University of Liège
Aerospace & Mechanical Engineering

Aircraft Structures
Plates – Reissner-Mindlin Theory

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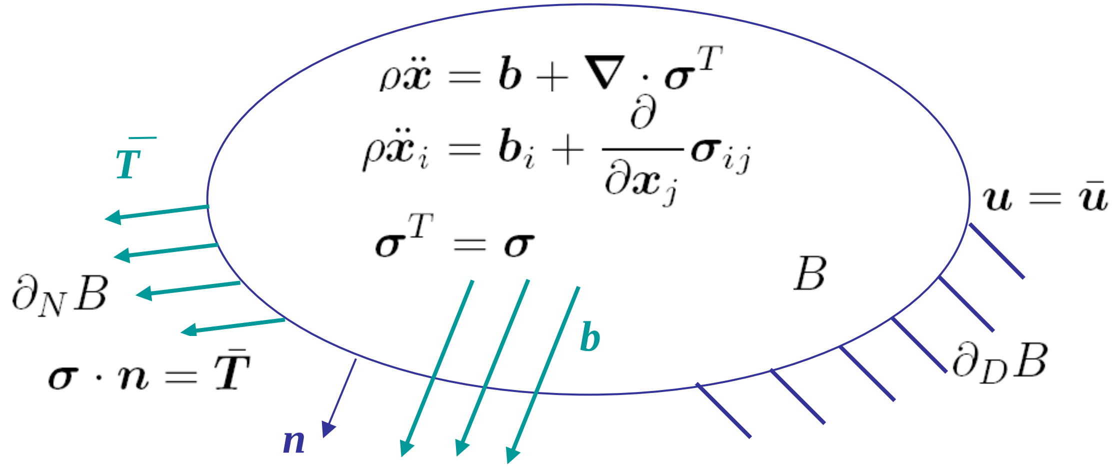
- Balance of body B
  - Momenta balance
    - Linear
    - Angular
  - Boundary conditions
    - Neumann
    - Dirichlet



- Small deformations with linear elastic, homogeneous & isotropic material
  - (Small) Strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla}\right)$, or $\begin{cases} \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{x}_i}\boldsymbol{u}_j + \frac{\partial}{\partial \boldsymbol{x}_j}\boldsymbol{u}_i\right) \\ \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\boldsymbol{u}_{j,i} + \boldsymbol{u}_{i,j}\right) \end{cases}$
  - Hooke's law $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon}$, or $\boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl}\boldsymbol{\varepsilon}_{kl}$

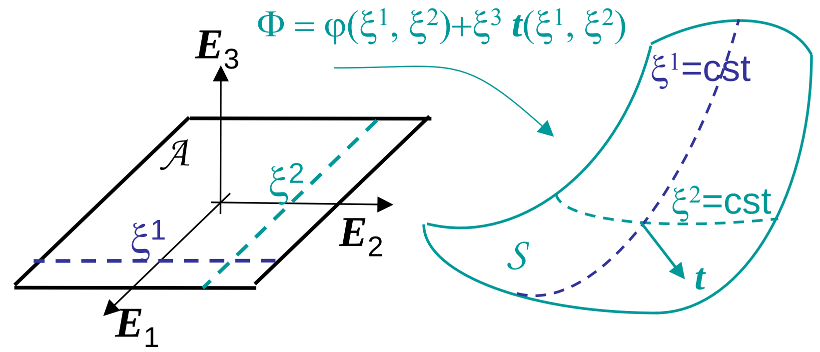
    with $$\mathcal{H}_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{1+\nu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$$

    $\lambda = K - 2\mu/3$ $\quad$ 2μ
  - Inverse law $\boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma}$

    with $$\mathcal{G}_{ijkl} = \frac{1+\nu}{E}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right) - \frac{\nu}{E}\delta_{ij}\delta_{kl}$$

- Description
  - In the reference frame $\boldsymbol{E}_i$
    - The plate is defined by

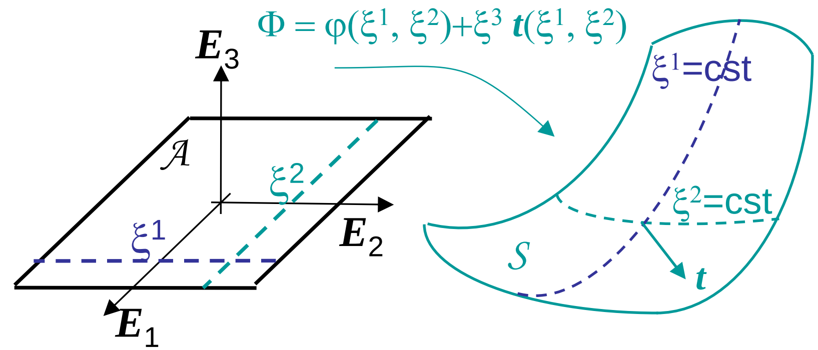
$$\boldsymbol{\xi} = \xi^I \boldsymbol{E}_I \quad \text{with} \quad \begin{cases} (\xi^1, \xi^2) \in \mathcal{A} \\ \xi^3 \in [-\frac{h_0}{2}; \frac{h_0}{2}] \end{cases}$$



  - Mapping of the plate
    - Neutral plane $\quad \boldsymbol{\varphi}_0\left(\xi^1, \xi^2\right) = \xi^\alpha \boldsymbol{E}_\alpha \quad$ α =1 or 2, I = 1, 2 or 3
    - Cross section $\quad \boldsymbol{t}_0(\xi^1, \xi^2)\,, \ \|\boldsymbol{t}\| = 1 \quad$ with $\boldsymbol{t}_0 = \boldsymbol{E}_3$
    - Initial plate S_0
      - $S_0 = \mathcal{A}$ x [-h_0/2 h_0/2], for a plate of initial thickness h_0
      - $\boldsymbol{X} = \boldsymbol{\Phi}_0\left(\xi^I\right) = \boldsymbol{\varphi}_0\left(\xi^\alpha\right) + \xi^3 \boldsymbol{t}_0(\xi^1, \xi^2)$
    - Deformed plate S
      - $\boldsymbol{x} = \boldsymbol{\Phi}\left(\xi^I\right) = \boldsymbol{\varphi}\left(\xi^\alpha\right) + \xi^3 \boldsymbol{t}(\xi^1, \xi^2)$

Plate kinematics

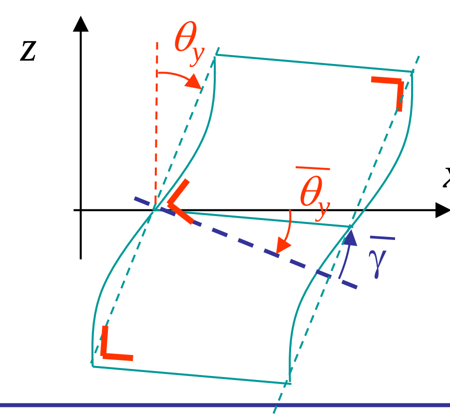
- Assumptions
  - Small deformations/displacements
    - $\mathcal{S} \simeq \mathcal{S}_0$
    - $\boldsymbol{\nabla} \simeq \boldsymbol{\nabla}_0$
    - $\int_{\mathcal{S}} \simeq \int_{\mathcal{S}_0}$
    - $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 + \boldsymbol{u}$ with $\|\boldsymbol{u}\| << \sqrt[3]{|\mathcal{S}_0|}$



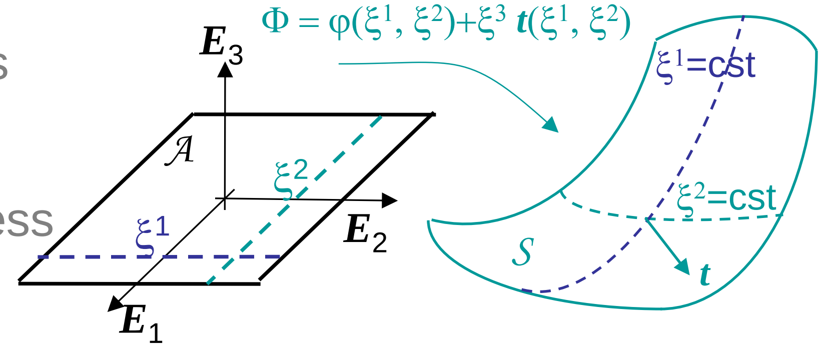
  - Kirchhoff-Love assumption (no shearing)
    - Normal is assumed to remain
      - Planar
      - Perpendicular to the neutral plane

$$\Rightarrow \begin{cases} \boldsymbol{t}_0 = \boldsymbol{E}_3 \\ \boldsymbol{t} = \dfrac{\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}}{\|\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}\|} \end{cases}$$

  - Reissner-Mindlin (shearing is allowed)
    - Normal is assumed to remain planar
    - But not perpendicular to neutral plane



- Idea
  - Avoiding discretization on the thickness
    - $\boldsymbol{u}$ and $\boldsymbol{t}$ constant on the thickness
  - Equations are integrated on the thickness



- Linear momentum equation
  - $\rho \ddot{\boldsymbol{x}} = \boldsymbol{b} + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T$

$$\Longrightarrow \int_{\mathcal{S}} \left\{ \rho \left(\ddot{\boldsymbol{\varphi}} + \xi^3 \ddot{\boldsymbol{t}} \right) \right\} dV = \int_{\mathcal{S}} \boldsymbol{b} dV + \int_{\mathcal{S}} \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T dV$$

  - Small transformations assumptions ($\mathcal{S} \simeq \mathcal{S}_0$, $\boldsymbol{\nabla} \simeq \boldsymbol{\nabla}_0$, $\int_{\mathcal{S}} \simeq \int_{\mathcal{S}_0}$)

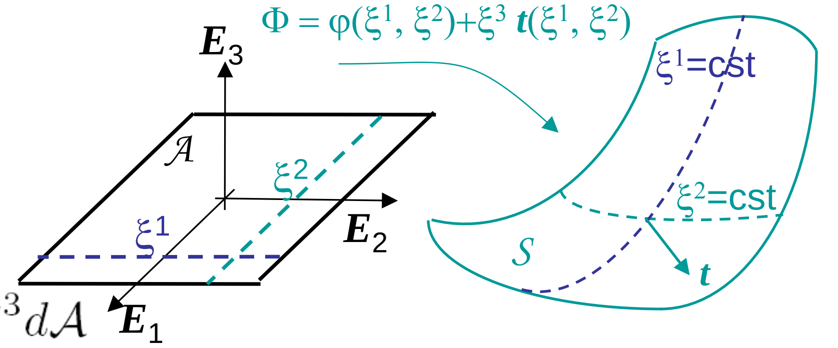
$$\Longrightarrow \int_{\mathcal{S}_0} \left\{ \rho_0 \left(\ddot{\boldsymbol{\varphi}} + \xi^3 \ddot{\boldsymbol{t}} \right) \right\} dV = \int_{\mathcal{S}_0} \boldsymbol{b} dV + \int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV$$

- Using $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 + \boldsymbol{u}$

$$\Longrightarrow \int_{\mathcal{S}_0} \left\{ \rho_0 \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}} \right) \right\} dV = \int_{\mathcal{S}_0} \boldsymbol{b} dV + \int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV$$

Resultant equilibrium equations

- Linear momentum equation (2)



  – Inertial term

$$\int_{\mathcal{S}_0} \{\rho_0 (\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}})\} dV =$$

$$\int_{\mathcal{A}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \rho_0 \ddot{\boldsymbol{u}} d\xi^3 d\mathcal{A} + \int_{\mathcal{A}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \rho_0 \xi^3 \ddot{\boldsymbol{t}} d\xi^3 d\mathcal{A}$$

  - As the main idea in plates is to consider $\boldsymbol{u}$ and $\boldsymbol{t}$ constant on the thickness

$$\Longrightarrow \int_{\mathcal{S}_0} \{\rho_0 (\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}})\} dV = \int_{\mathcal{A}} \rho_0 \ddot{\boldsymbol{u}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\xi^3 d\mathcal{A} + \int_{\mathcal{A}} \rho_0 \ddot{\boldsymbol{t}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \xi^3 d\xi^3$$

$$\Longrightarrow \int_{\mathcal{S}_0} \{\rho_0 (\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}})\} dV = \int_{\mathcal{A}} \rho_0 h_0 \ddot{\boldsymbol{u}} d\mathcal{A}$$

  – Volume loading term

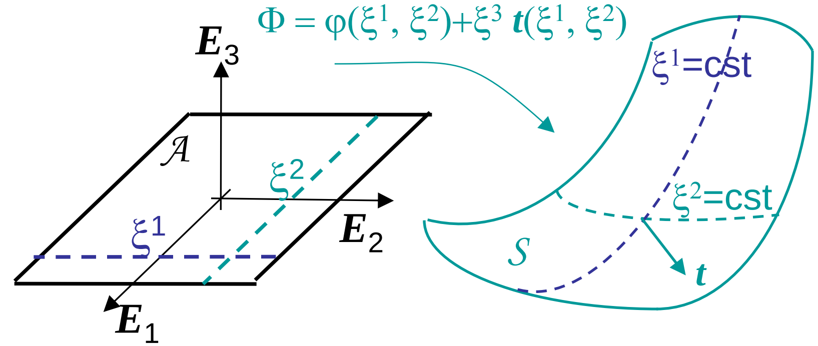
  - $\int_{\mathcal{S}_0} \boldsymbol{b} dV = \int_{\mathcal{A}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{b} d\xi^3 d\mathcal{A} \Longrightarrow \int_{\mathcal{S}_0} \boldsymbol{b} dV = \int_{\mathcal{A}} \bar{\boldsymbol{b}} d\mathcal{A}$

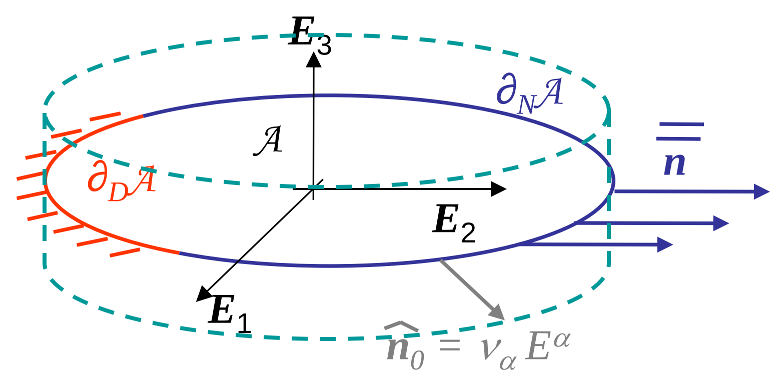
  - With the loading per unit area $\bar{\boldsymbol{b}}$.

Resultant equilibrium equations

- Linear momentum equation (3)
  - Stress term
    - Gauss theorem

$$\int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV = \int_{\partial \mathcal{S}_0} \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}_0 dS$$





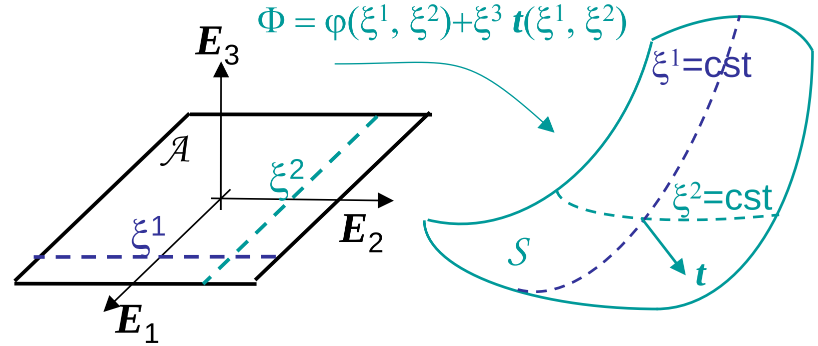
- Where $\hat{\boldsymbol{n}}_0$ is the normal to the plate surface (3D volume) in the reference configuration
- On top/bottom faces $\hat{\boldsymbol{n}}_0 = \pm \boldsymbol{E}^3$
- On lateral surface: $\hat{\boldsymbol{n}}_0 = \nu_\alpha \boldsymbol{E}^\alpha$

$$\int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV = \int_{\mathcal{A}} \boldsymbol{\sigma}\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \cdot \boldsymbol{E}^3 d\mathcal{A} + \int_{\partial\mathcal{A}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \nu_\alpha \boldsymbol{E}^\alpha d\xi^3 dl$$

  - Let us define the resultant stresses: $\boldsymbol{n}^\alpha = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^\alpha d\xi^3$

- Linear momentum equation (4)
  - Resultant stresses

$$\boldsymbol{n}^{\alpha} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^{\alpha} d\xi^3$$



- These are two vectors

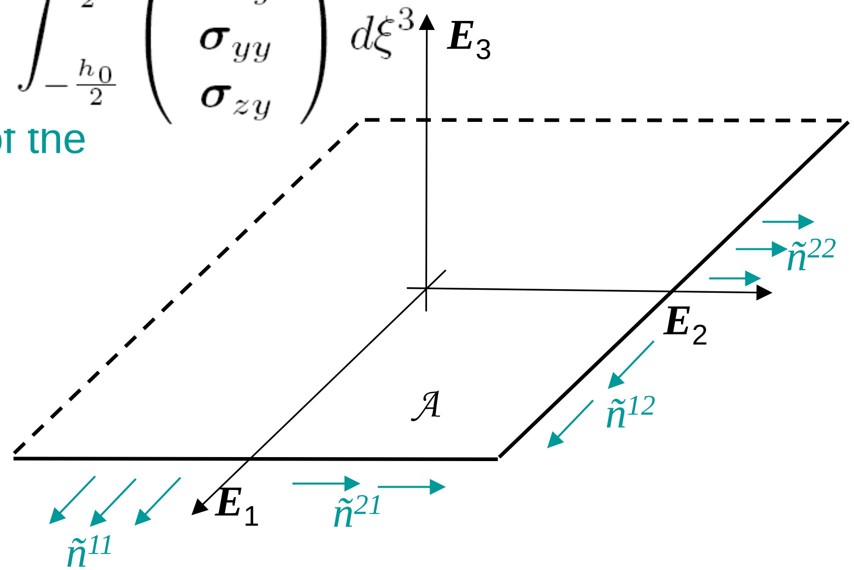
$$- \quad \boldsymbol{n}^{x} = \boldsymbol{n}^{1} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^{1} d\xi^3 = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yx} \\ \boldsymbol{\sigma}_{zx} \end{pmatrix} d\xi^3$$

$$- \quad \boldsymbol{n}^{y} = \boldsymbol{n}^{2} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^{2} d\xi^3 = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \begin{pmatrix} \boldsymbol{\sigma}_{xy} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{zy} \end{pmatrix} d\xi^3$$

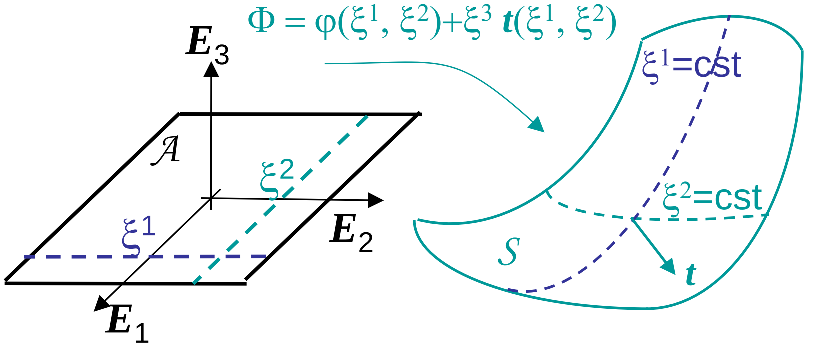
- Which correspond to the integration of the surface traction on the thickness

$$- \quad \tilde{n}^{i\alpha} = \boldsymbol{n}_i^{\alpha} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}_{i\alpha} d\xi^3$$

  - Symmetric 2x2 matrix + Out of plane component



- Linear momentum equation (5)
  - From



    - $\displaystyle\int_{\mathcal{S}_0} \left\{\rho_0 \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right)\right\} dV = \int_{\mathcal{A}} \rho_0 h_0 \ddot{\boldsymbol{u}} d\mathcal{A}$
    - $\displaystyle\int_{\mathcal{S}_0} \boldsymbol{b} dV = \int_{\mathcal{A}} \bar{\boldsymbol{b}} d\mathcal{A}$
    - $\displaystyle\int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV = \int_{\mathcal{A}} \boldsymbol{\sigma}|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \cdot \boldsymbol{E}^3 d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{n}^\alpha \nu_\alpha dl$
    - Applying Gauss theorem on last term leads to

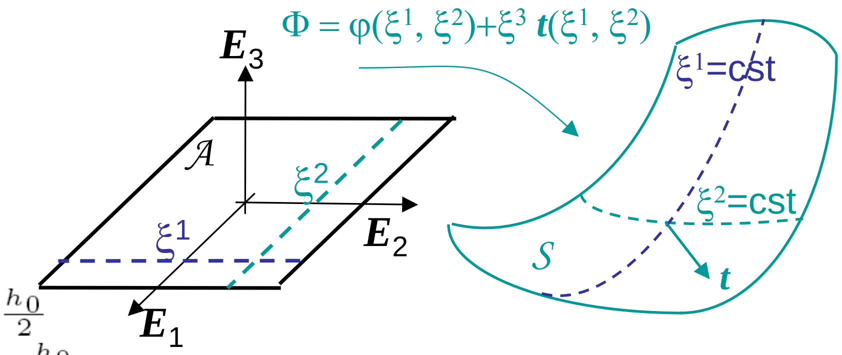
$$\int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV = \int_{\mathcal{A}} \boldsymbol{\sigma}|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \cdot \boldsymbol{E}^3 d\mathcal{A} + \int_{\mathcal{A}} \left(\boldsymbol{n}^\alpha\right)_{,\alpha} d\mathcal{A}$$

  - Resultant linear momentum equation

$$\int_{\mathcal{S}_0} \left\{\rho_0 \left(\ddot{\boldsymbol{\varphi}} + \xi^3 \ddot{\boldsymbol{t}}\right)\right\} dV = \int_{\mathcal{S}_0} \boldsymbol{b} dV + \int_{\mathcal{S}_0} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T dV$$

$$\Longrightarrow \int_{\mathcal{A}} \rho_0 h_0 \ddot{\boldsymbol{u}} d\mathcal{A} = \int_{\mathcal{A}} \left\{\bar{\boldsymbol{b}} + \boldsymbol{\sigma}|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \cdot \boldsymbol{E}^3 + \left(\boldsymbol{n}^\alpha\right)_{,\alpha}\right\} d\mathcal{A}$$

- Linear momentum equation (6)
  - Defining
    - Density per unit area $\bar{\rho}$
    - Volume loading per unit area $\bar{\boldsymbol{b}}$
    - Resultant loading $\bar{\boldsymbol{n}} = \bar{\boldsymbol{b}} + \left(\boldsymbol{\sigma}\cdot\boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}}$



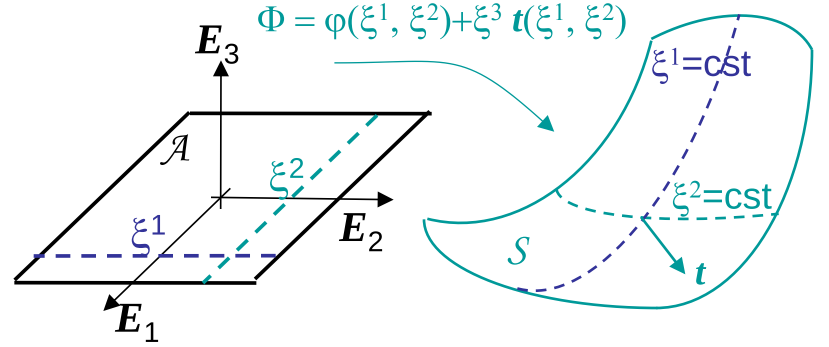
  - The linear momentum equation
    - $\rho\ddot{\boldsymbol{x}} = \boldsymbol{b} + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^T$

  becomes after being integrated on the volume

$$\int_{\mathcal{A}} \rho_0 h_0 \ddot{\boldsymbol{u}} d\mathcal{A} = \int_{\mathcal{A}} \left\{\bar{\boldsymbol{b}} + \boldsymbol{\sigma}\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}}\cdot\boldsymbol{E}^3 + \left(\boldsymbol{n}^\alpha\right)_{,\alpha}\right\} d\mathcal{A}$$

    - Is rewritten in the Cosserat plane A as $\boxed{\left(\boldsymbol{n}^\alpha\right)_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}\ddot{\boldsymbol{u}}}$
    - With the resultant stresses $\boldsymbol{n}^\alpha = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}\cdot\boldsymbol{E}^\alpha d\xi^3$
    - With the resultant loading $\bar{\boldsymbol{n}} = \bar{\boldsymbol{b}} + \left(\boldsymbol{\sigma}\cdot\boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}}$
  - But we have no equation for bending (yet)

- Angular momentum equation



  – $\rho \ddot{\boldsymbol{x}} = \boldsymbol{b} + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T$

$$\Longrightarrow \rho\Phi \wedge \ddot{\boldsymbol{x}} = \Phi \wedge \boldsymbol{b} + \Phi \wedge \left(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T\right)$$

$$\Longrightarrow \int_{\mathcal{S}} \rho \left(\boldsymbol{\varphi} + \xi^3 \boldsymbol{t}\right) \wedge \left(\ddot{\boldsymbol{\varphi}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV = \int_{\mathcal{S}} \left(\boldsymbol{\varphi} + \xi^3 \boldsymbol{t}\right) \wedge \boldsymbol{b} dV + \int_{\mathcal{S}} \left(\boldsymbol{\varphi} + \xi^3 \boldsymbol{t}\right) \wedge \left(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^T\right) dV$$

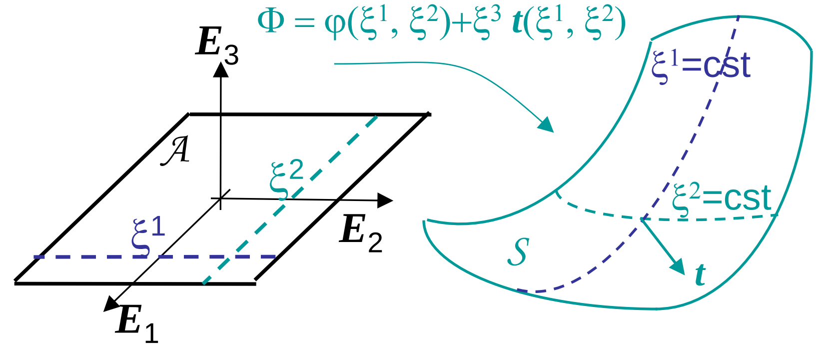
  – Small transformations assumptions ($\mathcal{S} \simeq \mathcal{S}_0$, $\boldsymbol{\nabla} \simeq \boldsymbol{\nabla}_0$, $\int_{\mathcal{S}} \simeq \int_{\mathcal{S}_0}$)

$$\Longrightarrow \int_{\mathcal{S}_0} \rho_0 \left(\boldsymbol{\varphi} + \xi^3 \boldsymbol{t}\right) \wedge \left(\ddot{\boldsymbol{\varphi}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV = \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi} + \xi^3 \boldsymbol{t}\right) \wedge \boldsymbol{b} dV + \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi} + \xi^3 \boldsymbol{t}\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV$$

    - Using $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 + \boldsymbol{u}$

$$\Longrightarrow \int_{\mathcal{S}_0} \rho_0 \left(\boldsymbol{\varphi}_0 + \boldsymbol{u} + \xi^3 \boldsymbol{t}\right) \wedge \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV = \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \boldsymbol{u} + \xi^3 \boldsymbol{t}\right) \wedge \boldsymbol{b} dV + \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \boldsymbol{u} + \xi^3 \boldsymbol{t}\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV$$

- Angular momentum equation (2)
  - Small transformations assumptions (2)



$$\int_{\mathcal{S}_0} \rho_0 \left(\boldsymbol{\varphi}_0 + \boldsymbol{u} + \xi^3 \boldsymbol{t}\right) \wedge \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV = \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \boldsymbol{u} + \xi^3 \boldsymbol{t}\right) \wedge \boldsymbol{b} dV + \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \boldsymbol{u} + \xi^3 \boldsymbol{t}\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV$$

- As second order terms can be neglected

$$\Rightarrow \int_{\mathcal{S}_0} \rho_0 \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0\right) \wedge \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV = \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0\right) \wedge \boldsymbol{b} dV + \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{t}_0\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV$$

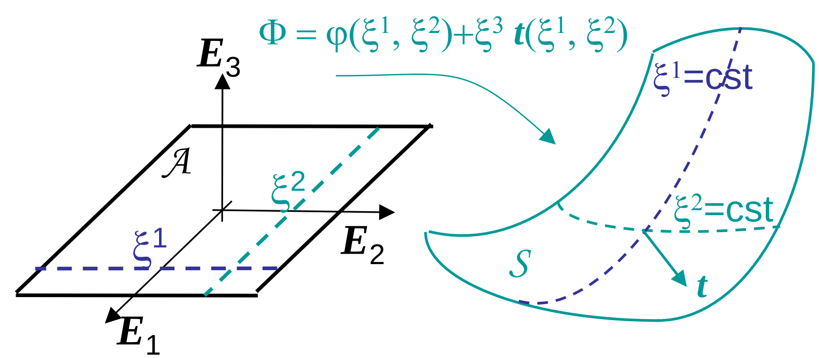
- With $\boldsymbol{t}_0 = \boldsymbol{E}_3$

- Angular momentum equation (3)
  - Inertial term
    - $$\int_{\mathcal{S}_0} \rho_0 \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3\right) \wedge \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV =$$
      $$\int_{\mathcal{A}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \left\{\rho_0 \boldsymbol{\varphi}_0 \wedge \ddot{\boldsymbol{u}} + \rho_0 \xi^3 \left(\boldsymbol{E}_3 \wedge \ddot{\boldsymbol{u}} + \boldsymbol{\varphi}_0 \wedge \ddot{\boldsymbol{t}}\right) + \rho_0 {\xi^3}^2 \boldsymbol{E}^3 \wedge \ddot{\boldsymbol{t}}\right\} d\xi^3 d\mathcal{A}$$
    - As the main idea in plates is to consider $\boldsymbol{u}$ and $\boldsymbol{t}$ constant on the thickness

$$\int_{\mathcal{S}_0} \rho_0 \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3\right) \wedge \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV = \int_{\mathcal{A}} \bar{\rho} \boldsymbol{\varphi}_0 \wedge \ddot{\boldsymbol{u}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{E}_3 \wedge \ddot{\boldsymbol{t}} I_p d\mathcal{A}$$

- With the density per unit area $\bar{\rho}$
- With the mass inertia

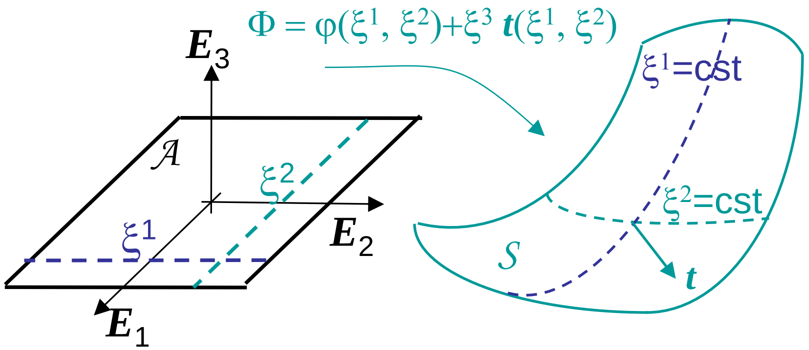
$$I_p = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \rho_0 {\xi^3}^2 d\xi^3 = \rho_0 \frac{h_0^3}{12}$$



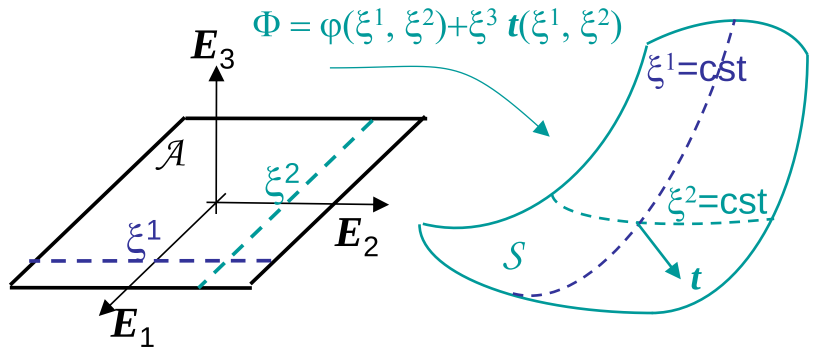
- Angular momentum equation (4)
  - Loading term
    - $$\int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3) \wedge \boldsymbol{b} dV = \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{b} d\xi^3 d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{E}_3 \wedge \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \xi^3 \boldsymbol{b} d\xi^3 d\mathcal{A}$$

    $$\Longrightarrow \int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3) \wedge \boldsymbol{b} dV = \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \bar{\boldsymbol{b}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{E}_3 \wedge \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \xi^3 \boldsymbol{b} d\xi^3 d\mathcal{A}$$

    - With the loading per unit area $\bar{\boldsymbol{b}}$



- Angular momentum equation (5)
  - Stress term
    - $$\int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3)\wedge(\boldsymbol{\nabla}_0\cdot\boldsymbol{\sigma}^T)\,dV = \int_{\mathcal{S}_0} \Phi_0\wedge(\boldsymbol{\nabla}_0\cdot\boldsymbol{\sigma}^T)\,dV$$



$$\Longrightarrow \left.\int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3)\wedge(\boldsymbol{\nabla}_0\cdot\boldsymbol{\sigma}^T)\,dV\right|_i = \int_{\mathcal{S}_0} e_{ijk}\Phi_{0\,j}\boldsymbol{\nabla}_{0\,l}\boldsymbol{\sigma}_{kl}\,dV$$

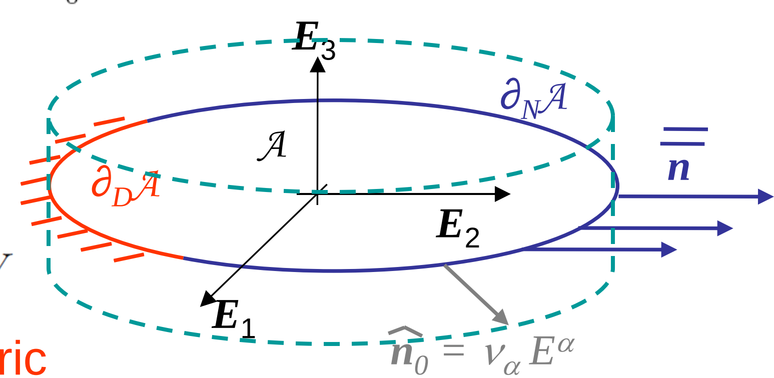
- Integration by parts

$$\left.\int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3)\wedge(\boldsymbol{\nabla}_0\cdot\boldsymbol{\sigma}^T)\,dV\right|_i = \int_{\mathcal{S}_0} e_{ijk}\boldsymbol{\nabla}_{0\,l}\left(\Phi_{0\,j}\boldsymbol{\sigma}_{kl}\right)dV - \int_{\mathcal{S}_0} e_{ijk}\overbrace{\left(\boldsymbol{\nabla}_{0\,l}\Phi_{0\,j}\right)}^{\delta_{jl}}\boldsymbol{\sigma}_{kl}\,dV$$

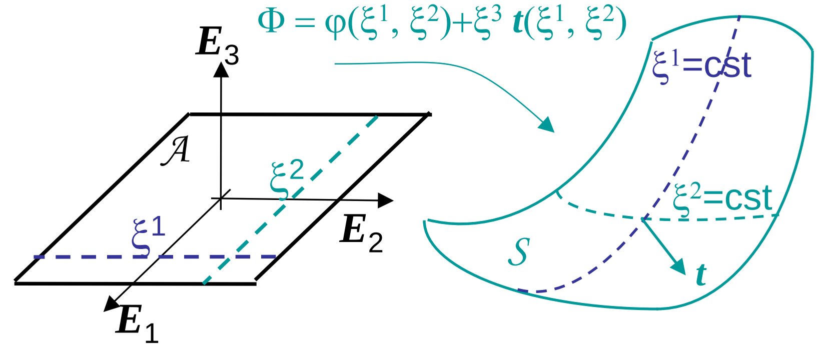
- Gauss theorem

$$\int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3)\wedge(\boldsymbol{\nabla}_0\cdot\boldsymbol{\sigma}^T)\,dV = \int_{\partial\mathcal{S}_0} e_{ijk}\Phi_{0\,j}\boldsymbol{\sigma}_{kl}\hat{\boldsymbol{n}}_{0l}\,dS - \int_{\mathcal{S}_0} \cancel{e_{ijk}\boldsymbol{\sigma}_{kj}}\,dV$$

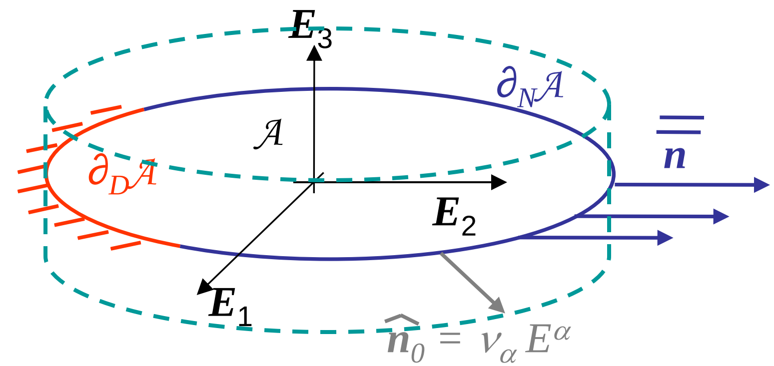
as σ symmetric



- Angular momentum equation (6)
  - Stress term (2)
    - $$\int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV = \int_{\partial\mathcal{S}_0} \Phi_0 \wedge \left(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}\right) dS$$



- Where $\widehat{\boldsymbol{n}}_0$ is the normal to the plate surface (3D volume) in the reference configuration
- On top/bottom faces $\widehat{\boldsymbol{n}}_0 = \pm\, \boldsymbol{E}^3$
- On lateral surface: $\widehat{\boldsymbol{n}}_0 = \nu_\alpha\, \boldsymbol{E}^\alpha$



$$\Rightarrow \int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV = $$

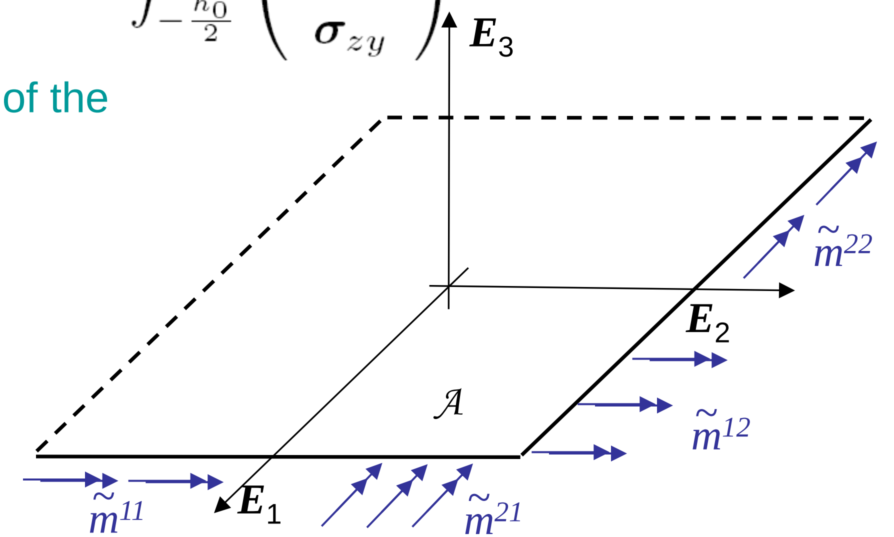
$$\int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^\alpha \nu_\alpha\right) d\xi^3 dl +$$

$$\int_{\mathcal{A}} \xi^3 \boldsymbol{E}_3 \wedge \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{E}_3 \wedge \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \xi^3 \boldsymbol{\sigma} \cdot \boldsymbol{E}^\alpha d\xi^3 \nu_\alpha dl$$

- Angular momentum equation (7)
  - Stress term (3)
    - $$\int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3) \wedge (\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T)\, dV \quad = \\ \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge (\boldsymbol{\sigma} \cdot \boldsymbol{E}^3)\big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} (\boldsymbol{\sigma} \cdot \boldsymbol{E}^\alpha \nu_\alpha)\, d\xi^3 dl + \\ \int_{\mathcal{A}} \xi^3 \boldsymbol{E}_3 \wedge (\boldsymbol{\sigma} \cdot \boldsymbol{E}^3)\big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{E}_3 \wedge \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \xi^3 \boldsymbol{\sigma} \cdot \boldsymbol{E}^\alpha d\xi^3 \nu_\alpha dl$$
    - Resultant bending stresses & resultant stresses
      - $\tilde{\boldsymbol{m}}^\alpha = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^\alpha \xi^3 d\xi^3$, $\boldsymbol{n}^\alpha = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^\alpha d\xi^3$

$$\Longrightarrow \int_{\mathcal{S}_0} (\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3) \wedge (\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T)\, dV \quad = \\ \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge (\boldsymbol{\sigma} \cdot \boldsymbol{E}^3)\big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \boldsymbol{n}^\alpha \nu_\alpha dl + \\ \int_{\mathcal{A}} \xi^3 \boldsymbol{E}_3 \wedge (\boldsymbol{\sigma} \cdot \boldsymbol{E}^3)\big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{E}_3 \wedge \tilde{\boldsymbol{m}}^\alpha \nu_\alpha dl$$

- Angular momentum equation (8)
  - Resultant bending stress
    - $\tilde{\boldsymbol{m}}^{\alpha} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^{\alpha} \xi^3 d\xi^3$
    - These are two vectors
      - $\tilde{\boldsymbol{m}}^{x} = \tilde{\boldsymbol{m}}^{1} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^{1} \xi^3 d\xi^3 = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yx} \\ \boldsymbol{\sigma}_{zx} \end{pmatrix} \xi^3 d\xi^3$
      - $\tilde{\boldsymbol{m}}^{y} = \tilde{\boldsymbol{m}}^{2} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^{2} \xi^3 d\xi^3 = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \begin{pmatrix} \boldsymbol{\sigma}_{xy} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{zy} \end{pmatrix} \xi^3 d\xi^3$
    - Which correspond to the integration of the surface couple on the thickness
      - $\tilde{m}^{i\alpha} = \tilde{\boldsymbol{m}}_i^{\alpha} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}_{i\alpha} \xi^3 d\xi^3$
      - Symmetric 2x2 matrix



- Angular momentum equation (9)
  - Stress term (4)
    - $$\int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV \quad = \\ \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \boldsymbol{n}^\alpha \nu_\alpha dl + \\ \int_{\mathcal{A}} \xi^3 \boldsymbol{E}_3 \wedge \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \int_{\partial\mathcal{A}} \boldsymbol{E}_3 \wedge \tilde{\boldsymbol{m}}^\alpha \nu_\alpha dl$$
    - Applying Gauss theorem

$$\int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV = \\ \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(\boldsymbol{n}^\alpha\right)_{,\alpha} d\mathcal{A} + \\ \int_{\mathcal{A}} \xi^3 \boldsymbol{E}_3 \wedge \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{E}_3 \wedge \left(\tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha} d\mathcal{A}$$

- Angular momentum equation (10)
  - From
    - $$\int_{\mathcal{S}_0} \rho_0 \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3\right) \wedge \left(\ddot{\boldsymbol{u}} + \xi^3 \ddot{\boldsymbol{t}}\right) dV = \int_{\mathcal{A}} \bar{\rho} \boldsymbol{\varphi}_0 \wedge \ddot{\boldsymbol{u}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{E}_3 \wedge \ddot{\boldsymbol{t}} I_p d\mathcal{A}$$
    - $$\int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3\right) \wedge \boldsymbol{b} dV = \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \bar{\boldsymbol{b}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{E}_3 \wedge \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \xi^3 \boldsymbol{b} d\xi^3 d\mathcal{A}$$
    - $$\int_{\mathcal{S}_0} \left(\boldsymbol{\varphi}_0 + \xi^3 \boldsymbol{E}_3\right) \wedge \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^T\right) dV =$$
      $$\int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(\boldsymbol{n}^\alpha\right)_{,\alpha} d\mathcal{A} +$$
      $$\int_{\mathcal{A}} \xi^3 \boldsymbol{E}_3 \wedge \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{E}_3 \wedge \left(\tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha} d\mathcal{A}$$
    - It comes

$$\int_{\mathcal{A}} \bar{\rho} \boldsymbol{\varphi}_0 \wedge \ddot{\boldsymbol{u}} d\mathcal{A} + \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \ddot{\boldsymbol{t}} I_p d\mathcal{A} = \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \bar{\boldsymbol{b}} d\mathcal{A} + \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \xi^3 \boldsymbol{b} d\xi^3 d\mathcal{A} +$$
$$\int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(\boldsymbol{n}^\alpha\right)_{,\alpha} d\mathcal{A} +$$
$$\boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \xi^3 \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \left(\tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha} d\mathcal{A}$$

- Angular momentum equation (11)
  - Resultant form
    - $$\int_{\mathcal{A}} \bar{\rho}\boldsymbol{\varphi}_0 \wedge \ddot{\boldsymbol{u}} d\mathcal{A} + \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \ddot{\boldsymbol{t}} I_p d\mathcal{A} = \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \bar{\boldsymbol{b}} d\mathcal{A} + \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \xi^3 \boldsymbol{b} d\xi^3 d\mathcal{A} +$$
      $$\int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_0 \wedge \left(\boldsymbol{n}^\alpha\right)_{,\alpha} d\mathcal{A} +$$
      $$\boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \xi^3 \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \left(\tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha} d\mathcal{A}$$
    - But the resultant linear momentum equation reads

$$\begin{cases} \left(\boldsymbol{n}^\alpha\right)_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}\ddot{\boldsymbol{u}} \\ \bar{\boldsymbol{n}} = \bar{\boldsymbol{b}} + \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \end{cases}$$

    - So the angular momentum equation reads

$$\boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \ddot{\boldsymbol{t}} I_p d\mathcal{A} = \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \xi^3 \boldsymbol{b} d\xi^3 d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha d\mathcal{A} +$$
$$\boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \xi^3 \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \left(\tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha} d\mathcal{A}$$

- Angular momentum equation (12)
  - Resultant form (2)
    - $$\boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \ddot{\boldsymbol{t}} I_p d\mathcal{A} = \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \xi^3 \boldsymbol{b} d\xi^3 d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha d\mathcal{A} + \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \xi^3 \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} d\mathcal{A} + \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \left(\tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha} d\mathcal{A}$$
    - Defining the applied torque $\bar{\tilde{\boldsymbol{m}}} = \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3 \xi^3\right)\Big|_{-\frac{h}{2}}^{\frac{h}{2}} + \int_{-\frac{h}{2}}^{\frac{h}{2}} \xi^3 \boldsymbol{b} d\xi^3$

$$\Longrightarrow \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \ddot{\boldsymbol{t}} I_p d\mathcal{A} = \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \bar{\tilde{\boldsymbol{m}}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha d\mathcal{A} + \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \left(\tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha} d\mathcal{A}$$

- Term which is preventing from uncoupling the equations is

$$\int_{\mathcal{A}} \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha d\mathcal{A}$$

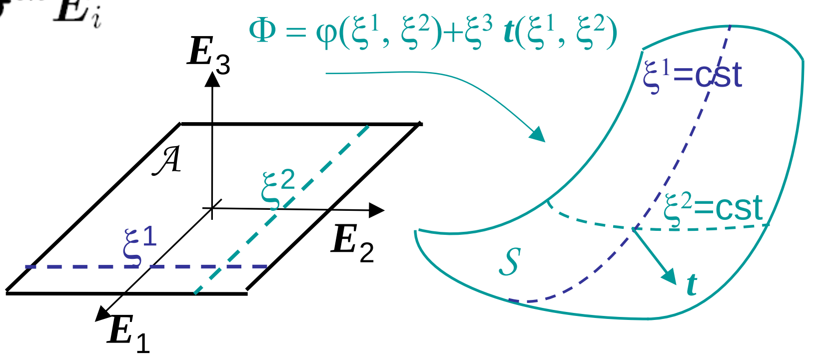
- Angular momentum equation (13)
  - Term $\int_{\mathcal{A}} \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^{\alpha} d\mathcal{A}$
    - Let us rewrite the Cauchy stress tensor in terms of its components

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{ij} \boldsymbol{E}_i \otimes \boldsymbol{E}_j$$

$$\Longrightarrow \boldsymbol{\sigma} \cdot \boldsymbol{E}^k = \boldsymbol{\sigma}^{ij} \boldsymbol{E}_i \otimes \boldsymbol{E}_j \cdot \boldsymbol{E}^k = \boldsymbol{\sigma}^{ik} \boldsymbol{E}_i$$

    - As Cauchy stress tensor is symmetrical

$$\boldsymbol{E}_k \wedge \boldsymbol{\sigma} \cdot \boldsymbol{E}^k = \boldsymbol{\sigma}^{ik} \boldsymbol{E}_k \wedge \boldsymbol{E}_i = 0$$



    - Using $\boldsymbol{E}_{\alpha} = \boldsymbol{\varphi}_{0,\alpha}$

$$\Longrightarrow \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{\sigma} \cdot \boldsymbol{E}^{\alpha} + \boldsymbol{E}_3 \wedge \boldsymbol{\sigma} \cdot \boldsymbol{E}^3 = 0$$

    - This new equation can be integrated on the thickness

$$\int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{\sigma} \cdot \boldsymbol{E}^{\alpha} d\xi^3 + \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{E}_3 \wedge \boldsymbol{\sigma} \cdot \boldsymbol{E}^3 d\xi^3 = 0$$

    - Defining the out-of-plane resultant stress $\boldsymbol{n}^3 = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^3 d\xi^3$

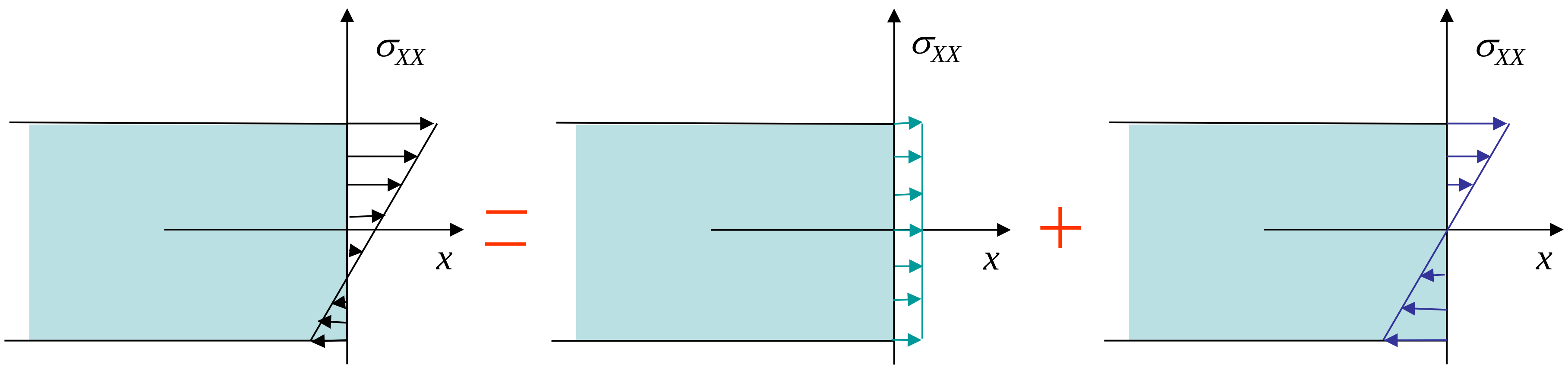
$$\Longrightarrow \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^{\alpha} = -\boldsymbol{E}_3 \wedge \boldsymbol{n}^3$$

- Angular momentum equation (14)
  - Resultant form (3)
    - $\boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \ddot{\boldsymbol{t}} I_p d\mathcal{A} = \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \bar{\tilde{\boldsymbol{m}}} d\mathcal{A} + \int_{\mathcal{A}} \boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha d\mathcal{A} + \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \left(\tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha} d\mathcal{A}$
    - $\boldsymbol{\varphi}_{0,\alpha} \wedge \boldsymbol{n}^\alpha = -\boldsymbol{E}_3 \wedge \boldsymbol{n}^3$

$$\Longrightarrow \quad \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \ddot{\boldsymbol{t}} I_p d\mathcal{A} = \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \bar{\tilde{\boldsymbol{m}}} d\mathcal{A} - \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \boldsymbol{n}^3 d\mathcal{A} + \boldsymbol{E}_3 \wedge \int_{\mathcal{A}} \left(\tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha} d\mathcal{A}$$

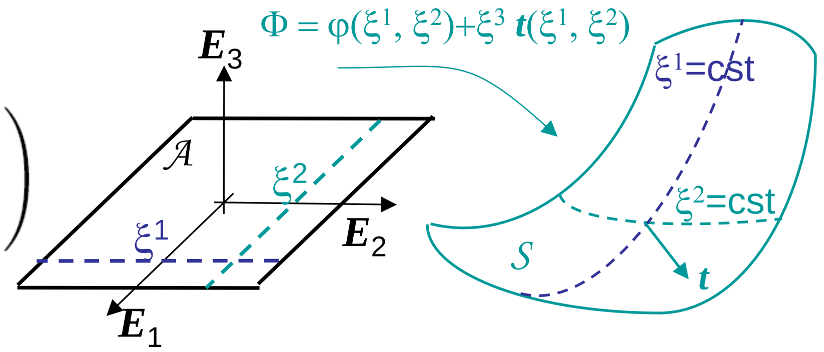
    - If λ is an undefined pressure applied through the thickness, the resultant angular momentum equation reads $\boxed{\ddot{\boldsymbol{t}} I_p = \bar{\tilde{\boldsymbol{m}}} - \left(\boldsymbol{n}^3 - \lambda \boldsymbol{E}_3\right) + \left(\tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha}}$
    - With
      - $\tilde{\boldsymbol{m}}^\alpha = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^\alpha \xi^3 d\xi^3$
      - $\bar{\tilde{\boldsymbol{m}}} = \left(\boldsymbol{\sigma} \cdot \boldsymbol{E}^3 \xi^3\right)\Big|_{-\frac{h_0}{2}}^{\frac{h_0}{2}} + \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \xi^3 \boldsymbol{b} d\xi^3$
      - $I_p = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \rho_0 {\xi^3}^2 d\xi^3 = \rho_0 \frac{h_0^3}{12}$

- Resultant equations summary
  - Linear momentum
    - $\boxed{\left(\boldsymbol{n}^{\alpha}\right)_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}\ddot{\boldsymbol{u}}}$
    - Resultant stresses $\boldsymbol{n}^{\alpha} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^{\alpha} d\xi^3$
  - Angular momentum
    - $\boxed{\ddot{\boldsymbol{t}} I_p = \tilde{\bar{\boldsymbol{m}}} - \left(\boldsymbol{n}^3 - \lambda \boldsymbol{E}_3\right) + \left(\tilde{\boldsymbol{m}}^{\alpha}\right)_{,\alpha}}$
    - Resultant bending stresses $\tilde{\boldsymbol{m}}^{\alpha} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^{\alpha} \xi^3 d\xi^3$
  - Interpretation



- What is missing is the link between the deformations and the stresses
  - Idea: as the discretization does not involve the thickness, the deformations should be evaluated at neutral plane too
  - Works only in linear elasticity

- Deformations
  - In small deformations, the tensor reads



$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\frac{\partial\left(\boldsymbol{x}-\boldsymbol{X}\right)}{\partial \boldsymbol{X}} + \left(\frac{\partial\left(\boldsymbol{x}-\boldsymbol{X}\right)}{\partial \boldsymbol{X}}\right)^T\right)$$

- But defining

$$\boldsymbol{\Delta t}\left(\xi^1,\, \xi^2\right) = \boldsymbol{t} - \boldsymbol{E}_3$$

leads to

$$\boldsymbol{x} - \boldsymbol{X} = \boldsymbol{\varphi}_0 + \boldsymbol{u} + \xi^3\boldsymbol{t} - \boldsymbol{\varphi}_0 - \xi^3\boldsymbol{E}_3 = \boldsymbol{u}\left(\xi^1,\, \xi^2\right) + \xi^3\boldsymbol{\Delta t}\left(\xi^1,\, \xi^2\right)$$

- A vector can always be written in terms of its components

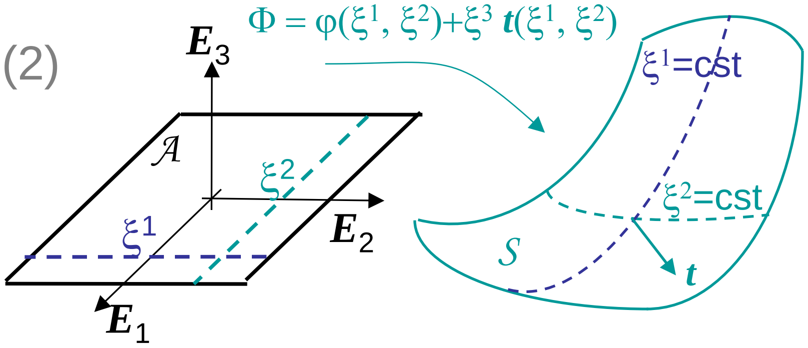
$$\boldsymbol{a} = \boldsymbol{a}\cdot\boldsymbol{E}_i\boldsymbol{E}^i = \boldsymbol{a}\cdot\boldsymbol{E}_\alpha\boldsymbol{E}^\alpha + \boldsymbol{a}\cdot\boldsymbol{E}_3\boldsymbol{E}^3$$

$$\Longrightarrow \boldsymbol{x}-\boldsymbol{X} = \boldsymbol{u}\left(\xi^1,\, \xi^2\right)\cdot\boldsymbol{E}_\alpha\boldsymbol{E}^\alpha + \boldsymbol{u}\left(\xi^1,\, \xi^2\right)\cdot\boldsymbol{E}_3\boldsymbol{E}^3 + \xi^3\boldsymbol{\Delta t}\left(\xi^1,\, \xi^2\right)\cdot\boldsymbol{E}_\alpha\boldsymbol{E}^\alpha + \xi^3\boldsymbol{\Delta t}\left(\xi^1,\, \xi^2\right)\cdot\boldsymbol{E}_3\boldsymbol{E}^3$$

- Deformations (2)
  - In small deformations, the tensor reads (2)
    - Relations on the normal
      - By definition $\boldsymbol{t}\cdot\boldsymbol{t}=1$
      - In small deformations

$$(\boldsymbol{E}_3+\boldsymbol{\Delta t})\cdot(\boldsymbol{E}_3+\boldsymbol{\Delta t}) = \boldsymbol{E}_3\cdot\boldsymbol{E}_3 + 2\boldsymbol{E}_3\cdot\boldsymbol{\Delta t} = 1$$

      - Which implies $\boldsymbol{\Delta t}\cdot\boldsymbol{E}_3=0$ & $\boldsymbol{\Delta t}_{,\alpha}\cdot\boldsymbol{E}_3=0$



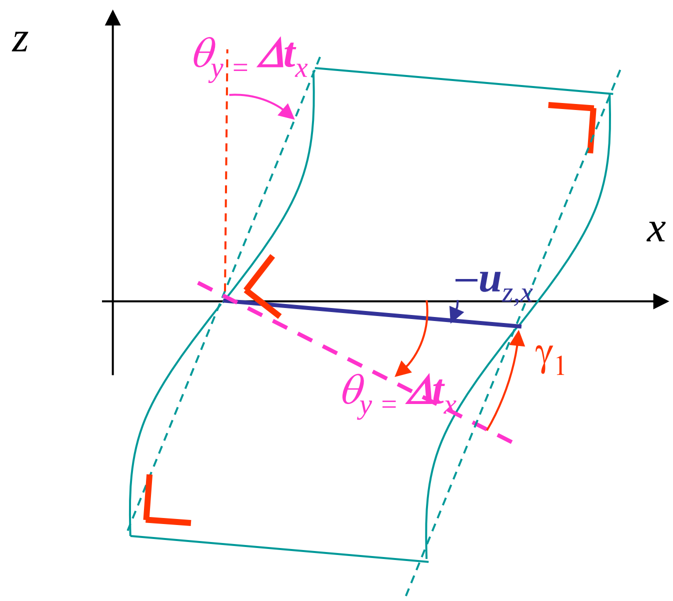
    - So relation

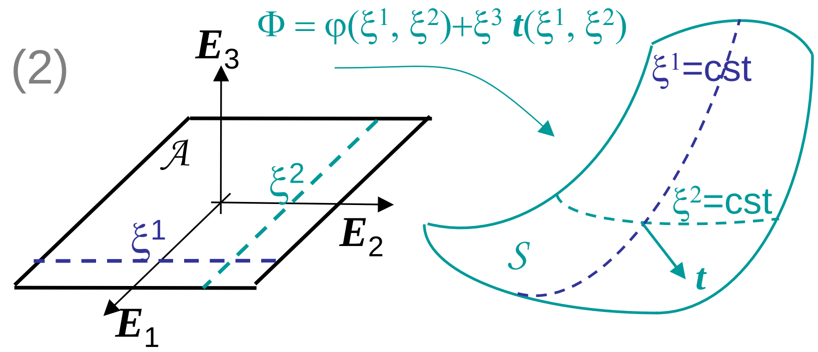
$$\boldsymbol{x}-\boldsymbol{X} = \boldsymbol{u}\left(\xi^1,\,\xi^2\right)\cdot\boldsymbol{E}_\alpha\boldsymbol{E}^\alpha+\boldsymbol{u}\left(\xi^1,\,\xi^2\right)\cdot\boldsymbol{E}_3\boldsymbol{E}^3+ \xi^3\boldsymbol{\Delta t}\left(\xi^1,\,\xi^2\right)\cdot\boldsymbol{E}_\alpha\boldsymbol{E}^\alpha+\cancel{\xi^3\boldsymbol{\Delta t}\left(\xi^1,\,\xi^2\right)\cdot\boldsymbol{E}_3\boldsymbol{E}^3}$$

$$\Longrightarrow \boldsymbol{x}-\boldsymbol{X} = \boldsymbol{u}\left(\xi^1,\,\xi^2\right)\cdot\boldsymbol{E}_\alpha\boldsymbol{E}^\alpha+\boldsymbol{u}\left(\xi^1,\,\xi^2\right)\cdot\boldsymbol{E}_3\boldsymbol{E}^3+\xi^3\boldsymbol{\Delta t}\left(\xi^1,\,\xi^2\right)\cdot\boldsymbol{E}_\alpha\boldsymbol{E}^\alpha$$

    - Interpretation
      - See next slide

- Deformations (3)
  - In small deformations, the tensor reads (2)
    - Interpretation





$$\Longrightarrow \boldsymbol{x} - \boldsymbol{X} = \boldsymbol{u}\left(\xi^1, \xi^2\right) \cdot \boldsymbol{E}_\alpha \boldsymbol{E}^\alpha + \boldsymbol{u}\left(\xi^1, \xi^2\right) \cdot \boldsymbol{E}_3 \boldsymbol{E}^3 + \xi^3 \boldsymbol{\Delta t}\left(\xi^1, \xi^2\right) \cdot \boldsymbol{E}_\alpha \boldsymbol{E}^\alpha$$

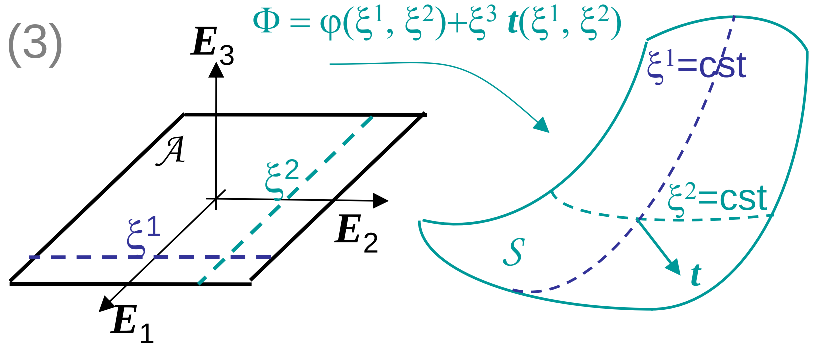
Motion of the neutral plane in its plane (2D in-plane problem): membrane mode

Motion of the neutral plane out-of-its-plane: bending mode

Change of the neutral plane direction resulting from
1) Bending
2) Out-of-plane shearing

- Deformations (4)
  - In small deformations, the tensor reads (3)
    - From

$$\boldsymbol{x} - \boldsymbol{X} = \boldsymbol{u}\left(\xi^1, \xi^2\right) \cdot \boldsymbol{E}_\alpha \boldsymbol{E}^\alpha + \boldsymbol{u}\left(\xi^1, \xi^2\right) \cdot \boldsymbol{E}_3 \boldsymbol{E}^3 + \xi^3 \boldsymbol{\Delta t}\left(\xi^1, \xi^2\right) \cdot \boldsymbol{E}_\alpha \boldsymbol{E}^\alpha$$



- As $\partial_{\boldsymbol{X}} = \partial_i \boldsymbol{E}^i = \partial_\beta \boldsymbol{E}^\beta + \partial_{\xi^3} \boldsymbol{E}^3$

  and as only the last term depends on ξ^3, the gradient reads

$$\frac{\partial\left(\boldsymbol{x} - \boldsymbol{X}\right)}{\partial \boldsymbol{X}} = \boldsymbol{u}_{,\beta}\left(\xi^1, \xi^2\right) \cdot \boldsymbol{E}_\alpha\, \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta + \boldsymbol{u}_{,\beta}\left(\xi^1, \xi^2\right) \cdot \boldsymbol{E}_3\, \boldsymbol{E}^3 \otimes \boldsymbol{E}^\beta + \xi^3 \boldsymbol{\Delta t}_{,\beta}\left(\xi^1, \xi^2\right) \cdot \boldsymbol{E}_\alpha \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta + \boldsymbol{\Delta t}\left(\xi^1, \xi^2\right) \cdot \boldsymbol{E}_\alpha \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^3$$

- So the deformations tensor $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\frac{\partial\left(\boldsymbol{x} - \boldsymbol{X}\right)}{\partial \boldsymbol{X}} + \left(\frac{\partial\left(\boldsymbol{x} - \boldsymbol{X}\right)}{\partial \boldsymbol{X}}\right)^T\right)$ reads

$$\boldsymbol{\varepsilon} = \frac{\boldsymbol{u}_{,\beta} \cdot \boldsymbol{E}_\alpha + \boldsymbol{u}_{,\alpha} \cdot \boldsymbol{E}_\beta}{2} \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta + \boldsymbol{u}_{,\beta} \cdot \boldsymbol{E}_3 \frac{\boldsymbol{E}^3 \otimes \boldsymbol{E}^\beta + \boldsymbol{E}^\beta \otimes \boldsymbol{E}^3}{2} + \xi^3 \frac{\boldsymbol{\Delta t}_{,\beta} \cdot \boldsymbol{E}_\alpha + \boldsymbol{\Delta t}_{,\alpha} \cdot \boldsymbol{E}_\beta}{2} \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta + \boldsymbol{\Delta t} \cdot \boldsymbol{E}_\alpha \frac{\boldsymbol{E}^\alpha \otimes \boldsymbol{E}^3 + \boldsymbol{E}^3 \otimes \boldsymbol{E}^\alpha}{2}$$

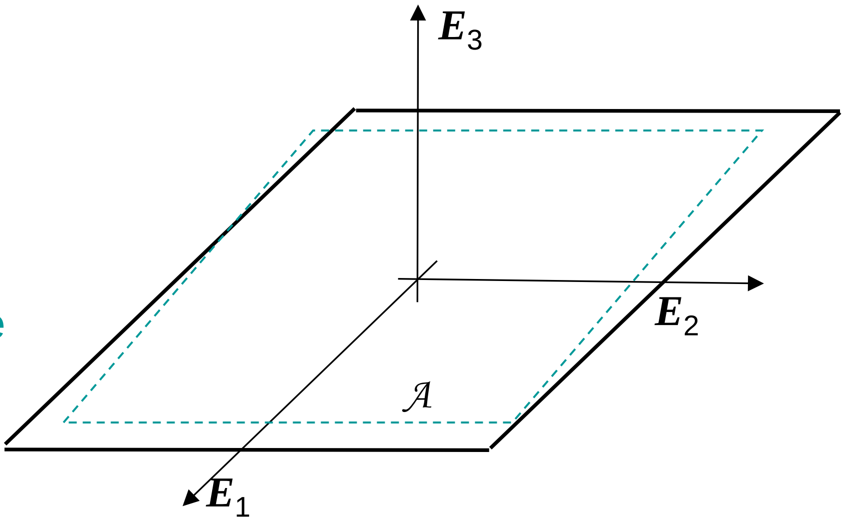
- Deformations (5)
  - Deformation modes

$$\boldsymbol{\varepsilon} = \frac{\boldsymbol{u}_{,\beta}\cdot\boldsymbol{E}_\alpha + \boldsymbol{u}_{,\alpha}\cdot\boldsymbol{E}_\beta}{2}\boldsymbol{E}^\alpha\otimes\boldsymbol{E}^\beta + \boldsymbol{u}_{,\beta}\cdot\boldsymbol{E}_3\,\frac{\boldsymbol{E}^3\otimes\boldsymbol{E}^\beta + \boldsymbol{E}^\beta\otimes\boldsymbol{E}^3}{2} +$$
$$\xi^3\frac{\boldsymbol{\Delta t}_{,\beta}\cdot\boldsymbol{E}_\alpha + \boldsymbol{\Delta t}_{,\alpha}\cdot\boldsymbol{E}_\beta}{2}\boldsymbol{E}^\alpha\otimes\boldsymbol{E}^\beta + \boldsymbol{\Delta t}\cdot\boldsymbol{E}_\alpha\,\frac{\boldsymbol{E}^\alpha\otimes\boldsymbol{E}^3 + \boldsymbol{E}^3\otimes\boldsymbol{E}^\alpha}{2}$$

- Interpretation: membrane mode
  - $\varepsilon_{\alpha\beta} = \frac{\boldsymbol{u}_{,\beta}\cdot\boldsymbol{E}_\alpha + \boldsymbol{u}_{,\alpha}\cdot\boldsymbol{E}_\beta}{2} \Longrightarrow \varepsilon_{\alpha\beta} = \frac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2}$
  - Location in the deformation tensor

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \bullet & \bullet & . \\ \bullet & \bullet & . \\ . & . & . \end{pmatrix}$$

  - Corresponds to the in-plane deformations of the neutral plane



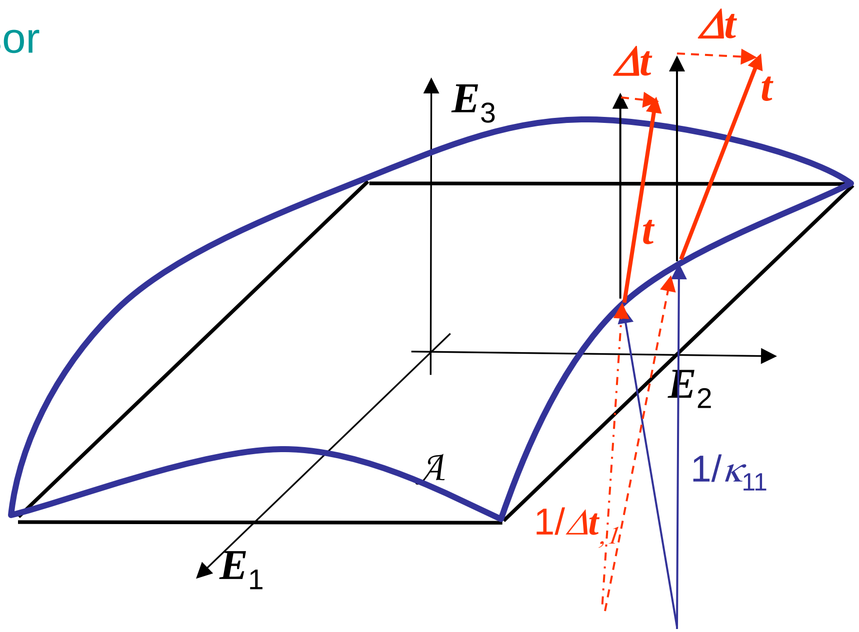
- Deformations (6)
  - Deformation modes (2)

$$\boldsymbol{\varepsilon} = \frac{\boldsymbol{u}_{,\beta}\cdot\boldsymbol{E}_\alpha + \boldsymbol{u}_{,\alpha}\cdot\boldsymbol{E}_\beta}{2}\boldsymbol{E}^\alpha\otimes\boldsymbol{E}^\beta + \boldsymbol{u}_{,\beta}\cdot\boldsymbol{E}_3\frac{\boldsymbol{E}^3\otimes\boldsymbol{E}^\beta + \boldsymbol{E}^\beta\otimes\boldsymbol{E}^3}{2} +$$
$$\xi^3\frac{\boldsymbol{\Delta t}_{,\beta}\cdot\boldsymbol{E}_\alpha + \boldsymbol{\Delta t}_{,\alpha}\cdot\boldsymbol{E}_\beta}{2}\boldsymbol{E}^\alpha\otimes\boldsymbol{E}^\beta + \boldsymbol{\Delta t}\cdot\boldsymbol{E}_\alpha\frac{\boldsymbol{E}^\alpha\otimes\boldsymbol{E}^3 + \boldsymbol{E}^3\otimes\boldsymbol{E}^\alpha}{2}$$

- Interpretation: bending mode
  - $\kappa_{\alpha\beta} = \frac{\boldsymbol{\Delta t}_{,\beta}\cdot\boldsymbol{E}_\alpha + \boldsymbol{\Delta t}_{,\alpha}\cdot\boldsymbol{E}_\beta}{2} \Longrightarrow \kappa_{\alpha\beta} = \frac{\boldsymbol{\Delta t}_{\alpha,\beta} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2}$
  - Location in the deformation tensor

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \bullet & \bullet & . \\ \bullet & \bullet & . \\ . & . & . \end{pmatrix}$$

  - Corresponds to the final curvature of the neutral plane

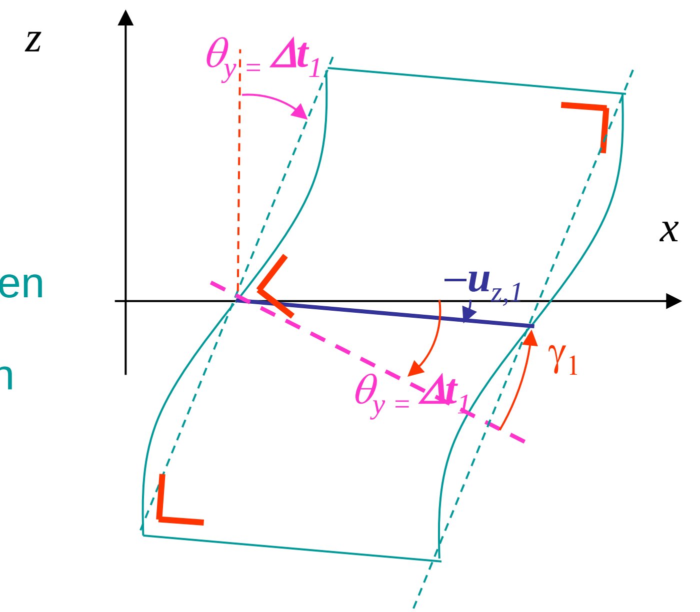


- Deformations (7)
  - Deformation modes (3)

$$\boldsymbol{\varepsilon} = \frac{\boldsymbol{u}_{,\beta}\cdot\boldsymbol{E}_\alpha + \boldsymbol{u}_{,\alpha}\cdot\boldsymbol{E}_\beta}{2}\boldsymbol{E}^\alpha\otimes\boldsymbol{E}^\beta + \boldsymbol{u}_{,\beta}\cdot\boldsymbol{E}_3\frac{\boldsymbol{E}^3\otimes\boldsymbol{E}^\beta + \boldsymbol{E}^\beta\otimes\boldsymbol{E}^3}{2} +$$
$$\xi^3\frac{\boldsymbol{\Delta t}_{,\beta}\cdot\boldsymbol{E}_\alpha + \boldsymbol{\Delta t}_{,\alpha}\cdot\boldsymbol{E}_\beta}{2}\boldsymbol{E}^\alpha\otimes\boldsymbol{E}^\beta + \boldsymbol{\Delta t}\cdot\boldsymbol{E}_\alpha\frac{\boldsymbol{E}^\alpha\otimes\boldsymbol{E}^3 + \boldsymbol{E}^3\otimes\boldsymbol{E}^\alpha}{2}$$

  - Interpretation: Through-the-thickness shearing
    - $2\delta_\alpha = \gamma_\alpha = \boldsymbol{u}_{,\alpha}\cdot\boldsymbol{E}_3 + \boldsymbol{\Delta t}\cdot\boldsymbol{E}_\alpha \Longrightarrow \gamma_\alpha = \boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_\alpha$
    - Location in the deformation tensor

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \cdot & \cdot & \bullet \\ \cdot & \cdot & \bullet \\ \bullet & \bullet & \cdot \end{pmatrix}$$



    - Corresponds to the average angle between the neutral plane normal and the direction vector $\boldsymbol{t}$ (initially the same)

- Deformations (8)
  - Deformation modes (4)

$$\boldsymbol{\varepsilon} = \frac{\boldsymbol{u}_{,\beta}\cdot\boldsymbol{E}_\alpha + \boldsymbol{u}_{,\alpha}\cdot\boldsymbol{E}_\beta}{2}\boldsymbol{E}^\alpha\otimes\boldsymbol{E}^\beta + \boldsymbol{u}_{,\beta}\cdot\boldsymbol{E}_3\frac{\boldsymbol{E}^3\otimes\boldsymbol{E}^\beta + \boldsymbol{E}^\beta\otimes\boldsymbol{E}^3}{2} + \xi^3\frac{\boldsymbol{\Delta t}_{,\beta}\cdot\boldsymbol{E}_\alpha + \boldsymbol{\Delta t}_{,\alpha}\cdot\boldsymbol{E}_\beta}{2}\boldsymbol{E}^\alpha\otimes\boldsymbol{E}^\beta + \boldsymbol{\Delta t}\cdot\boldsymbol{E}_\alpha\frac{\boldsymbol{E}^\alpha\otimes\boldsymbol{E}^3 + \boldsymbol{E}^3\otimes\boldsymbol{E}^\alpha}{2}$$

- Interpretation: Through-the-thickness elongation
  - In this model there is no through-the-thickness elongation

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \bullet \end{pmatrix}$$

  - Actually the plate is in plane-σ state, meaning there is such a deformation
    - » To be introduced: $\xi^3\,\boldsymbol{t}$ should be substituted by $\lambda_h(\xi^3)\,\boldsymbol{t}$ in the shell kinematics
    - » We have to introduce it to get the plane-σ effect
    - » In small deformations this term would lead to second order effects on other components

- Deformations (9)
  - Final expression
    - We had

$$\boldsymbol{\varepsilon} = \frac{\boldsymbol{u}_{,\beta} \cdot \boldsymbol{E}_\alpha + \boldsymbol{u}_{,\alpha} \cdot \boldsymbol{E}_\beta}{2} \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta + \boldsymbol{u}_{,\beta} \cdot \boldsymbol{E}_3 \frac{\boldsymbol{E}^3 \otimes \boldsymbol{E}^\beta + \boldsymbol{E}^\beta \otimes \boldsymbol{E}^3}{2} + \xi^3 \frac{\boldsymbol{\Delta t}_{,\beta} \cdot \boldsymbol{E}_\alpha + \boldsymbol{\Delta t}_{,\alpha} \cdot \boldsymbol{E}_\beta}{2} \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta + \boldsymbol{\Delta t} \cdot \boldsymbol{E}_\alpha \frac{\boldsymbol{E}^\alpha \otimes \boldsymbol{E}^3 + \boldsymbol{E}^3 \otimes \boldsymbol{E}^\alpha}{2}$$

  - With

$$\begin{cases} \varepsilon_{\alpha\beta} = \dfrac{\boldsymbol{u}_{,\beta} \cdot \boldsymbol{E}_\alpha + \boldsymbol{u}_{,\alpha} \cdot \boldsymbol{E}_\beta}{2} \\ \kappa_{\alpha\beta} = \dfrac{\boldsymbol{\Delta t}_{,\beta} \cdot \boldsymbol{E}_\alpha + \boldsymbol{\Delta t}_{,\alpha} \cdot \boldsymbol{E}_\beta}{2} \\ 2\delta_\alpha = \gamma_\alpha = \boldsymbol{u}_{,\alpha} \cdot \boldsymbol{E}_3 + \boldsymbol{\Delta t} \cdot \boldsymbol{E}_\alpha \end{cases}$$

    - And through-the-thickness elongation $\lambda_h(\xi^3)\, \boldsymbol{t}$

$$\Longrightarrow \boldsymbol{\varepsilon} = \varepsilon_{\alpha\beta} \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta + \delta_\alpha \left(\boldsymbol{E}^3 \otimes \boldsymbol{E}^\alpha + \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^3 \right) + \xi^3 \kappa_{\alpha\beta} \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta + \lambda_h \boldsymbol{E}^3 \otimes \boldsymbol{E}^3$$

- Hooke's law
  - Small strain tensor

$$\boldsymbol{\varepsilon} = \varepsilon_{\alpha\beta}\boldsymbol{E}^{\alpha}\otimes\boldsymbol{E}^{\beta}+\delta_{\alpha}\left(\boldsymbol{E}^{3}\otimes\boldsymbol{E}^{\alpha}+\boldsymbol{E}^{\alpha}\otimes\boldsymbol{E}^{3}\right)+\xi^{3}\kappa_{\alpha\beta}\boldsymbol{E}^{\alpha}\otimes\boldsymbol{E}^{\beta}+\lambda_{h}\boldsymbol{E}^{3}\otimes\boldsymbol{E}^{3}$$

  - There are 4 contributions
    - Membrane mode
      - $\boldsymbol{\sigma}_{\varepsilon}^{ij} = \mathcal{H}^{ij\alpha\beta}\varepsilon_{\alpha\beta}$ as frame is orthonormal $\mathcal{H}^{ijkl} = \mathcal{H}_{ijkl}$ (notation abuse)

with $\mathcal{H}_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{1+\nu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$

$$\Longrightarrow \boldsymbol{\sigma}_{\varepsilon}^{ij} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta^{ij}\varepsilon_{\alpha\alpha} + \frac{E}{1+\nu}\varepsilon_{\alpha\beta}\delta^{i\alpha}\delta^{j\beta}$$

      - As $\varepsilon_{\alpha\beta} = \frac{\boldsymbol{u}_{\alpha,\beta}+\boldsymbol{u}_{\beta,\alpha}}{2}$ the non-zero components are

$$\begin{cases} \boldsymbol{\sigma}_{\varepsilon}^{\alpha\beta} = \frac{E\nu}{(1+\nu)(1-2\nu)}\boldsymbol{u}_{\gamma,\gamma}\delta^{\alpha\beta} + \frac{E}{1+\nu}\frac{\boldsymbol{u}_{\alpha,\beta}+\boldsymbol{u}_{\beta,\alpha}}{2} \\ \boldsymbol{\sigma}_{\varepsilon}^{33} = \frac{E\nu}{(1+\nu)(1-2\nu)}\boldsymbol{u}_{\gamma,\gamma} \end{cases}$$

- Hooke's law (2)
  - There are 4 contributions (2)
    - Bending mode
      - $\boldsymbol{\sigma}_{\kappa}^{ij} = \mathcal{H}^{ij\alpha\beta}\xi^3\kappa_{\alpha\beta}$

        with $\mathcal{H}_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{1+\nu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$

        $$\Longrightarrow \boldsymbol{\sigma}_{\kappa}^{ij} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta^{ij}\xi^3\kappa_{\alpha\alpha} + \frac{E}{1+\nu}\xi^3\kappa_{\alpha\beta}\delta^{i\alpha}\delta^{j\beta}$$

      - As $\kappa_{\alpha\beta} = \frac{\boldsymbol{\Delta t}_{\alpha,\beta} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2}$ the non-zero components are

        $$\begin{cases} \boldsymbol{\sigma}_{\kappa}^{\alpha\beta} = \frac{E\nu}{(1+\nu)(1-2\nu)}\xi^3\boldsymbol{\Delta t}_{\gamma,\gamma}\delta^{\alpha\beta} + \frac{E}{1+\nu}\xi^3\frac{\boldsymbol{\Delta t}_{\alpha,\beta} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} \\ \boldsymbol{\sigma}_{\kappa}^{33} = \frac{E\nu}{(1+\nu)(1-2\nu)}\xi^3\boldsymbol{\Delta t}_{\gamma,\gamma} \end{cases}$$

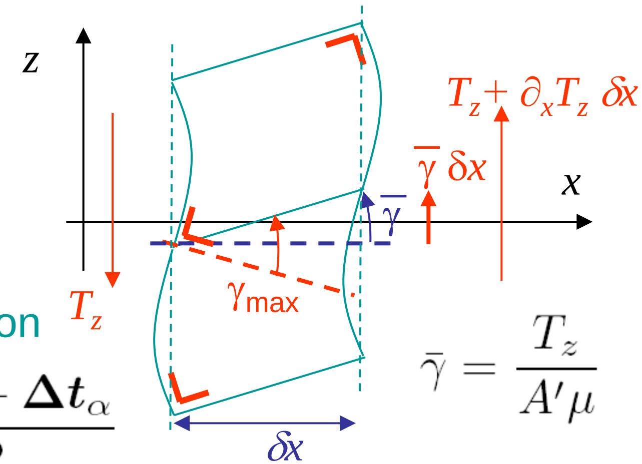
- Hooke's law (3)
  - There are 4 contributions (3)
    - Shearing mode
      - $\boldsymbol{\sigma}_{\delta}^{ij} = \mathcal{H}^{ij3\alpha}\delta_{\alpha} + \mathcal{H}^{ij\alpha 3}\delta_{\alpha}$

        with $\mathcal{H}_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{1+\nu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$

        $\Longrightarrow \boldsymbol{\sigma}_{\delta}^{ij} = \frac{E}{1+\nu}\delta_{\alpha}\left(\delta^{i\alpha}\delta^{j3} + \delta^{i3}\delta^{j\alpha}\right)$
      - As $2\delta_{\alpha} = \gamma_{\alpha} = \boldsymbol{u}_{,\alpha}\cdot\boldsymbol{E}_3 + \boldsymbol{\Delta t}\cdot\boldsymbol{E}_{\alpha}$

        we can deduce the non-zero components

        $$\boldsymbol{\sigma}_{\delta}^{\alpha 3} = \boldsymbol{\sigma}_{\delta}^{3\alpha} = \frac{E}{1+\nu}\frac{\boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_{\alpha}}{2}$$
      - As we did for beams, we have to account for the non-uniformity of γ by a shear section reduction $\boldsymbol{\sigma}_{\delta}^{\alpha 3} = \boldsymbol{\sigma}_{\delta}^{3\alpha} = \frac{E}{1+\nu}\frac{A'}{A}\frac{\boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_{\alpha}}{2}$



$$\bar{\gamma} = \frac{T_z}{A'\mu}$$

- Hooke's law (4)
  - There are 4 contributions (4)
    - Through-the thickness elongation
      - $\boldsymbol{\sigma}_{\lambda}^{ij} = \mathcal{H}^{ij33}\lambda_h$

        with $\mathcal{H}_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{1+\nu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$

        $$\Longrightarrow \boldsymbol{\sigma}_{\lambda}^{ij} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta^{ij}\lambda_h + \frac{E}{1+\nu}\lambda_h\delta^{i3}\delta^{j3}$$

      - We can deduce the non-zero components

$$\begin{cases} \boldsymbol{\sigma}_{\lambda}^{\alpha\beta} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta^{\alpha\beta}\lambda_h \\ \boldsymbol{\sigma}_{\lambda}^{33} = \left(\frac{E\nu}{(1+\nu)(1-2\nu)} + \frac{E}{1+\nu}\right)\lambda_h \end{cases}$$

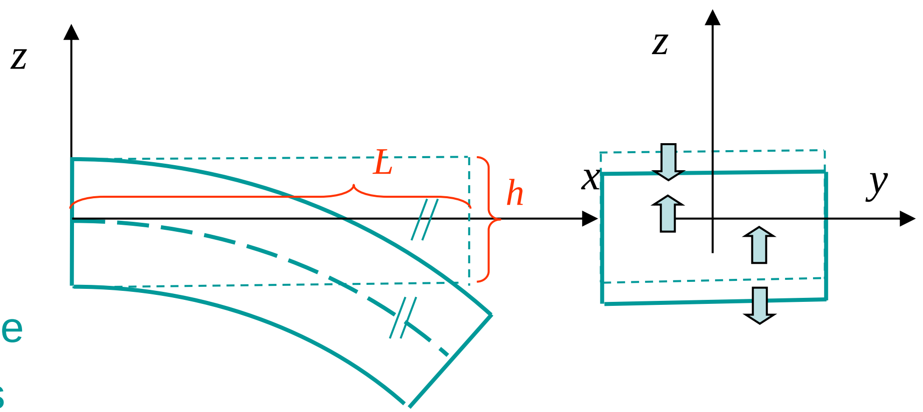
$$\Longrightarrow \boldsymbol{\sigma}_{\lambda}^{33} = E\frac{1-\nu}{(1+\nu)(1-2\nu)}\lambda_h$$

- Resultant stresses
  - Plane-σ state
    - Contributions

$$
\left\{
\begin{aligned}
\boldsymbol{\sigma}_{\varepsilon}^{33} &= \frac{E\nu}{(1+\nu)(1-2\nu)}\boldsymbol{u}_{\gamma,\gamma} \\
\boldsymbol{\sigma}_{\kappa}^{33} &= \frac{E\nu}{(1+\nu)(1-2\nu)}\xi^3 \boldsymbol{\Delta t}_{\gamma,\gamma} \\
\boldsymbol{\sigma}_{\lambda}^{33} &= E\frac{1-\nu}{(1+\nu)(1-2\nu)}\lambda_h
\end{aligned}
\right.
$$

$$\Rightarrow \lambda_h = \frac{\nu}{\nu - 1}\left(\boldsymbol{u}_{\gamma,\gamma} + \xi^3 \boldsymbol{\Delta t}_{\gamma,\gamma}\right)$$

- Elongation depends on ξ^3
  - Part is stretched and part is compressed
  - For pure bending the change of sign is on the neutral axis



- Average trough the thickness elongation

$$\frac{1}{h_0}\int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \lambda_h d\xi^3 = \frac{1}{h_0}\int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \frac{\nu}{\nu-1}\boldsymbol{u}_{\gamma,\gamma} d\xi^3$$

  - Depends on the membrane mode only

- Resultant stresses (2)
  - Plane σ-stated (2)
    - Values αβ can now be deduced
      - $$\begin{cases} \boldsymbol{\sigma}_{\varepsilon}^{\alpha\beta} = \frac{E\nu}{(1+\nu)(1-2\nu)} \boldsymbol{u}_{\gamma,\gamma} \delta^{\alpha\beta} + \frac{E}{1+\nu} \frac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2} \\ \boldsymbol{\sigma}_{\kappa}^{\alpha\beta} = \frac{E\nu}{(1+\nu)(1-2\nu)} \xi^3 \boldsymbol{\Delta t}_{\gamma,\gamma} \delta^{\alpha\beta} + \frac{E}{1+\nu} \xi^3 \frac{\boldsymbol{\Delta t}_{\alpha,\beta} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} \\ \boldsymbol{\sigma}_{\lambda}^{\alpha\beta} = \frac{E\nu}{(1+\nu)(1-2\nu)} \delta^{\alpha\beta} \lambda_h \quad \text{with} \quad \lambda_h = \frac{\nu}{\nu - 1} \left(\boldsymbol{u}_{\gamma,\gamma} + \xi^3 \boldsymbol{\Delta t}_{\gamma,\gamma} \right) \end{cases}$$
    - Can be rewritten as a through-the-thickness constant term and a linear term
      - $\boldsymbol{\sigma}^{\alpha\beta} = \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} + \xi^3 \boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta}$
      - With $$\begin{cases} \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} = \frac{E\nu}{(1-\nu^2)} \boldsymbol{u}_{\gamma,\gamma} \delta^{\alpha\beta} + \frac{E}{1+\nu} \frac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2} \\ \boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta} = \frac{E\nu}{(1-\nu^2)} \boldsymbol{\Delta t}_{\gamma,\gamma} \delta^{\alpha\beta} + \frac{E}{1+\nu} \frac{\boldsymbol{\Delta t}_{\alpha,\beta} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} \end{cases}$$
    - Out-of-plane shearing remains the same
      - $$\boldsymbol{\sigma}_{\delta}^{\alpha 3} = \boldsymbol{\sigma}_{\delta}^{3\alpha} = \frac{E}{1+\nu} \frac{A'}{A} \frac{\boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_{\alpha}}{2}$$

- Resultant stresses (3)
  - From stress fields
    - $\boldsymbol{\sigma}^{\alpha\beta} = \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} + \xi^3 \boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta}$
    - $\boldsymbol{\sigma}_{\delta}^{\alpha 3} = \boldsymbol{\sigma}_{\delta}^{3\alpha} = \dfrac{E}{1+\nu} \dfrac{A'}{A} \dfrac{\boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_{\alpha}}{2}$
  - Membrane resultant stresses
    - $\boldsymbol{n}^{\alpha} = \displaystyle\int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^{\alpha} d\xi^3$

$$\Longrightarrow \boldsymbol{n}^{\alpha} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}^{\alpha I} \boldsymbol{E}_I d\xi^3 = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} \boldsymbol{E}_{\beta} d\xi^3 + \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}_{\delta}^{\alpha 3} \boldsymbol{E}_3 d\xi^3$$

$$\Longrightarrow \boldsymbol{n}^{\alpha} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} d\xi^3 \boldsymbol{E}_{\beta} + \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}_{\delta}^{\alpha 3} d\xi^3 \boldsymbol{E}_3 = \tilde{n}^{\alpha\beta} \boldsymbol{E}_{\beta} + \tilde{q}^{\alpha} \boldsymbol{E}_3$$

In plane membrane stress

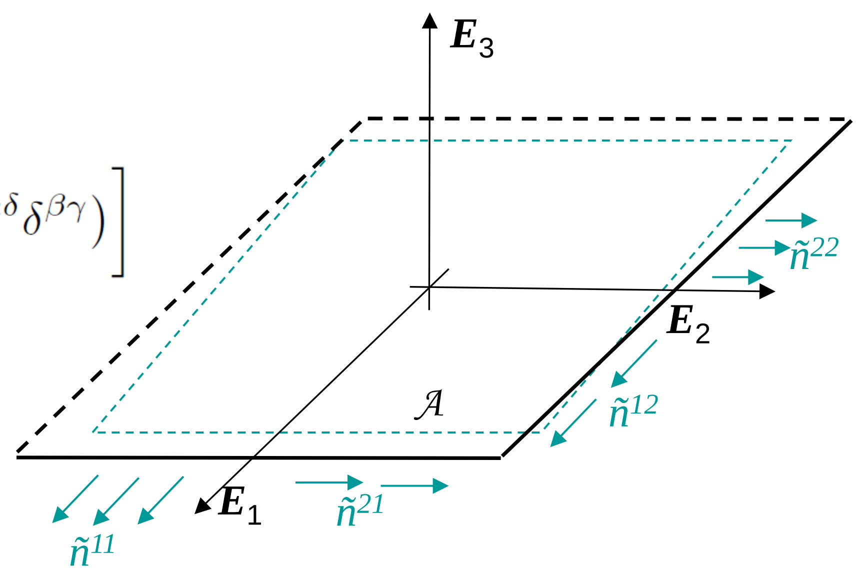
Out-of plane shear stress

Material law

- Resultant stresses (4)
  - Membrane resultant stresses (2)
    - $$\boldsymbol{n}^{\alpha} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} d\xi^3 \boldsymbol{E}_{\beta} + \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}_{\delta}^{\alpha 3} d\xi^3 \boldsymbol{E}_3 = \tilde{n}^{\alpha\beta} \boldsymbol{E}_{\beta} + \tilde{q}^{\alpha} \boldsymbol{E}_3$$
    - In plane component: membrane stress
      - As $\boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} = \frac{E\nu}{(1-\nu^2)} \boldsymbol{u}_{\gamma,\gamma} \delta^{\alpha\beta} + \frac{E}{1+\nu} \frac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2}$ is cst with ξ^3

        $$\Rightarrow \tilde{n}^{\alpha\beta} = h_0 \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} = \frac{E\nu h_0}{(1-\nu^2)} \boldsymbol{u}_{\gamma,\gamma} \delta^{\alpha\beta} + \frac{E h_0}{1+\nu} \frac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2} = \mathcal{H}_n^{\alpha\beta\gamma\delta} \varepsilon_{\alpha\beta}$$
      - Defining the membrane Hooke tensor
        » $$\mathcal{H}_n^{\alpha\beta\gamma\delta} = \frac{h_0 E}{1-\nu^2} \left[\nu \delta^{\alpha\beta} \delta^{\gamma\delta} + \frac{1-\nu}{2} \left(\delta^{\alpha\gamma} \delta^{\beta\delta} + \delta^{\alpha\delta} \delta^{\beta\gamma} \right) \right]$$
        » As $\varepsilon_{\alpha\beta} = \frac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2}$

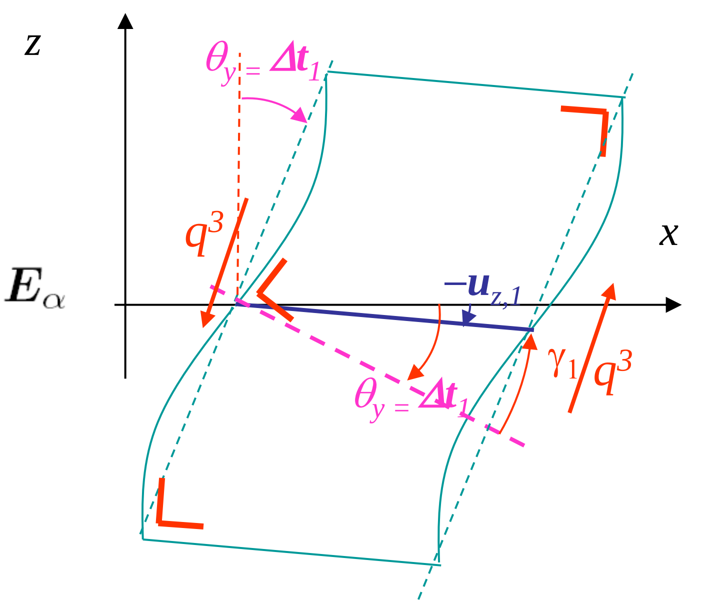
        $$\Rightarrow \boxed{\tilde{n}^{\alpha\beta} = \mathcal{H}_n^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta}}$$
      - 2D plane-σ problem



- # Resultant stresses (5)
  - Membrane resultant stresses (3)
    - $$\boldsymbol{n}^{\alpha} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} d\xi^3 \boldsymbol{E}_{\beta} + \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}_{\delta}^{\alpha 3} d\xi^3 \boldsymbol{E}_3 = \tilde{n}^{\alpha\beta} \boldsymbol{E}_{\beta} + \tilde{q}^{\alpha} \boldsymbol{E}_3$$
    - Out-of-plane component: shear stress
      - As $\boldsymbol{\sigma}_{\delta}^{\alpha 3} = \boldsymbol{\sigma}_{\delta}^{3\alpha} = \frac{E}{1+\nu} \frac{A'}{A} \frac{\boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_{\alpha}}{2}$ is cst with ξ^3

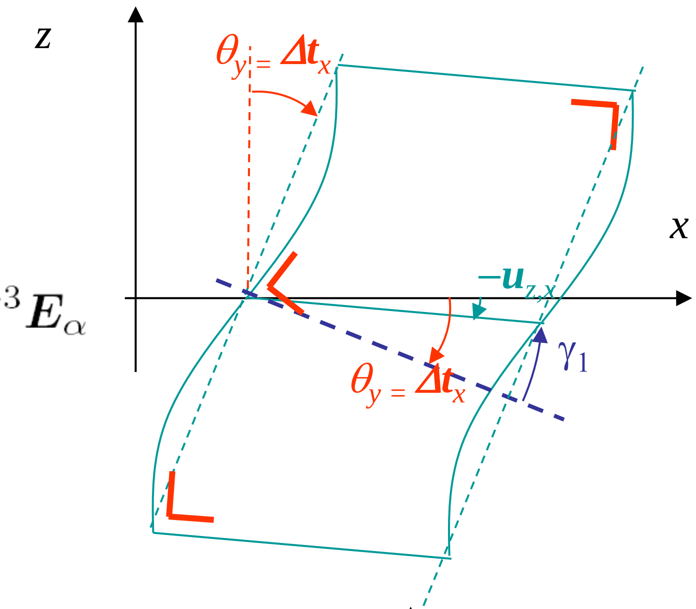
        $$\Longrightarrow \tilde{q}^{\alpha} = h_0 \boldsymbol{\sigma}_{\delta}^{\alpha 3} = \frac{E h_0}{1+\nu} \frac{A'}{A} \frac{\boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_{\alpha}}{2}$$
      - Defining the shearing Hooke tensor
        - » $\mathcal{H}_q^{\alpha\beta} = \frac{E h_0}{1+\nu} \frac{A'}{A} \delta^{\alpha\beta}$
        - » As $2\delta_{\alpha} = \gamma_{\alpha} = \boldsymbol{u}_{,\alpha} \cdot \boldsymbol{E}_3 + \boldsymbol{\Delta t} \cdot \boldsymbol{E}_{\alpha}$

        $$\Longrightarrow \boxed{\tilde{q}^{\alpha} = \mathcal{H}_q^{\alpha\beta} \delta_{\beta} = \frac{1}{2} \mathcal{H}_q^{\alpha\beta} \gamma_{\beta}}$$
      - Corresponds to the average shearing of a beam

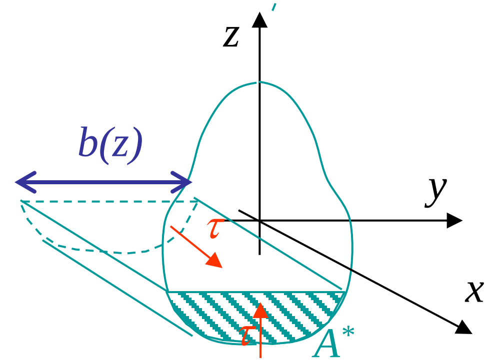


- Resultant stresses (6)
  - Out-of-plane resultant stresses
    - We defined $\boldsymbol{n}^3 = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^3 d\xi^3$

$$\Longrightarrow \boldsymbol{n}^3 = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}^{\alpha 3} d\xi^3 \boldsymbol{E}_\alpha = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}_\delta^{\alpha 3} d\xi^3 \boldsymbol{E}_\alpha$$



- Owing to previous definitions
  - » $\boldsymbol{n}^3 = h_0 \boldsymbol{\sigma}_{\tilde{q}}^{\alpha 3} \boldsymbol{E}_\alpha = \tilde{q}^\alpha \boldsymbol{E}_\alpha$
  - » With $\tilde{q}^\alpha = \mathcal{H}_q^{\alpha\beta} \delta_\beta = \frac{1}{2} \mathcal{H}_q^{\alpha\beta} \gamma_\beta$



- Corresponds to the shearing symmetrical to the out-of-plane shearing

- Resultant stresses (7)
  - From stress fields (2)
    - $\boldsymbol{\sigma}^{\alpha\beta} = \boldsymbol{\sigma}_{\tilde{n}}^{\alpha\beta} + \xi^3 \boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta}$
    - $\boldsymbol{\sigma}_{\delta}^{\alpha 3} = \boldsymbol{\sigma}_{\delta}^{3\alpha} = \frac{E}{1+\nu} \frac{A'}{A} \frac{\boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_{\alpha}}{2}$
  - Bending resultant stresses
    - $\tilde{\boldsymbol{m}}^{\alpha} = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma} \cdot \boldsymbol{E}^{\alpha} \xi^3 d\xi^3 = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \boldsymbol{\sigma}^{\alpha I} \boldsymbol{E}_I \xi^3 d\xi^3 = \boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta} \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \left(\xi^3\right)^2 d\xi^3 \boldsymbol{E}_{\beta}$

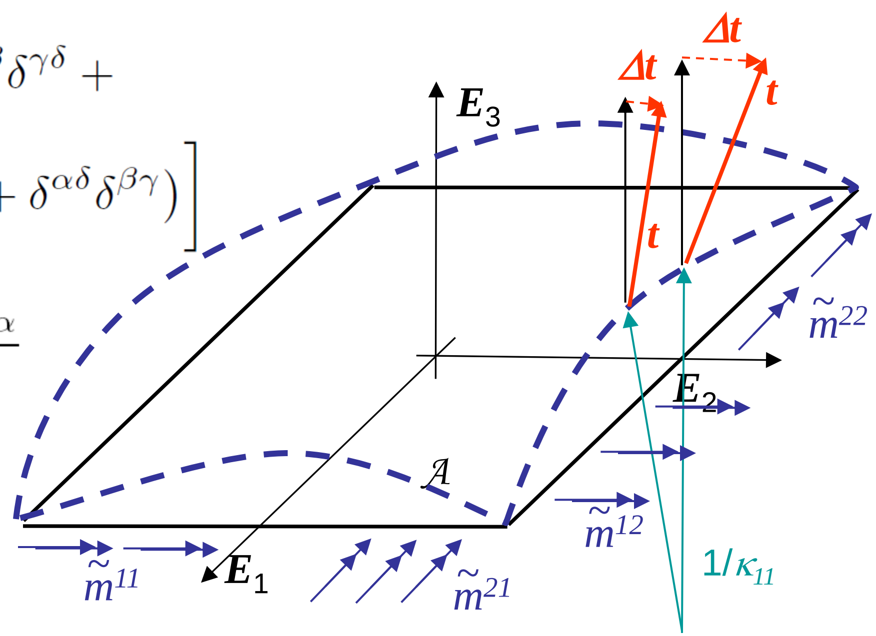
$$\Longrightarrow \tilde{\boldsymbol{m}}^{\alpha} = \frac{h_0^3}{12} \boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta} \boldsymbol{E}_{\beta} = \tilde{m}^{\alpha\beta} \boldsymbol{E}_{\beta}$$

Bending stress

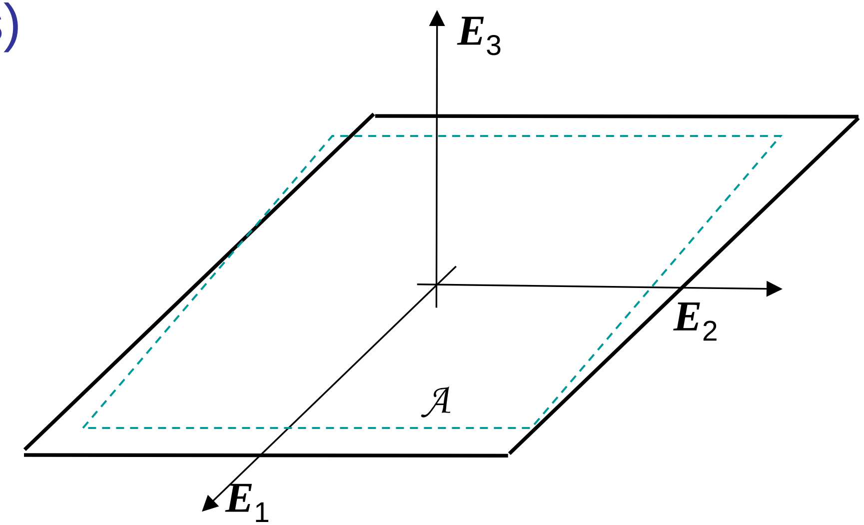
- Resultant stresses (8)
  - Bending resultant stresses (2)
    - $\tilde{\boldsymbol{m}}^{\alpha} = \frac{h_0^3}{12}\boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta}\boldsymbol{E}_{\beta} = \tilde{m}^{\alpha\beta}\boldsymbol{E}_{\beta}$
      - As $\boldsymbol{\sigma}_{\tilde{m}}^{\alpha\beta} = \frac{E\nu}{(1-\nu^2)}\boldsymbol{\Delta t}_{\gamma,\gamma}\delta^{\alpha\beta} + \frac{E}{1+\nu}\frac{\boldsymbol{\Delta t}_{\alpha,\beta}+\boldsymbol{\Delta t}_{\beta,\alpha}}{2}$

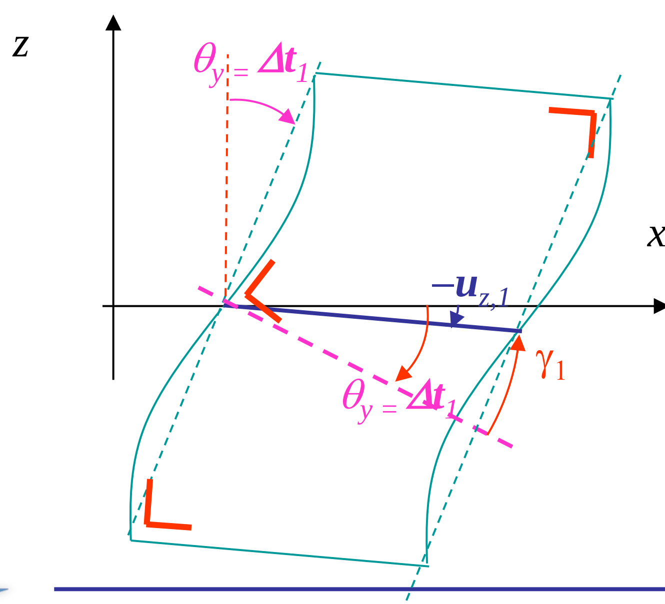
        $$\Longrightarrow \tilde{m}^{\alpha\beta} = \frac{h_0^3 E\nu}{12\,(1-\nu^2)}\boldsymbol{\Delta t}_{\gamma,\gamma}\delta^{\alpha\beta} + \frac{h_0^3 E}{12\,(1+\nu)}\frac{\boldsymbol{\Delta t}_{\alpha,\beta}+\boldsymbol{\Delta t}_{\beta,\alpha}}{2}$$
      - Defining the bending Hooke tensor
        - » $\mathcal{H}_m^{\alpha\beta\gamma\delta} = \frac{h_0^3 E}{12\,(1-\nu^2)}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta} + \frac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta}+\delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$
        - » As $\kappa_{\alpha\beta} = \frac{\boldsymbol{\Delta t}_{\alpha,\beta}+\boldsymbol{\Delta t}_{\beta,\alpha}}{2}$

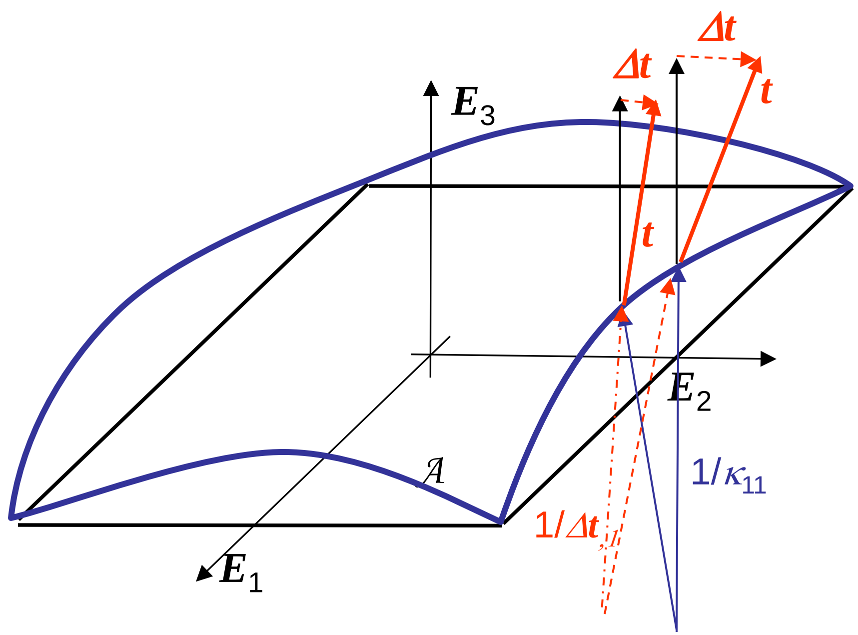
        $$\Longrightarrow \boxed{\tilde{m}^{\alpha\beta} = \mathcal{H}_m^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta}}$$
      - 2D bending problem



- Deformations (small transformations)
  - In plane membrane
    - $\boxed{\varepsilon_{\alpha\beta} = \dfrac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2}}$
  - Curvature
    - $\boxed{\kappa_{\alpha\beta} = \dfrac{\boldsymbol{\Delta t}_{\alpha,\beta} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2}}$
  - Out-of-plane sliding
    - $\boxed{\gamma_{\alpha} = \boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_{\alpha}}$

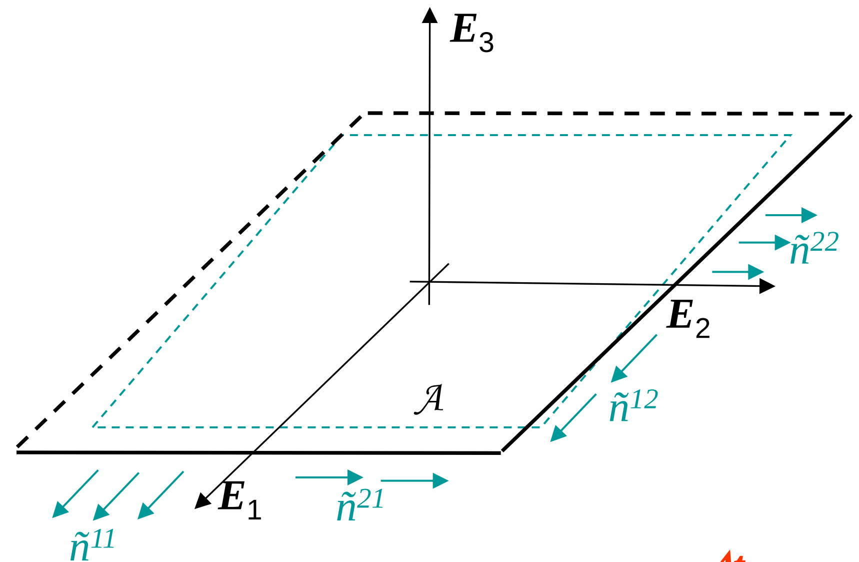


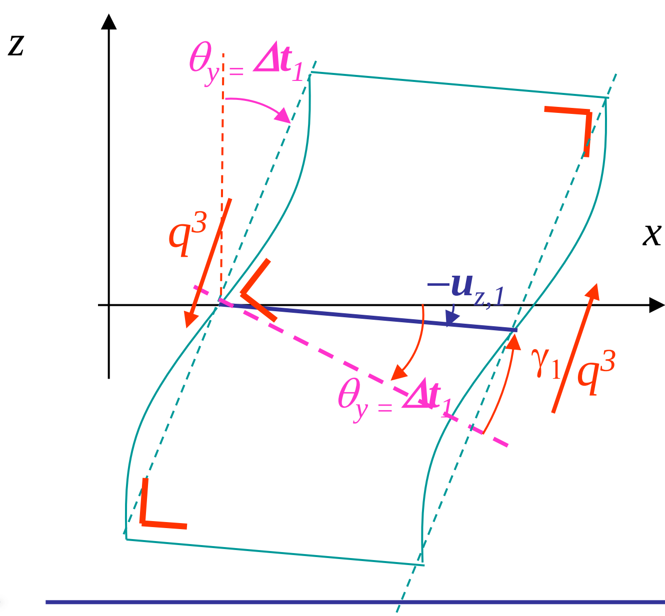


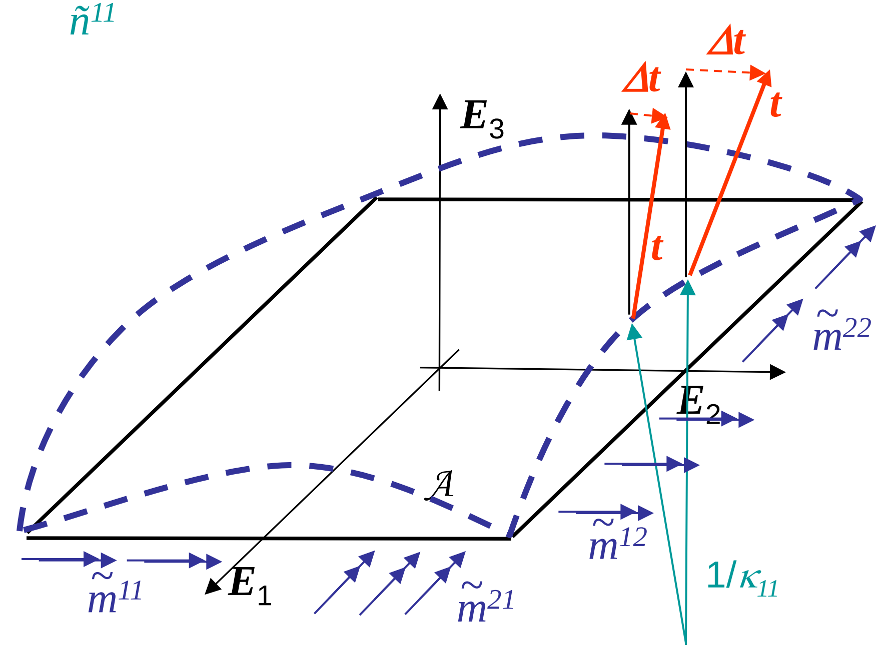


Reissner-Mindlin equations summary

- Resultant stresses in linear elasticity
  - Membrane stress
    - $\boxed{\tilde{n}^{\alpha\beta} = \mathcal{H}_n^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta}}$
  - Bending stress
    - $\boxed{\tilde{m}^{\alpha\beta} = \mathcal{H}_m^{\alpha\beta\gamma\delta} \kappa_{\gamma\delta}}$
  - Out-of-plane shear stress
    - $\boxed{\tilde{q}^{\alpha} = \frac{1}{2} \mathcal{H}_q^{\alpha\beta} \gamma_\beta}$

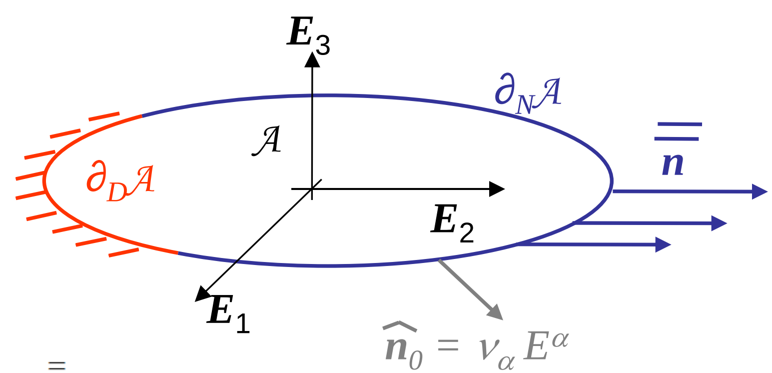






- Resultant equations
  - Membrane mode
    - $\boxed{\left(\boldsymbol{n}^{\alpha}\right)_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}\ddot{\boldsymbol{u}}}$
    - $\boldsymbol{n}^{\alpha} = \tilde{n}^{\alpha\beta}\boldsymbol{E}_{\beta} + \tilde{q}^{\alpha}\boldsymbol{E}_{3}$
      - $\tilde{n}^{\alpha\beta} = \mathcal{H}_n^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}$ with $\varepsilon_{\alpha\beta} = \dfrac{\boldsymbol{u}_{\alpha,\beta} + \boldsymbol{u}_{\beta,\alpha}}{2}$
      - $\tilde{q}^{\alpha} = \dfrac{1}{2}\mathcal{H}_q^{\alpha\beta}\gamma_{\beta}$ with $\gamma_{\alpha} = \boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_{\alpha}$
    - Clearly, the solution can be directly computed in plane Oxy (constant $\mathcal{H}_n$)
      - $\mathcal{H}_n^{\alpha\beta\gamma\delta}\dfrac{\boldsymbol{u}_{\gamma,\delta\alpha} + \boldsymbol{u}_{\delta,\gamma\alpha}}{2} + \bar{\boldsymbol{n}}_{\beta} = \bar{\rho}\ddot{\boldsymbol{u}}_{\beta}$
      - Boundary conditions
        - » Dirichlet $\boldsymbol{u}_{\alpha} = \bar{\boldsymbol{u}}_{\alpha}$
        - » Neumann

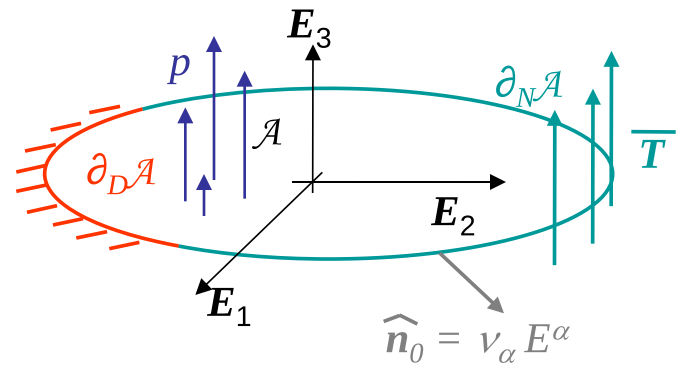
$$\boldsymbol{n}_{\beta}^{\alpha}\nu_{\alpha} = \mathcal{H}_n^{\alpha\beta\gamma\delta}\frac{\boldsymbol{u}_{\gamma,\delta} + \boldsymbol{u}_{\delta,\gamma}}{2}\nu_{\alpha} = \bar{\bar{\boldsymbol{n}}}_{\beta}$$



- Remaining equation along E^3: $\mathcal{H}_q^{\alpha\beta}\dfrac{\boldsymbol{u}_{3,\beta\alpha} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} + \bar{\boldsymbol{n}}_3 = \bar{\rho}\ddot{\boldsymbol{u}}_3$

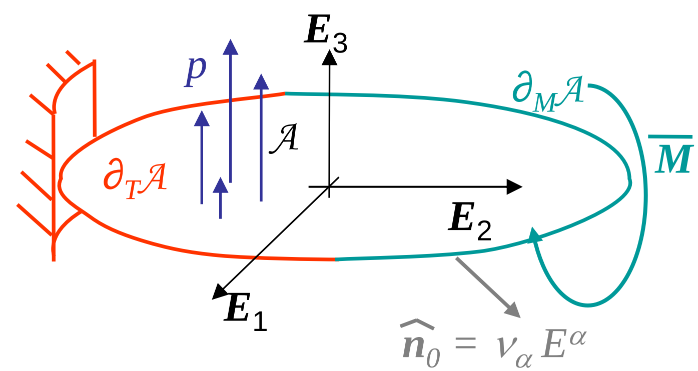
- Resultant equations (2)
  - Bending mode
    - $\boxed{\ddot{\boldsymbol{t}} I_p = \bar{\tilde{\boldsymbol{m}}} - \left(\boldsymbol{n}^3 - \lambda \boldsymbol{E}_3\right) + \left(\tilde{\boldsymbol{m}}^\alpha\right)_{,\alpha}}$
    - $\tilde{\boldsymbol{m}}^\alpha = \tilde{m}^{\alpha\beta} \boldsymbol{E}_\beta$ & $\boldsymbol{n}^3 = \tilde{q}^\alpha \boldsymbol{E}_\alpha$
      - $\tilde{m}^{\alpha\beta} = \mathcal{H}_m^{\alpha\beta\gamma\delta} \kappa_{\gamma\delta}$ with $\kappa_{\alpha\beta} = \dfrac{\boldsymbol{\Delta t}_{\alpha,\beta} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2}$
      - $\tilde{q}^\alpha = \dfrac{1}{2} \mathcal{H}_q^{\alpha\beta} \gamma_\beta$ with $\gamma_\alpha = \boldsymbol{u}_{3,\alpha} + \boldsymbol{\Delta t}_\alpha$
    - Solution is obtained by projecting into the plane Oxy (constant $\mathcal{H}_q$, $\mathcal{H}_m$)
      - $I_p \ddot{\boldsymbol{\Delta t}}_\alpha = \bar{\tilde{\boldsymbol{m}}}_\alpha - \dfrac{1}{2} \mathcal{H}_q^{\alpha\beta} \left(\boldsymbol{u}_{3,\beta} + \boldsymbol{\Delta t}_\beta\right) + \mathcal{H}_m^{\alpha\beta\gamma\delta} \dfrac{\boldsymbol{\Delta t}_{\gamma,\delta\beta} + \boldsymbol{\Delta t}_{\delta,\gamma\beta}}{2}$
      - 2 equations (α=1, 2) with 3 unknowns ($\Delta \boldsymbol{t}_1$, $\Delta \boldsymbol{t}_2$, $\boldsymbol{u}_3$)
      - Use remaining equation $\mathcal{H}_q^{\alpha\beta} \dfrac{\boldsymbol{u}_{3,\beta\alpha} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} + \bar{\boldsymbol{n}}_3 = \bar{\rho} \ddot{\boldsymbol{u}}_3$

- Resultant equations (3)
  - Bending mode (2)
    - 3 equations with 3 unknowns
      - $I_p \ddot{\boldsymbol{\Delta t}}_\alpha = \bar{\tilde{\boldsymbol{m}}}_\alpha - \frac{1}{2}\mathcal{H}_q^{\alpha\beta}\left(\boldsymbol{u}_{3,\beta} + \boldsymbol{\Delta t}_\beta\right) + \mathcal{H}_m^{\alpha\beta\gamma\delta}\frac{\boldsymbol{\Delta t}_{\gamma,\delta\beta} + \boldsymbol{\Delta t}_{\delta,\gamma\beta}}{2}$
      - $\mathcal{H}_q^{\alpha\beta}\frac{\boldsymbol{u}_{3,\beta\alpha} + \boldsymbol{\Delta t}_{\beta,\alpha}}{2} + \bar{\boldsymbol{n}}_3 = \bar{\rho}\ddot{\boldsymbol{u}}_3$



    - To be completed by BCs
      - Low order constrains
        - » Displacement $\boldsymbol{u}_3 = \bar{\boldsymbol{u}}_3$ or
        - » Shearing

$$\mathcal{H}_q^{\alpha\beta}\frac{\boldsymbol{u}_{3,\beta} + \boldsymbol{\Delta t}_\beta}{2}\nu_\alpha = \bar{T}$$



      - High order
        - » Rotation $\boldsymbol{\Delta t} = \bar{\boldsymbol{\Delta t}}$ or
        - » Bending

$$\tilde{\boldsymbol{m}}_\beta^\alpha \nu_\alpha = \mathcal{H}_m^{\alpha\beta\gamma\delta}\frac{\boldsymbol{\Delta t}_{\gamma,\delta} + \boldsymbol{\Delta t}_{\delta,\gamma}}{2}\nu_\alpha = \bar{\boldsymbol{M}}_\beta$$

- Membrane problem
  - Similar to 2D elasticity
- Bending problem
  - 3 degree of freedom by nodes
  - Shear locking
    - For reduced thickness ($h/L \rightarrow 0$) the structure is too stiff
    - This results from the fact that for thin thickness $\boldsymbol{u}_{3,\alpha} \rightarrow \Delta \boldsymbol{t}_{\alpha}$ physically
      - Bernoulli assumption for beams
    - Then we have extra constrains but no new degree of freedom
    - The solution found is then zero deformation
  - In order to avoid shear locking
    - Different techniques
    - High order elements
    - Shear strains γ evaluated at particular points (assumed strain method)
      - These values can be formulated in terms of the displacements/rotations degrees of freedom
    - Internal degrees of freedom (enhanced assumed strain method)
    - ….

References

- Lecture notes
  - Aircraft Structures for engineering students, T. H. G. Megson, Butterworth-Heinemann, An imprint of Elsevier Science, 2003, ISBN 0 340 70588 4
- Other references
  - Papers

    - Simo JC, Fox DD. On a stress resultant geometrically exact shell model. Part I: formulation and optimal parametrization. *Computer Methods in Applied Mechanics and Engineering* 1989; **72**:267–304.
    - Simo JC, Fox DD, Rifai MS. On a stress resultant geometrically exact shell model. Part II: the linear theory, computational aspects. *Computer Methods in Applied Mechanics and Engineering* 1989; **73**:53–92.