University of Liège
Aerospace & Mechanical Engineering

Aircraft Structures
Kirchhoff-Love Plates

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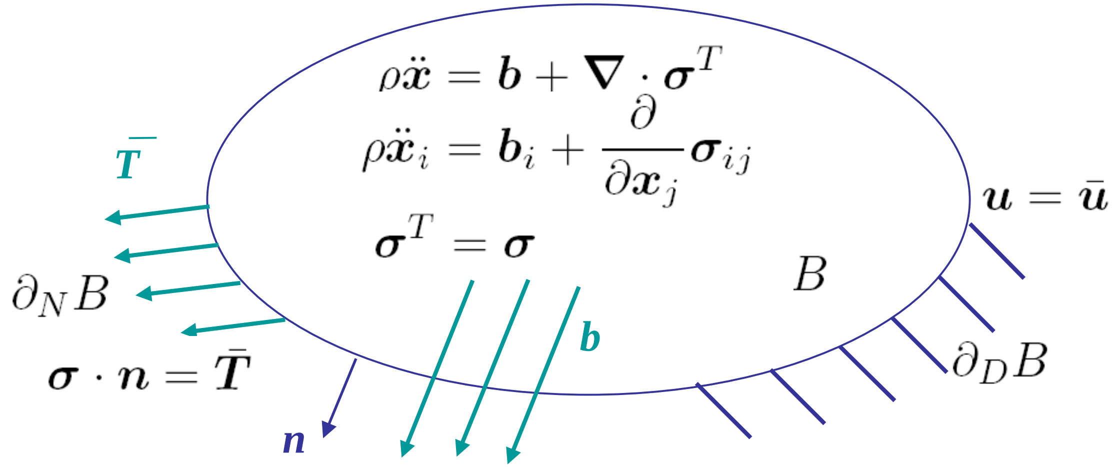
Elasticity

- Balance of body B
  - Momenta balance
    - Linear
    - Angular
  - Boundary conditions
    - Neumann
    - Dirichlet



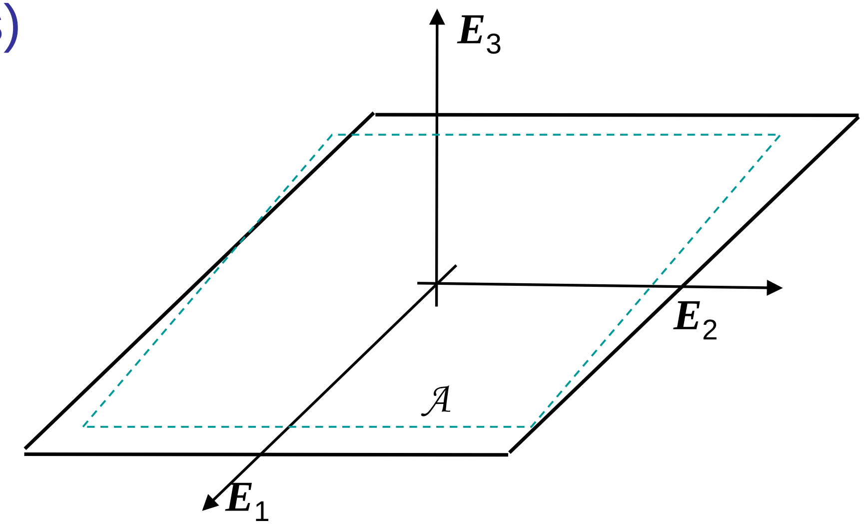
- Small deformations with linear elastic, homogeneous & isotropic material
  - (Small) Strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla}\right)$, or $\begin{cases} \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{x}_i}\boldsymbol{u}_j + \frac{\partial}{\partial \boldsymbol{x}_j}\boldsymbol{u}_i\right) \\ \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\boldsymbol{u}_{j,i} + \boldsymbol{u}_{i,j}\right) \end{cases}$
  - Hooke's law $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon}$, or $\boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl}\boldsymbol{\varepsilon}_{kl}$

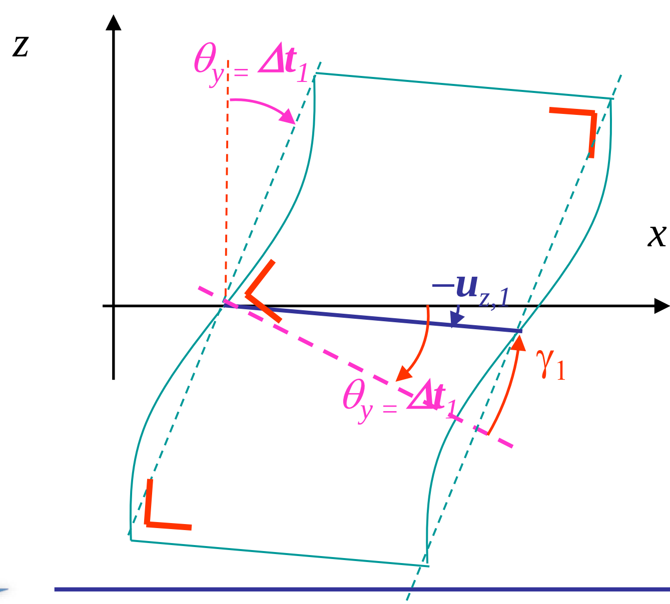
    with $$\mathcal{H}_{ijkl} = \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda = K - 2\mu/3}\delta_{ij}\delta_{kl} + \underbrace{\frac{E}{1+\nu}}_{2\mu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$$
  - Inverse law $\boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma}$

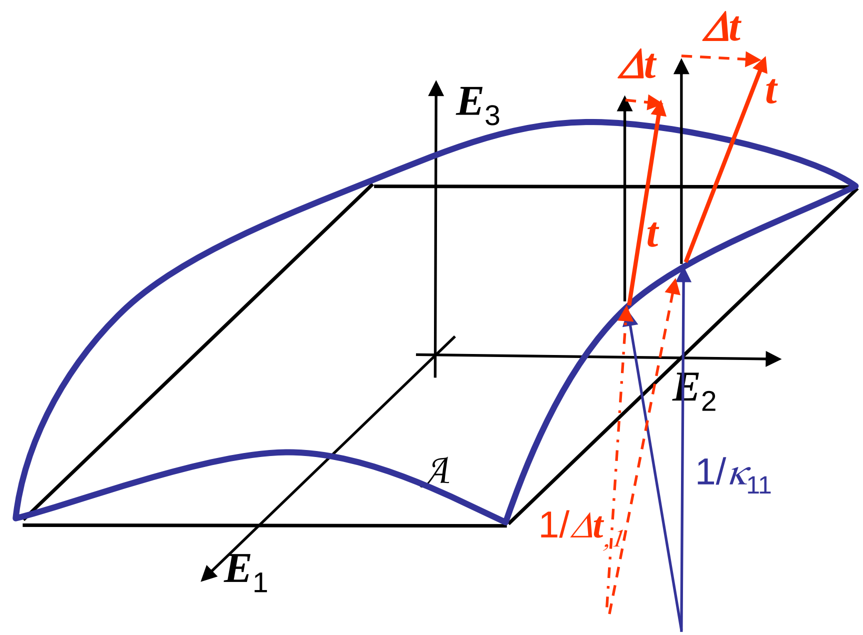
    with $$\mathcal{G}_{ijkl} = \frac{1+\nu}{E}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right) - \frac{\nu}{E}\delta_{ij}\delta_{kl}$$

Reissner-Mindlin equations summary

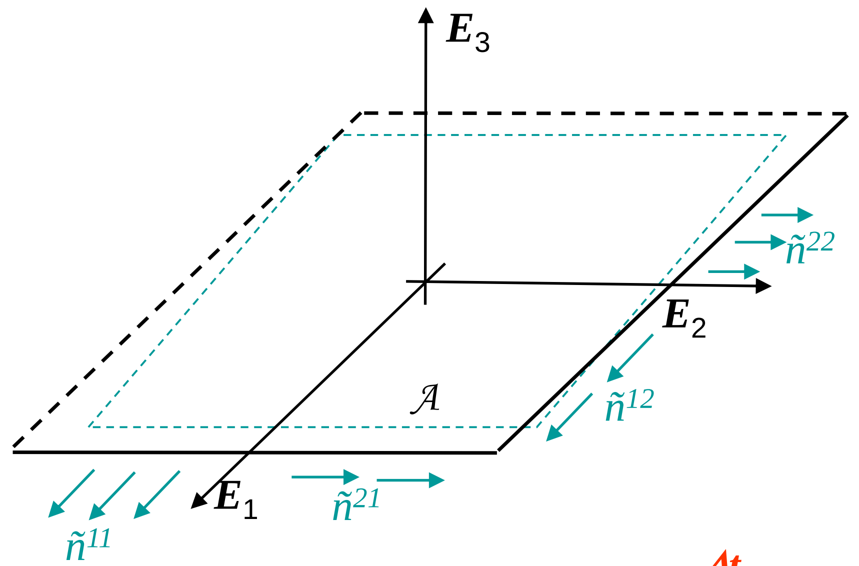
- Deformations (small transformations)
  - In plane membrane
    - $\boxed{\varepsilon_{\alpha\beta} = \dfrac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2}}$
  - Curvature
    - $\boxed{\kappa_{\alpha\beta} = \dfrac{\boldsymbol{\Delta t}_{\alpha,\beta} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2}}$
  - Out-of-plane sliding
    - $\boxed{\gamma_\alpha = \boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_\alpha}$

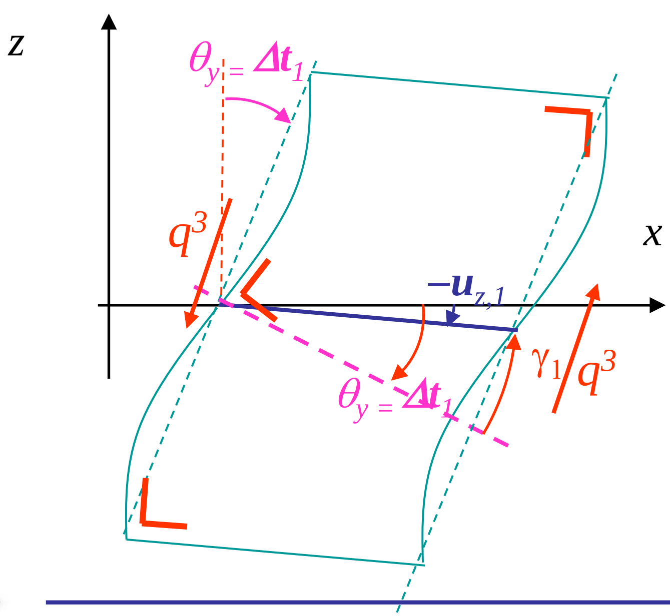


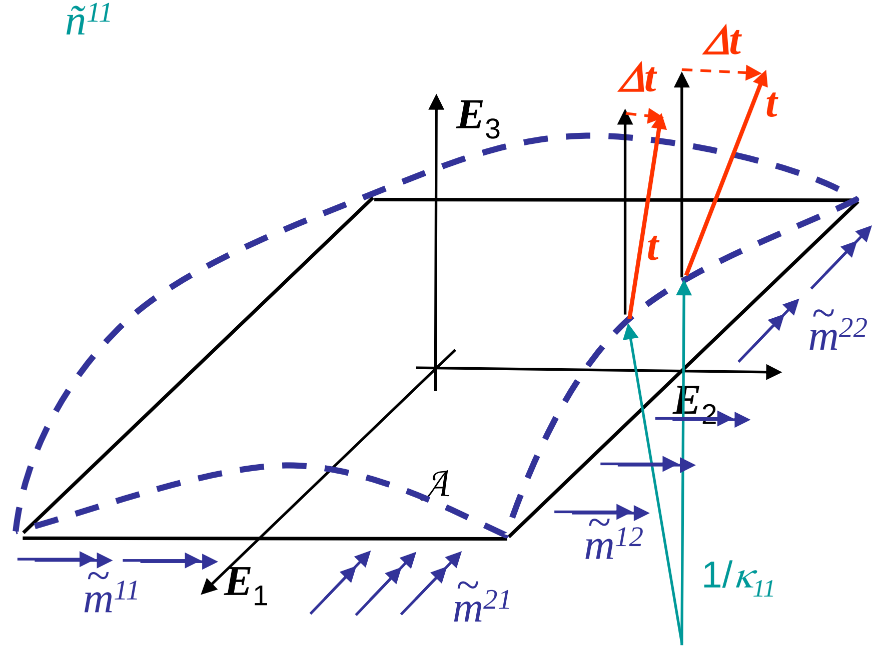




- Resultant stresses in linear elasticity
  - Membrane stress
    - $\boxed{\tilde{n}^{\alpha\beta} = \mathcal{H}_n^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}}$
  - Bending stress
    - $\boxed{\tilde{m}^{\alpha\beta} = \mathcal{H}_m^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta}}$
  - Out-of-plane shear stress
    - $\boxed{\tilde{q}^{\alpha} = \frac{1}{2}\mathcal{H}_q^{\alpha\beta}\gamma_{\beta}}$



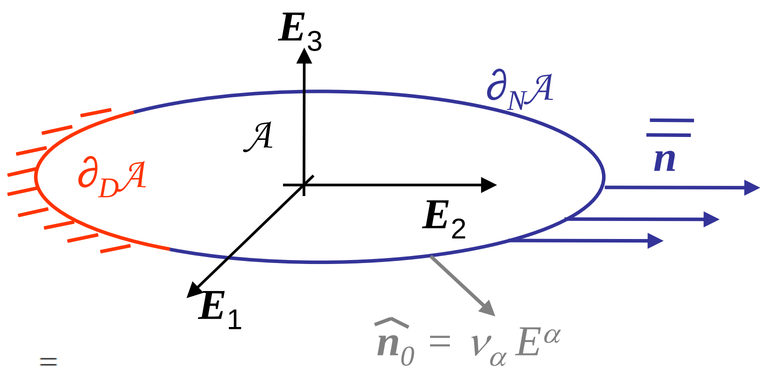




- Resultant Hooke tensor in linear elasticity
  - Membrane mode
    - $\mathcal{H}_n^{\alpha\beta\gamma\delta} = \frac{h_0 E}{1-\nu^2}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta} + \frac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$
  - Bending mode
    - $\mathcal{H}_m^{\alpha\beta\gamma\delta} = \frac{h_0^3 E}{12\left(1-\nu^2\right)}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta} + \frac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$
  - Shear mode
    - $\mathcal{H}_q^{\alpha\beta} = \frac{E h_0}{1+\nu}\frac{A'}{A}\delta^{\alpha\beta}$

- Resultant equations
  - Membrane mode
    - $\boxed{\left(\boldsymbol{n}^{\alpha}\right)_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}\ddot{\boldsymbol{u}}}$
    - $\boldsymbol{n}^{\alpha} = \tilde{n}^{\alpha\beta}\boldsymbol{E}_{\beta} + \tilde{q}^{\alpha}\boldsymbol{E}_3$
      - $\tilde{n}^{\alpha\beta} = \mathcal{H}_n^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}$ with $\varepsilon_{\alpha\beta} = \dfrac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2}$
      - $\tilde{q}^{\alpha} = \dfrac{1}{2}\mathcal{H}_q^{\alpha\beta}\gamma_{\beta}$ with $\gamma_{\alpha} = \boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_{\alpha}$
    - Clearly, the solution can be directly computed in plane Oxy (constant $\mathcal{H}_n$)
      - $\mathcal{H}_n^{\alpha\beta\gamma\delta}\dfrac{\boldsymbol{u}_{\gamma,\delta\alpha} + \boldsymbol{u}_{\delta,\gamma\alpha}}{2} + \bar{\boldsymbol{n}}_{\beta} = \bar{\rho}\ddot{\boldsymbol{u}}_{\beta}$
      - Boundary conditions
        - » Dirichlet $\boldsymbol{u}_{\alpha} = \bar{\boldsymbol{u}}_{\alpha}$
        - » Neumann

$$\boldsymbol{n}_{\beta}^{\alpha}\nu_{\alpha} = \mathcal{H}_n^{\alpha\beta\gamma\delta}\frac{\boldsymbol{u}_{\gamma,\delta} + \boldsymbol{u}_{\delta,\gamma}}{2}\nu_{\alpha} = \bar{\bar{\boldsymbol{n}}}_{\beta}$$



- Remaining equation along E^3: $\mathcal{H}_q^{\alpha\beta}\dfrac{\boldsymbol{u}_{3,\beta\alpha} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} + \bar{\boldsymbol{n}}_3 = \bar{\rho}\ddot{\boldsymbol{u}}_3$

- Resultant equations (2)
  - Bending mode
    - $\boxed{\ddot{\boldsymbol{t}} I_p = \bar{\tilde{\boldsymbol{m}}} - \left(\boldsymbol{n}^3 - \lambda \boldsymbol{E}_3\right) + \left(\tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha}}$
    - $\tilde{\boldsymbol{m}}^\alpha = \tilde{m}^{\alpha\beta} \boldsymbol{E}_\beta$ & $\boldsymbol{n}^3 = \tilde{q}^\alpha \boldsymbol{E}_\alpha$
      - $\tilde{m}^{\alpha\beta} = \mathcal{H}_m^{\alpha\beta\gamma\delta} \kappa_{\gamma\delta}$ with $\kappa_{\alpha\beta} = \frac{\boldsymbol{\Delta t}_{\alpha,\beta} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2}$
      - $\tilde{q}^\alpha = \frac{1}{2} \mathcal{H}_q^{\alpha\beta} \gamma_\beta$ with $\gamma_\alpha = \boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_\alpha$
    - Solution is obtained by projecting into the plane Oxy (constant $\mathcal{H}_q$, $\mathcal{H}_m$)
      - $I_p \ddot{\boldsymbol{\Delta t}}_\alpha = \bar{\tilde{\boldsymbol{m}}}_\alpha - \frac{1}{2} \mathcal{H}_q^{\alpha\beta} \left(\boldsymbol{u}_{3,\beta} + \boldsymbol{\Delta t}_\beta\right) + \mathcal{H}_m^{\alpha\beta\gamma\delta} \frac{\boldsymbol{\Delta t}_{\gamma,\delta\beta} + \boldsymbol{\Delta t}_{\delta,\gamma\beta}}{2}$
      - 2 equations (α=1, 2) with 3 unknowns ($\Delta \boldsymbol{t}_1$, $\Delta \boldsymbol{t}_2$, $\boldsymbol{u}_3$)
      - Use remaining equation $\mathcal{H}_q^{\alpha\beta} \frac{\boldsymbol{u}_{3,\beta\alpha} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} + \bar{\boldsymbol{n}}_3 = \bar{\rho} \ddot{\boldsymbol{u}}_3$

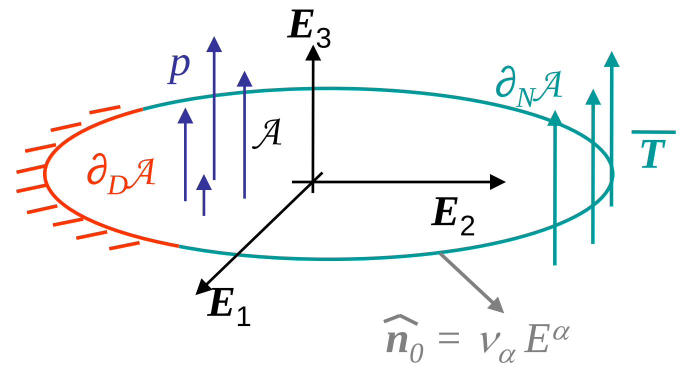
Reissner-Mindlin equations summary

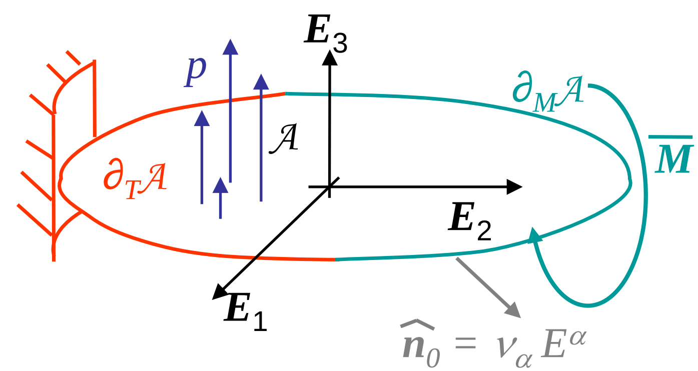
- Resultant equations (3)
  - Bending mode (2)
    - 3 equations with 3 unknowns
      - $I_p \ddot{\boldsymbol{\Delta t}}_\alpha = \bar{\tilde{\boldsymbol{m}}}_\alpha - \frac{1}{2}\mathcal{H}_q^{\alpha\beta}\left(\boldsymbol{u}_{3,\beta} + \boldsymbol{\Delta t}_\beta\right) + \mathcal{H}_m^{\alpha\beta\gamma\delta}\frac{\boldsymbol{\Delta t}_{\gamma,\delta\beta} + \boldsymbol{\Delta t}_{\delta,\gamma\beta}}{2}$
      - $\mathcal{H}_q^{\alpha\beta}\frac{\boldsymbol{u}_{3,\beta\alpha} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} + \bar{\boldsymbol{n}}_3 = \bar{\rho}\ddot{\boldsymbol{u}}_3$
    - To be completed by BCs
      - Low order constrains
        - » Displacement $\boldsymbol{u}_3 = \bar{\boldsymbol{u}}_3$ or
        - » Shearing

$$\mathcal{H}_q^{\alpha\beta}\frac{\boldsymbol{u}_{3,\beta} + \boldsymbol{\Delta t}_\beta}{2}\nu_\alpha = \bar{T}$$

      - High order
        - » Rotation $\boldsymbol{\Delta t} = \bar{\boldsymbol{\Delta t}}$ or
        - » Bending

$$\tilde{\boldsymbol{m}}_\beta^\alpha \nu_\alpha = \mathcal{H}_m^{\alpha\beta\gamma\delta}\frac{\boldsymbol{\Delta t}_{\gamma,\delta} + \boldsymbol{\Delta t}_{\delta,\gamma}}{2}\nu_\alpha = \bar{\boldsymbol{M}}_\beta$$

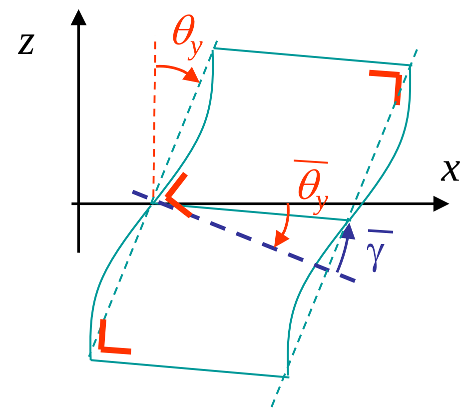




- Resultant equations (4)
  - Remarks
    - Compare bending equations
      - $I_p \ddot{\boldsymbol{\Delta t}}_\alpha = \bar{\tilde{\boldsymbol{m}}}_\alpha - \frac{1}{2}\mathcal{H}_q^{\alpha\beta}\left(\boldsymbol{u}_{3,\beta} + \boldsymbol{\Delta t}_\beta\right) + \mathcal{H}_m^{\alpha\beta\gamma\delta}\frac{\boldsymbol{\Delta t}_{\gamma,\delta\beta} + \boldsymbol{\Delta t}_{\delta,\gamma\beta}}{2}$
      - $\mathcal{H}_q^{\alpha\beta}\frac{\boldsymbol{u}_{3,\beta\alpha} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} + \bar{\boldsymbol{n}}_3 = \bar{\rho}\ddot{\boldsymbol{u}}_3$
      - With Timoshenko beam equations

$$\frac{\partial}{\partial_x}\left(EI\frac{\partial\theta_y}{\partial x}\right) - \mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right) = 0$$

$$\frac{\partial}{\partial x}\left(\mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right)\right) = -f$$

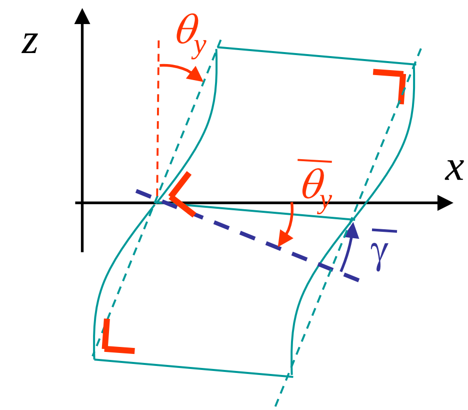


- Membrane and bending equations are uncoupled
  - No initial curvature
  - Small deformations (equilibrium on non curved configuration)

- Kirchhoff-Love assumption
  - Beam analogy
    - Timoshenko beam equations

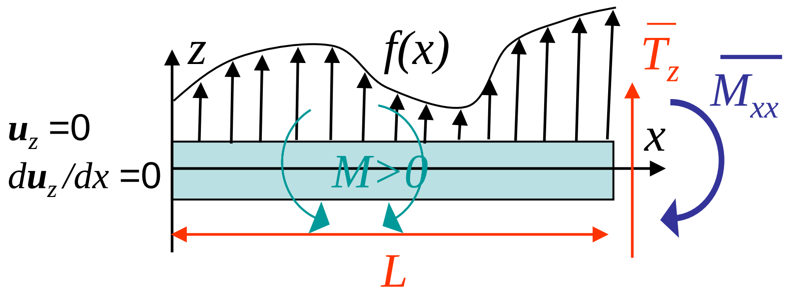
$$\frac{\partial}{\partial_x}\left(EI\frac{\partial\theta_y}{\partial x}\right) - \mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right) = 0$$

$$\frac{\partial}{\partial x}\left(\mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right)\right) = -f$$



- Euler-Bernoulli equations

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f\left(x\right)$$



- It has been assumed that

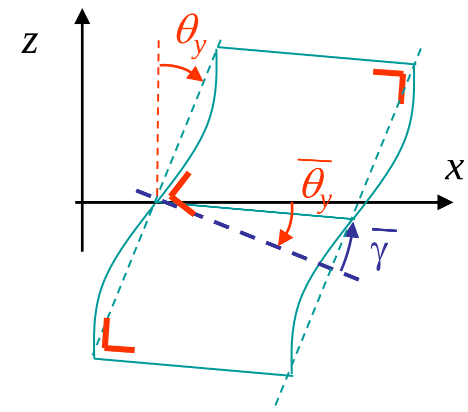
  the cross-section remains planar
  and perpendicular to the neutral axis
- $\boldsymbol{u}_{z,x} = -\theta_y$
- Validity:

  $(EI)/(L^2 A'\mu) << 1$

- The same assumption can be made for plates

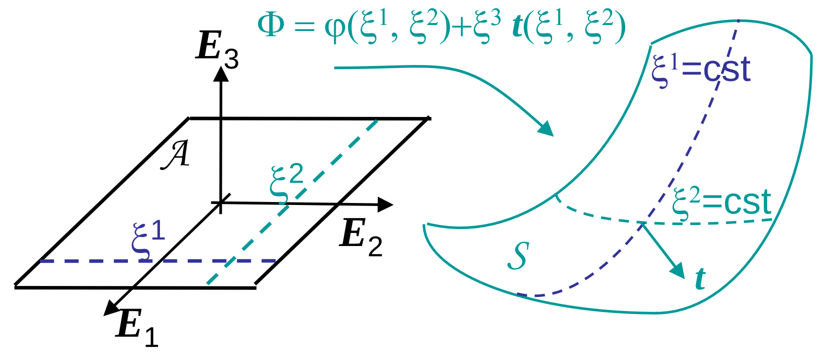
Shear effect

- Kirchhoff-Love assumption (2)
  - As
    - $$\mathcal{H}_m^{\alpha\beta\gamma\delta} = \overbrace{\frac{h_0^3 E}{12\left(1-\nu^2\right)}}^{D}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta} + \frac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$$
    - $$\mathcal{H}_q^{\alpha\beta} = \frac{E h_0}{1+\nu}\frac{A'}{A}\delta^{\alpha\beta}$$



  - Kirchhoff-Love assumption requires
    - $$\frac{D(1+\nu)A}{L^2 E h_0 A'} = \frac{h_0^2 A}{12L^2(1-\nu)A'} \ll 1$$
    - Where L is a characteristic distance

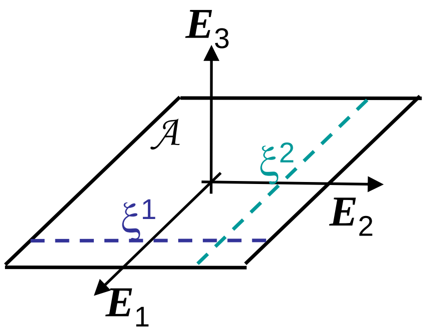
- Description
  - In the reference frame $\boldsymbol{E}_i$
    - The plate is defined by



$$\boldsymbol{\xi} = \xi^I \boldsymbol{E}_I \quad \text{with} \begin{cases} (\xi^1, \xi^2) \in \mathcal{A} \\ \xi^3 \in [-\frac{h_0}{2}; \frac{h_0}{2}] \end{cases}$$

  - Mapping of the (deformed) plate
    - Neutral plane $\boldsymbol{\varphi}_0\left(\xi^1, \xi^2\right) = \xi^\alpha \boldsymbol{E}_\alpha$ α =1 or 2, I = 1, 2 or 3
    - Cross section $\boldsymbol{t}_0(\xi^1, \xi^2)\,,\ \|\boldsymbol{t}\| = 1$ with $\boldsymbol{t}_0 = \boldsymbol{E}_3$
    - Initial plate S_0
      - $S_0 = \mathcal{A}$ x [-h_0/2 h_0/2], for a plate of initial thickness h_0
      - $\boldsymbol{X} = \boldsymbol{\Phi}_0\left(\xi^I\right) = \boldsymbol{\varphi}_0\left(\xi^\alpha\right) + \xi^3 \boldsymbol{t}_0(\xi^1, \xi^2)$
    - Deformed plate S
      - $\boldsymbol{x} = \boldsymbol{\Phi}\left(\xi^I\right) = \boldsymbol{\varphi}\left(\xi^\alpha\right) + \xi^3 \boldsymbol{t}(\xi^1, \xi^2)$

- Kirchhoff-Love plate
  - Assumptions
    - Kirchhoff (cross section remains plane)



$$\boldsymbol{t}_0 = \boldsymbol{E}_3 \quad \& \quad \boldsymbol{t} = \frac{\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}}{\|\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}\|}$$

    - Small deformations $\mathcal{S} \simeq \mathcal{S}_0, \quad \boldsymbol{\nabla} \simeq \boldsymbol{\nabla}_0, \quad \int_{\mathcal{S}} \simeq \int_{\mathcal{S}_0}$
    - Displacement field $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 + \boldsymbol{u}$ with $\|\boldsymbol{u}\| << \sqrt[3]{|\mathcal{S}_0|}$
  - Kinematics

$$\begin{cases} \boldsymbol{\varphi}_{,1} = \boldsymbol{E}_1 + \boldsymbol{u}_{1,1}\boldsymbol{E}_1 + \boldsymbol{u}_{2,1}\boldsymbol{E}_2 + \boldsymbol{u}_{3,1}\boldsymbol{E}_3 \\ \boldsymbol{\varphi}_{,2} = \boldsymbol{E}_2 + \boldsymbol{u}_{1,2}\boldsymbol{E}_1 + \boldsymbol{u}_{2,2}\boldsymbol{E}_2 + \boldsymbol{u}_{3,2}\boldsymbol{E}_3 \end{cases}$$

$$\begin{cases} \boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2} \simeq (1 + \boldsymbol{u}_{1,1} + \boldsymbol{u}_{2,2})\,\boldsymbol{E}_3 - \boldsymbol{u}_{3,1}\boldsymbol{E}_1 - \boldsymbol{u}_{3,2}\boldsymbol{E}_2 \\ \|\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}\| \simeq 1 + \boldsymbol{u}_{1,1} + \boldsymbol{u}_{2,2} \end{cases}$$

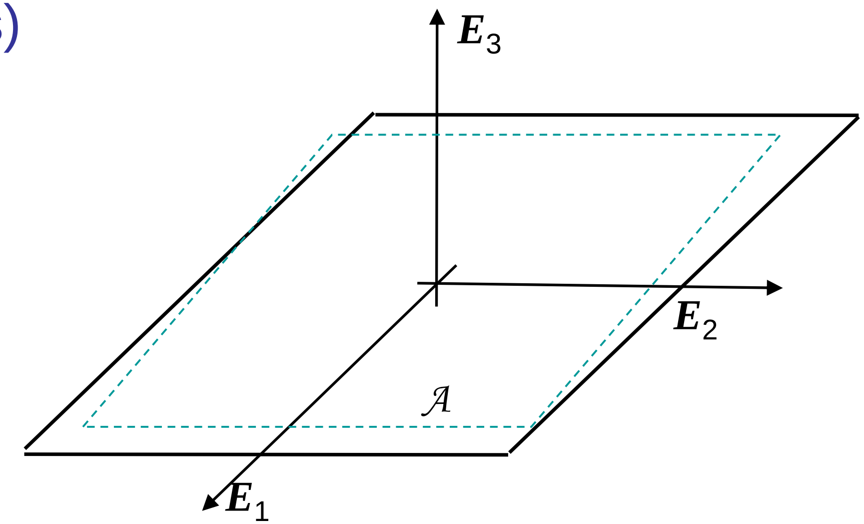
$$\begin{cases} \boldsymbol{t} \simeq \boldsymbol{E}_3 - \boldsymbol{u}_{3,1}\boldsymbol{E}_1 - \boldsymbol{u}_{3,2}\boldsymbol{E}_2 = \boldsymbol{E}_3 - \boldsymbol{u}_{3,\alpha}\boldsymbol{E}_\alpha \\ \boldsymbol{t}_{,\beta} = -\boldsymbol{u}_{3,\alpha\beta}\boldsymbol{E}_\alpha \end{cases}$$

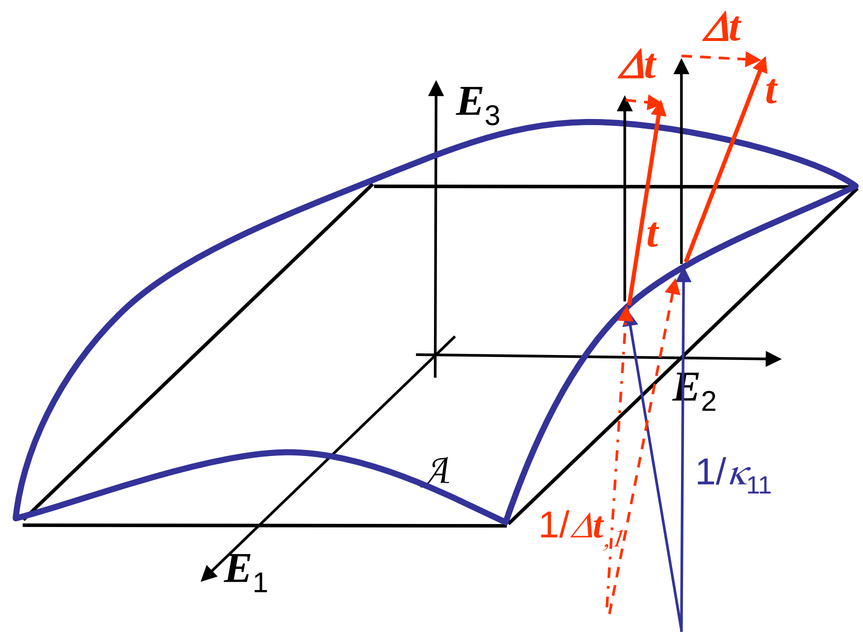
$$\begin{cases} \boldsymbol{\Delta t} = -\boldsymbol{u}_{3,\alpha}\boldsymbol{E}_\alpha \\ \boldsymbol{\Delta t}_{,\beta} = -\boldsymbol{u}_{3,\alpha\beta}\boldsymbol{E}_\alpha \end{cases}$$

- Deformations (small transformations)
  - Reissner-Mindlin
    - $\boxed{\varepsilon_{\alpha\beta} = \frac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2}}$
    - $\boxed{\kappa_{\alpha\beta} = \frac{\boldsymbol{\Delta t}_{\alpha,\beta} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2}}$
    - $\boxed{\gamma_\alpha = \boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_\alpha}$
  - Using

$$\begin{cases} \boldsymbol{\Delta t} = -\boldsymbol{u}_{3,\alpha}\boldsymbol{E}_\alpha \\ \boldsymbol{\Delta t}_{,\beta} = -\boldsymbol{u}_{3,\alpha\beta}\boldsymbol{E}_\alpha \end{cases}$$

$$\Rightarrow \begin{cases} \varepsilon_{\alpha\beta} = \frac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2} \\ \kappa_{\alpha\beta} = -\boldsymbol{u}_{3,\alpha\beta} \\ \gamma_\alpha = 0 \end{cases}$$

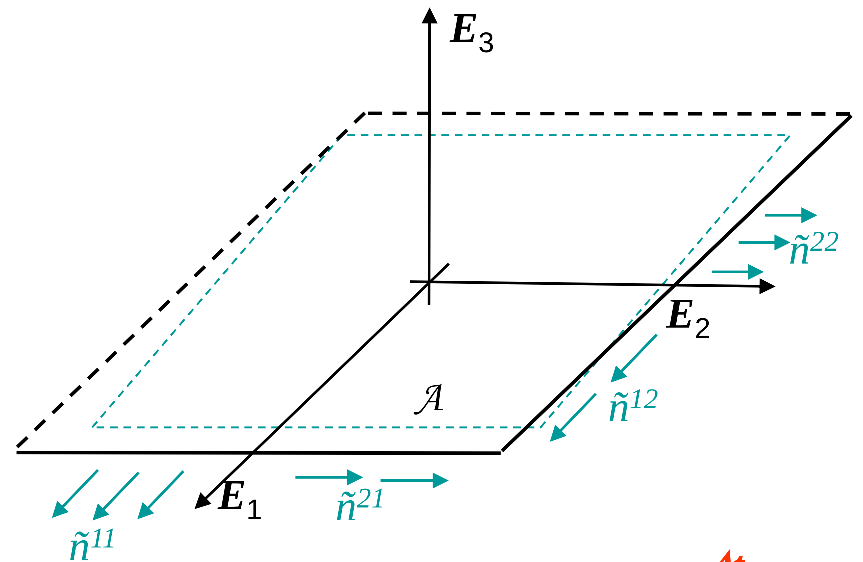


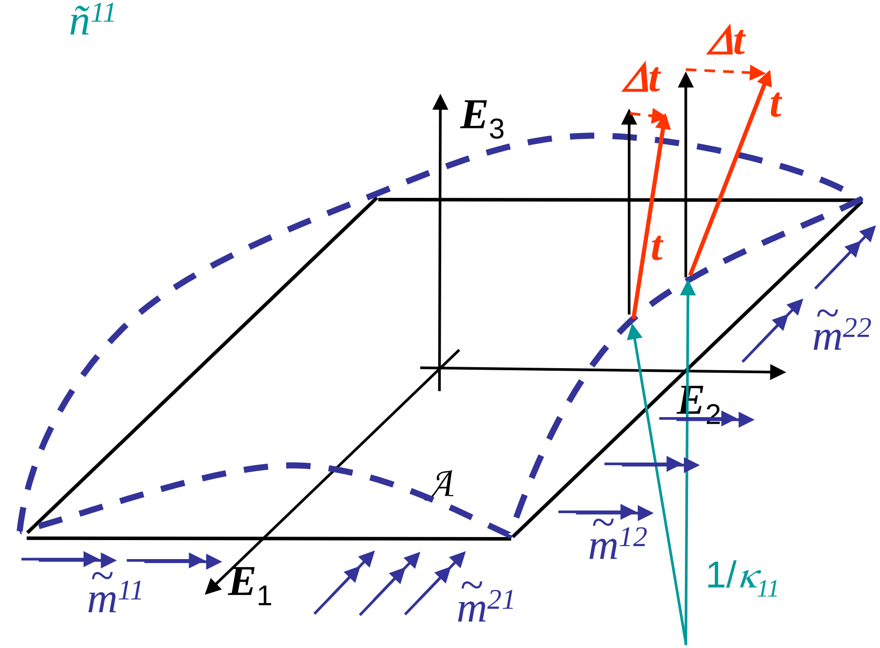


- Resultant stresses
  - Reissner-Mindlin
    - $\boxed{\tilde{n}^{\alpha\beta} = \mathcal{H}_n^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}}$
    - $\boxed{\tilde{m}^{\alpha\beta} = \mathcal{H}_m^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta}}$
    - $\boxed{\tilde{q}^{\alpha} = \frac{1}{2}\mathcal{H}_q^{\alpha\beta}\gamma_{\beta}}$
  - Using

$$\left\{\begin{aligned} \varepsilon_{\alpha\beta} &= \frac{\boldsymbol{u}_{\alpha,\beta}+\boldsymbol{u}_{\beta,\alpha}}{2} \\ \kappa_{\alpha\beta} &= -\boldsymbol{u}_{3,\alpha\beta} \\ \gamma_{\alpha} &= 0 \end{aligned}\right.$$

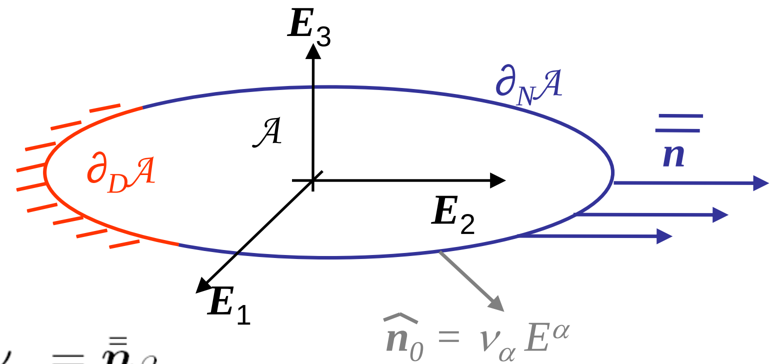
$$\Rightarrow \left\{\begin{aligned} \tilde{n}^{\alpha\beta} &= \mathcal{H}_n^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} \\ \tilde{m}^{\alpha\beta} &= \mathcal{H}_m^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta} \\ \tilde{q}^{\alpha} &= 0 \end{aligned}\right.$$





- Resultant equations
  - Membrane mode
    - $\boxed{\left(\boldsymbol{n}^{\alpha}\right)_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}\ddot{\boldsymbol{u}}}$
    - $\boldsymbol{n}^{\alpha} = \tilde{n}^{\alpha\beta}\boldsymbol{E}_{\beta} + \tilde{q}^{\alpha}\boldsymbol{E}_{3}$
      - $\tilde{n}^{\alpha\beta} = \mathcal{H}_n^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}$ with $\varepsilon_{\alpha\beta} = \dfrac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2}$
      - $\tilde{q}^{\alpha} = 0$
    - Clearly, the solution can be directly computed in plane Oxy (constant $\mathcal{H}_n$)
      - $\mathcal{H}_n^{\alpha\beta\gamma\delta}\dfrac{\boldsymbol{u}_{\gamma,\delta\alpha} + \boldsymbol{u}_{\delta,\gamma\alpha}}{2} + \bar{\boldsymbol{n}}_{\beta} = \bar{\rho}\ddot{\boldsymbol{u}}_{\beta}$
      - Boundary conditions
        - » Neumann $\boldsymbol{u}_{\alpha} = \bar{\boldsymbol{u}}_{\alpha}$
        - » Dirichlet

$$\boldsymbol{n}_{\beta}^{\alpha}\nu_{\alpha} = \mathcal{H}_n^{\alpha\beta\gamma\delta}\frac{\boldsymbol{u}_{\gamma,\delta\alpha} + \boldsymbol{u}_{\delta,\gamma\alpha}}{2}\nu_{\alpha} = \bar{\bar{\boldsymbol{n}}}_{\beta}$$



    - Same membrane problem as for Reissner-Mindlin
    - Remark: $\mathcal{H}_n^{\alpha\beta\gamma\delta} = \dfrac{h_0 E}{1-\nu^2}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta} + \dfrac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$

- Resultant equations (2)
  - Bending mode
    - $\boxed{\ddot{\boldsymbol{t}} I_p = \bar{\tilde{\boldsymbol{m}}} - \left(\boldsymbol{n}^3 - \lambda \boldsymbol{E}_3\right) + \left(\tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha}}$
    - $\tilde{\boldsymbol{m}}^\alpha = \tilde{m}^{\alpha\beta} \boldsymbol{E}_\beta$ & $\boldsymbol{n}^3 = \tilde{q}^\alpha \boldsymbol{E}_\alpha$
      - $\tilde{m}^{\alpha\beta} = \mathcal{H}_m^{\alpha\beta\gamma\delta} \kappa_{\gamma\delta}$ with $\kappa_{\alpha\beta} = -\boldsymbol{u}_{3,\alpha\beta}$
      - $\tilde{q}^\alpha = 0$
    - Solution is obtained by projecting into the plane Oxy (constant $\mathcal{H}_m$)
      - $-I_p \ddot{\boldsymbol{u}}_{3,\alpha} = \bar{\tilde{\boldsymbol{m}}}_\alpha - \mathcal{H}_m^{\alpha\beta\gamma\delta} \boldsymbol{u}_{3,\gamma\delta\beta}$
      - 1 higher-order equation with 1 unknown ($\boldsymbol{u}_3$)
      - With $\mathcal{H}_m^{\alpha\beta\gamma\delta} = \frac{h_0^3 E}{12\left(1-\nu^2\right)} \left[\nu \delta^{\alpha\beta} \delta^{\gamma\delta} + \frac{1-\nu}{2} \left(\delta^{\alpha\gamma} \delta^{\beta\delta} + \delta^{\alpha\delta} \delta^{\beta\gamma}\right)\right]$
    - Compare to Euler-Bernoulli beams

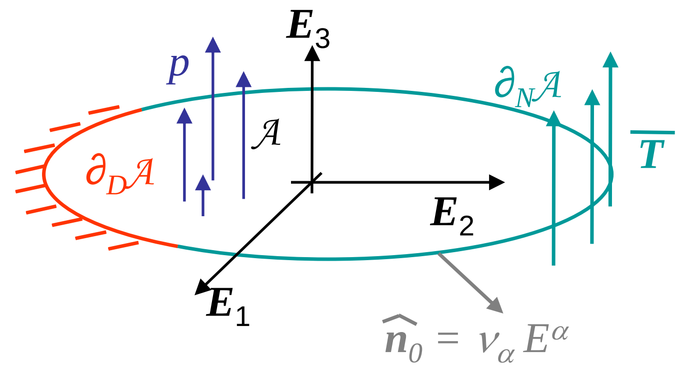
$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f\left(x\right)$$

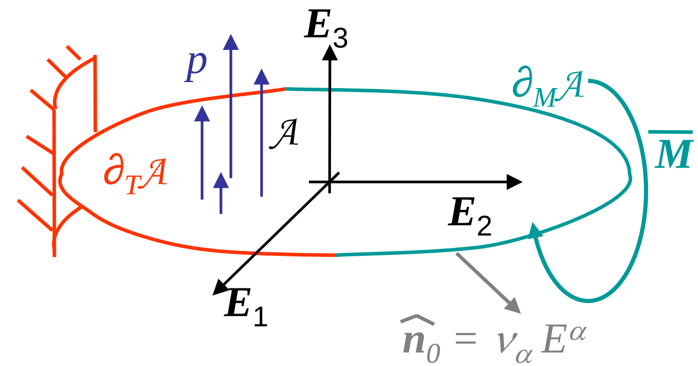
  - How to include pressure effect?

- Applied pressure
  - 1 higher-order equation with 1 unknown
    - $-I_p \ddot{\boldsymbol{u}}_{3,\alpha} = \bar{\tilde{\boldsymbol{m}}}_\alpha - \mathcal{H}_m^{\alpha\beta\gamma\delta} \boldsymbol{u}_{3,\gamma\delta\beta}$
  - In case of pressure loading p
    - Assume static problem (and constant $\mathcal{H}_m$)
    - $\bar{\tilde{\boldsymbol{m}}} = \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3 \xi^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} + \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \xi^3 \boldsymbol{b} d\xi^3$

      has no component along α in case

      of pressure loading





  - Use energy conservation
    - Work of external forces

$$\delta W_{\text{ext}} = \int_{\mathcal{A}} p \delta \boldsymbol{u}_3 d\mathcal{A} + \int_{\partial_N \mathcal{A}} \bar{T} \delta \boldsymbol{u}_3 - \int_{\partial_M \mathcal{A}} \bar{M} \delta \boldsymbol{u}_{3,\alpha} \nu_\alpha dl$$

    - Bending part of internal energy (as problems are uncoupled)?

- Applied pressure (2)
  - Internal energy
    - Last lecture we considered the decomposition
      - $\boldsymbol{\varepsilon} = \varepsilon_{\alpha\beta}\boldsymbol{E}^{\alpha}\otimes\boldsymbol{E}^{\beta} + \delta_{\alpha}\left(\boldsymbol{E}^{3}\otimes\boldsymbol{E}^{\alpha} + \boldsymbol{E}^{\alpha}\otimes\boldsymbol{E}^{3}\right) + \xi^{3}\kappa_{\alpha\beta}\boldsymbol{E}^{\alpha}\otimes\boldsymbol{E}^{\beta} + \lambda_{h}\boldsymbol{E}^{3}\otimes\boldsymbol{E}^{3}$

        $\gamma_{\alpha}/2$
      - $\boldsymbol{\sigma}^{\alpha\beta} = \boldsymbol{\sigma}^{\alpha\beta}_{\tilde{n}} + \xi^{3}\boldsymbol{\sigma}^{\alpha\beta}_{\tilde{m}}$

        with $\boldsymbol{\sigma}^{\alpha\beta}_{\tilde{m}} = \frac{E\nu}{(1-\nu^{2})}\kappa_{\gamma\gamma}\delta^{\alpha\beta} + \frac{E}{1+\nu}\frac{\kappa_{\alpha\beta}+\kappa_{\beta\alpha}}{2}$
    - Bending contribution to internal energy
      - $\delta E_{\text{int}} = \int_{\mathcal{A}}\int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} (\xi^{3})^{2}\boldsymbol{\sigma}_{\tilde{m}} : \kappa_{ij}\boldsymbol{E}^{I}\otimes\boldsymbol{E}^{J} d\xi^{3} d\mathcal{A}$

        $\Longrightarrow \delta E_{\text{int}} = \int_{\mathcal{A}}\int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} (\xi^{3})^{2} d\xi^{3}\boldsymbol{\sigma}^{\alpha\beta}_{\tilde{m}}\kappa_{\alpha\beta} d\mathcal{A} = \int_{\mathcal{A}} \frac{h_0^{3}}{12}\boldsymbol{\sigma}^{\alpha\beta}_{\tilde{m}}\kappa_{\alpha\beta} d\mathcal{A}$
      - As $\tilde{\boldsymbol{m}}^{\alpha} = \frac{h_0^{3}}{12}\boldsymbol{\sigma}^{\alpha\beta}_{\tilde{m}}\boldsymbol{E}_{\beta} = \tilde{m}^{\alpha\beta}\boldsymbol{E}_{\beta}$

        $\Longrightarrow \delta E_{\text{int}} = \int_{\mathcal{A}} \tilde{m}^{\alpha\beta}\delta\kappa_{\alpha\beta} d\mathcal{A}$
      - Remark other contributions (for the uncoupled cases) follow

        » $\delta E_{\text{int}} = \int_{\mathcal{A}} \tilde{m}^{\alpha\beta}\delta\kappa_{\alpha\beta} d\mathcal{A} + \int_{\mathcal{A}} \tilde{n}^{\alpha\beta}\delta\varepsilon_{\alpha\beta} d\mathcal{A} + \cancel{\int_{\mathcal{A}} \tilde{q}^{\alpha}\delta\gamma_{\alpha} d\mathcal{A}}$

        For Kirchhoff plates

- Applied pressure (3)
  - Energy conservation
    - $\delta E_{\text{int}} = \int_{\mathcal{A}} \tilde{m}^{\alpha\beta} \delta\kappa_{\alpha\beta} d\mathcal{A}$
    - $\delta W_{\text{ext}} = \int_{\mathcal{A}} p\delta \boldsymbol{u}_3 d\mathcal{A} + \int_{\partial_N \mathcal{A}} \bar{T} \delta \boldsymbol{u}_3 - \int_{\partial_M \mathcal{A}} \bar{M} \delta \boldsymbol{u}_{3,\alpha} \nu_\alpha dl$
    - As $\kappa_{\alpha\beta} = -\boldsymbol{u}_{3,\alpha\beta}$

$$\Longrightarrow -\int_{\mathcal{A}} \tilde{m}^{\alpha\beta} \delta \boldsymbol{u}_{3,\alpha\beta} d\mathcal{A} = \int_{\mathcal{A}} p\delta \boldsymbol{u}_3 d\mathcal{A} + \int_{\partial_N \mathcal{A}} \bar{T} \delta \boldsymbol{u}_3 dl - \int_{\partial_M \mathcal{A}} \bar{M} \delta \boldsymbol{u}_{3,\alpha} \nu_\alpha dl$$

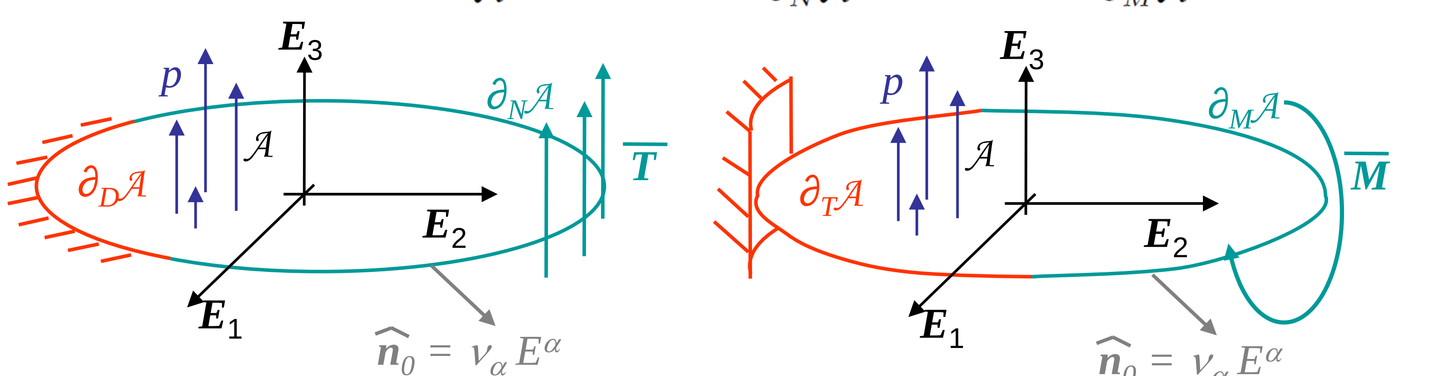
- Applied pressure (4)
  - Resulting bending equations

$$-\int_{\mathcal{A}} \tilde{m}^{\alpha\beta}\delta \boldsymbol{u}_{3,\alpha\beta}\, d\mathcal{A} = \int_{\mathcal{A}} p\,\delta\boldsymbol{u}_3\, d\mathcal{A} + \int_{\partial_N\mathcal{A}} \bar{T}\delta\boldsymbol{u}_3\, dl - \int_{\partial_M\mathcal{A}} \bar{M}\delta\boldsymbol{u}_{3,\alpha}\nu_\alpha\, dl$$

- Double integration by parts

$$-\int_{\partial\mathcal{A}} \tilde{m}^{\alpha\beta}\delta \boldsymbol{u}_{3,\alpha}\nu_\beta\, dl + \int_{\mathcal{A}} \tilde{m}^{\alpha\beta}_{,\beta}\delta \boldsymbol{u}_{3,\alpha}\, d\mathcal{A} = \int_{\mathcal{A}} p\,\delta\boldsymbol{u}_3\, d\mathcal{A} + \int_{\partial_N\mathcal{A}} \bar{T}\delta\boldsymbol{u}_3\, dl - \int_{\partial_M\mathcal{A}} \bar{M}\delta\boldsymbol{u}_{3,\alpha}\nu_\alpha\, dl$$

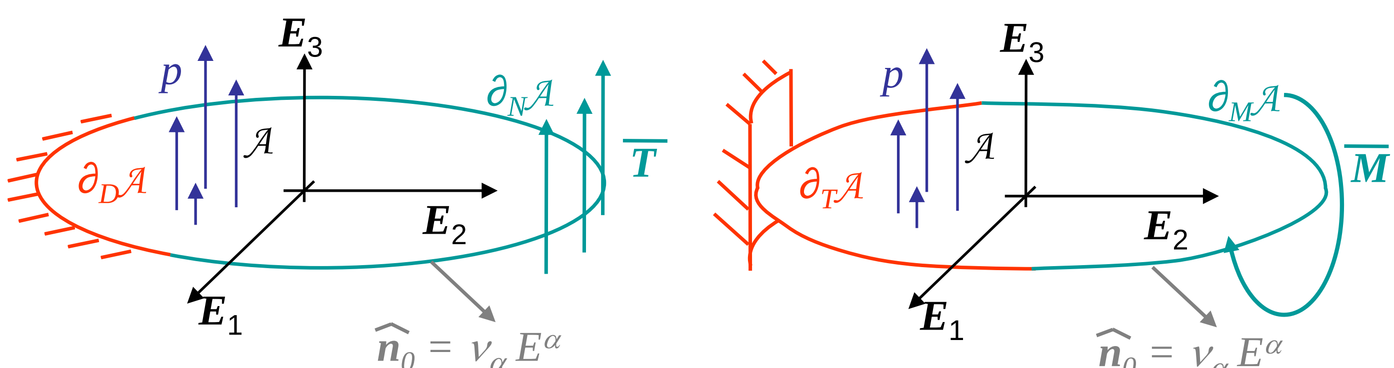
$$-\int_{\partial\mathcal{A}} \tilde{m}^{\alpha\beta}\delta \boldsymbol{u}_{3,\alpha}\nu_\beta\, dl + \int_{\partial\mathcal{A}} \tilde{m}^{\alpha\beta}_{,\beta}\delta \boldsymbol{u}_{3}\nu_\alpha\, dl - \int_{\mathcal{A}} \tilde{m}^{\alpha\beta}_{,\beta\alpha}\delta \boldsymbol{u}_{3}\, d\mathcal{A} = \int_{\mathcal{A}} p\,\delta\boldsymbol{u}_3\, d\mathcal{A} + \int_{\partial_N\mathcal{A}} \bar{T}\delta\boldsymbol{u}_3\, dl - \int_{\partial_M\mathcal{A}} \bar{M}\delta\boldsymbol{u}_{3,\alpha}\nu_\alpha\, dl$$



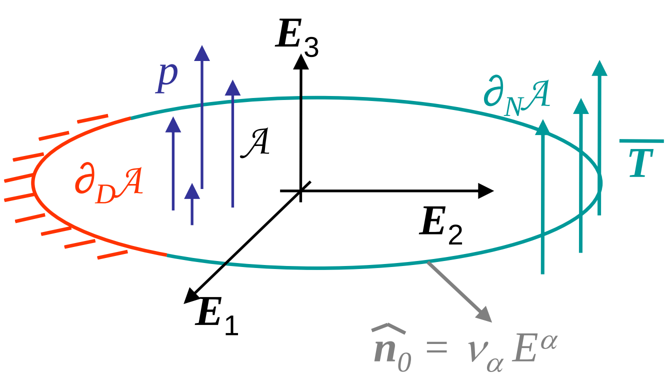
- Applied pressure (5)
  - Resulting bending equations (2)

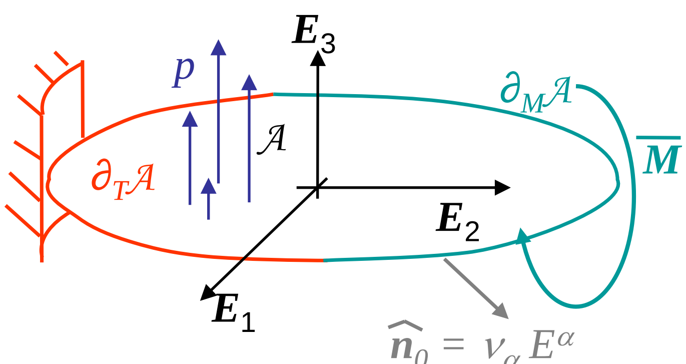
$$-\int_{\partial\mathcal{A}} \tilde{m}^{\alpha\beta}\delta \boldsymbol{u}_{3,\alpha}\nu_\beta dl + \int_{\partial\mathcal{A}} \tilde{m}^{\alpha\beta}_{,\beta}\delta\boldsymbol{u}_3\nu_\alpha dl - \int_{\mathcal{A}} \tilde{m}^{\alpha\beta}_{,\beta\alpha}\delta\boldsymbol{u}_3 d\mathcal{A} =$$
$$\int_{\mathcal{A}} p\delta\boldsymbol{u}_3 d\mathcal{A} + \int_{\partial_N\mathcal{A}} \bar{T}\delta\boldsymbol{u}_3 dl - \int_{\partial_M\mathcal{A}} \bar{M}\delta\boldsymbol{u}_{3,\alpha}\nu_\alpha dl$$

- Using $\tilde{m}^{\alpha\beta} = \mathcal{H}_m^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta}$, $\kappa_{\alpha\beta} = -\boldsymbol{u}_{3,\alpha\beta}$ and essential boundary conditions
  - On $\mathcal{A}$: $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\alpha\beta} = p$
  - On $\partial_N\mathcal{A}$: $-\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\beta}\nu_\alpha = \bar{T}$
  - On $\partial_M\mathcal{A}$: $-\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)\nu_\beta = \bar{M}\nu_\alpha$

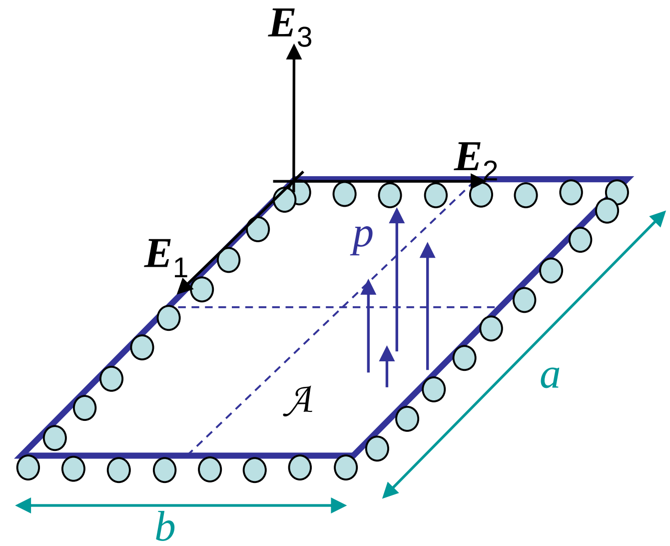


- Applied pressure (6)
  - Comparison with Euler-Bernoulli beams
    - Plates / Beams
      - On $\mathcal{A}$: $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\alpha\beta} = p$ $\quad\quad$ $\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f(x)$
      - On $\partial_N\mathcal{A}$: $-\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\beta}\nu_\alpha = \bar{T}$ $\quad\quad$ $-\frac{\partial}{\partial x}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\Big|_{0,\,L} = \bar{T}_z\big|_{0,\,L}$
      - On $\partial_M\mathcal{A}$: $-\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)\nu_\beta = \bar{M}\nu_\alpha$ $\quad\quad$ $-EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\Big|_{0,\,L} = \bar{M}_{xx}\big|_{0,\,L}$





- Example: simply supported plate



  – Equations
    - $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\alpha\beta} = p$

    with $\mathcal{H}_m^{\alpha\beta\gamma\delta} = \overbrace{\frac{h_0^3 E}{12\left(1-\nu^2\right)}}^{D}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta} + \frac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$

    - Bending couple can be rewritten

$$\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} = D\nu\delta^{\alpha\beta}\left(\boldsymbol{u}_{3,11} + \boldsymbol{u}_{3,22}\right) + D\left(1-\nu\right)\boldsymbol{u}_{3,\alpha\beta}$$

$$\Longrightarrow \left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\alpha\beta} = D\nu\left(\partial_{xx} + \partial_{yy}\right)\left(\boldsymbol{u}_{3,11} + \boldsymbol{u}_{3,22}\right) + D\left(1-\nu\right)\left(\boldsymbol{u}_{3,1111} + 2\boldsymbol{u}_{3,1122} + \boldsymbol{u}_{3,2222}\right)$$

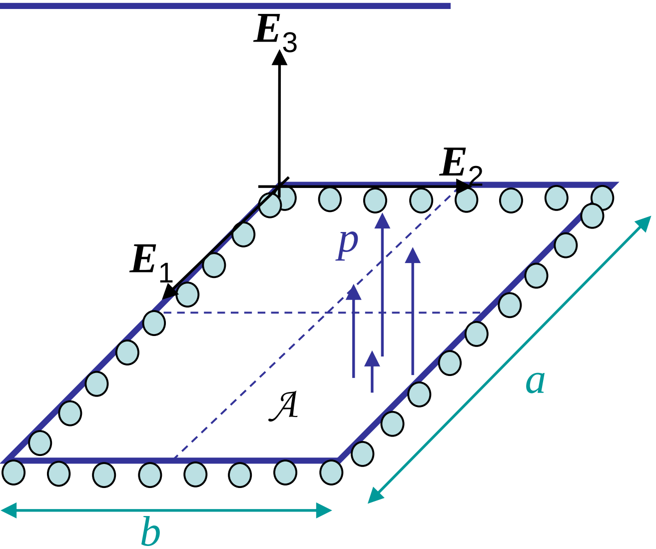
    - General equation: $D\left(\boldsymbol{u}_{3,1111} + 2\boldsymbol{u}_{3,1122} + \boldsymbol{u}_{3,2222}\right) = p\left(x,\, y\right)$

  – BCs for $x = 0$ & $x = a$
    - $\boldsymbol{u}_3 = 0$
    - No torque $\Longrightarrow \tilde{m}^{11} = -D\left(\boldsymbol{u}_{3,11} + \nu\boldsymbol{u}_{3,22}\right)$
      – As on these edges $\boldsymbol{u}_{3,22} = 0 \Longrightarrow \boldsymbol{u}_{3,11} = 0$

- Simply supported plate: Methodology
  - Using BCs for $x = 0, a$ & for $y = 0, b$
    - $\boldsymbol{u}_3 = 0$ $\boldsymbol{u}_3 = 0$
    - $\boldsymbol{u}_{3,11} = 0$ $\boldsymbol{u}_{3,22} = 0$
    - Solution can be represented as a Fourier series

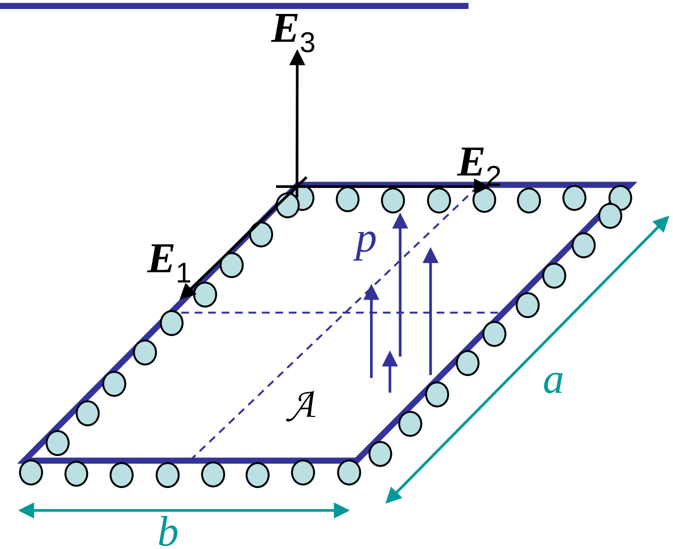
$$\boldsymbol{u}_3 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

    - Coefficients A_{mn} such that $D\left(\boldsymbol{u}_{3,1111} + 2\boldsymbol{u}_{3,1122} + \boldsymbol{u}_{3,2222}\right) = p\left(x,\, y\right)$



  - Solution for
    - $p(x, y)$?
    - $p(x,y) = p_0$?

- Simply supported plate: General solution
  - Applied pressure can also be expended as a Fourier series
    - $p(x,\,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$
    - Coefficient a_{mn} can be evaluated using
      - $\int_0^a \sin\frac{m\pi x}{a}\sin\frac{m'\pi x}{a}dx = \frac{a}{2}\delta_{mm'}$
      - $\int_0^b \sin\frac{n\pi x}{b}\sin\frac{n'\pi x}{b}dx = \frac{b}{2}\delta_{nn'}$

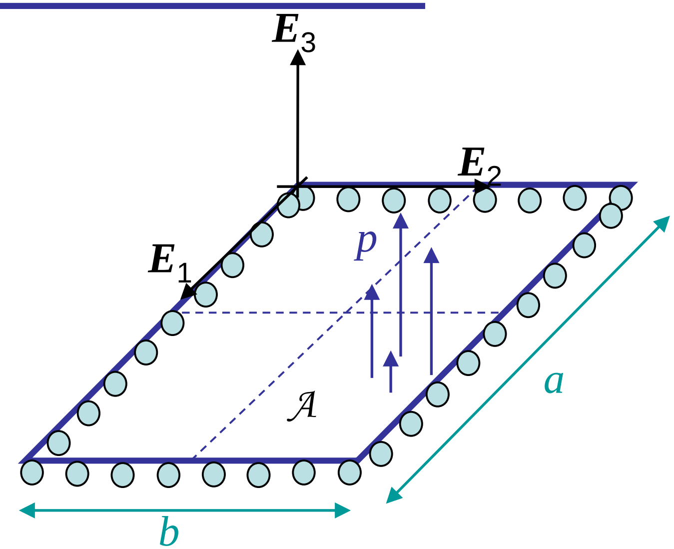


$$\Rightarrow \int_0^a\int_0^b p(x,\,y)\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}dxdy =$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^a\int_0^b a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}dxdy = \frac{ab}{4}a_{m'n'}$$

$$\Rightarrow a_{mn} = \frac{4}{ab}\int_0^a\int_0^b p(x,\,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$

- Simply supported plate: General solution (2)
  - Equations
    - $$\begin{cases} p\left(x,\, y\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ \boldsymbol{u}_3 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ D\left(\boldsymbol{u}_{3,1111} + 2\boldsymbol{u}_{3,1122} + \boldsymbol{u}_{3,2222}\right) = p\left(x,\, y\right) \end{cases}$$



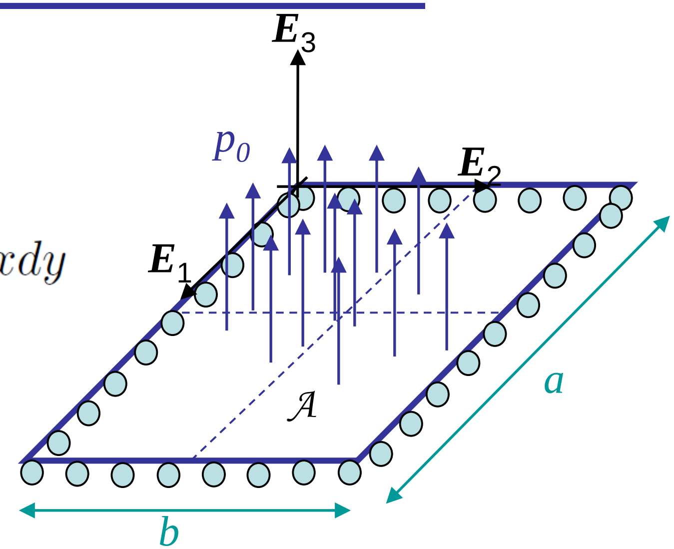
  - What remains to be defined are the A_{mn} (as a_{mn} are known)
    - Balance equation yields

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{A_{mn}\left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right] - \frac{a_{mn}}{D}\right\} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = 0$$

    - Should remain true for any x, y

$$\Longrightarrow A_{mn} = \frac{1}{\pi^4 D}\frac{a_{mn}}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2}$$

- Simply supported plate: Constant pressure p_0



  – Coefficients a_{mn}
    - $$a_{mn} = \frac{4}{ab}\int_0^a\int_0^b p(x,\, y) \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} dx dy$$

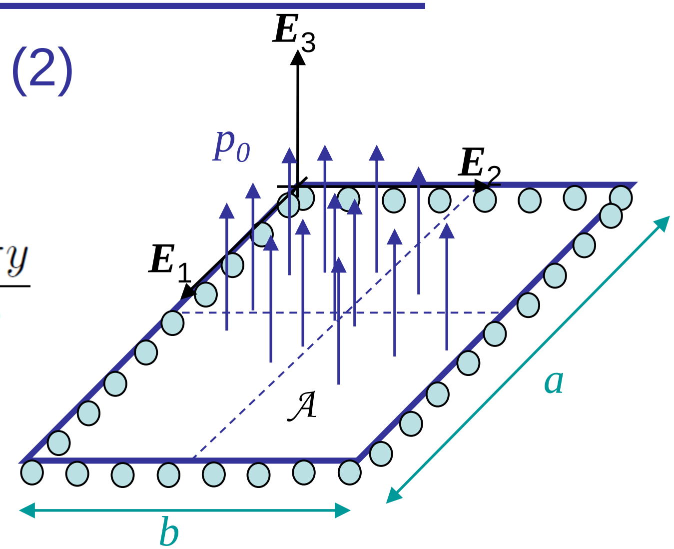
$$\Rightarrow\ a_{mn} = \frac{4p_0}{ab}\int_0^a \sin\frac{m\pi x}{a} dx \int_0^b \sin\frac{n\pi y}{b} dy$$

      – For m or n even: $a_{mn} = 0$
      – For m & n odd: $a_{mn} = \frac{16p_0}{\pi^2 mn}$

  – Coefficients A_{mn} for m & n odd:
    - $$A_{mn} = \frac{1}{\pi^4 D}\frac{a_{mn}}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} = \frac{16p_0}{D\pi^6 mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}$$

  – Solution of the problem
    - $$\boldsymbol{u}_3 = \sum_{m=1,\,3,\,5}^{\infty}\sum_{n=1,\,3,\,5}^{\infty}\frac{16p_0}{D\pi^6 mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

- Simply supported plate: Constant pressure p_0 (2)
  - Solution of the problem



$$\boldsymbol{u}_3 = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{16 p_0}{D\pi^6 mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

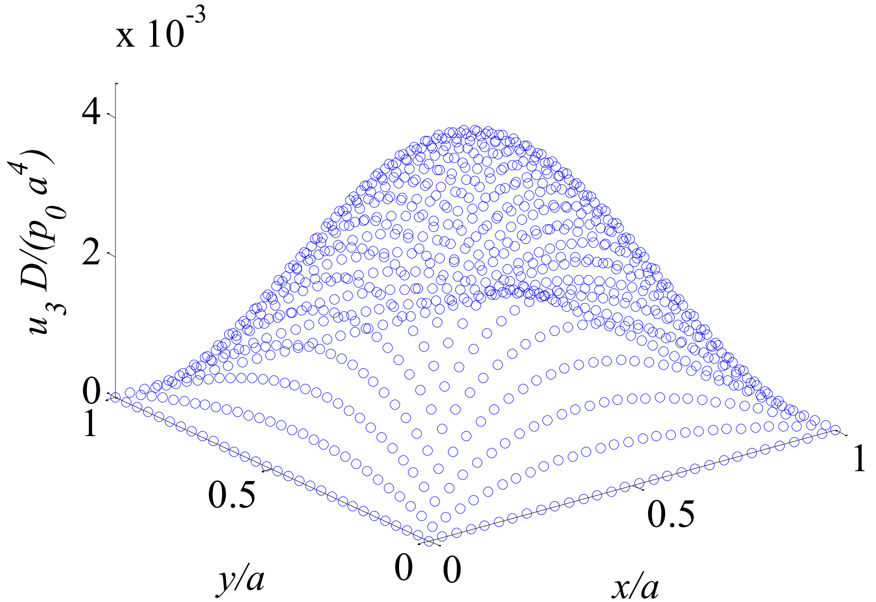
  - For a square plate

$$\boldsymbol{u}_3 = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{16 p_0 a^4}{D\pi^6 mn \left(m^2 + n^2\right)^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a}$$

$\frac{A_{mn} D}{p_0 a^4}$	$m=1$	$m=3$	$m=5$	$m=7$
$n=1$	$4.161\ 10^{-3}$	$0.055\ 10^{-3}$	$0.005\ 10^{-3}$	$0.001\ 10^{-3}$
$n=3$	Sym.	$0.006\ 10^{-3}$	$0.001\ 10^{-3}$	$0.002\ 10^{-4}$
$n=5$	Sym.	Sym.	$0.003\ 10^{-4}$	$0.001\ 10^{-4}$
$n=7$	Sym.	Sym.	Sym.	$0.004\ 10^{-5}$

- Simply supported plate: Constant pressure p_0 (3)
  - For a square plate (2)
    - $$\boldsymbol{u}_3 = \sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{16 p_0 a^4}{D\pi^6 mn\left(m^2+n^2\right)^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}$$
    - Maximum deflection at $x=y=a/2$

$$\boldsymbol{u}_3\left(\frac{a}{2},\frac{a}{2}\right) = \frac{p_0 a^4}{D}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{16}{\pi^6 mn\left(m^2+n^2\right)^2}\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}$$
$$\simeq 0.0042\frac{p_0 a^4}{D}$$



- Simply supported plate: Constant pressure p_0 (4)
  - Resultant stress couple
    - From $\boldsymbol{u}_3 = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{16 p_0}{D\pi^6 mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

      &
    - Bending $\tilde{m}^{\alpha\beta} = -\mathcal{H}_m^{\alpha\beta\gamma\delta} \boldsymbol{u}_{3,\gamma\delta} = -D\left[\nu\delta^{\alpha\beta}\left(\boldsymbol{u}_{3,11} + \boldsymbol{u}_{3,22}\right) + (1-\nu)\,\boldsymbol{u}_{3,\alpha\beta}\right]$
      - $\tilde{m}^{11} = -D\left(\boldsymbol{u}_{3,11} + \nu \boldsymbol{u}_{3,22}\right)$

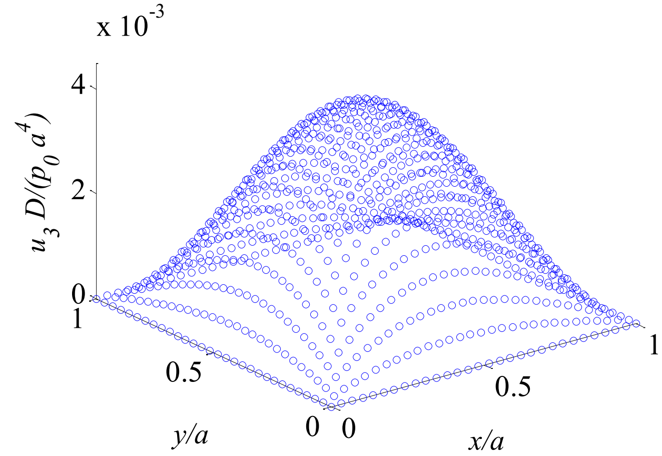
        $$\Longrightarrow \tilde{m}^{11} = \frac{16 p_0}{\pi^4} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\frac{m^2}{a^2} + \nu\frac{n^2}{b^2}}{mn\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$
      - $\tilde{m}^{22} = -D\left(\boldsymbol{u}_{3,22} + \nu \boldsymbol{u}_{3,11}\right)$

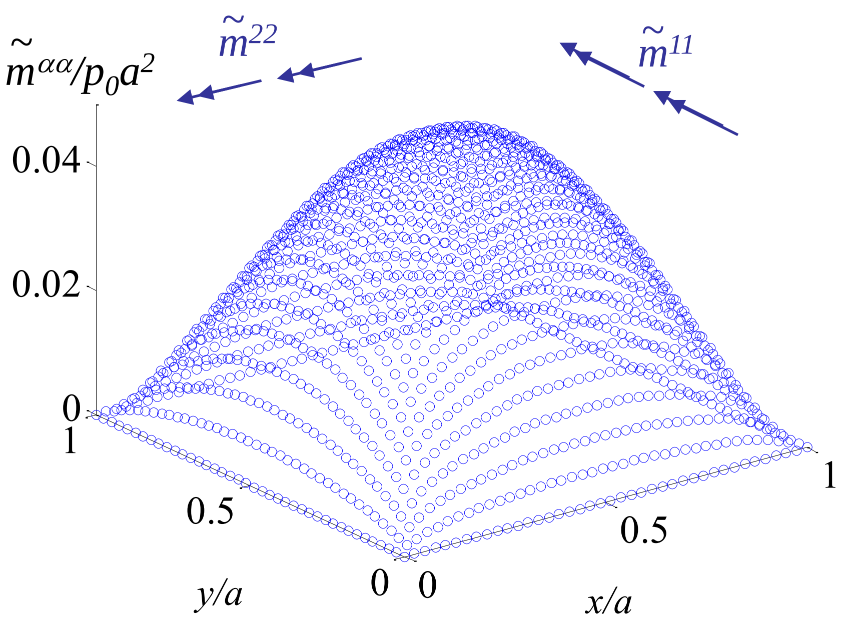
        $$\Longrightarrow \tilde{m}^{22} = \frac{16 p_0}{\pi^4} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\nu\frac{m^2}{a^2} + \frac{n^2}{b^2}}{mn\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$
    - Twisting
      - $\tilde{m}^{12} = -D(1-\nu)\,\boldsymbol{u}_{3,12}$

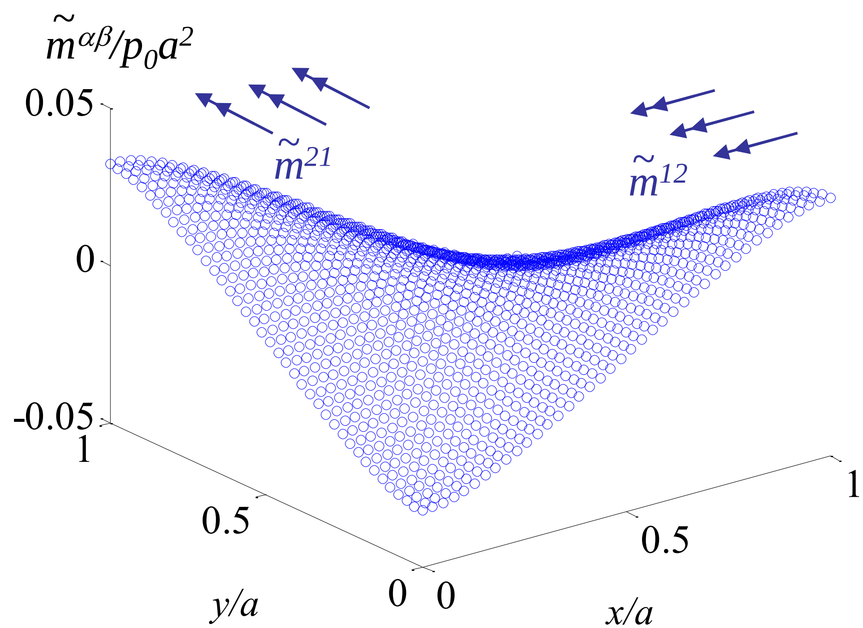
        $$\Longrightarrow \tilde{m}^{12} = -\frac{16 p_0 (1-\nu)}{\pi^4} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{1}{ab\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$$

- Simply supported plate: Constant pressure p_0 (5)
  - Resultant stress couple (2)
    - For a square plate and ν=0.3

$$\tilde{m}^{11}\left(\frac{a}{2},\frac{a}{2}\right)=\tilde{m}^{22}\left(\frac{a}{2},\frac{a}{2}\right)=\frac{16p_0a^2}{\pi^4}$$
$$\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{m^2+\nu n^2}{mn\left(m^2+n^2\right)^2}\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}$$
$$=0.0477p_0a^2$$





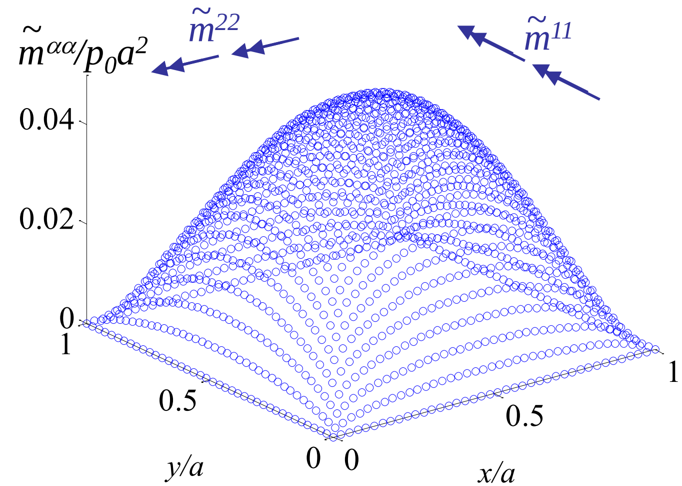


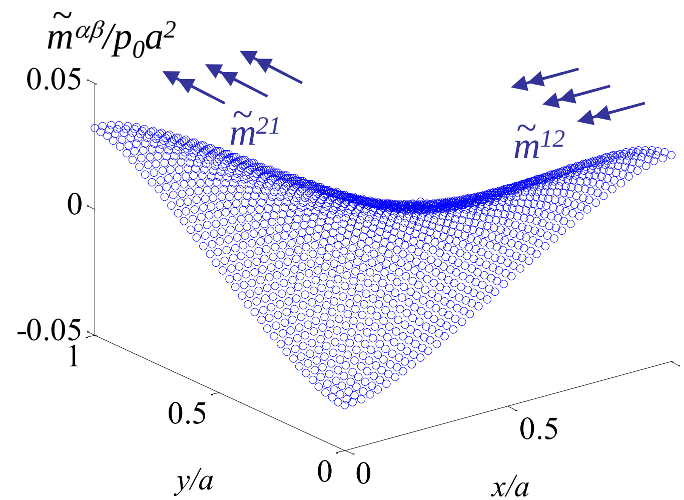
- Simply supported plate: Constant pressure p_0 (6)
  - Stresses
    - Using

$$\tilde{m}^{i\alpha} = \tilde{\boldsymbol{m}}_i^{\alpha} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}_{i\alpha} \xi^3 d\xi^3$$

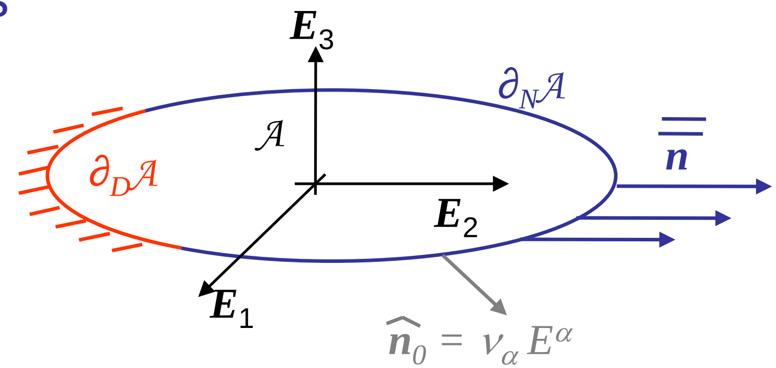
    - Direct stresses due to bending
      - $\boldsymbol{\sigma}_{11} = \dfrac{12\tilde{m}^{11}\xi^3}{h_0^3}$
      - $\boldsymbol{\sigma}_{22} = \dfrac{12\tilde{m}^{22}\xi^3}{h_0^3}$
    - Shear stress due to torsion
      - $\boldsymbol{\sigma}_{12} = \dfrac{12\tilde{m}^{12}\xi^3}{h_0^3}$
  - For a square plate (ν=0.3)

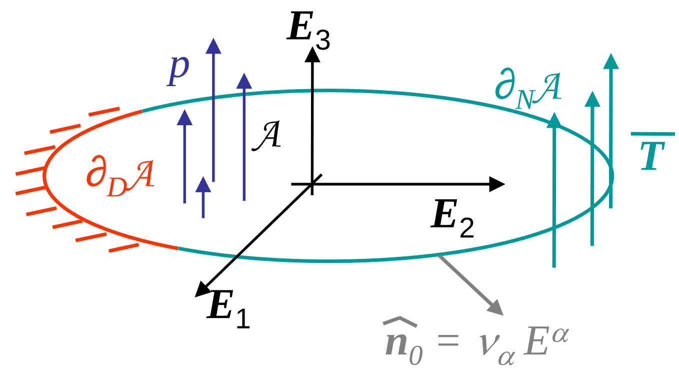
$$\boldsymbol{\sigma}_{11}\left(\frac{a}{2}, \frac{a}{2}, \frac{h_0}{2}\right) = \boldsymbol{\sigma}_{22}\left(\frac{a}{2}, \frac{a}{2}, \frac{h_0}{2}\right) = \frac{6\tilde{m}^{11}}{h_0^2} = 0.287\frac{p_0 a^2}{h_0^2}$$

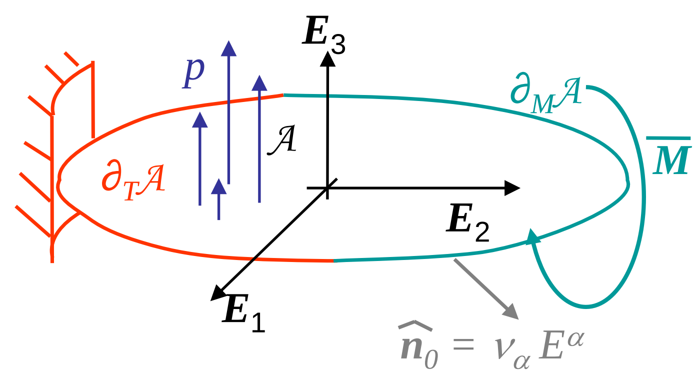




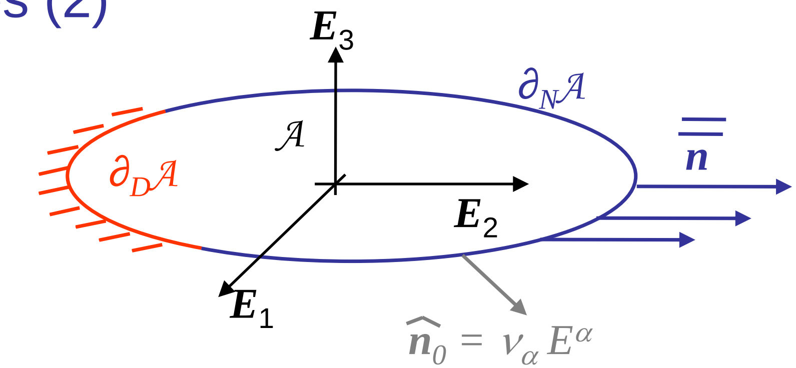
- Uncoupled theory of Kirchhoff-Love plates
  - We found for tension
    - $\mathcal{H}_n^{\alpha\beta\gamma\delta}\dfrac{\boldsymbol{u}_{\gamma,\delta\alpha}+\boldsymbol{u}_{\delta,\gamma\alpha}}{2}+\bar{\boldsymbol{n}}_\beta=\bar{\rho}\ddot{\boldsymbol{u}}_\beta$
    - With $\mathcal{H}_n^{\alpha\beta\gamma\delta}=\dfrac{h_0 E}{1-\nu^2}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta}+\dfrac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta}+\delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$
    - Completed by appropriate BCs
  - We found in bending
    - $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\alpha\beta}=p$
    - With $\mathcal{H}_m^{\alpha\beta\gamma\delta}=\dfrac{h_0^3 E}{12\left(1-\nu^2\right)}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta}+\dfrac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta}+\delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$
    - Completed by appropriate BCs
      - Low order
      - High order

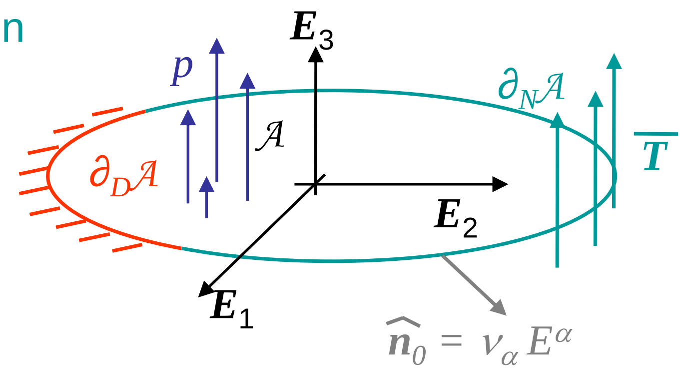


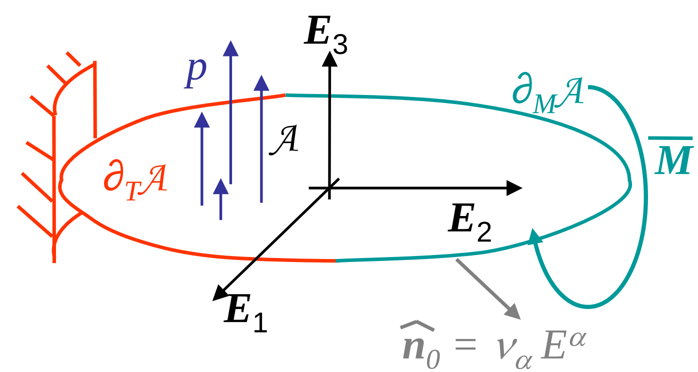




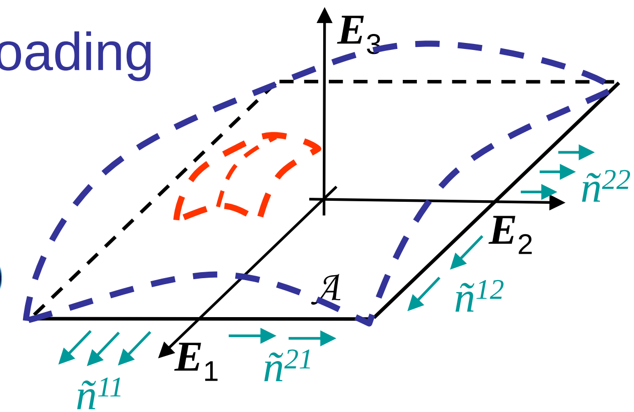
- Uncoupled theory of Kirchhoff-Love plates (2)
  - Uncoupled theory
    - $\mathcal{H}_n^{\alpha\beta\gamma\delta}\dfrac{\boldsymbol{u}_{\gamma,\delta\alpha}+\boldsymbol{u}_{\delta,\gamma\alpha}}{2}+\bar{\boldsymbol{n}}_\beta=\bar{\rho}\ddot{\boldsymbol{u}}_\beta$
    - $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\alpha\beta}=p$
  - Results from the fact that
    - Equilibrium is written on the initial configuration
      - Small deformations assumption
    - This initial configuration is planar
  - As equilibrium should be written
    - In deformed configuration, which has a curvature
    - Tension will affect bending
    - Second order theory







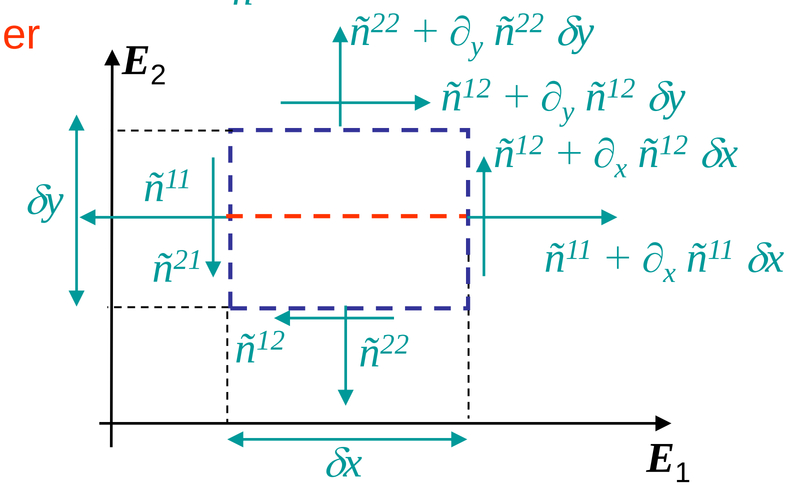
- Second order theory: Tension without external loading
  - Equilibrium in x direction
    - $\left(\tilde{n}^{11}+\partial_x \tilde{n}^{11}\delta x\right)\delta y \cos\left(\boldsymbol{u}_{3,1}+\boldsymbol{u}_{3,11}\delta x\right)-$
      $\tilde{n}^{11}\delta y \cos \boldsymbol{u}_{3,1}+\left(\tilde{n}^{12}+\partial_y \tilde{n}^{12}\delta y\right)\delta x-\tilde{n}^{12}\delta x=0$



  - Cosines lead to terms in $1+O(\boldsymbol{u}_{3,x})^2$ Lead to 3rd order
    - Second order theory becomes

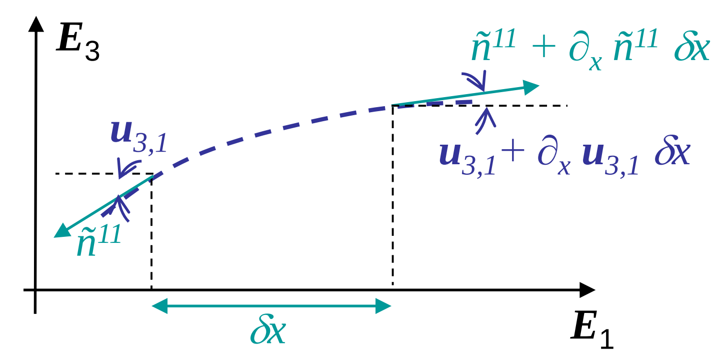
$$\left(\tilde{n}^{11}+\partial_x \tilde{n}^{11}\delta x\right)\delta y-\tilde{n}^{11}\delta y+$$
$$\left(\tilde{n}^{12}+\partial_y \tilde{n}^{12}\delta y\right)\delta x-\tilde{n}^{12}\delta x=0$$

$$\Longrightarrow \partial_x \tilde{n}^{11}+\partial_y \tilde{n}^{12}=0$$



    - Similarly

$$\partial_x \tilde{n}^{12}+\partial_y \tilde{n}^{22}=0$$



  - Resulting equations
    - $\tilde{n}^{\alpha\beta}_{,\alpha}=0$
    - As for first order theory

- Second order theory: Bending
  - Tension and shearing have a resultant loading in the z direction
  - Due to x tension

$$\left(\tilde{n}^{11} + \partial_x \tilde{n}^{11} \delta x\right) \delta y \sin\left(\boldsymbol{u}_{3,1} + \boldsymbol{u}_{3,11} \delta x\right) - \tilde{n}^{11} \delta y \sin \boldsymbol{u}_{3,1}$$

    - Second order theory: sin (x) $\Longrightarrow$ x

$$\left(\cancel{\tilde{n}}^{11} + \partial_x \tilde{n}^{11} \delta x\right) \delta y \boldsymbol{u}_{3,1} + \left(\tilde{n}^{11} + \cancel{\partial_x \tilde{n}^{11}} \delta x\right) \delta y \boldsymbol{u}_{3,11} \delta x - \tilde{n}^{11} \delta \cancel{y} \boldsymbol{u}_{3,1}$$

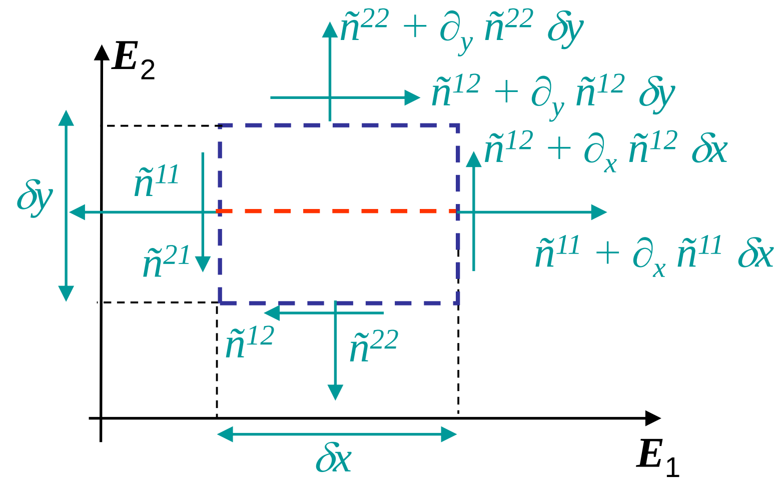
    - After simplification (and removing δ^3 terms)

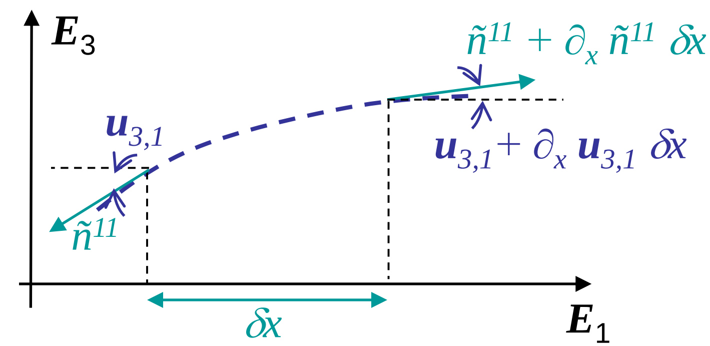
$$\left(\tilde{n}^{11} \boldsymbol{u}_{3,11} + \partial_x \tilde{n}^{11} \boldsymbol{u}_{3,1}\right) \delta x \delta y$$

  - Due to y tension
    - Similarly

$$\left(\tilde{n}^{22} \boldsymbol{u}_{3,22} + \partial_y \tilde{n}^{22} \boldsymbol{u}_{3,2}\right) \delta x \delta y$$

  - Due to yx shearing?





Tension effect on bending

- Second order theory: Bending (2)
  - Tension and shearing have a resultant loading in the z direction (2)
  - Due to yx shearing
    - $$\left(\tilde{n}^{21} + \partial_y \tilde{n}^{21} \delta y\right) \delta x \sin\left(\boldsymbol{u}_{3,1} + \boldsymbol{u}_{3,12} \delta y\right) - \tilde{n}^{21} \delta x \sin \boldsymbol{u}_{3,1}$$
    - Second order theory: sin (x) $\Longrightarrow$ x

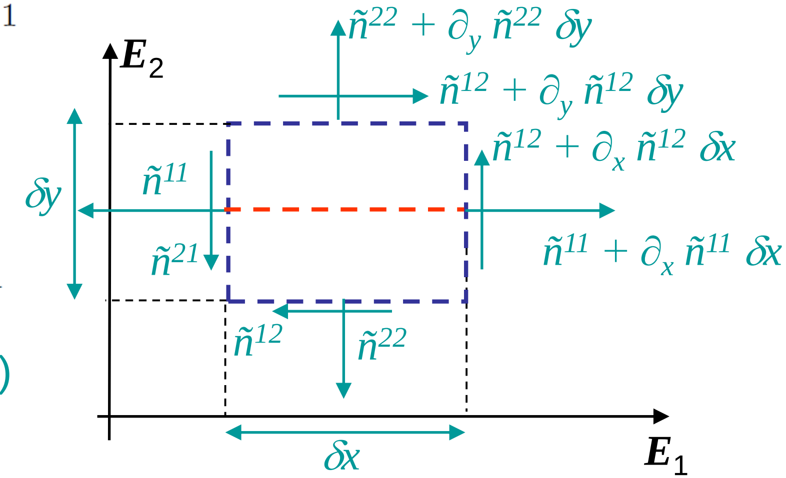
$$\left(\cancel{\tilde{n}^{21}} + \partial_y \tilde{n}^{21} \delta y\right) \delta x \boldsymbol{u}_{3,1} + \left(\tilde{n}^{21} + \cancel{\partial_y \tilde{n}^{21} \delta y}\right) \delta x \boldsymbol{u}_{3,12} \delta y - \cancel{\tilde{n}^{21} \delta x \boldsymbol{u}_{3,1}}$$

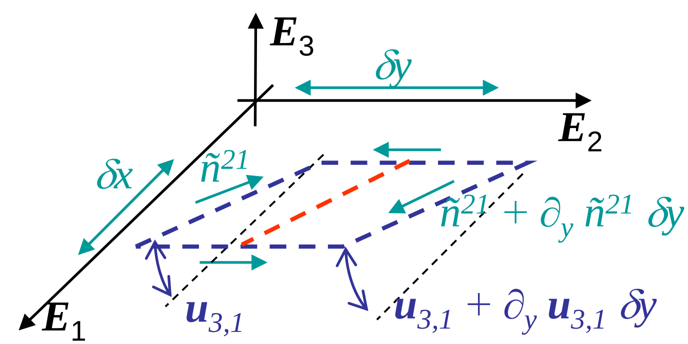
    - After simplification (and removing δ^3 terms)

$$\left(\partial_y \tilde{n}^{21} \boldsymbol{u}_{3,1} + \tilde{n}^{21} \boldsymbol{u}_{3,12}\right) \delta x \delta y$$

  - Due to xy shearing
    - Similarly

$$\left(\partial_x \tilde{n}^{12} \boldsymbol{u}_{3,2} + \tilde{n}^{12} \boldsymbol{u}_{3,21}\right) \delta x \delta y$$





- Second order theory: Bending (3)
  - Tension and shearing have a resultant loading in the z direction
    - Value on surface $\delta x \delta y$:

$$\left(\tilde{n}^{11}\boldsymbol{u}_{3,11} + \partial_x \tilde{n}^{11}\boldsymbol{u}_{3,1} + \tilde{n}^{22}\boldsymbol{u}_{3,22} + \partial_y \tilde{n}^{22}\boldsymbol{u}_{3,2} + \partial_y \tilde{n}^{21}\boldsymbol{u}_{3,1} + \tilde{n}^{21}\boldsymbol{u}_{3,12} + \partial_x \tilde{n}^{12}\boldsymbol{u}_{3,2} + \tilde{n}^{12}\boldsymbol{u}_{3,21}\right)\delta x \delta y$$

    - But tension equilibrium reads

$$\begin{cases} \partial_x \tilde{n}^{11} + \partial_y \tilde{n}^{12} = 0 \\ \partial_x \tilde{n}^{12} + \partial_y \tilde{n}^{22} = 0 \end{cases}$$

    - Final expression

$$\left(\tilde{n}^{11}\boldsymbol{u}_{3,11} + \tilde{n}^{22}\boldsymbol{u}_{3,22} + \tilde{n}^{21}\boldsymbol{u}_{3,12} + \tilde{n}^{12}\boldsymbol{u}_{3,21}\right)\delta x \delta y$$

$$\Longrightarrow \tilde{n}^{\alpha\beta}\boldsymbol{u}_{3,\alpha\beta}\delta x \delta y$$

    - Using tension equilibrium twice more

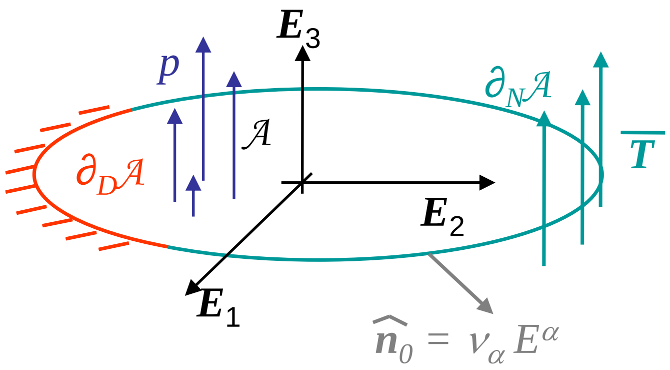
$$\tilde{n}^{\alpha\beta}\boldsymbol{u}_{3,\alpha\beta}\delta x \delta y = \left[\left(\tilde{n}^{\alpha\beta}\boldsymbol{u}_{3,\beta}\right)_{,\alpha} - \tilde{n}^{\alpha\beta}_{,\alpha}\boldsymbol{u}_{3,\beta}\right]\delta x \delta y = \left(\tilde{n}^{\alpha\beta}\boldsymbol{u}_{3,\beta}\right)_{,\alpha}\delta x \delta y$$

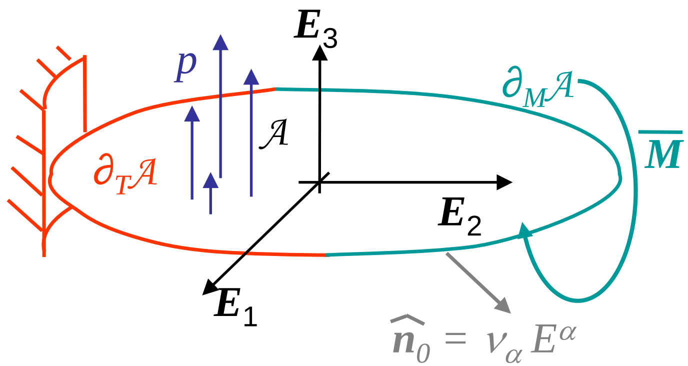
$$\Longrightarrow \tilde{n}^{\alpha\beta}\boldsymbol{u}_{3,\alpha\beta}\delta x \delta y = \left[\left(\tilde{n}^{\alpha\beta}\boldsymbol{u}_{3}\right)_{,\alpha\beta} - \left(\tilde{n}^{\alpha\beta}_{,\beta}\boldsymbol{u}_{3}\right)_{,\alpha}\right]\delta x \delta y = \left(\tilde{n}^{\alpha\beta}\boldsymbol{u}_{3}\right)_{,\alpha\beta}\delta x \delta y$$

- Second order theory: Bending (4)
  - First order theory
    - $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\alpha\beta} = p$
  - Second order theory
    - Tension & shearing lead to a force
      $\left(\tilde{n}^{\alpha\beta}\boldsymbol{u}_3\right)_{,\alpha\beta}\delta x\delta y$
      along z on a surface $\delta x\delta y$
    - This corresponds to a pressure
      $\left(\tilde{n}^{\alpha\beta}\boldsymbol{u}_3\right)_{,\alpha\beta}$
    - New resulting equation

$$\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\alpha\beta} = p + \left(\tilde{n}^{\alpha\beta}\boldsymbol{u}_3\right)_{,\alpha\beta}$$

$$\Longrightarrow \left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} - \tilde{n}^{\alpha\beta}\boldsymbol{u}_3\right)_{,\alpha\beta} = p$$





- Second order theory: Energy
  - Equation

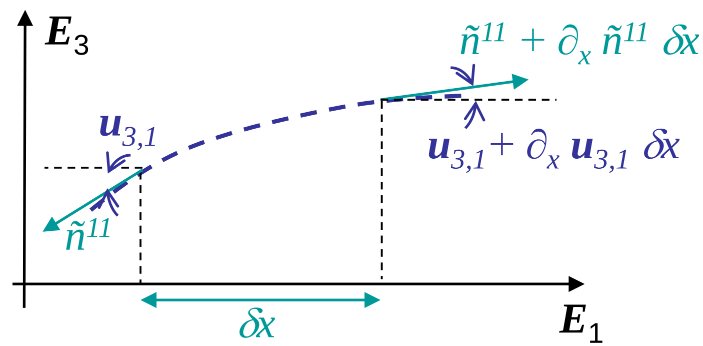
$$\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} - \tilde{n}^{\alpha\beta}\boldsymbol{u}_3\right)_{,\alpha\beta} = p$$

  - Bending energy
    - We saw that $\delta E_{\text{int}} = \int_{\mathcal{A}} \tilde{m}^{\alpha\beta}\delta\kappa_{\alpha\beta}d\mathcal{A}$ with $\begin{cases}\kappa_{\alpha\beta} = -\boldsymbol{u}_{3,\alpha\beta} \\ \tilde{m}^{\alpha\beta} = \mathcal{H}_m^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta}\end{cases}$
    - As $\tilde{m}^{\alpha\beta} = -\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} = -D\left[\nu\delta^{\alpha\beta}\left(\boldsymbol{u}_{3,11} + \boldsymbol{u}_{3,22}\right) + (1-\nu)\,\boldsymbol{u}_{3,\alpha\beta}\right]$

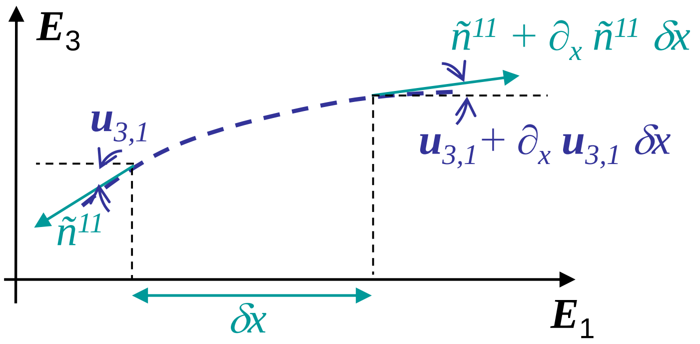
$$\Longrightarrow E_{\text{int, bending}} = \int_{\mathcal{A}} \frac{D}{2}\left[\nu\left(\boldsymbol{u}_{3,11} + \boldsymbol{u}_{3,22}\right)^2 + (1-\nu)\left(\boldsymbol{u}_{3,11}^2 + 2\boldsymbol{u}_{3,12}^2 + \boldsymbol{u}_{3,22}^2\right)\right]d\mathcal{A}$$

$$\Longrightarrow E_{\text{int, bending}} = \int_{\mathcal{A}} \frac{D}{2}\left[\boldsymbol{u}_{3,11}^2 + \boldsymbol{u}_{3,22}^2 + 2(1-\nu)\,\boldsymbol{u}_{3,12}^2 + 2\nu\boldsymbol{u}_{3,11}\boldsymbol{u}_{3,22}\right]d\mathcal{A}$$

    - But part of the bending results from tension



- Second order theory: Energy (2)
  - Bending energy (2)
    - Part of the bending resulting from tension
    - This corresponds to a pressure $\left(\tilde{n}^{\alpha\beta}\boldsymbol{u}_3\right)_{,\alpha\beta}$



$$\Rightarrow \delta W_{\text{ext, traction}} = \int_{\mathcal{A}} \tilde{n}^{\alpha\beta}\boldsymbol{u}_{3,\alpha\beta}\,\delta\boldsymbol{u}_3 d\mathcal{A} = \int_{\partial\mathcal{A}} \tilde{n}^{\alpha\beta}\boldsymbol{u}_{3,\alpha}\,\delta\boldsymbol{u}_3 \nu_\beta dl - \int_{\mathcal{A}} \tilde{n}^{\alpha\beta}\boldsymbol{u}_{3,\alpha}\,\delta\boldsymbol{u}_{3,\beta} d\mathcal{A}$$

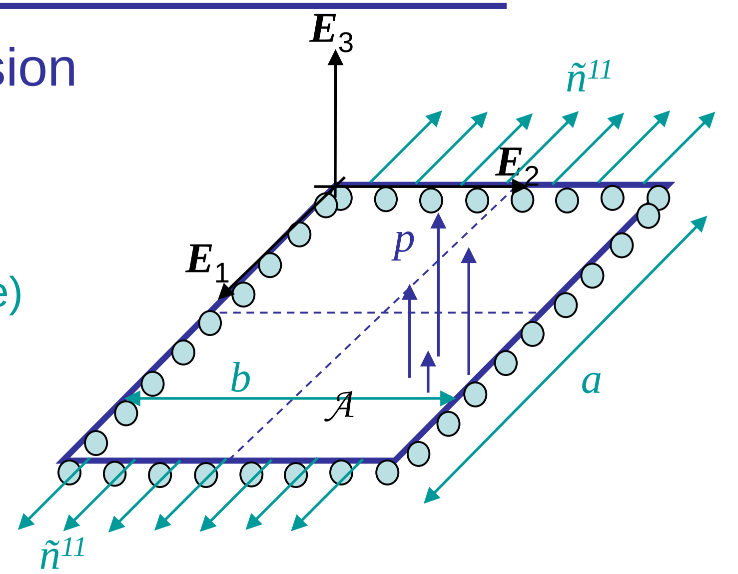
- As $\begin{cases} \partial_x \tilde{n}^{11} + \partial_y \tilde{n}^{12} = 0 \\ \partial_x \tilde{n}^{12} + \partial_y \tilde{n}^{22} = 0 \end{cases}$

  - Assuming BCs are such that there is no contribution to energy ($\boldsymbol{u}_3$=0 or $\boldsymbol{u}_{3,\alpha}$ = 0), and if $\tilde{n}$ is constant with $\boldsymbol{u}_3$,

$$E_{\text{int, traction}} = \frac{1}{2}\int_{\mathcal{A}} \left[\tilde{n}^{11}\boldsymbol{u}_{3,1}{}^2 + \tilde{n}^{22}\boldsymbol{u}_{3,2}{}^2 + 2\tilde{n}^{12}\boldsymbol{u}_{3,1}\boldsymbol{u}_{3,2}\right] d\mathcal{A}$$

  - Total bending energy

$$\Rightarrow E_{\text{int}} = \int_{\mathcal{A}} \frac{D}{2}\left[\boldsymbol{u}_{3,11}^2 + \boldsymbol{u}_{3,22}^2 + 2(1-\nu)\boldsymbol{u}_{3,12}^2 + 2\nu\boldsymbol{u}_{3,11}\boldsymbol{u}_{3,22}\right] d\mathcal{A} + \frac{1}{2}\int_{\mathcal{A}} \left[\tilde{n}^{11}\boldsymbol{u}_{3,1}{}^2 + \tilde{n}^{22}\boldsymbol{u}_{3,2}{}^2 + 2\tilde{n}^{12}\boldsymbol{u}_{3,1}\boldsymbol{u}_{3,2}\right] d\mathcal{A}$$

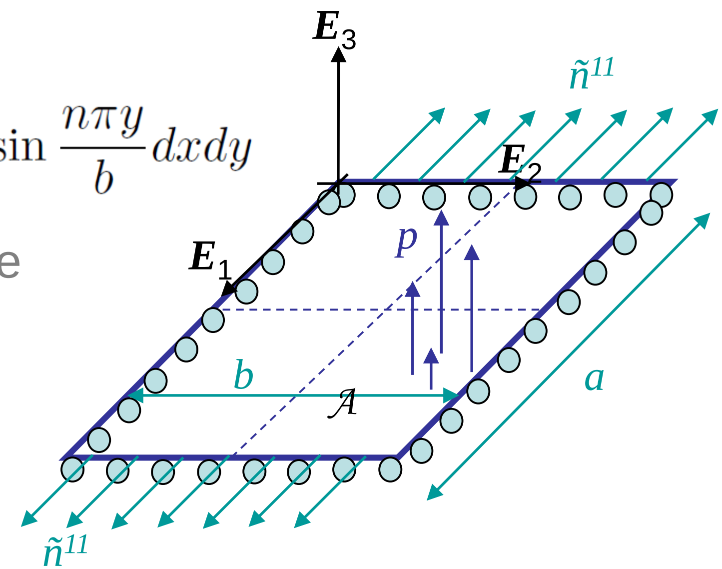


- Example: simply supported plate with tension
  - BCs for $x = 0, a$ & for $y = 0, b$
    - $\boldsymbol{u}_3 = 0$ $\boldsymbol{u}_3 = 0$
    - $\boldsymbol{u}_{3,11} = 0$ $\boldsymbol{u}_{3,22} = 0$ (no torque)
  - Constant tension along Ox
  - Equations
    - $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} - \tilde{n}^{\alpha\beta}\boldsymbol{u}_3\right)_{,\alpha\beta} = p$
    - Can be developed using
      - $\tilde{m}^{\alpha\beta} = -\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} = -D\left[\nu\delta^{\alpha\beta}\left(\boldsymbol{u}_{3,11} + \boldsymbol{u}_{3,22}\right) + \left(1-\nu\right)\boldsymbol{u}_{3,\alpha\beta}\right]$

        with $D = \dfrac{h_0^3 E}{12\left(1-\nu^2\right)}$
      - $\left(\tilde{n}^{\alpha\beta}\boldsymbol{u}_3\right)_{,\alpha\beta} = \tilde{n}^{11}\boldsymbol{u}_{3,11}$
    - Final equation
      - $D\left(\boldsymbol{u}_{3,1111} + 2\boldsymbol{u}_{3,1122} + \boldsymbol{u}_{3,2222}\right) - \tilde{n}^{11}\boldsymbol{u}_{3,11} = p$
  - Solution for
    - $p(x,y) = p_0$?

Tension effect on bending

- Simply supported plate with tension & constant pressure p_0
  - Pressure can be expended in a Fourier series
    - $$p(x,\, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$
    - With $a_{mn} = \frac{4}{ab} \int_0^a \int_0^b p(x,\, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy$
  - Coefficients a_{mn} in case of constant pressure
    - For m or n even: $a_{mn} = 0$
    - For m & n odd: $a_{mn} = \frac{16 p_0}{\pi^2 mn}$



$$\Rightarrow \quad p(x,\, y) = \frac{16 p_0}{\pi^2} \sum_{m=1,\,3,\,5}^{\infty} \sum_{n=1,\,3,\,5}^{\infty} \frac{1}{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

  - Final equation to be solved

$$\boldsymbol{u}_{3,1111} + 2\boldsymbol{u}_{3,1122} + \boldsymbol{u}_{3,2222} - \frac{\tilde{n}^{11}}{D} \boldsymbol{u}_{3,11} = \frac{16 p_0}{D\pi^2} \sum_{m=1,\,3,\,5}^{\infty} \sum_{n=1,\,3,\,5}^{\infty} \frac{1}{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

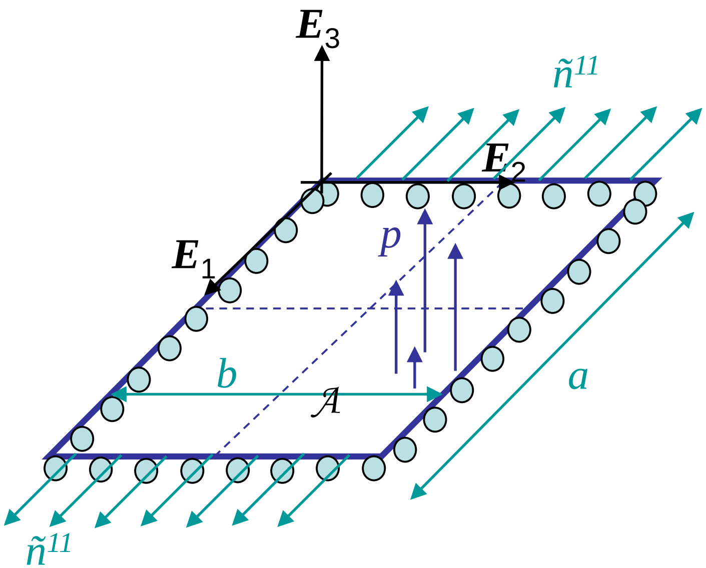
- Simply supported plate with tension & constant pressure p_0 (2)
  - Governing equation to be solved

$$\boldsymbol{u}_{3,1111}+2\boldsymbol{u}_{3,1122}+\boldsymbol{u}_{3,2222}-\frac{\tilde{n}^{11}}{D}\boldsymbol{u}_{3,11}=\frac{16p_0}{D\pi^2}\sum_{m=1,\,3,\,5}^{\infty}\sum_{n=1,\,3,\,5}^{\infty}\frac{1}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

  - Considering BCs

for $x = 0, a$ & for $y = 0, b$

$$\begin{cases}\boldsymbol{u}_3=0\\ \boldsymbol{u}_{3,11}=0\end{cases}\qquad\begin{cases}\boldsymbol{u}_3=0\\ \boldsymbol{u}_{3,22}=0\end{cases}$$



The solution can take the form

- $$\boldsymbol{u}_3=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$
- The governing equation leads to a system of equations with A_{mn} as unknowns

$$\Rightarrow \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\left[\left(\frac{m^4\pi^4}{a^4}+2\frac{m^2n^2\pi^4}{a^2b^2}+\frac{n^4\pi^4}{b^4}+\frac{\tilde{n}^{11}m^2\pi^2}{Da^2}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right]=$$
$$\frac{16p_0}{D\pi^2}\sum_{m=1,\,3,\,5}^{\infty}\sum_{n=1,\,3,\,5}^{\infty}\frac{1}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

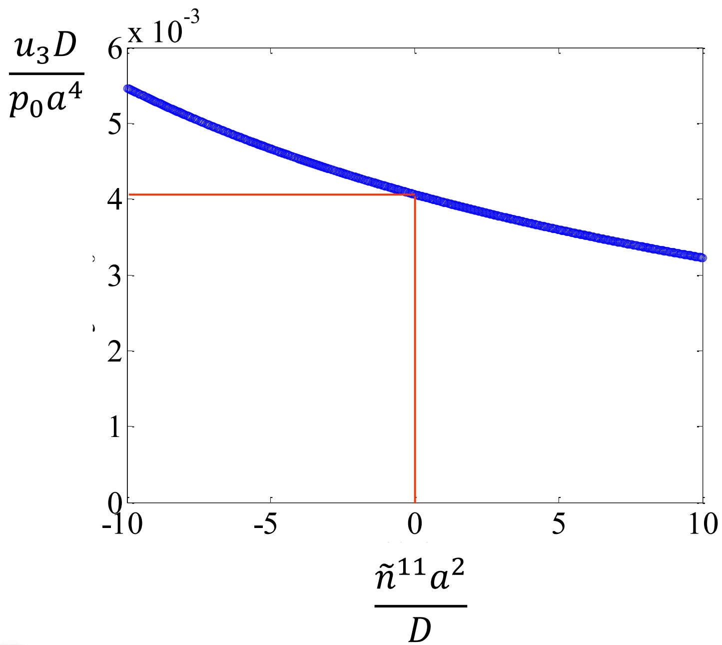
- Simply supported plate with tension & constant pressure p_0 (3)
  - System of equations
    - $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left[\left(\frac{m^4\pi^4}{a^4}+2\frac{m^2n^2\pi^4}{a^2b^2}+\frac{n^4\pi^4}{b^4}+\frac{\tilde{n}^{11}m^2\pi^2}{Da^2}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right]=$
      $\frac{16p_0}{D\pi^2}\sum_{m=1,\,3,\,5}^{\infty}\sum_{n=1,\,3,\,5}^{\infty}\frac{1}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$
    - By identification
      - For m or n even: $A_{mn} = 0$
      - For m & n odd: $A_{mn}=\frac{16p_0}{D\pi^6 mn}\frac{1}{\left[\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+\frac{\tilde{n}^{11}m^2}{Da^2\pi^2}\right]}$
  - With the solution stated as
    - $\boldsymbol{u}_3=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

$$\Rightarrow \boldsymbol{u}_3=\sum_{m=1,\,3,\,5}^{\infty}\sum_{n=1,\,3,\,5}^{\infty}\frac{16p_0}{D\pi^6 mn\left[\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+\frac{\tilde{n}^{11}m^2}{Da^2\pi^2}\right]}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

Tension effect on bending

- Simply supported plate with tension & constant pressure p_0 (4)
  - Solution
    - $$\boldsymbol{u}_3 = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{16 p_0}{D\pi^6 mn \left[\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 + \frac{\tilde{n}^{11} m^2}{D a^2 \pi^2}\right]} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$
  - For a square plate $a=b$

$$\boldsymbol{u}_3\left(\frac{a}{2}, \frac{a}{2}\right) = \frac{p_0 a^4}{D} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{16}{\pi^6 mn \left[\left(m^2 + n^2\right)^2 + \frac{\tilde{n}^{11} m^2 a^2}{D\pi^2}\right]} \sin\frac{m\pi}{2} \sin\frac{n\pi}{2}$$

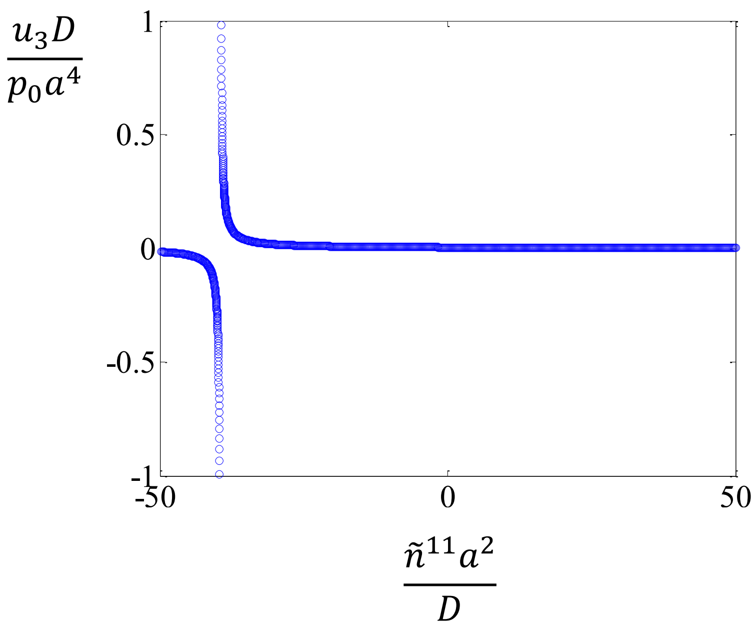


Tension increases the bending stiffness

- Simply supported plate with tension & constant pressure p_0 (5)
  - For a square plate $a=b$ (2)
    - Central displacement:

$$\boldsymbol{u}_3\left(\frac{a}{2},\,\frac{a}{2}\right)=\frac{p_0 a^4}{D}\sum_{m=1,\,3,\,5}^{\infty}\sum_{n=1,\,3,\,5}^{\infty}\frac{16}{\pi^6 mn\left[\left(m^2+n^2\right)^2+\frac{\tilde{n}^{11}m^2a^2}{D\pi^2}\right]}\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}$$

    - If we increase the compression range



Instability in compression

See next lecture

- Simply supported plate with tension & arbitrary pressure p
  - For constant pressure we found
    - $$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left[\left(\frac{m^4\pi^4}{a^4}+2\frac{m^2n^2\pi^4}{a^2b^2}+\frac{n^4\pi^4}{b^4}+\frac{\tilde{n}^{11}m^2\pi^2}{Da^2}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right]= \frac{16p_0}{D\pi^2}\sum_{m=1,\,3,\,5}^{\infty}\sum_{n=1,\,3,\,5}^{\infty}\frac{1}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$
  - For an arbitrary pressure we cannot particularize the a_{mn}
    - $$p(x,\,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

      with $a_{mn}=\frac{4}{ab}\int_0^a\int_0^b p(x,\,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi x}{b}dxdy$
  - General solution
    - From $\boldsymbol{u}_3=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$
    - Equation $D\left(\boldsymbol{u}_{3,1111}+2\boldsymbol{u}_{3,1122}+\boldsymbol{u}_{3,2222}\right)-\tilde{n}^{11}\boldsymbol{u}_{3,11}=p$ leads to

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left[\left(\frac{m^4\pi^4}{a^4}+2\frac{m^2n^2\pi^4}{a^2b^2}+\frac{n^4\pi^4}{b^4}+\frac{\tilde{n}^{11}m^2\pi^2}{Da^2}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right]= \frac{1}{D}\sum_{m}^{\infty}\sum_{n}^{\infty} a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\Longrightarrow A_{mn}=\frac{a_{mn}}{\pi^4 D\left[\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+\frac{\tilde{n}^{11}m^2}{\pi^2Da^2}\right]}$$

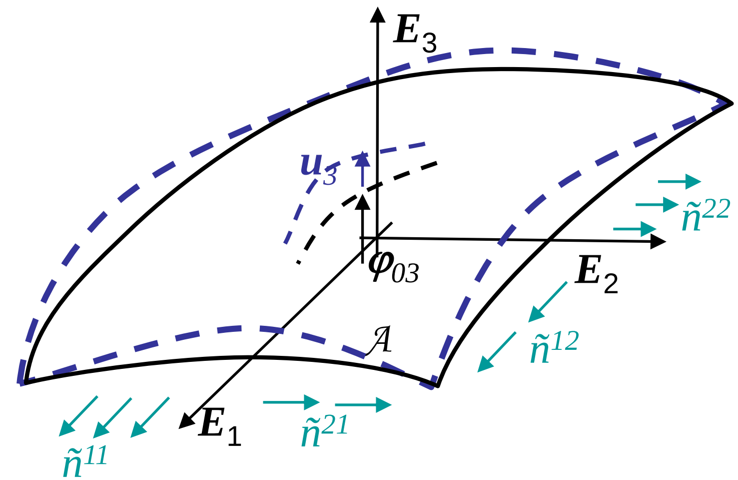
- Plate with initial curvature
  - This corresponds to
    - An initial mapping along $\boldsymbol{E}_3$: $\boldsymbol{\varphi}_{03}$
    - Deflection $\boldsymbol{u}_3$ is measured from this mapping
    - Final mapping along $\boldsymbol{E}_3$: $\boldsymbol{\varphi}_3 = \boldsymbol{\varphi}_{03} + \boldsymbol{u}_3$
  - Assumption
    - Initial mapping $\boldsymbol{\varphi}_{03}$ is small compared to the shell thickness

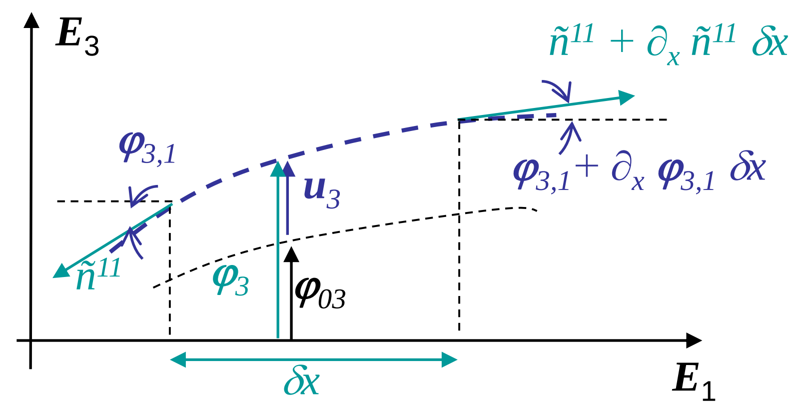
      $\Longrightarrow$ momentum is still computed from $\boldsymbol{u}_{3,\alpha\beta}$

      $\Longrightarrow$ $\tilde{m}^{\alpha\beta} = -\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}$
  - There is a tension resultant along $\boldsymbol{E}_3$
    - Previous analysis

      $$\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} - \tilde{n}^{\alpha\beta}\boldsymbol{u}_3\right)_{,\alpha\beta} = p$$
    - But here this effect is proportional to $\boldsymbol{\varphi}_{3,\alpha\beta}$

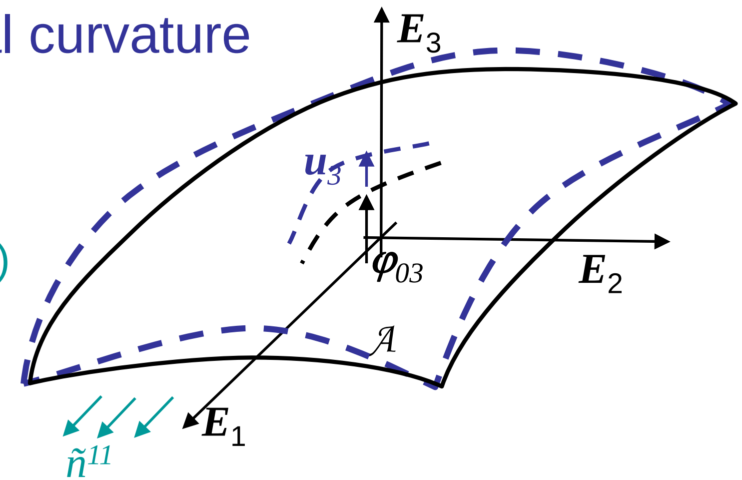
$$\Longrightarrow \left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} - \tilde{n}^{\alpha\beta}\boldsymbol{u}_3 - \tilde{n}^{\alpha\beta}\boldsymbol{\varphi}_{03}\right)_{,\alpha\beta} = p$$



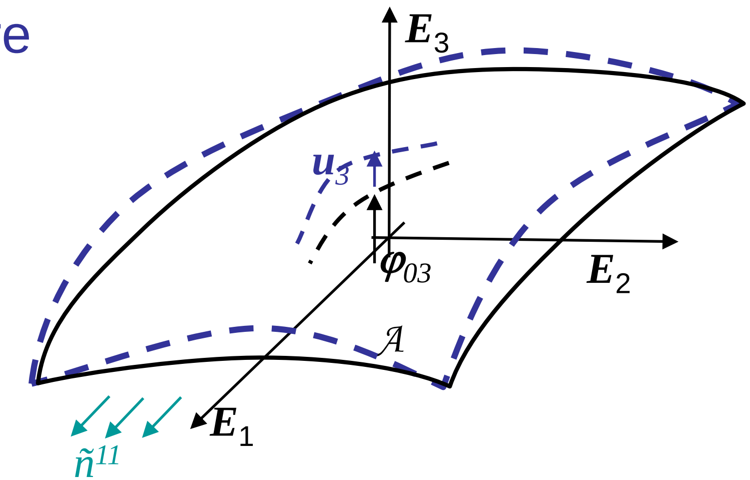


- Example: simply supported plate with initial curvature
  - BCs for $x = 0, a$ & for $y = 0, b$
    - $\boldsymbol{u}_3 = 0$ $\quad\quad$ $\boldsymbol{u}_3 = 0$
    - $\boldsymbol{u}_{3,11} = 0$ $\quad\quad$ $\boldsymbol{u}_{3,22} = 0$ (no torque)
  - Constant tension along Ox
  - Equations
    - $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} - \tilde{n}^{\alpha\beta}\boldsymbol{u}_3 - \tilde{n}^{\alpha\beta}\boldsymbol{\varphi}_{03}\right)_{,\alpha\beta} = p$
    - Can be developed using
      - $\tilde{m}^{\alpha\beta} = -\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} = -D\left[\nu\delta^{\alpha\beta}\left(\boldsymbol{u}_{3,11} + \boldsymbol{u}_{3,22}\right) + (1-\nu)\,\boldsymbol{u}_{3,\alpha\beta}\right]$

        with $D = \dfrac{h_0^3 E}{12\left(1-\nu^2\right)}$
      - $\left(\tilde{n}^{\alpha\beta}\boldsymbol{u}_3\right)_{,\alpha\beta} = \tilde{n}^{11}\boldsymbol{u}_{3,11}$ & $\left(\tilde{n}^{\alpha\beta}\boldsymbol{\varphi}_{03}\right)_{,\alpha\beta} = \tilde{n}^{11}\boldsymbol{\varphi}_{03,11}$
    - Final equation
      - $D\left(\boldsymbol{u}_{3,1111} + 2\boldsymbol{u}_{3,1122} + \boldsymbol{u}_{3,2222}\right) - \tilde{n}^{11}\boldsymbol{u}_{3,11} = p + \tilde{n}^{11}\boldsymbol{\varphi}_{03,11}$
  - Solution for
    - $p(x,y) = 0$?



- Simply supported plate with initial curvature
  - We can expend the initial mapping in a Fourier series



$$\varphi_{03}(x,\, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\Longrightarrow \quad \tilde{n}^{11}\varphi_{03,11}(x,\, y) = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2\pi^2\tilde{n}^{11}}{a^2} b_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

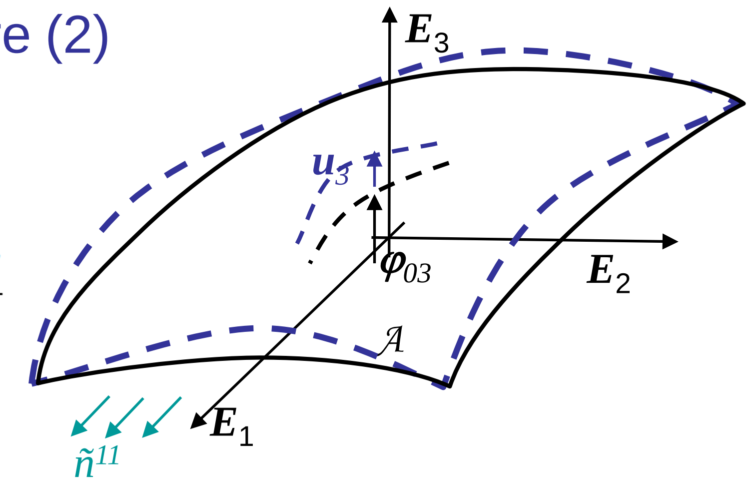
  - Considering BCs: for $x = 0, a$ & for $y = 0, b$

$$\begin{cases} \boldsymbol{u}_3 = 0 \\ \boldsymbol{u}_{3,11} = 0 \end{cases} \qquad \begin{cases} \boldsymbol{u}_3 = 0 \\ \boldsymbol{u}_{3,22} = 0 \end{cases}$$

The solution can take the form

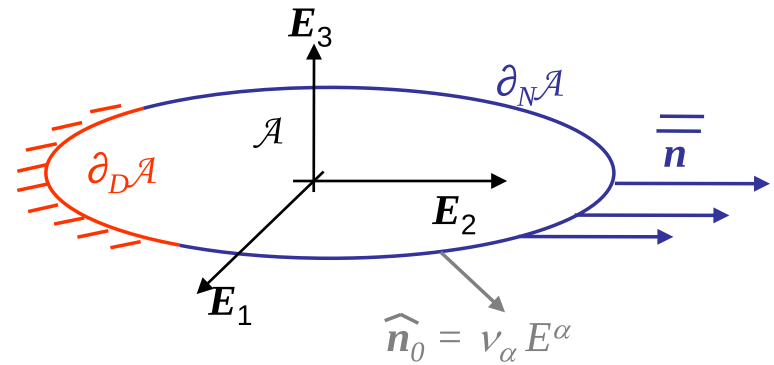
- $$\boldsymbol{u}_3 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

- Simply supported plate with initial curvature (2)
  - As we have
    - $\tilde{n}^{11}\boldsymbol{\varphi}_{03,11}(x,\, y) =$
      $$-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2\pi^2\tilde{n}^{11}}{a^2}b_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$
    - $$\boldsymbol{u}_3 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$
  - The governing equation
    $$D\left(\boldsymbol{u}_{3,1111} + 2\boldsymbol{u}_{3,1122} + \boldsymbol{u}_{3,2222}\right) - \tilde{n}^{11}\boldsymbol{u}_{3,11} = p + \tilde{n}^{11}\boldsymbol{\varphi}_{03,11}$$
    becomes
    $$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\left[\left(\frac{m^4\pi^4}{a^4} + 2\frac{m^2n^2\pi^4}{a^2b^2} + \frac{n^4\pi^4}{b^4} + \frac{\tilde{n}^{11}m^2\pi^2}{Da^2}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right] = -\frac{1}{D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2\pi^2\tilde{n}^{11}}{a^2}b_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$
    $$\Longrightarrow A_{mn} = -\frac{b_{mn}m^2\tilde{n}^{11}}{a^2\pi^2D\left[\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 + \frac{\tilde{n}^{11}m^2}{\pi^2Da^2}\right]}$$



- Due to the initial curvature, a tension produces a z-deflection

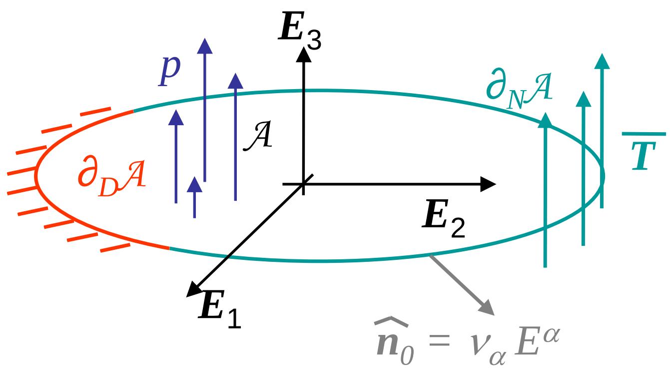
- Membrane mode
  - On $\mathcal{A}$: $\mathcal{H}_n^{\alpha\beta\gamma\delta}\dfrac{\boldsymbol{u}_{\gamma,\delta\alpha}+\boldsymbol{u}_{\delta,\gamma\alpha}}{2}+\bar{\boldsymbol{n}}_\beta=\bar{\rho}\ddot{\boldsymbol{u}}_\beta$
    - With $\mathcal{H}_n^{\alpha\beta\gamma\delta}=\dfrac{h_0E}{1-\nu^2}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta}+\dfrac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta}+\delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$

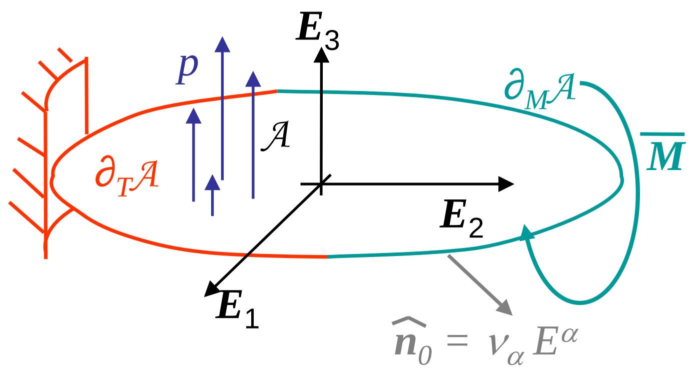


  - Completed by appropriate BCs
    - Dirichlet $\boldsymbol{u}_\alpha=\bar{\boldsymbol{u}}_\alpha$
    - Neumann $\boldsymbol{n}_\beta^\alpha\nu_\alpha=\mathcal{H}_n^{\alpha\beta\gamma\delta}\dfrac{\boldsymbol{u}_{\gamma,\delta\alpha}+\boldsymbol{u}_{\delta,\gamma\alpha}}{2}\nu_\alpha=\bar{\bar{\boldsymbol{n}}}_\beta$

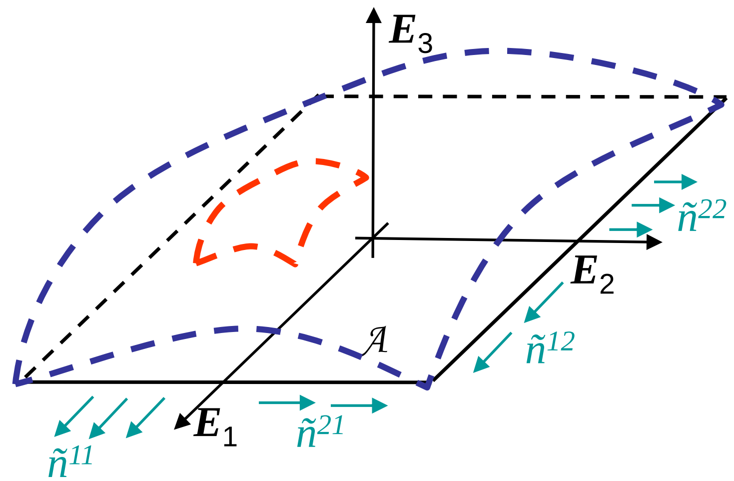
- Bending mode
  - On $\mathcal{A}$: $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\alpha\beta} = p$
    - With $\mathcal{H}_m^{\alpha\beta\gamma\delta} = \overbrace{\frac{h_0^3 E}{12\left(1-\nu^2\right)}}^{D}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta} + \frac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$
  - Completed by appropriate BCs
    - Low order
      - On $\partial_N\mathcal{A}$: $-\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\beta}\nu_\alpha = \bar{T}$
      - On $\partial_D\mathcal{A}$: $\boldsymbol{u}_3 = \bar{\boldsymbol{u}}_3$
    - High order
      - On $\partial_T\mathcal{A}$: $\boldsymbol{\Delta t} = \bar{\boldsymbol{\Delta t}}$

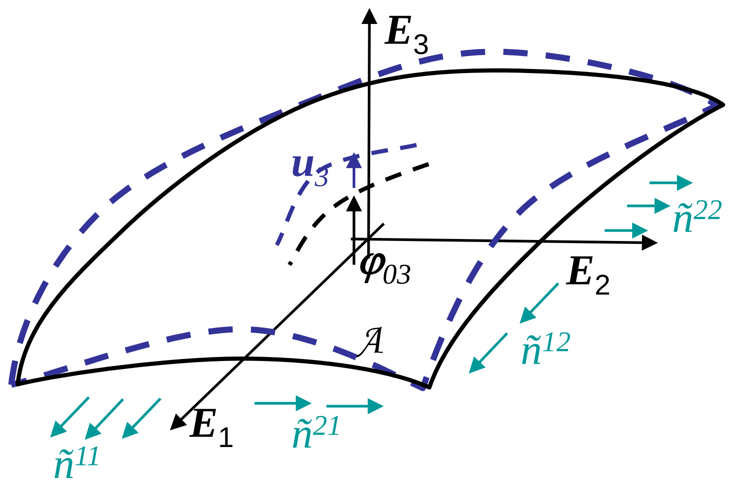
        with $\boldsymbol{\Delta t} = -\boldsymbol{u}_{3,\alpha}\boldsymbol{E}_\alpha$
      - On $\partial_M\mathcal{A}$: $-\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)\nu_\beta = \bar{M}\nu_\alpha$





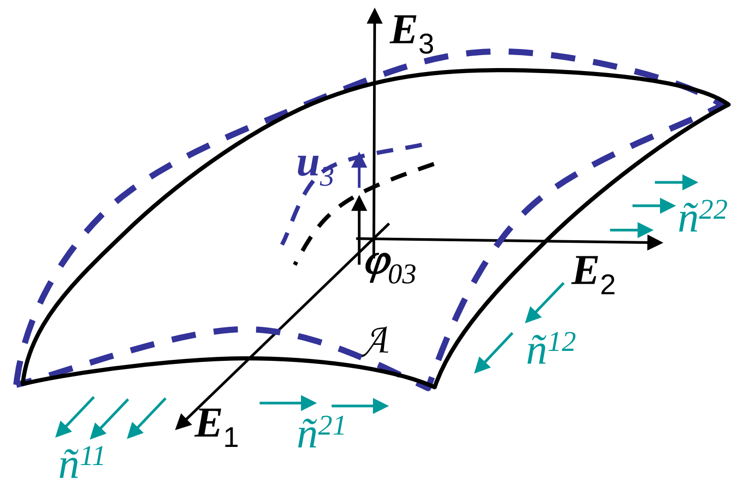
- Membrane-bending coupling
  - The first order theory is uncoupled
  - For second order theory
    - On $\mathcal{A}$: $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} - \tilde{n}^{\alpha\beta}\boldsymbol{u}_3\right)_{,\alpha\beta} = p$
    - Tension increases the bending stiffness of the plate
  - In case of small initial curvature (κ >>)
    - On $\mathcal{A}$:
      $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} - \tilde{n}^{\alpha\beta}\boldsymbol{u}_3 - \tilde{n}^{\alpha\beta}\boldsymbol{\varphi}_{03}\right)_{,\alpha\beta} = p$
    - Tension induces bending effect
- General theories
  - For not small initial curvature:
    - Linear shells
  - To fully account for tension effect
    - Non-Linear shells





- FE implementation
  - High order equation

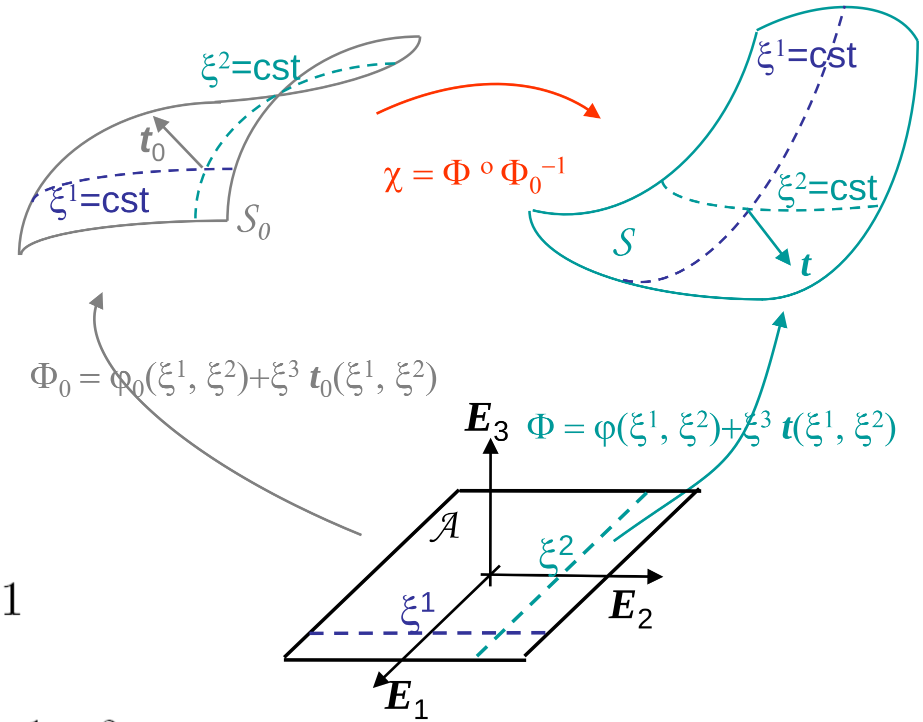
$$\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} - \tilde{n}^{\alpha\beta}\boldsymbol{u}_3 - \tilde{n}^{\alpha\beta}\boldsymbol{\varphi}_{03}\right)_{,\alpha\beta} = p$$



  - Requires C^1 finite-elements
    - Shape functions too complex
    - Can be enforced weakly
  - Not common, usually Reissner-Mindlin plates are implemented

- Shell kinematics
  - In the reference frame $\boldsymbol{E}_i$
    - The shell is described by

$$\boldsymbol{\xi} = \xi^I \boldsymbol{E}_I \quad \text{with} \begin{cases} (\xi^1,\, \xi^2) \in \mathcal{A} \\ \xi^3 \in [-\frac{h}{2};\, \frac{h}{2}] \end{cases}$$



  - Initial configuration S_0 mapping
    - Neutral plane $\boldsymbol{\varphi}_0\left(\xi^1,\, \xi^2\right)$
    - Cross section $\boldsymbol{t}_0(\xi^1,\, \xi^2)\,,\ \|\boldsymbol{t}\| = 1$
    - Thin body

$$\boldsymbol{x}_0 = \boldsymbol{\Phi}_0\left(\xi^I\right) = \boldsymbol{\varphi}_0\left(\xi^\alpha\right) + \xi^3 \boldsymbol{t}_0(\xi^1,\, \xi^2)$$

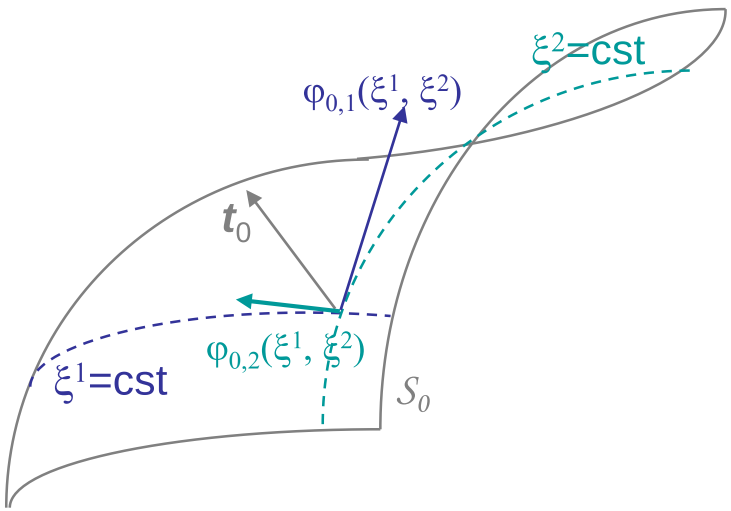
  - Deformed configuration S mapping
    - Thin body $\boldsymbol{x} = \boldsymbol{\Phi}\left(\xi^I\right) = \boldsymbol{\varphi}\left(\xi^\alpha\right) + \xi^3 \boldsymbol{t}(\xi^1,\, \xi^2)$

  - Two-point deformation mapping $\boldsymbol{\chi} = \boldsymbol{\Phi} \circ \boldsymbol{\Phi}_0^{-1}$

- Shell kinematics (2)
  - Deformation gradient
    - Two-point deformation mapping $\boldsymbol{\chi} = \boldsymbol{\Phi} \circ \boldsymbol{\Phi}_0^{-1}$
    - Two-point deformation gradient $\mathbf{F} = \boldsymbol{\nabla}\boldsymbol{\Phi} \circ \left[\boldsymbol{\nabla}\boldsymbol{\Phi}_0\right]^{-1}$
      - Small strain deformation gradient $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^T\right) - \mathbf{I}$
  - It is more convenient to evaluate these tensors in a convected basis
    - Example $\boldsymbol{g}_{0I}$ basis convected to S_0
      - As S_0 is described by $\boldsymbol{x}_0 = \boldsymbol{\Phi}_0\left(\xi^I\right) = \boldsymbol{\varphi}_0\left(\xi^\alpha\right) + \xi^3\boldsymbol{t}_0(\xi^1, \xi^2)$
      - One has $\boldsymbol{\nabla}\boldsymbol{\Phi}_0 = \boldsymbol{g}_{0I} \otimes \boldsymbol{E}^I$ with the convected basis

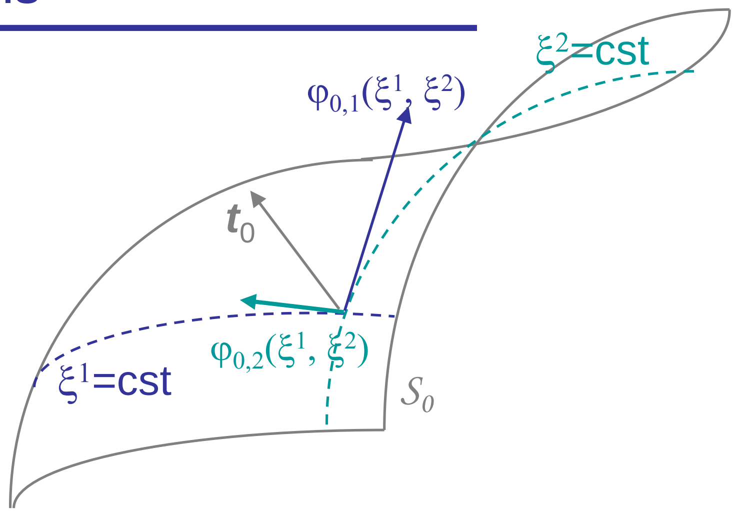
$$\begin{cases} \boldsymbol{g}_{0\alpha} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^\alpha} = \boldsymbol{\varphi}_{0,\alpha} + \xi^3 \boldsymbol{t}_{0,\alpha} \\ \boldsymbol{g}_{03} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^3} = \boldsymbol{t}_0 \end{cases}$$

      - The picture shows the basis for $\xi^3 = 0$
        - » $\boldsymbol{g}_{0\alpha}\,(\xi^3=0) = \boldsymbol{\varphi}_{0,\alpha}$

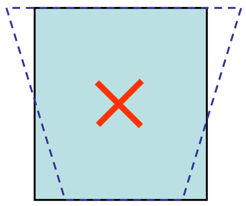


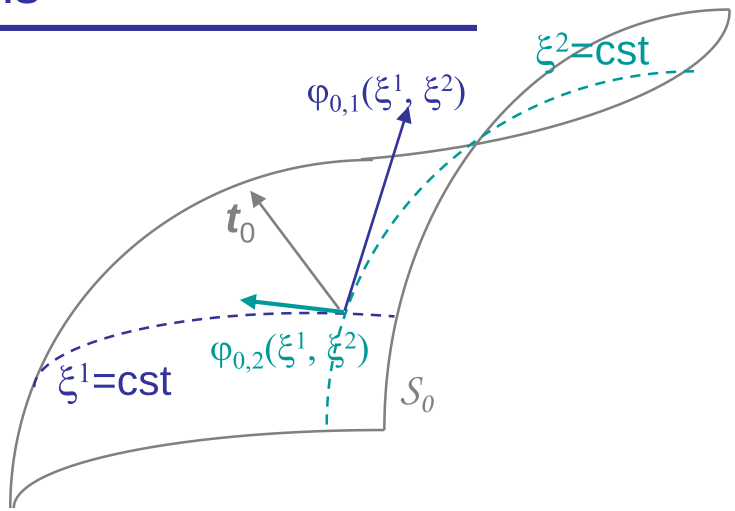
- Shell kinematics (3)
  - Convected basis $\boldsymbol{g}_{0I}$ to $\mathcal{S}_0$

$$\begin{cases} \boldsymbol{g}_{0\alpha} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^\alpha} = \boldsymbol{\varphi}_{0,\alpha} + \xi^3 \boldsymbol{t}_{0,\alpha} \\ \boldsymbol{g}_{03} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^3} = \boldsymbol{t}_0 \end{cases}$$



- If $\boldsymbol{t}_{0,\alpha} \neq 0$ (there is a curvature)
  - Tension and bending are coupled
  - This leads to locking with a FE integration: stiffness of the FE $\rightarrow \infty$
  - Locking can be solved by
    - » Adding internal degrees of freedom (EAS) $\Longrightarrow$ more expensive
    - » Using only one Gauss point (reduced integration) $\Longrightarrow$ hourglass modes (zero energy spurious deformation modes)
    - » See next lecture
  - Both methods lead to complex computational schemes





- Shell kinematics (4)
  - Convected basis $\boldsymbol{g}_{0I}$ to $\mathcal{S}_0$ (2)

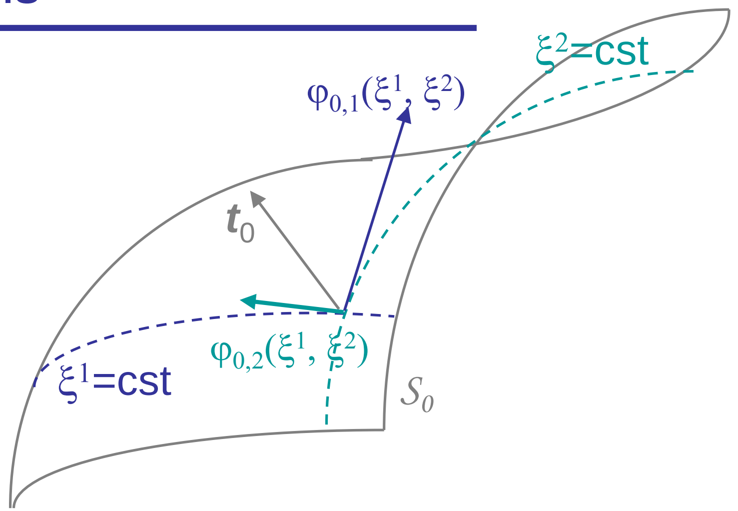
$$\begin{cases} \boldsymbol{g}_{0\alpha} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^\alpha} = \boldsymbol{\varphi}_{0,\alpha} + \xi^3 \boldsymbol{t}_{0,\alpha} \\ \boldsymbol{g}_{03} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^3} = \boldsymbol{t}_0 \end{cases}$$

- The basis is not orthonormal
  - A vector component is still defined as $a_I = \boldsymbol{a} \cdot \boldsymbol{g}_{0I}$
  - So can $\boldsymbol{a} = a_I \boldsymbol{g}_{0I}$ be written?
    - » If so $\boldsymbol{a} \cdot \boldsymbol{g}_{0I} = a_J \boldsymbol{g}_{0J} \cdot \boldsymbol{g}_{0I} \neq a_I$ which is not consistent

$\neq \delta_{IJ}$

- Shell kinematics (5)
  - Convected basis $\boldsymbol{g}_{0I}$ to $\mathcal{S}_0$ (3)

$$\left\{\begin{array}{l} \boldsymbol{g}_{0\alpha} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^\alpha} = \boldsymbol{\varphi}_{0,\alpha} + \xi^3 \boldsymbol{t}_{0,\alpha} \\ \boldsymbol{g}_{03} = \dfrac{\partial \boldsymbol{\Phi}_0}{\partial \xi^3} = \boldsymbol{t}_0 \end{array}\right.$$



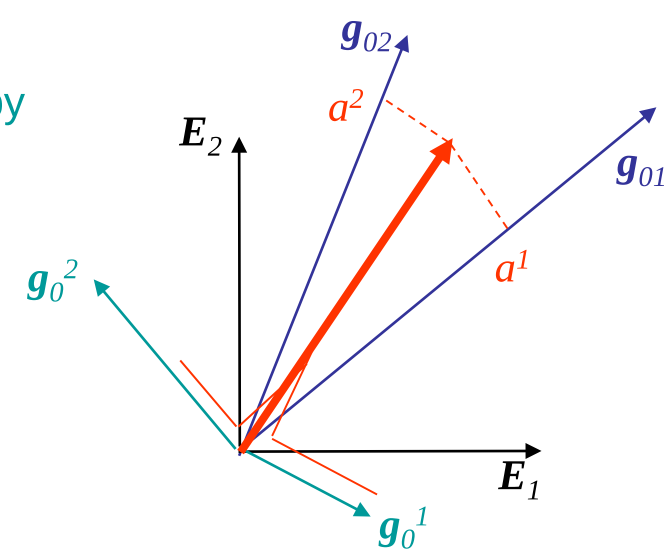
- The basis is not orthonormal (2)
  - A conjugate basis $\boldsymbol{g}_0^I$ has to be defined
    - » Such that $\boldsymbol{g}_{0I} \cdot \boldsymbol{g}_0^J = \delta_{IJ}$
    - » So vector components are defined by

$$a_I = \boldsymbol{a} \cdot \boldsymbol{g}_{0I} \ \& \ a^I = \boldsymbol{a} \cdot \boldsymbol{g}_0^I$$

    - » The vector can be represented by

$$\boldsymbol{a} = a^J \boldsymbol{g}_{0J} = a_I \boldsymbol{g}_0^I$$

as
$$\left\{\begin{array}{l} \boldsymbol{a} \cdot \boldsymbol{g}_{0I} = a_J \boldsymbol{g}_0^J \cdot \boldsymbol{g}_{0I} = a_I \\ \boldsymbol{a} \cdot \boldsymbol{g}_0^I = a^J \boldsymbol{g}_{0J} \cdot \boldsymbol{g}_0^I = a^I \end{array}\right.$$



  - For plates $\left\{\begin{array}{l} \boldsymbol{g}_{0\alpha} = \boldsymbol{\varphi}_{0,\alpha} = \boldsymbol{E}_\alpha = \boldsymbol{E}^\alpha \\ \boldsymbol{g}_{03} = \boldsymbol{t}_0 = \boldsymbol{E}_3 = \boldsymbol{E}^3 \end{array}\right.$

References

- Lecture notes
  - Aircraft Structures for engineering students, T. H. G. Megson, Butterworth-Heinemann, An imprint of Elsevier Science, 2003, ISBN 0 340 70588 4
- Other references
  - Papers

    - Simo JC, Fox DD. On a stress resultant geometrically exact shell model. Part I: formulation and optimal parametrization. *Computer Methods in Applied Mechanics and Engineering* 1989; **72**:267–304.
    - Simo JC, Fox DD, Rifai MS. On a stress resultant geometrically exact shell model. Part II: the linear theory, computational aspects. *Computer Methods in Applied Mechanics and Engineering* 1989; **73**:53–92.