{H}_m^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta} \text{ with } \mathcal{H}_m^{\alpha\beta\gamma\delta} = \frac{h_0^3 E}{12(1-\nu^2)}\left[\nu\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\beta}\boldsymbol{\varphi}_0^{,\gamma}\cdot\boldsymbol{\varphi}_0^{,\delta} + \frac{1-\nu}{2}\left(\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\gamma}\boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\varphi}_0^{,\delta} + \boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\delta}\boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\varphi}_0^{,\gamma}\right)\right] \end{aligned}\right.$$

See later for interpretation

- Resultant stresses closed form (3)
  - Membrane resultant stress components (3)
    - $\boldsymbol{n}^{\alpha} = n^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + q^{\alpha}\boldsymbol{t}_0$, $\boldsymbol{t}_{0,\alpha} = \lambda_{\alpha}^{\beta}\boldsymbol{\varphi}_{0,\beta}$

      with $\begin{cases} q^{\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0} \boldsymbol{\sigma}_{\delta}^{\alpha 3} j_0 d\xi^3 = \tilde{q}^{\alpha} \\ \boldsymbol{\sigma}_{\delta}^{\alpha 3} = \boldsymbol{\sigma}_{\delta}^{3\alpha} = \dfrac{E}{2(1+\nu)}\dfrac{A'}{A}\boldsymbol{\varphi}_0^{,\alpha} \cdot \boldsymbol{\varphi}_0^{,\gamma}\gamma_{\gamma} \end{cases}$

    - As $\displaystyle\int_{h_0} j_0 d\xi^3 \simeq \bar{j}_0 h_0$

      $\Rightarrow$ $\tilde{q}^{\alpha} = q^{\alpha} = \mathcal{H}_q^{\alpha\beta}\delta_{\beta} = \dfrac{1}{2}\mathcal{H}_q^{\alpha\beta}\gamma_{\beta}$ with $\mathcal{H}_q^{\alpha\beta} = \dfrac{Eh_0}{1+\nu}\dfrac{A'}{A}\boldsymbol{\varphi}_0^{,\alpha} \cdot \boldsymbol{\varphi}_0^{,\beta}$

- Resultant stresses closed form (4)
  - From stress fields
    - $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\alpha\beta} \boldsymbol{g}_{0\alpha} \otimes \boldsymbol{g}_{0\beta} + \boldsymbol{\sigma}^{\alpha 3} \left(\boldsymbol{g}_{0\alpha} \otimes \boldsymbol{t}_0 + \boldsymbol{t}_0 \otimes \boldsymbol{g}_{0\alpha} \right)$

      $\Longrightarrow \quad \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^{\gamma} = \boldsymbol{\sigma}^{\alpha\gamma} \boldsymbol{g}_{0\alpha} + \boldsymbol{\sigma}^{3\gamma} \boldsymbol{t}_0$

    With $\boldsymbol{\sigma}^{\alpha\beta} = \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} + \xi^3 \boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta}$ & $\boldsymbol{\sigma}_{\delta}^{\alpha 3} = \boldsymbol{\sigma}_{\delta}^{3\alpha} = \frac{E}{2(1+\nu)} \frac{A'}{A} \boldsymbol{\varphi}_0^{,\alpha} \cdot \boldsymbol{\varphi}_0^{,\gamma} \gamma_{\gamma}$
  - Bending resultant stress
    - $\tilde{\boldsymbol{m}}^{\alpha} = \frac{1}{\bar{j}_0} \int_{h_0} \boldsymbol{\sigma} \cdot \boldsymbol{g}_0^{\alpha} j_0 \xi^3 d\xi^3$

      $\Longrightarrow \tilde{\boldsymbol{m}}^{\alpha} = \frac{1}{\bar{j}_0} \int_{h_0} \left[\boldsymbol{\sigma}^{\beta\alpha} \boldsymbol{g}_{0\beta} + \boldsymbol{\sigma}_{\delta}^{\alpha 3} \boldsymbol{t}_0 \right] j_0 \xi^3 d\xi^3$

    - As $\boldsymbol{g}_{0\alpha} = \frac{\partial \boldsymbol{\Phi}_0}{\partial \xi^{\alpha}} = \boldsymbol{\varphi}_{0,\alpha} + \xi^3 \boldsymbol{t}_{0,\alpha}$

      $\Longrightarrow \tilde{\boldsymbol{m}}^{\alpha} = \frac{1}{\bar{j}_0} \int_{h_0} \left[\boldsymbol{\sigma}^{\beta\alpha} \boldsymbol{\varphi}_{0,\beta} + \boldsymbol{\sigma}_{\delta}^{\alpha 3} \boldsymbol{t}_0 \right] j_0 \xi^3 d\xi^3 + \frac{1}{\bar{j}_0} \int_{h_0} \boldsymbol{\sigma}^{\beta\alpha} \boldsymbol{t}_{0,\beta} j_0 {\xi^3}^2 d\xi^3$

- Resultant stresses closed form (5)
  - Bending resultant stress (2)
    - $\tilde{\boldsymbol{m}}^{\alpha} = \frac{1}{\bar{j}_0} \int_{h_0} \left[\boldsymbol{\sigma}^{\beta\alpha} \boldsymbol{\varphi}_{0,\beta} + \boldsymbol{\sigma}_{\delta}^{\alpha 3} \boldsymbol{t}_0 \right] j_0 \xi^3 d\xi^3 + \frac{1}{\bar{j}_0} \int_{h_0} \boldsymbol{\sigma}^{\beta\alpha} \boldsymbol{t}_{0,\beta} j_0 {\xi^3}^2 d\xi^3$

      With $\boldsymbol{\sigma}^{\alpha\beta} = \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} + \xi^3 \boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta}$ & $\boldsymbol{\sigma}_{\delta}^{\alpha 3} = \boldsymbol{\sigma}_{\delta}^{3\alpha} = \frac{E}{2(1+\nu)} \frac{A'}{A} \boldsymbol{\varphi}_0^{;\alpha} \cdot \boldsymbol{\varphi}_0^{;\gamma} \gamma_{\gamma}$
  - Bending resultant stress components
    - We previously defined $\tilde{\boldsymbol{m}}^{\alpha} = \tilde{m}^{\beta\alpha} \boldsymbol{\varphi}_{0,\beta} + \tilde{m}^{3\alpha} \boldsymbol{t}_0$

$$\Rightarrow \begin{cases} \tilde{m}^{\beta\alpha} = \frac{1}{\bar{j}_0} \int_{h_0} \left[\xi^3 \boldsymbol{\sigma}^{\beta\alpha} + {\xi^3}^2 \boldsymbol{\sigma}^{\gamma\alpha} \boldsymbol{t}_{0,\gamma} \cdot \boldsymbol{\varphi}_0^{;\beta} \right] j_0 d\xi^3 \\ \tilde{m}^{3\alpha} = \frac{1}{\bar{j}_0} \int_{h_0} \boldsymbol{\sigma}_{\delta}^{\alpha 3} j_0 \xi^3 d\xi^3 \end{cases}$$

New term compared to plates

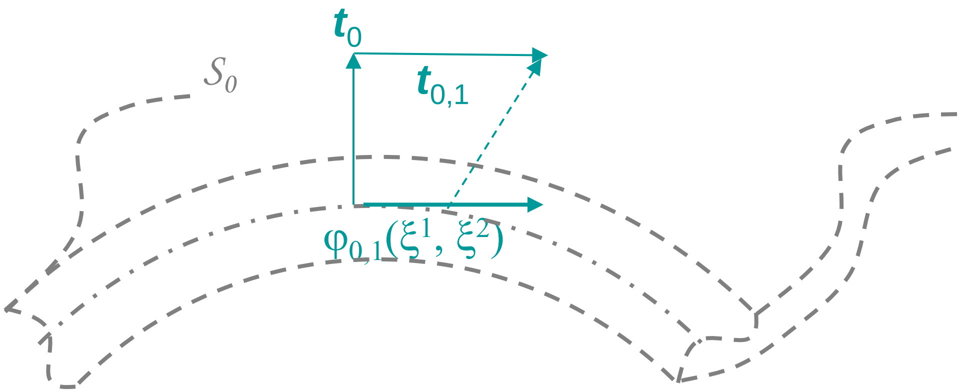
- Resultant stresses closed form (6)
  - Bending resultant stress components (2)
    - $\tilde{\boldsymbol{m}}^{\alpha} = \tilde{m}^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + \tilde{m}^{3\alpha}\boldsymbol{t}_0$

$$\begin{cases} \tilde{m}^{\beta\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0}\left[\xi^3\boldsymbol{\sigma}^{\beta\alpha} + {\xi^3}^2\boldsymbol{\sigma}^{\gamma\alpha}\boldsymbol{t}_{0,\gamma}\cdot\boldsymbol{\varphi}_0^{,\beta}\right]j_0 d\xi^3 \\ \tilde{m}^{3\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0}\boldsymbol{\sigma}_{\delta}^{\alpha 3} j_0 \xi^3 d\xi^3 \end{cases}$$

?

    - Due to the curvature $\boldsymbol{t}_{0,\alpha}$ has component in the basis $\boldsymbol{\varphi}_0{}^{,\beta}$

$\Longrightarrow$ $\boldsymbol{t}_{0,\alpha} = \lambda_{\alpha}^{\beta}\boldsymbol{\varphi}_{0,\beta}$ no component along $\boldsymbol{t}_0$ as $0 = \partial_\alpha\left(\boldsymbol{t}_0\cdot\boldsymbol{t}_0\right) = 2\boldsymbol{t}_{0,\alpha}\cdot\boldsymbol{t}_0$



$$\Longrightarrow \tilde{m}^{\beta\alpha} = \frac{1}{\bar{j}_0}\int_{h_0}\left[\xi^3\boldsymbol{\sigma}^{\beta\alpha} + {\xi^3}^2\lambda_{\gamma}^{\beta}\boldsymbol{\sigma}^{\gamma\alpha}\right]j_0 d\xi^3$$

- Resultant stresses closed form (7)
  - Bending resultant stress components (3)
    - $\tilde{\boldsymbol{m}}^{\alpha} = \tilde{m}^{\beta\alpha}\boldsymbol{\varphi}_{0,\beta} + \tilde{m}^{3\alpha}\boldsymbol{t}_0 \quad , \quad \boldsymbol{t}_{0,\alpha} = \lambda_{\alpha}^{\beta}\boldsymbol{\varphi}_{0,\beta}$

$$\begin{cases} \tilde{m}^{\beta\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0}\left[\xi^3\boldsymbol{\sigma}^{\beta\alpha} + {\xi^3}^2\lambda_{\gamma}^{\beta}\boldsymbol{\sigma}^{\gamma\alpha}\right] j_0 d\xi^3 \\ \tilde{m}^{3\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0}\boldsymbol{\sigma}_{\delta}^{\alpha 3} j_0 \xi^3 d\xi^3 \end{cases}$$

    - As $\boldsymbol{\sigma}^{\alpha\beta} = \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} + \xi^3\boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta} \quad , \quad \boldsymbol{\sigma}_{\delta}^{\alpha 3} = \boldsymbol{\sigma}_{\delta}^{3\alpha} = \dfrac{E}{2(1+\nu)}\dfrac{A'}{A}\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\gamma}\gamma_{\gamma}$

      & $\displaystyle\int_{h_0} j_0\rho_0\xi^3 d\xi^3 = 0$, and neglecting term in $(\xi^3)^3$ leads to, for constant density,

$$\Rightarrow \begin{cases} \tilde{m}^{\beta\alpha} = \dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0}\boldsymbol{\sigma}_{\tilde{m}}^{\beta\alpha} j_0 {\xi^3}^2 d\xi^3 + \lambda_{\gamma}^{\beta}\dfrac{1}{\bar{j}_0}\displaystyle\int_{h_0}\boldsymbol{\sigma}_{\tilde{n}}^{\gamma\alpha} j_0 {\xi^3}^2 d\xi^3 = \tilde{m}'^{\beta\alpha} + \lambda_{\gamma}^{\beta}\dfrac{h_0^2}{12}\tilde{n}^{\gamma\alpha} \\ \tilde{m}^{3\alpha} = 0 \end{cases}$$

With

$$\begin{cases} \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} = \dfrac{E\nu}{(1-\nu^2)}\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\beta}\boldsymbol{\varphi}_0^{,\gamma}\cdot\boldsymbol{\varphi}_0^{,\delta}\varepsilon_{\gamma\delta} + \dfrac{E}{1+\nu}\varepsilon_{\gamma\delta}\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\gamma}\boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\varphi}_0^{,\delta} \\ \boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta} = \dfrac{E\nu}{(1-\nu^2)}\boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\beta}\boldsymbol{\varphi}_0^{,\gamma}\cdot\boldsymbol{\varphi}_0^{,\delta}\kappa_{\gamma\delta} + \boldsymbol{\varphi}_0^{,\alpha}\cdot\boldsymbol{\varphi}_0^{,\gamma}\boldsymbol{\varphi}_0^{,\beta}\cdot\boldsymbol{\varphi}_0^{,\delta}\kappa_{\gamma\delta} \end{cases}$$