University of Liège
Aerospace & Mechanical Engineering

Aircraft Structures
Overview

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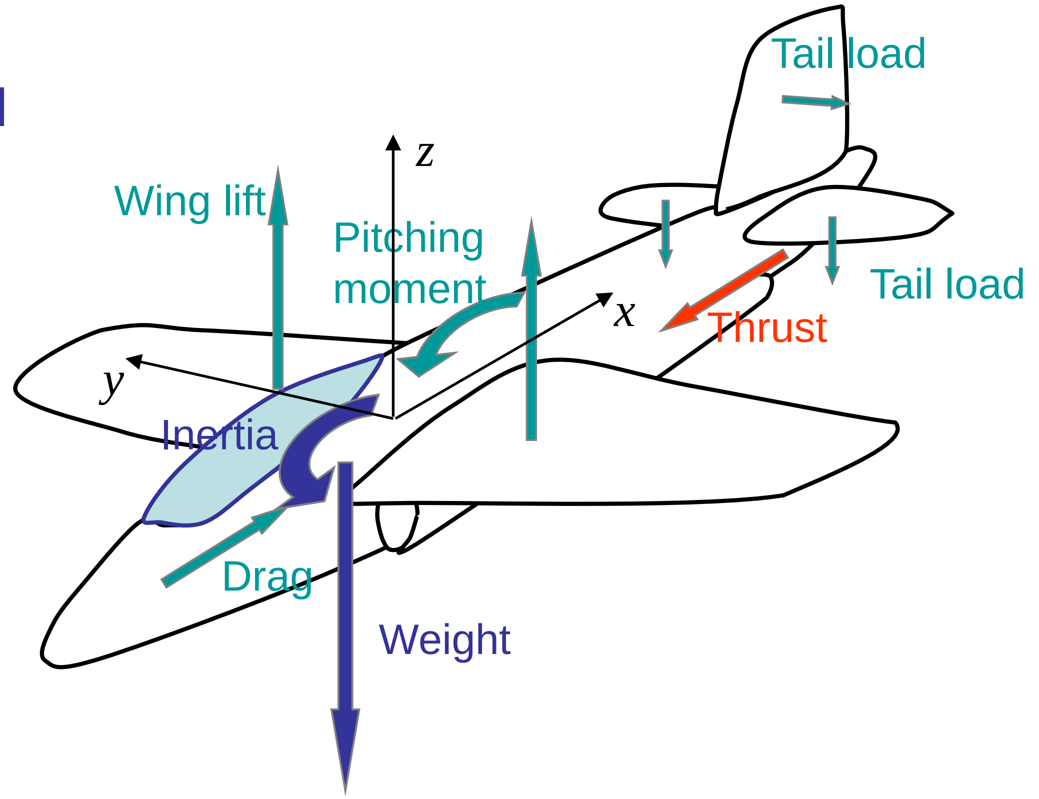
Loading

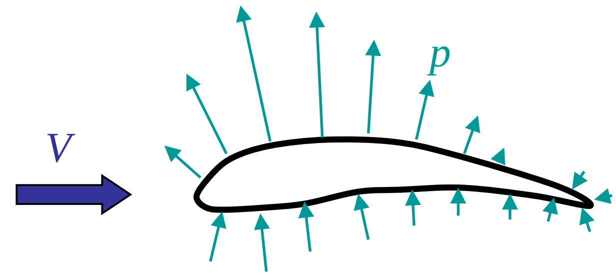
- Primary purpose of the structure
  - To transmit and resist the applied loads
  - To provide an aerodynamic shape
  - To protect passengers, payload, systems

- The structure has to withstand
  - Aerodynamic loadings
  - Thrust
  - Weight and inertial loadings
  - Pressurization cycle
  - Shocks at landing, …

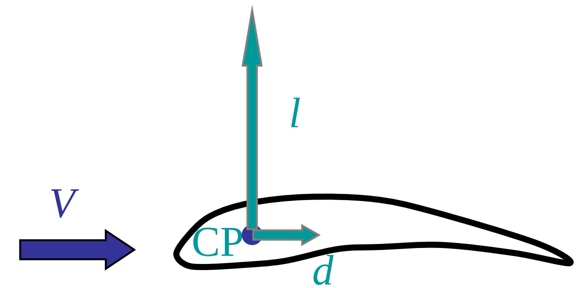


Aerodynamic loading

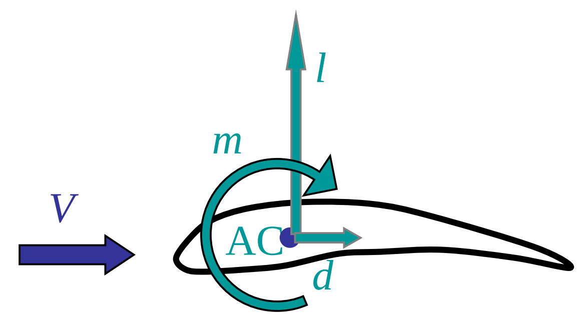
- Example: wing loading
  - Pressure distribution on an airfoil
    - Results from angle of attack and/or camber



  - This distribution can be modeled by
    - A lift (per unit length)
    - A drag (per unit length)
    - Applied at the Center of Pressure (CP)

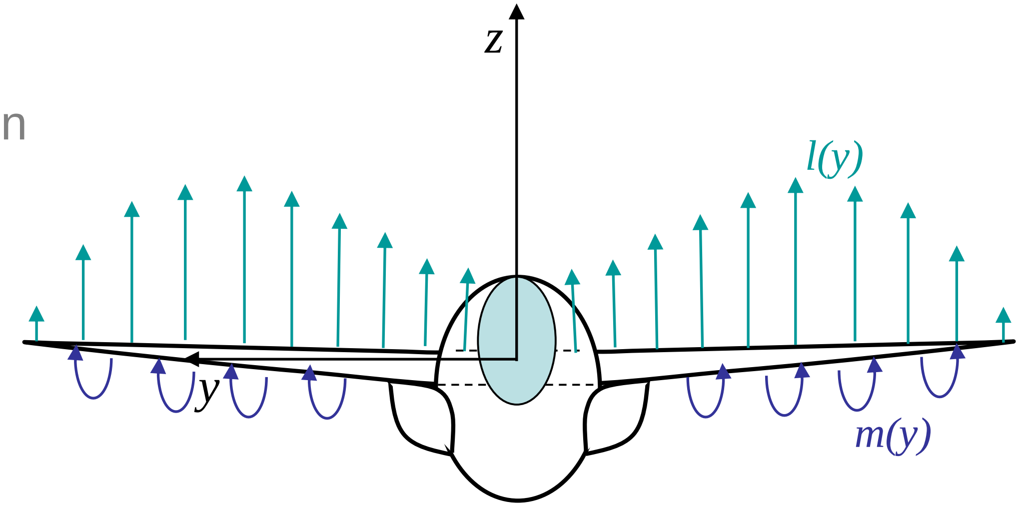


  - As the CP moves with the angle of attack, this is more conveniently modeled by
    - Lift and drag
    - A constant moment
    - Applied at the fixed Aerodynamic Center (AC)
      - Can actually move due to compression effects

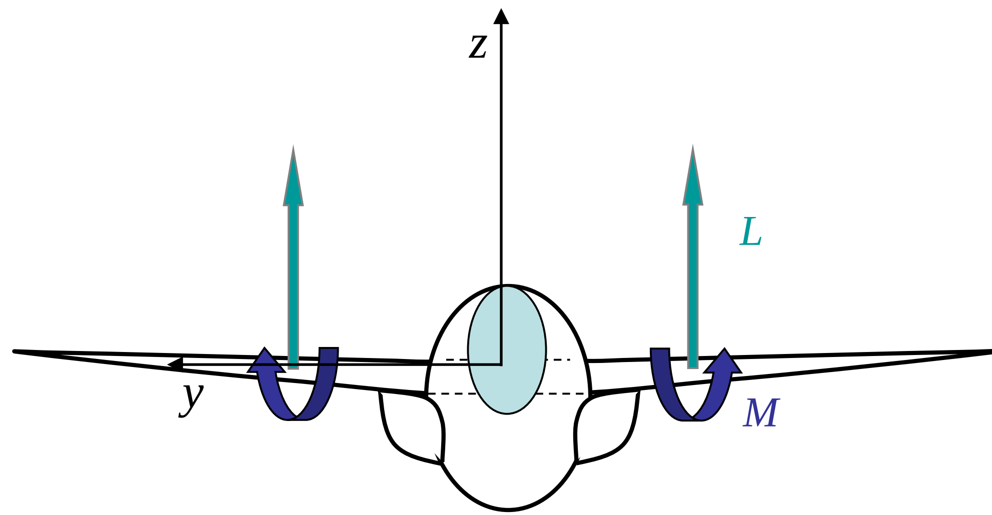


  - As the structural axis is not always at the CP
    - There is a torsion of the wing (particularly when ailerons are actuated)
  - There is always flexion

- Example: wing loading (2)
  - The lift distribution depends on
    - Sweep angle
    - Taper ratio
    - …

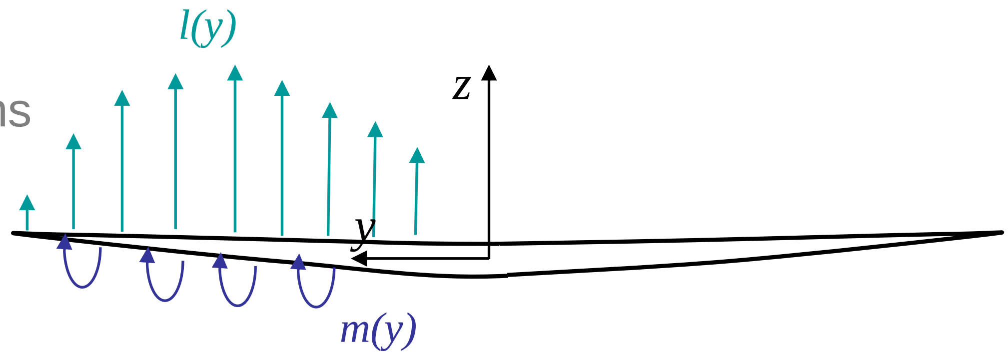


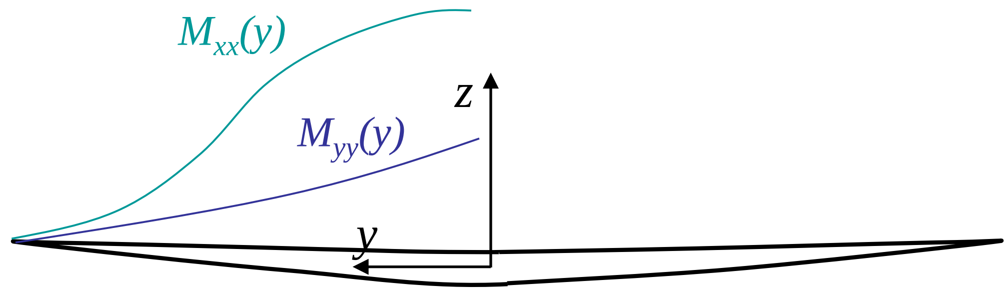
  - Load can be modeled by
    - Lift and moment
    - Applied on the aerodynamic center



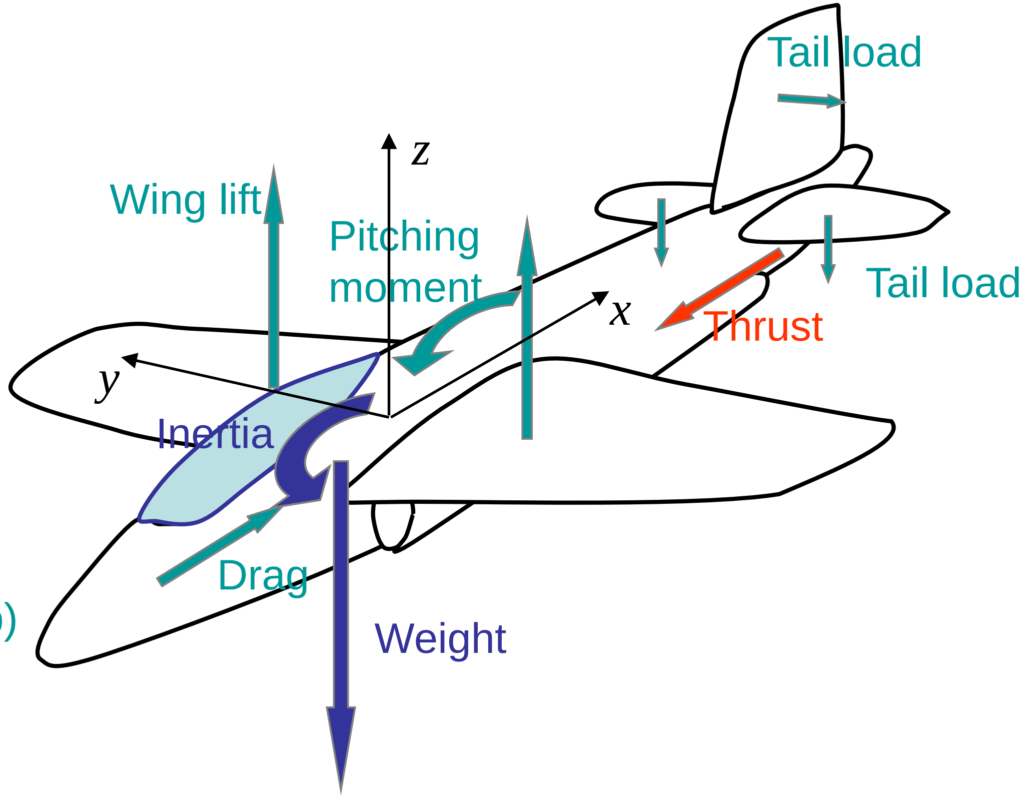
Aerodynamic loading

- Example: wing loading (3)
  - The lift and moment distributions result into
    - A bending moment
      - Due to *l(y)*
    - A torsion
      - Due to *m(y)*
      - Due to the fact that *l(y)* is not applied on the structural axis
    - Which depend on
      - Velocity
      - Altitude
      - Maneuver
      - Surface control actuation
      - Configuration (flaps down or up)
      - Gust
      - Take off weight

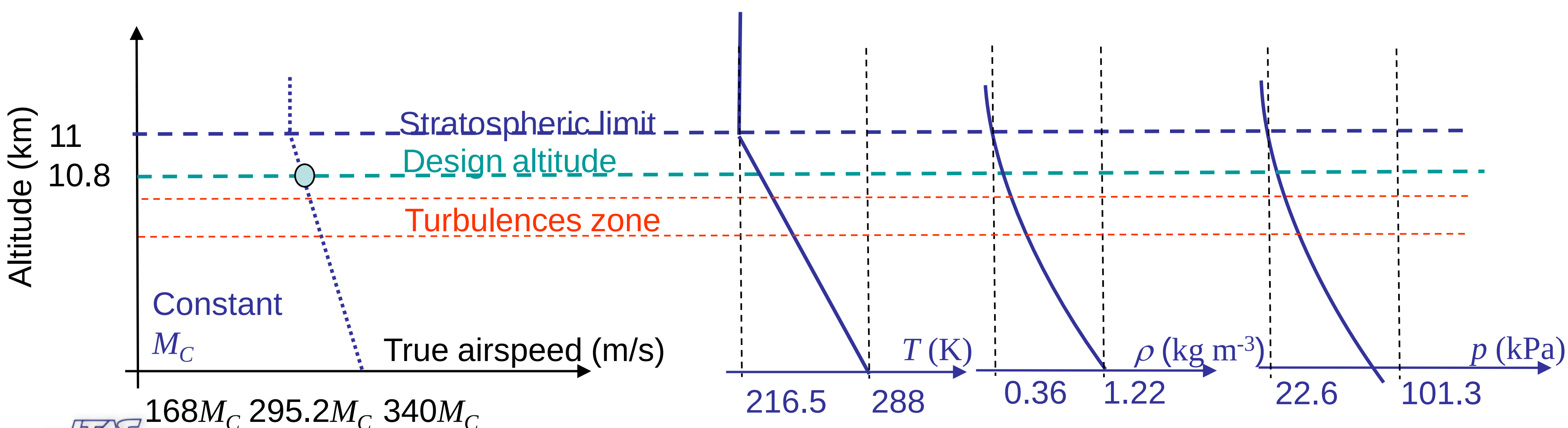




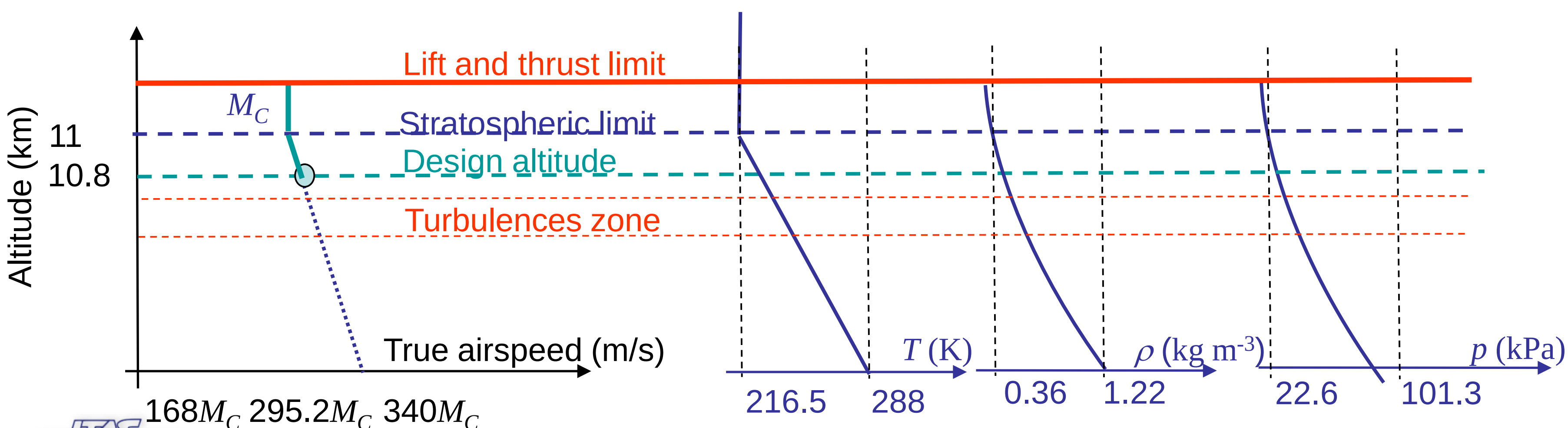
- Load intensity
  - Global loading can be represented by the load factor n (in g-unit)
    - n corresponds to the ratio between
      - The resulting aerodynamic loads perpendicular to the aircraft x-axis
      - The weight
    - When flying: $n \sim L / W$
    - Steady flight: $n = 1$
    - Pullout: n > 1
  - Loading factor depends on
    - Velocity
    - Altitude
    - Maneuver
    - Surface control actuation
    - Configuration (flaps down or up)
    - Gust
    - Take off weight



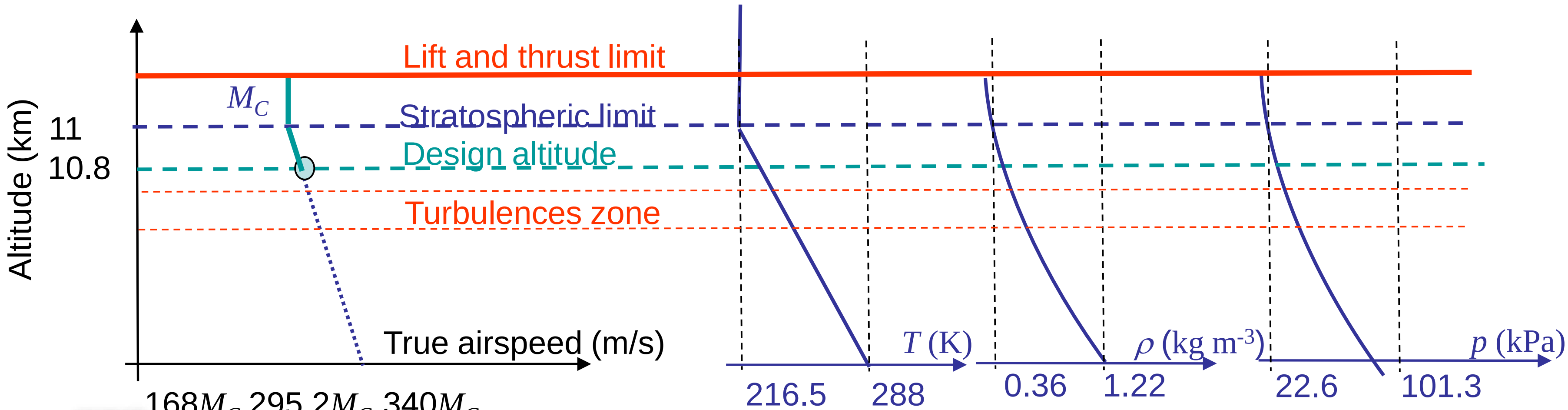
- Placard diagram (Altitude-Velocity dependency)
  - Design altitude
    - High enough to reduce drag (as density decreases with the altitude)
    - Above turbulence zone
  - Design cruise Mach (M_C)
    - Usually maximum operating Mach:
      Mach obtained at maximum engine thrust ⟹ $M_C = M_{mo} \sim 1.06\ M_{cruise}$
    - Temperature evolves linearly with altitude until the stratosphere



- Placard diagram (2)
  - Above the design altitude
    - Although density is reduced, the compressibility effects do not allow flying at higher Mach
    - The plane will fly at the same M_C number
  - Ceiling
    - At high altitude the density is too small
      - The wing cannot produce the required lift
      - The engines cannot produce the required thrust

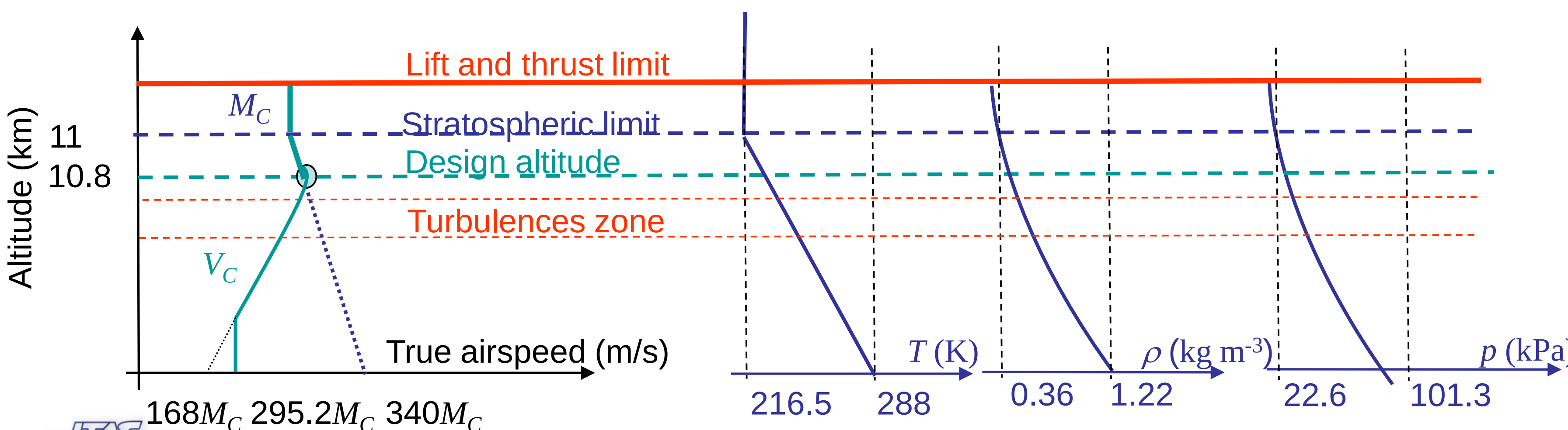


- Placard diagram (3)
  - 1957, Lockheed U2
    - Ceiling 21 km (70000 ft)
    - Only one engine
    - AR ~ 10
    - Stall speed close to maximum speed

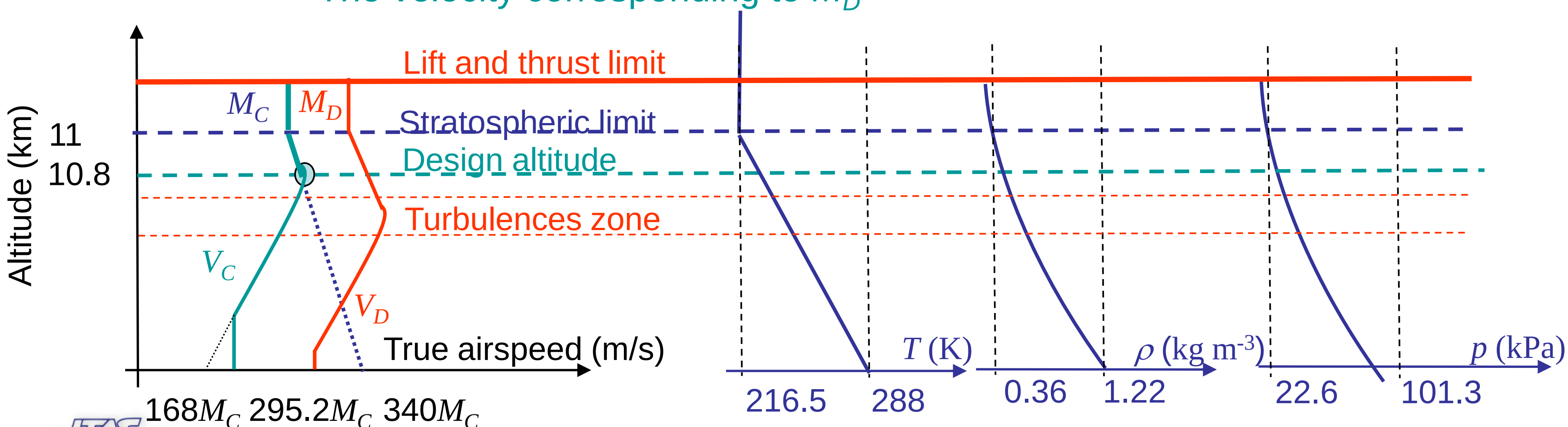


- Placard diagram (4)
  - Below design altitude, when getting closer to the sea level
    - Density increases
      - Engines cannot deliver enough thrust to maintain M_C (drag increases with ρ)
      - Drag has to be kept constant

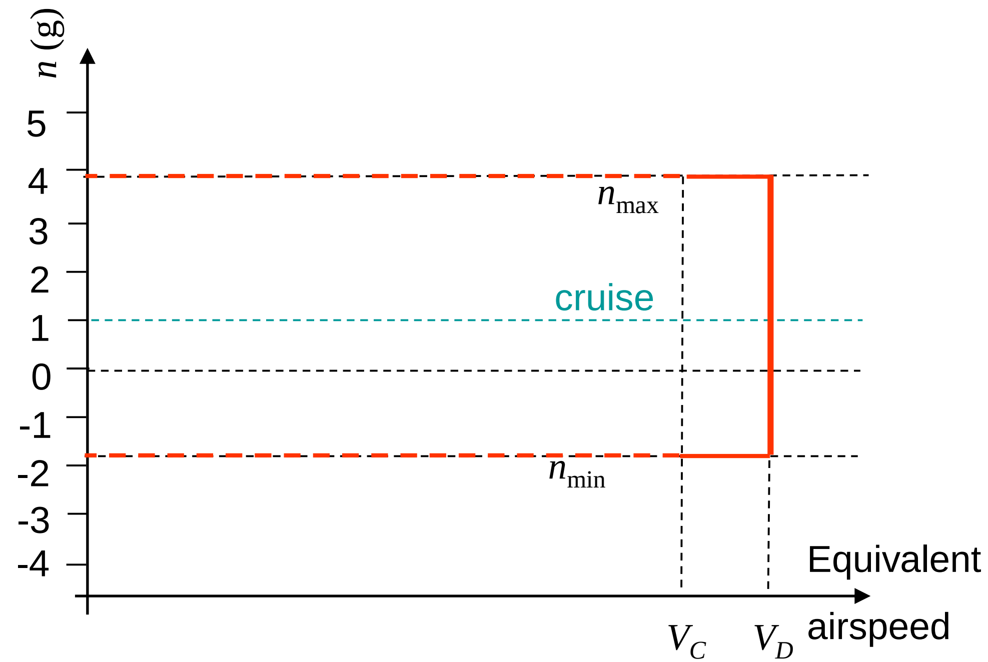
        ⟹ $\rho V_{\mathrm{True}}{}^2/2$ constant (V_{True} is the true airspeed)
      - From the dynamical pressure $\rho V_{\mathrm{True}}{}^2/2$, the equivalent velocity at sea level can be deduced: $V_e = V_{\mathrm{True}} (\rho / \rho_0)^{1/2}$ (ρ_0 = density at sea level)
    - Equivalent velocity is constant ⟹ true airspeed is decreasing
      - There can be an operational limit as take off speed



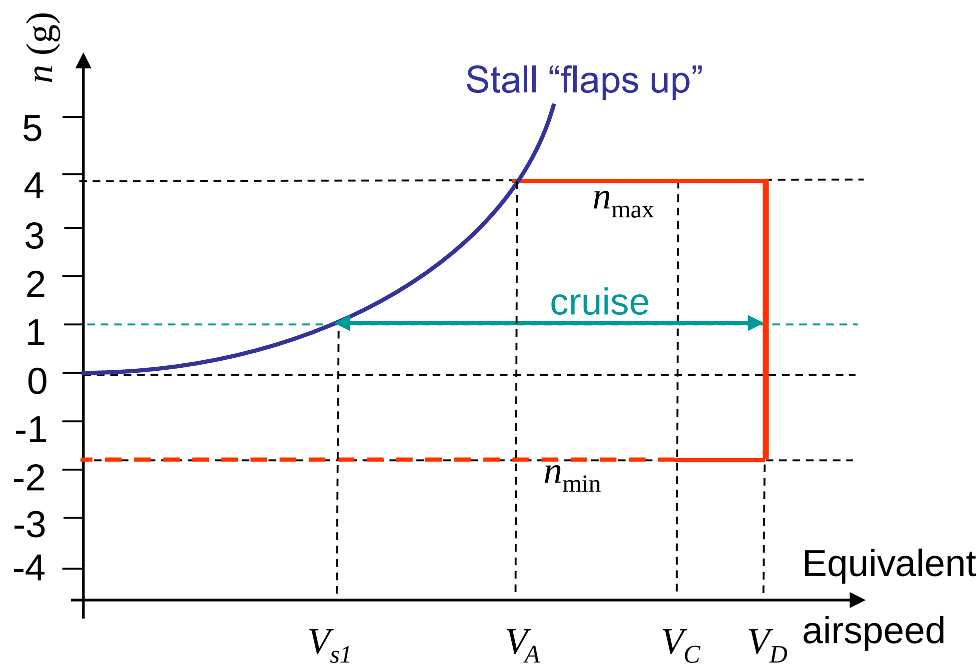
- Placard diagram (5)
  - Maximum velocity?
  - During a dive the plane can go faster than the design mach cruise
    - Design dive Mach (FAR) is defined as the minimum between
      - 1.25 M_C
      - Mach actually obtained after a 20-second dive at 7.5° followed by a 1.5-g pullout ⟹ M_D ~1.07 M_C
    - Above design altitude the maximum velocity is limited by M_D constant
    - Below design altitude the maximum dive velocity V_D is the minimum of
      - 1.25 V_C
      - The dive velocity (20-second dive at …) ~ 1.15 V_C
      - The velocity corresponding to M_D



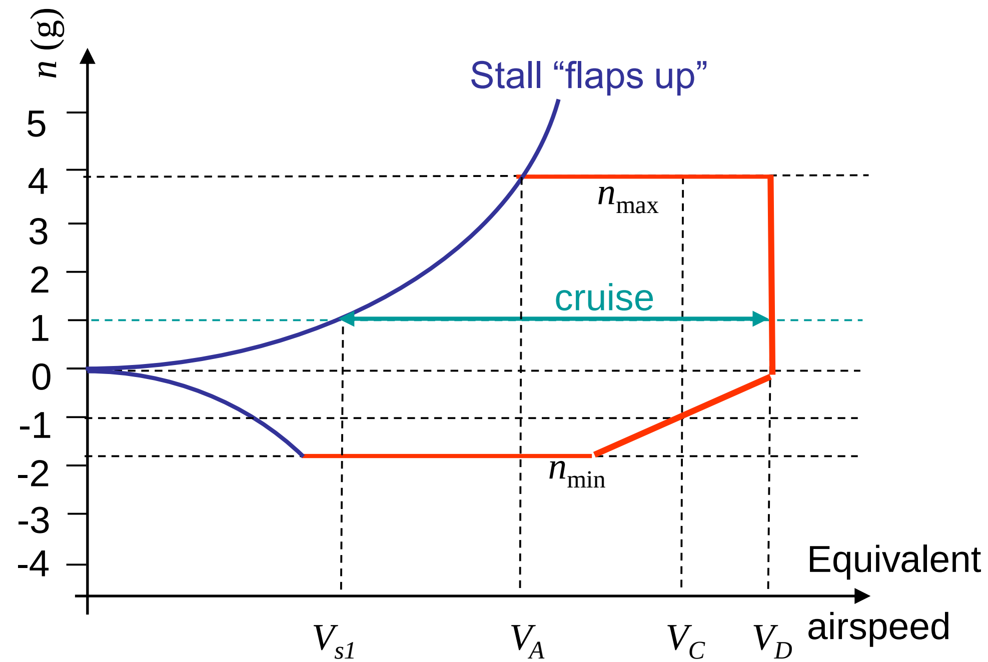
- Maneuver envelope (Velocity-load factor dependency)
  - Extreme load factors
    - Light airplanes (W < 50000 lb)
      - From -1.8 to minimum of
        - » 2.1 + 24000 lb/(W [lb] + 10000 lb)
        - » 3.8
    - Airliners (W > 50000 lb)
      - From -1 to 2.5
    - Acrobatic airplanes
      - From -3 to 6
  - Two design velocities
    - These are equivalent velocities
    - Design dive velocity V_D
      - The plane cannot fly faster
    - Design cruise velocity V_C
  - Are these load limits relevant if the plane fly slower than V_C ?



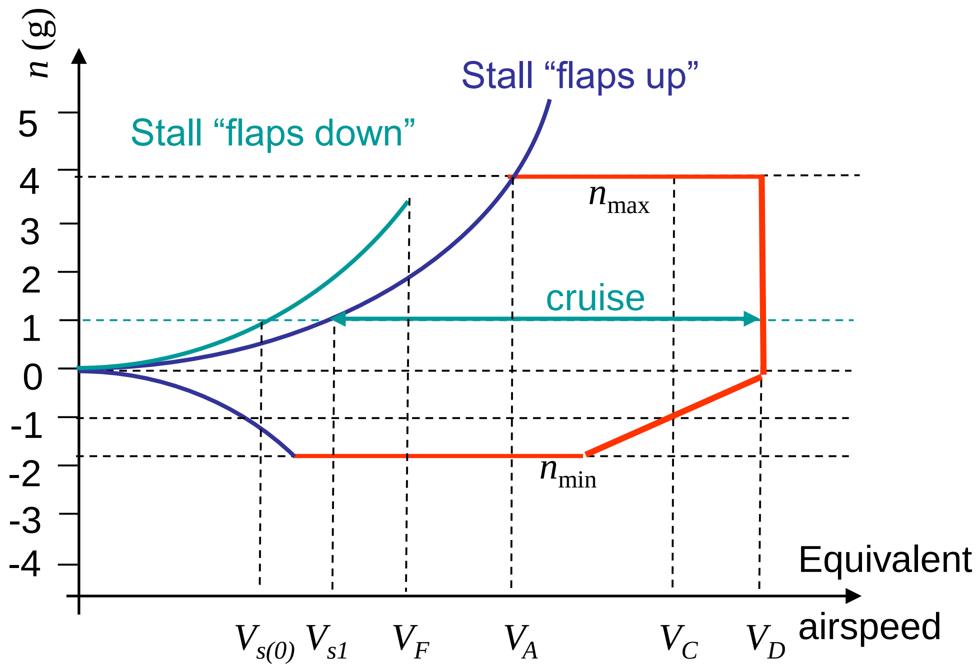
- Maneuver envelope (2)
  - At velocity lower than design cruise V_C
    - A pullout is limited by the maximum lift the plane can withstand before stalling
      - In terms of equivalent velocity and maximum lift coefficient flaps up, the maximum load factor becomes: $n = \frac{L}{W} = \frac{\rho_0 V_e^2 S C_{L\,\max,1}}{2W}$
      - V_A: Intersection between stall line and $n_{\max}$
        - » This is the maximum velocity at which maximum deflection of controls is authorized
      - V_{s1}: Intersection between stall line and n = 1
        - » This is the stall velocity in cruise (flaps up)
      - FAR requirement
        - » $V_A > V_{s1}\, n^{1/2}$ but
        - » V_A can be limited to V_C



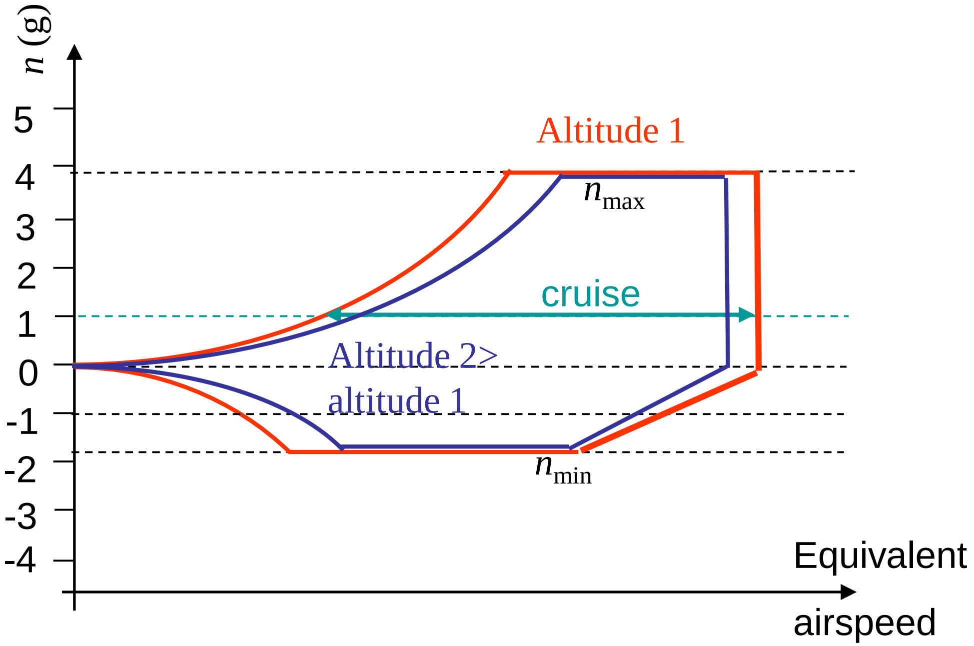
- Maneuver envelope (3)
  - Negative load factor
    - At low velocities
      - Same thing than for pullout: stall limits the load factor
    - At high velocities
      - When diving only a pullout is meaningful
      - Linear interpolation between
        - » V_e=V_D & n=0
        - » V_e=V_C & n=-1



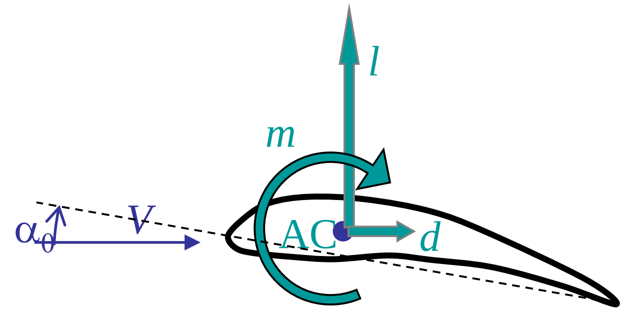
- Maneuver envelope (4)
  - Configuration flaps down
    - The maximum lift coefficient changes, so the load factor
      - Landing configuration $n = \frac{L}{W} = \frac{\rho_0 V_e^2 S C_{L\,\max,0}}{2W}$
      - Takeoff configuration $n = \frac{L}{W} = \frac{\rho_0 V_e^2 S C_{L\,\max}}{2W}$
    - Stall velocities
      - V_s: take off
      - V_{s0}: landing
      - V_{s1}: flaps up
    - V_F: velocity below which the flaps can be down (structural limit)
    - FAR requirements
      - $V_F > 1.6\ V_{s1}$ in take off configuration (MTOW)
      - $V_F > 1.8\ V_{s1}$ in approach configuration (weight)
      - $V_F > 1.8\ V_{s0}$ at landing configuration (weight)



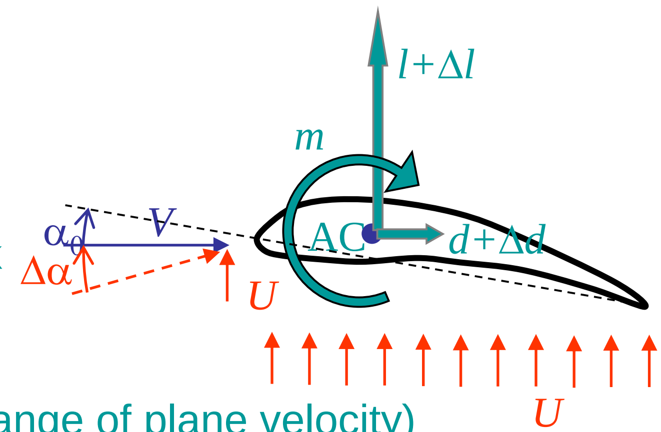
- Maneuver envelope (5)
  - Altitude dependency
    - Use of equivalent velocity reduces the effect of altitude
    - But the envelope still depends on the altitude
      - With the altitude the speed of sounds decreases and density is reduced
        - » For a given equivalent velocity the compressibility effects are higher (higher Mach number) and the maximum lift coefficient decreases
      - The computed V_D will be lower as limited by M_D constant
    - One flight envelope is therefore valid for an altitude range
    - Another factor which is altitude-dependant, and that should also be considered, is the gust factor



- Gust effect
  - Airfoil in still air
    - Airplane velocity V
    - Attack angle α_0



  - Sudden vertical gust U
    - The plane keeps temporarily the same
      - Velocity V
      - Attitude α_0
    - Due to the vertical velocity the angle of attack becomes $\alpha = \alpha_0 + \Delta\alpha \simeq \alpha_0 + \frac{U}{V}$



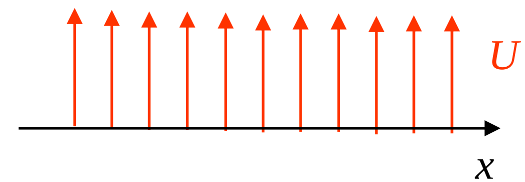
    - Resulting increase of plane lift (neglecting change of plane velocity)

$$\Delta L \simeq \frac{\rho V^2 S \partial_\alpha C_{L\,\text{plane}} \Delta\alpha}{2} \simeq \frac{\rho V S C_{L\alpha\,\text{plane}} U}{2}$$

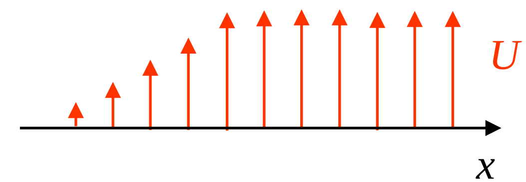
  - Increase in load factor
    - As $\rho U V = \rho_0 U_e V_e \Longrightarrow$

$$\boxed{\Delta n \simeq \frac{\rho_0 V_e S C_{L\alpha\,\text{plane}} U_e}{2W}}$$

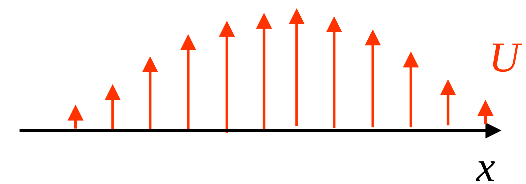
- Gust effect (2)
  - Realistic vertical gust
    - The plane do not really see a sudden vertical gust



    - A real vertical gust can be modeled as graded
      - Ramp



      - Cosine



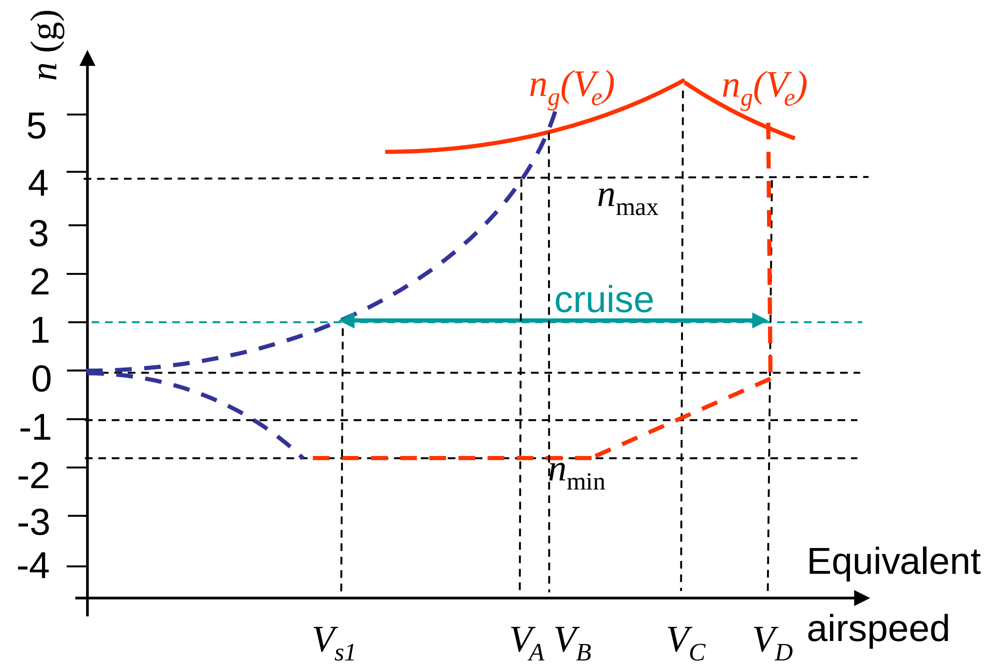
    - Modern methods consider power spectrum analysis
  - Gust alleviation factor: Before gust has reached its maximum value
    - The aircraft has developed a vertical velocity ⟹ reduces the severity
    - The aircraft might be pitching ⟹ effect on the loading (increase of decrease)
    - Elastic deformations of the structure ⟹ might increase the severity
  - So $\Delta n \simeq \frac{\rho_0 V_e S C_{L\alpha \text{plane}} U_e}{2W}$ becomes $\boxed{\Delta n \simeq \frac{\rho_0 V_e S F C_{L\alpha \text{plane}} U_e}{2W}}$
    - F is the gust alleviation factor (<1)

- Gust alleviation factor
  - Expression $\Delta n \simeq \frac{\rho_0 V_e S F C_{L\alpha\text{plane}} U_e}{2W}$ is difficult to be evaluated
  - FAR simple rule $n_g = 1 + \frac{F C_{L\alpha\text{plane}} U_e V_e S}{498W}$
    - W plane weight in lb
    - V_e equivalent plane velocity in knots (1 knots = 1.852 km /h)
    - Gust alleviation factor $F = \frac{0.88\mu}{5.3+\mu}$
    - Airplane weigh ratio $\mu = \frac{2W}{\rho C_{L\alpha\text{plane}} c g S}$
    - c mean aerodynamic chord
    - U_e equivalent gust velocity in ft/s
      - Is interpolated from statistical values at different altitudes and for different planes velocities
      - V_B: Velocity when maximum load factor is governed by gust (see next slide)

U_e in ft/s	$V_e = V_B$	$V_e = V_C$	$V_e = V_D$
Sea level	± 56	± 56	± 28
15000 ft	± 44	± 44	± 22
60000 ft	±20.86	±20.86	±10.43

- Gust envelope
  - Gust load factor
    - $n_g = 1 + \frac{F C_{L\alpha\,\text{plane}} U_e V_e S}{498 W}$
    - This gives two branches for $n_g(V_e)$ for $U_e > 0$
    - V_B is the intersection between
      - The stall curve
      - $n_g(V_e)$
    - This means that if
      - $V_e < V_B$ the plane might stall in case of gust
      - So V_B is minimum speed to enter a gust region
    - FAR requirement
      - V_B can be $< V_{s1} [n_g(V_C)]^{1/2}$
      - $V_C > V_B + 1.32 U_e$
      - $V_B > V_{s_1} \sqrt{1 + \frac{\rho_0 V_c S F C_{L_{\alpha_{plane}}}}{2W} U_e}$

U_e in ft/s	$V_e = V_B$	$V_e = V_C$	$V_e = V_D$
Sea level	± 56	± 56	± 28
15000 ft	± 44	± 44	± 22
60000 ft	±20.86	±20.86	±10.43



- Gust envelope (2)
  - Gust load factor
    - $n_g = 1 + \frac{F C_{L\alpha\,\text{plane}} U_e V_e S}{498W}$
    - This gives two branches for $n_g(V_e)$ for $U_e < 0$
  - Gust envelope is the linear interpolation between
    - Positive stall
    - $n_g(V_B)$
    - $n_g(V_C)$
    - $n_g(V_D)$

U_e in ft/s	$V_e = V_B$	$V_e = V_C$	$V_e = V_D$
Sea level	± 56	± 56	± 28
15000 ft	± 44	± 44	± 22
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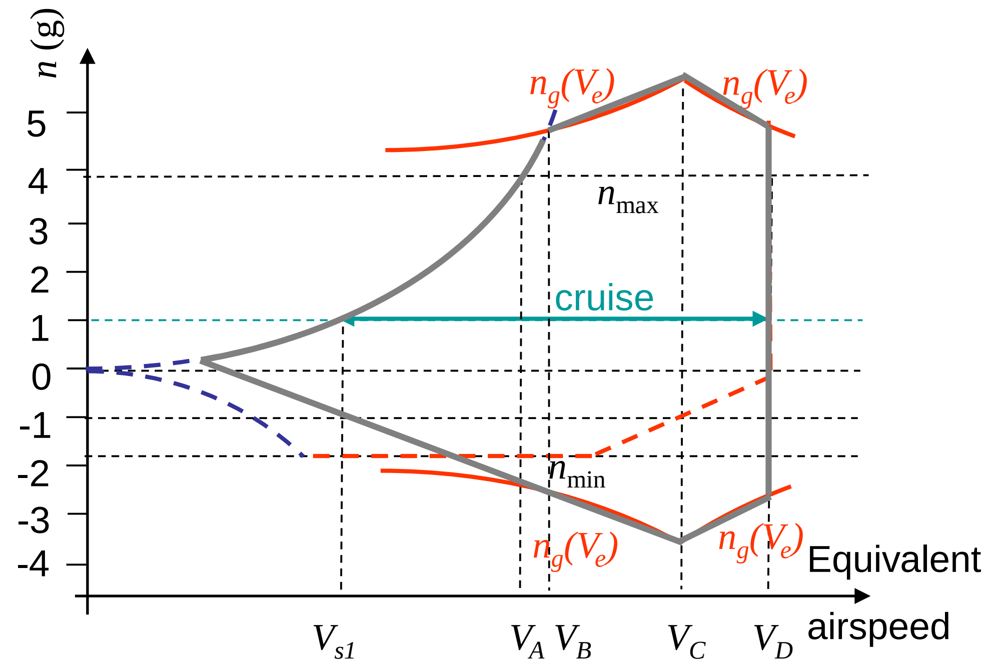
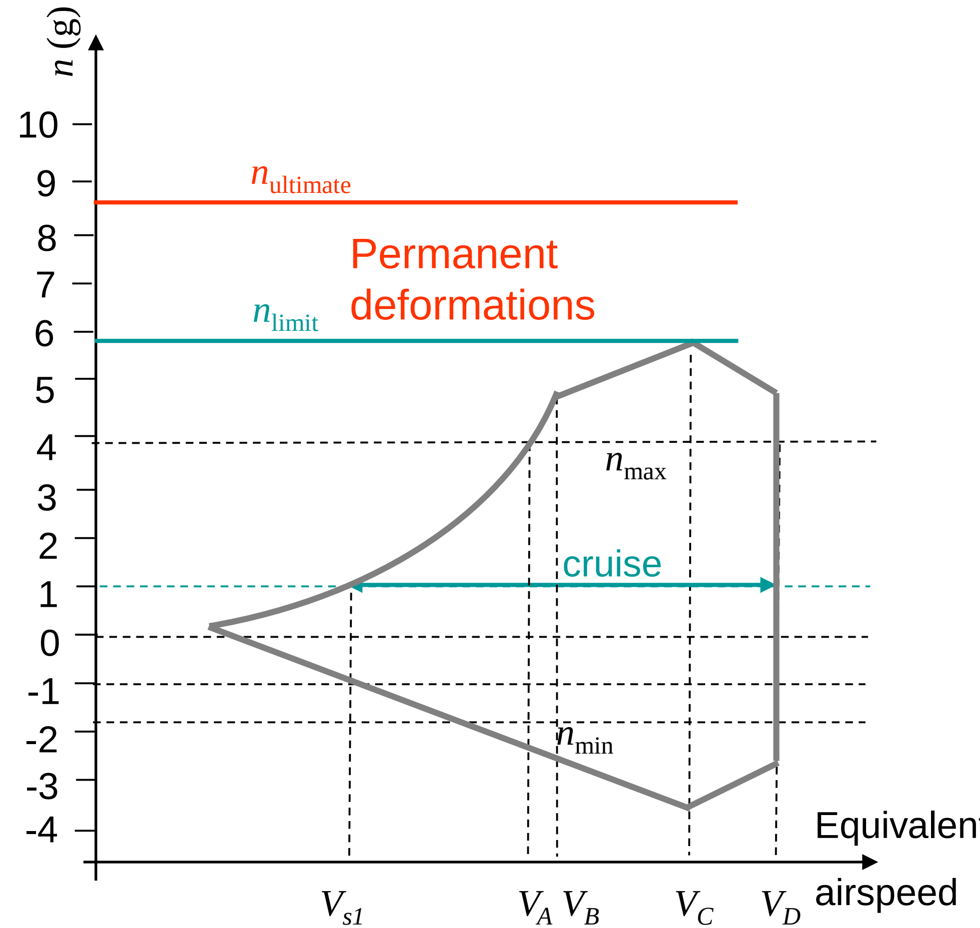


- Design load factors
  - Limit load factor n_{limit}
    - Maximum expected load during service (from gust envelope)
    - The plane cannot experience permanent deformations
  - Ultimate load factor n_{ultimate}
    - Limit load times a security factor (1.5)
    - The plane can experience permanent deformations
    - The structure must be able to withstand the ultimate load for 3 seconds without failure



Structure

- First structure designs
  - A wood internal structure smoothed by fabrics
  - A plywood structure was also used for the fuselage

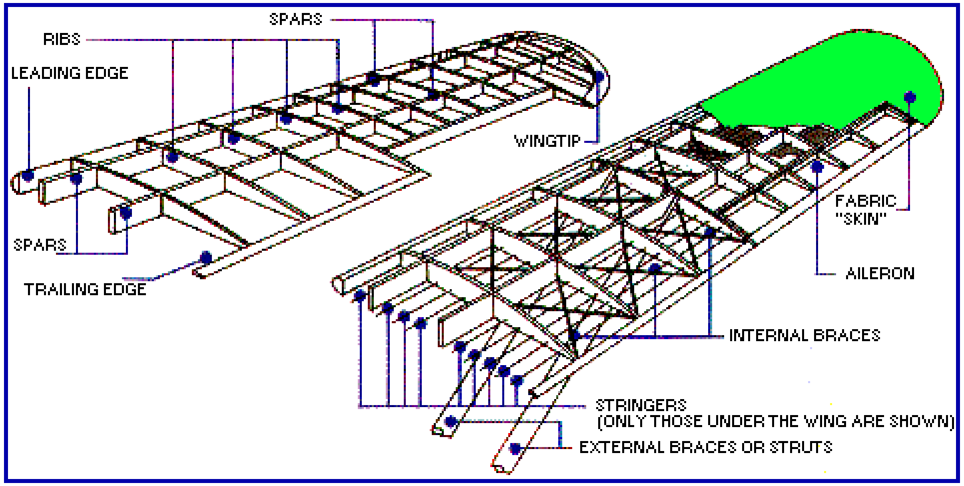


Figure 1-5 Wood-and-fabric-type wing structure

- Was wood a good choice?
  - Specific mechanical properties of wood are favorable to aluminum alloy

	Yield or tensile strength* [MPa]	Young [MPa]	Density $[kg \cdot m^{-3}]$	Ratio Young-Density	Ratio Strength-Density
Wood	100*	14000	640	21.9	0.156
Structural steel	200	210000	7800	26.9	0.025
Aluminum	75	70000	2700	8.9	0.027
High strength steel alloy A514	690	210000	7800	26.9	0.088
Aluminum alloy 2014	400	73000	2700	9.3	0.148
Titanium alloy 6Al-4V	830	118000	4510	26.17	0.184
Carbon fiber reinforced plastic	1400* (theoretical)	130000	1800	72.2	0.777

Structure

- Was wood a good choice (2)?
  - Drawbacks of wood
    - Moisture absorption changed shape and dimensions
    - Glued structures affected by humidity
    - Strongly anisotropic
    - Oversee import
    - Not suited to stress concentration
  - Wood-fabric structures
    - Were not always waterproof
      - Picture Fokker Dr.I
    - Did not allow to build high-aspect ratio wing
      - Most of the planes were biplanes or triplanes with lower lift/drag ratio

Photo Courtesy Hans Franke

- Was wood a good choice (3)?
  - Nowadays, only light aircrafts are built using this concept (ex: Mudry)

  - In 1915, Junkers constructed a steel plane
    - Cantilevered wing
    - Steel is too heavy (specific tensile strength too low)

- Duralumin
  - 1909, Alfred Wilm, Germany
    - An aluminum alloy containing
      - 3.5 per cent copper
      - 0.5 per cent magnesium
      - Silicon and iron as impurities

      spontaneously hardened after quenching from about 480°C.
  - This alloy had interesting specific mechanical properties
    - Yield 230 Mpa but
    - Density only 2700 $kg \cdot m^{-3}$
  - The question was
    - How to efficiently use this duralumin?

Structure

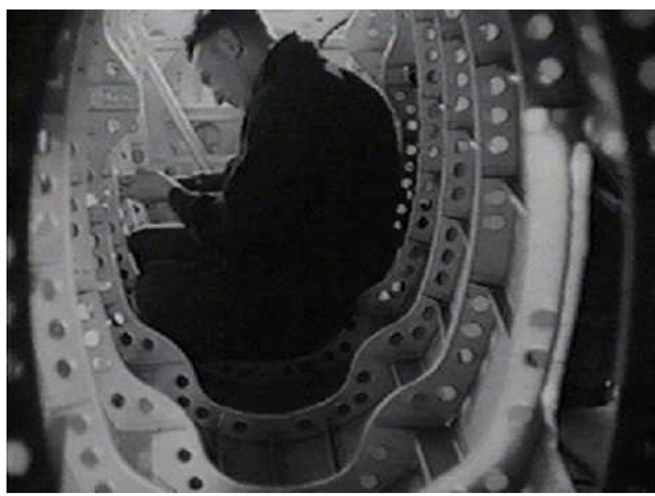
- Monocoque
  - Instead of
    - Using a frame as main structure and
    - Covering it with thin metal sheets
  - The skin of the structure can be such that it resists the load by itself
    - Lighter than framed structures
    - Sport cars (carbon fiber)
    - Soda can (aluminum)
      - As long as it is filled, it is resistant
      - Empty, it is subjected to buckling
  - These structures are subject to buckling and cannot be used for an aircraft

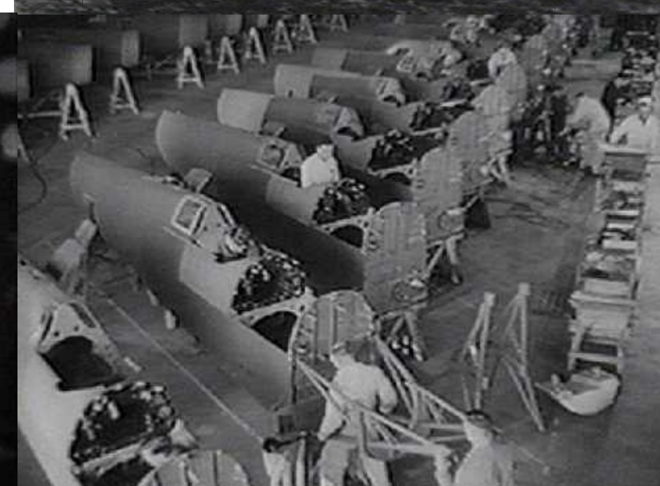


Structure

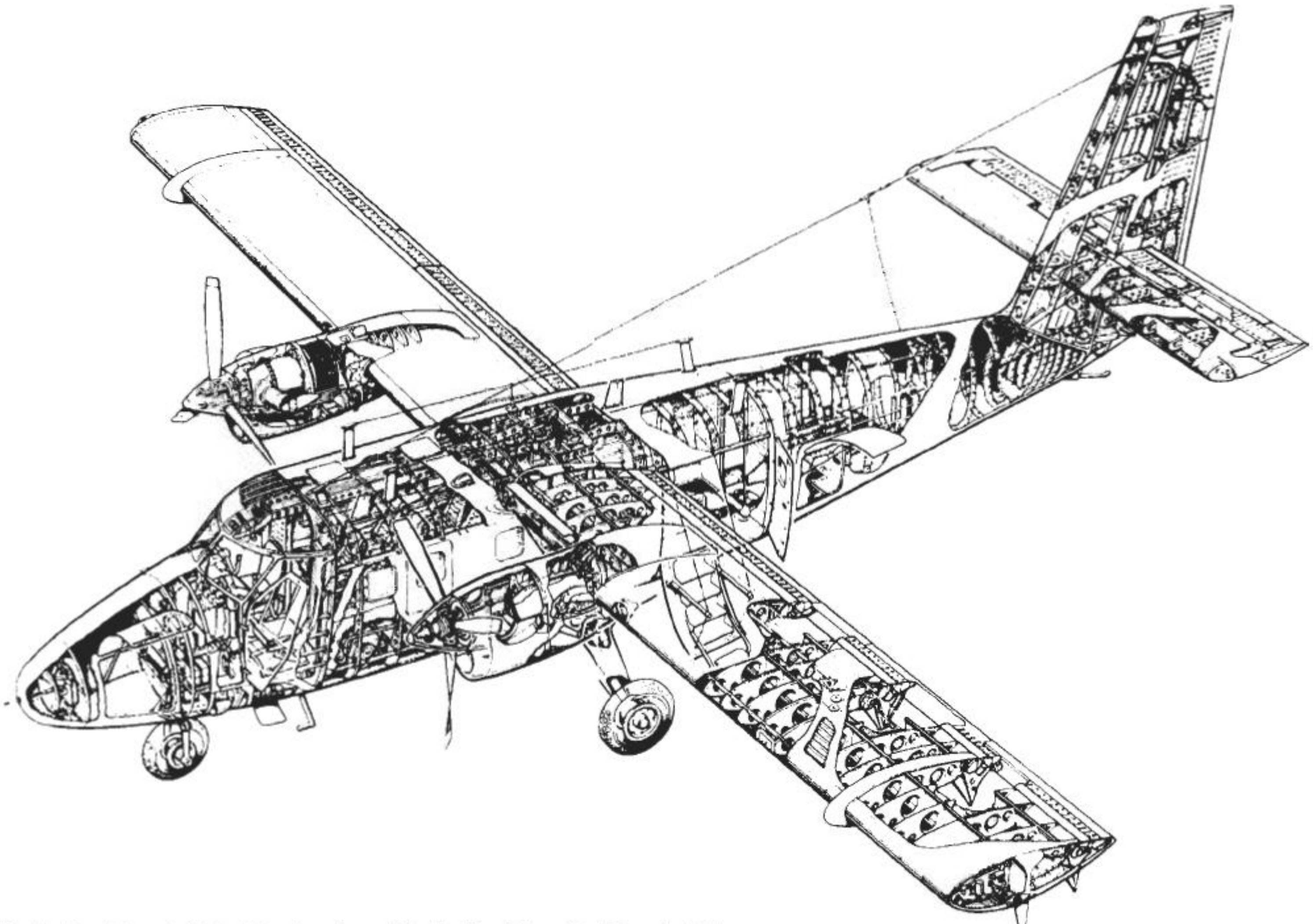
- Semi-monocoque
  - Monocoques are subject to buckling
  - The skin of the shell is usually supported by
    - Longitudinal stiffening members
    - Transverse frames

    to enable it to resist bending, compressive and torsional loads without buckling
  - These stiffeners are fixed to the skin instead of putting a skin on a structural frame
- First semi-monocoque aircrafts were made of duralumin (example: spitfire)





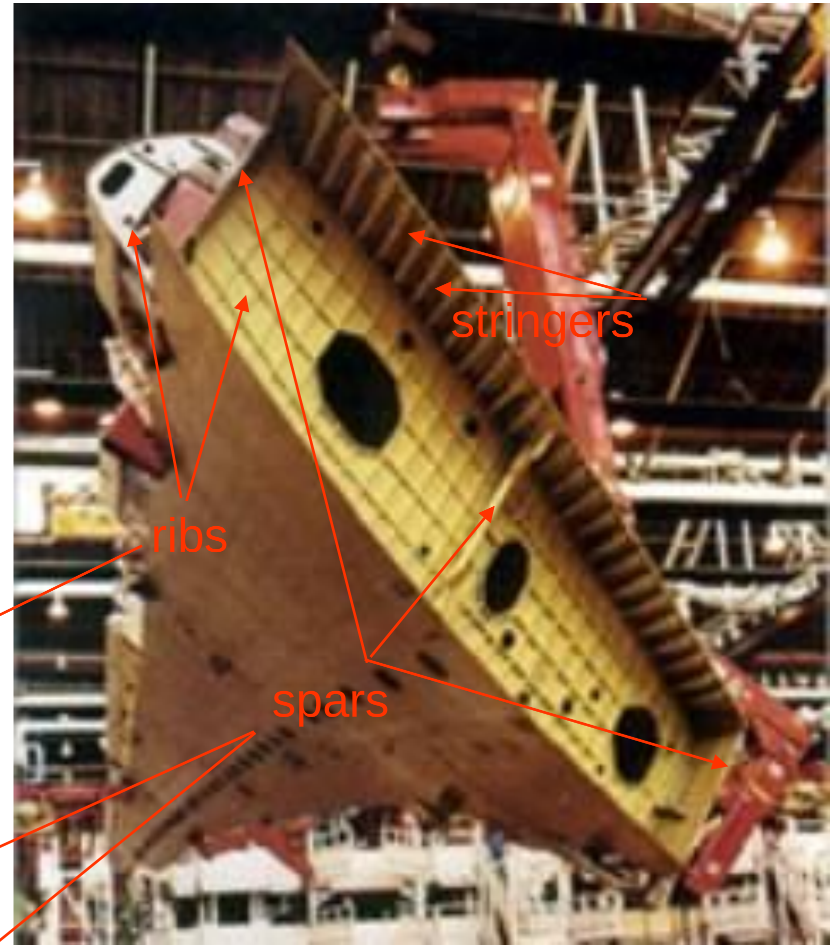
- Global view

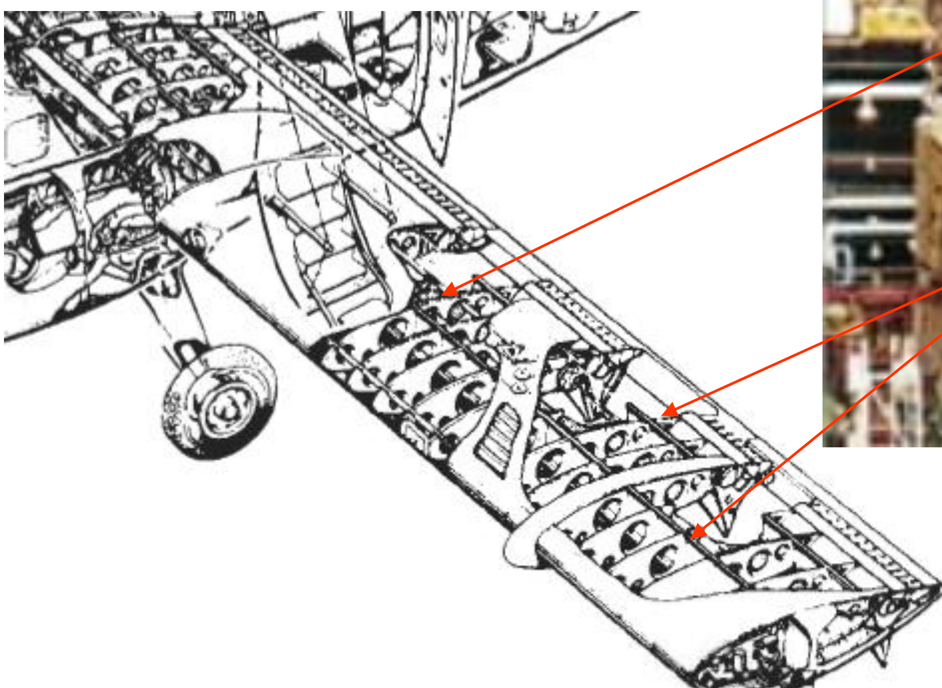


De Havilland Canada Twin Otter (courtesy of De Havilland Aircraft of Canada Ltd.).

Semi-monocoque structure

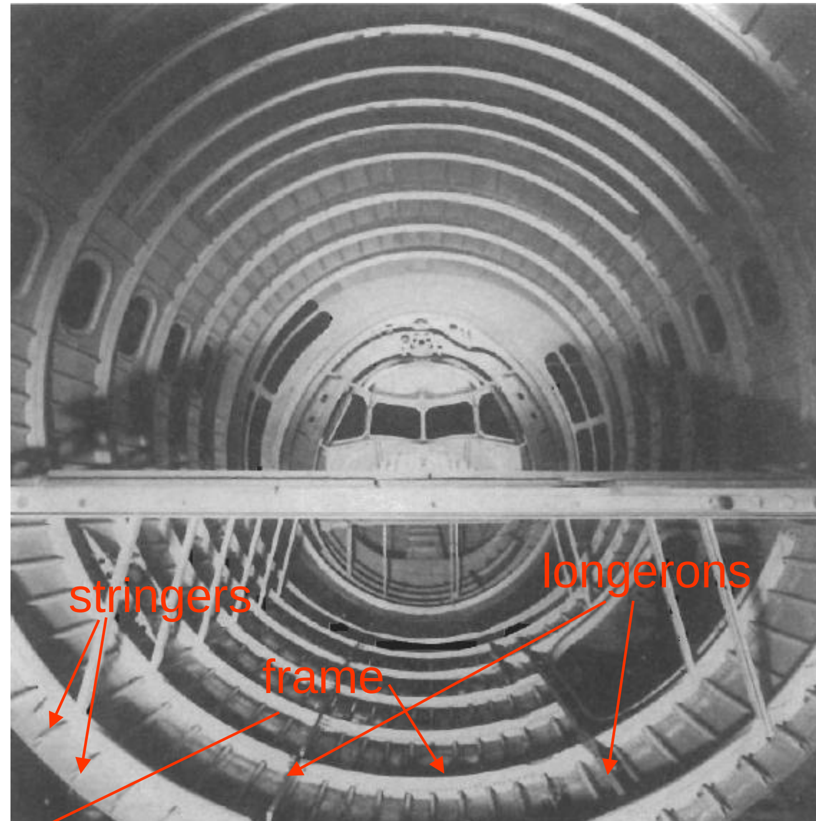
- Wing: Box-beam structure
  - 2 or 3 spars
  - Ribs
  - Stringers fixed to the skin
  - Transport aircrafts
    - Skin >~ 1. mm
    - Ribs >~ 0.5 mm
    - Spars >~ 1. mm





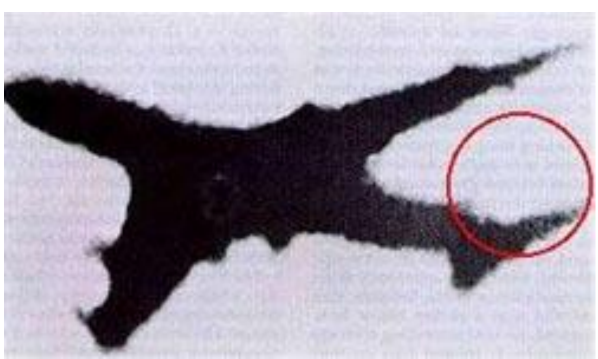
Semi-monocoque structure

- Fuselage
  - Circular if pressurized
  - Longerons
  - Stringers
  - Frames or formers
  - Bulkheads (see next slide)



Semi-monocoque structure

- Fuselage (2)
  - Circular if pressurized
  - Longerons
  - Stringers
  - Frames or formers
  - Bulkheads
    - Reinforcement at
      - Wing root
      - Empennage fixation
      - Engine fixation
      - …
    - Pressurization
      - Between cabin and tailfin
      - B747, Japan Airline 123: bulkhead repaired with a single row of rivets instead of two



Design method

- Structural integrity of the airframe
  - Must be ensured in the event of
    - Failure of a single primary structural element
    - Partial damage occurrence in extensive structures (e.g. skin panels)
    - Crack propagation
      - Adequate residual strength and stiffness
      - Slow rate of crack propagation
  - Design for a specified life in terms of
    - Operational hours
    - Number of flight cycles (ground-air-ground)

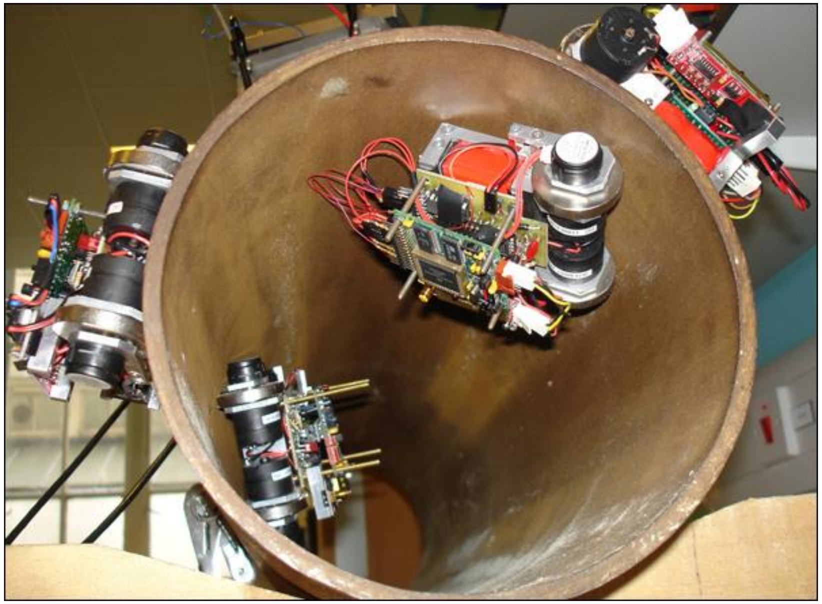
Fatigue design

- « Infinite life design »
  - $\sigma_a < \sigma_e$: « infinite » life
  - Economically deficient
- « Safe life design »
  - No crack before a determined number of cycles
    - At the end of the expected life the component is changed even if no failure has occurred
    - Emphasis on prevention of crack initiation
    - Approach theoretical in nature
      - Assumes initial crack free structures
  - Use of $\sigma_a - N_f$ curves (stress life)
    - Add factor of safety
  - Components of rotating structures vibrating with the flow cycles (blades)
    - Once cracks form, the remaining life is very short due to the high frequency of loading

Fatigue design

- « Fail safe design »
  - Even if an individual member of a component fails, there should be sufficient structural integrity to operate safely
  - Load paths and crack arresters
  - Mandate periodic inspection
  - Accent on crack growth rather than crack initiation
  - Example: 1988, B737, Aloha Airlines 243
    - 2 fuselage plates not glued
    - Sea water ⟹ rust and volume increased
    - Fatigue of the rivets
    - The crack followed a predefined path allowing a safe operation

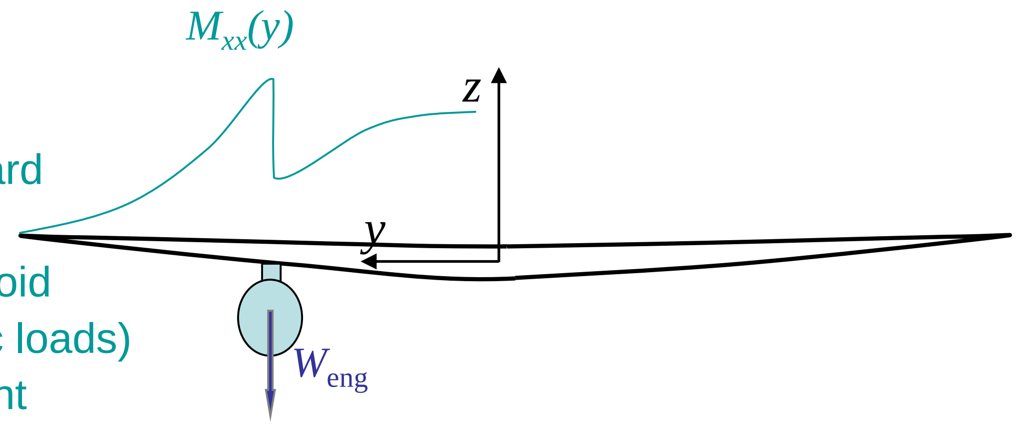
- « Damage tolerant design »
  - Assume cracks are present from the beginning of service
  - Characterize the significance of fatigue cracks on structural performance
    - Control initial crack sizes through manufacturing processes and (non-destructive) inspections
    - Estimate crack growth rates during service (Paris-Erdogan) & plan conservative inspection intervals (e.g. every so many years, number of flights)
    - Verify crack growth during these inspections
    - Predict end of life (a_f)
    - Remove old structures from service before predicted end-of-life (fracture) or implement repair-rehabilitation strategy



  - Non-destructive inspections
    - Optical
    - X-rays
    - Ultrasonic (reflection on crack surface)

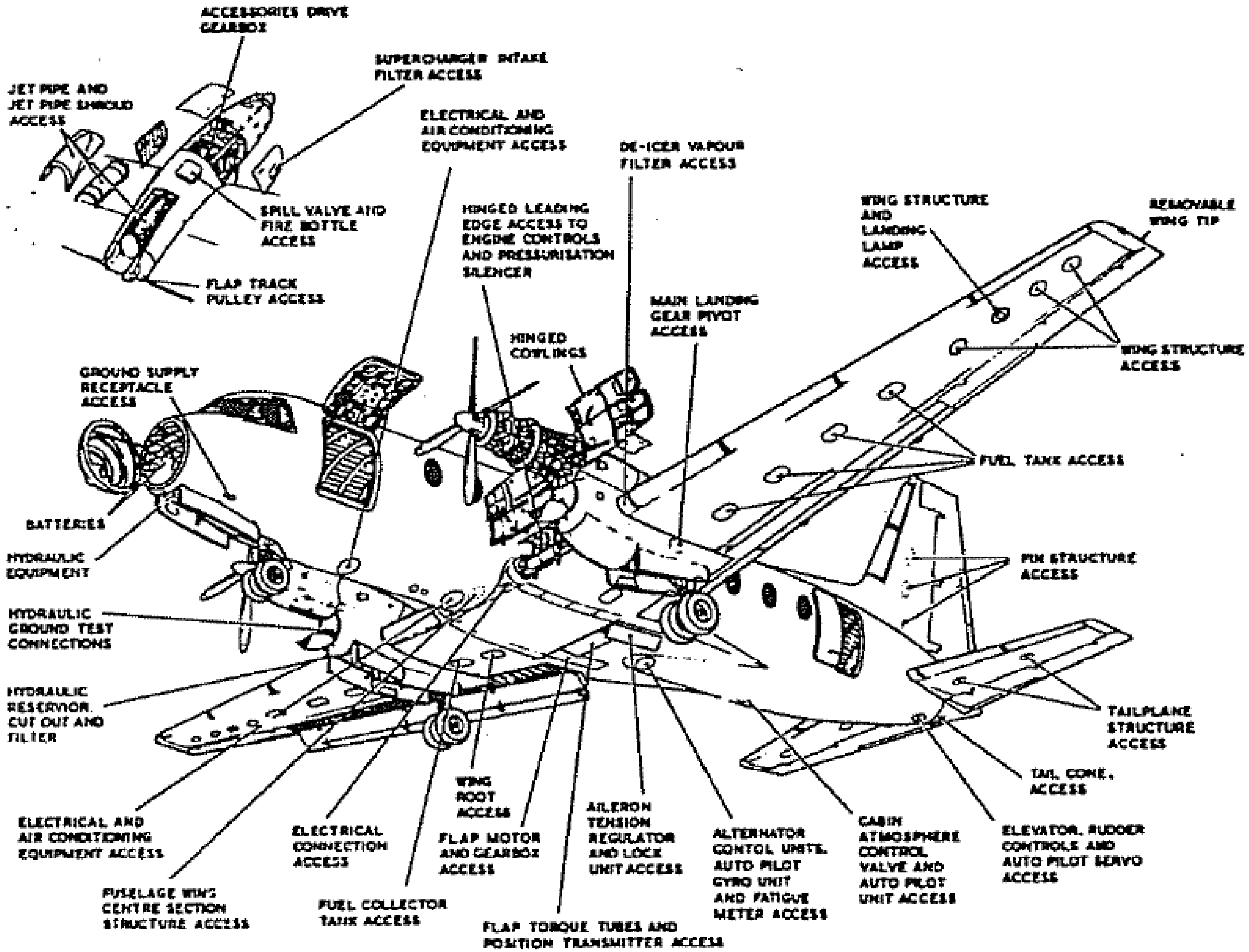
Design criteria

- Minimum structural weight
  - Wing
    - Fixed items & fuel tank outboard of wing (reduce wing loading)
    - 1-m free of fuel at wing tip (avoid fire risk in case of electrostatic loads)
    - Heavy mass at the wing in front of the structural axis (reduce aeroelastic issues)
    - Use the same ribs to support landing gear, flaps, engine
    - If possible wing in one part (throughout the fuselage for mid-wing)
  - Landing gear
    - Commonly attached to the wing
    - Should not induce bending nor shearing larger than in flight
      - Close to the root
      - Just forward of flexural axis



- Minimum structural weight (2)
  - Fuselage
    - Heavy masses near the CG (reduce the inertia loads)
    - Limited number of bulkheads
  - Empennages
    - Far from the wing (to reduce the aerodynamic loading)
    - Supported by an existing bulkhead
  - Other
    - Simple structures (avoid rollers, …)

- Ease of maintenance and inspection



Materials

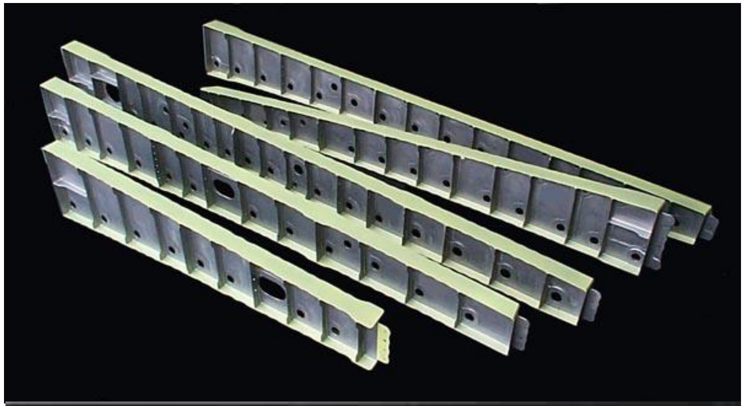
- Aluminum alloys
  - Duralumin (2xxx)
    - 4-7% Cu, 0.5-1.5% Mg, 0.2-2% Mn, 0.3% Si, 0.2-1% Fe
    - Picture: F15 horizontal stabilizer skin
  - Magnesium-Silicon alloy (6xxx)
    - 0.1-0.4% Cu, 0.5-1.5% Mg, 0.1-0.4% Mn, 0.3-2% Si, 0.1-0.7% Fe
  - Aluminum-Zinc-Magnesium alloy (7xxx)
    - 1-2.5% Cu, 1-7% Zn, 1-3% Mg, 0.3% Si
  - Used on fuselage and wing, also for rivets, …

	Yield [MPa]	Weldability	Machinability	Corrosion resistance	Fatigue properties
2024-T351	270	No	Average	Poor	Excellent
6061 T6	240	Excellent	Good	Good	Good
7075 T651	400	No	Average	Average	Good

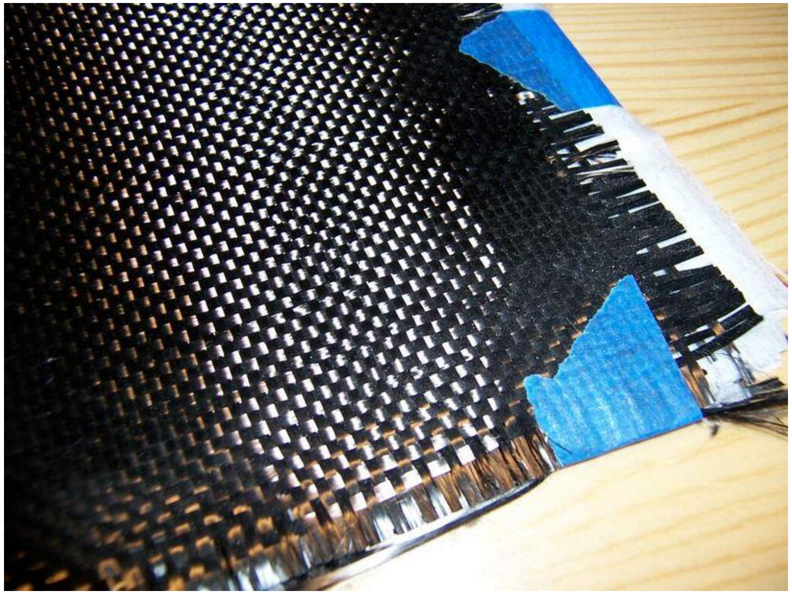
Materials

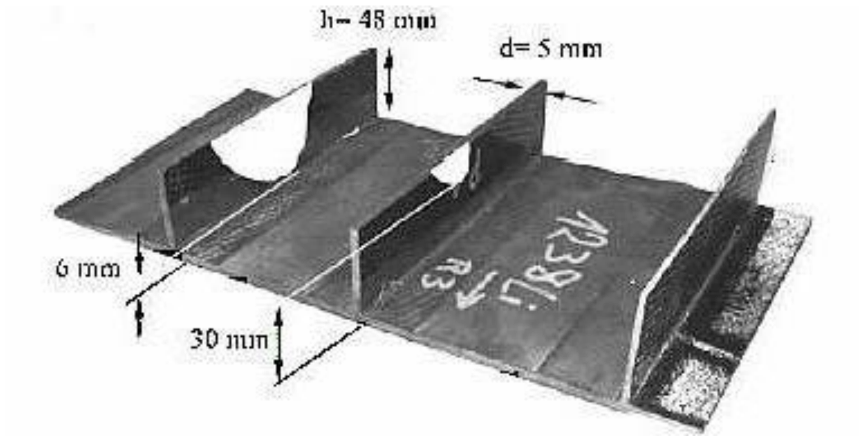
- Steel
  - Iron
    - Specific strength too low
  - Ultra-high-tensile strength carbon alloys
    - Brittleness
    - Not easily machinable, nor to weld
  - Maraging steel
    - Low carbon (<0.03%)
    - 17-19% Ni, 8-9% Co, 3-3.5 Mo, 0.15-0.25% Ti
    - High Yield strength (1400 MPa)
    - Compared to carbon-alloy
      - Higher toughness
      - Easier to machine and to weld
      - Better corrosion resistance
      - 3x more expensive
    - Aircraft arrester hook, undercarriage, …
    - Can be used at elevated temperature (400°C)

- Titanium alloy
  - High specific strength
    - Example Ti 6Al-4V
      - Yield 830 MPa, density 4510 $\text{kg}\cdot\text{m}^{-3}$
  - Properties
    - High toughness
    - Good fatigue resistance
    - Good corrosion resistance
      - Except at high T° and salt environment
    - Good Machinability and can be welded
    - Retains strength at high T° (500°C)
  - High primary and fabrication cost
    - 7X higher than aluminum alloys
  - Uses
    - Military aircrafts
      - Picture: F22 wing spars (Ti 6Al-4V)
    - Slat and flap tracks
      - Picture: B757 flap track (Ti 10V-2Fe-3Al)
    - Undercarriage



- Composite
  - Fibers in a matrix
    - Fibers: polymers, metals or ceramics
    - Matrix: polymers, metals or ceramics
    - Fibers orientation: unidirectional, woven, random
  - Carbon Fiber Reinforced Plastic
    - Carbon woven fibers in epoxy resin
      - Picture: carbon fibers
    - Tensile strength: 1400 MPa
    - Density: 1800 $kg \cdot m^{-3}$
    - A laminate is a stack of CFRP plies
      - Picture: skin with stringers

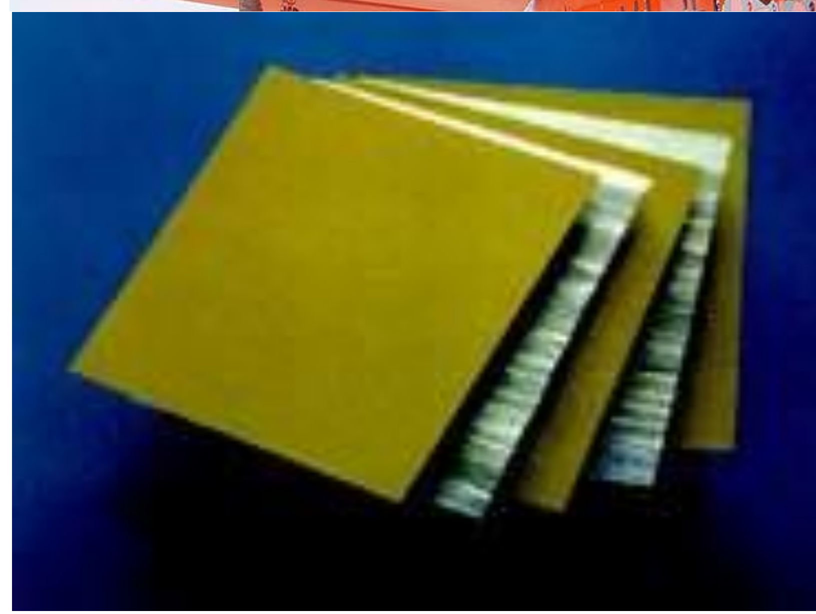


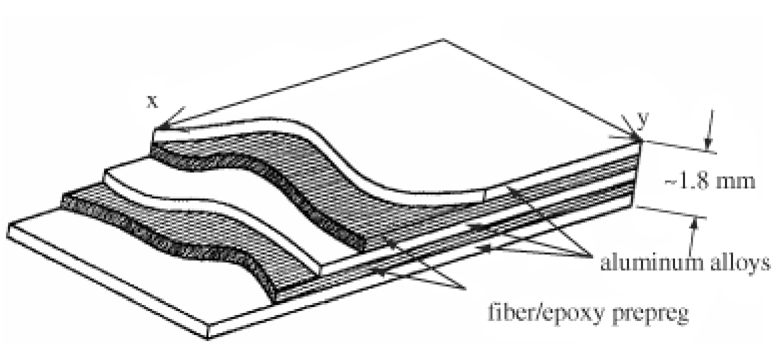


- Composite (2)
  - Wing, fuselage, …
  - Typhoon: CFRP
    - 70% of the skin
    - 40% of total weight
  - B787:
    - Fuselage all in CFRP

Materials

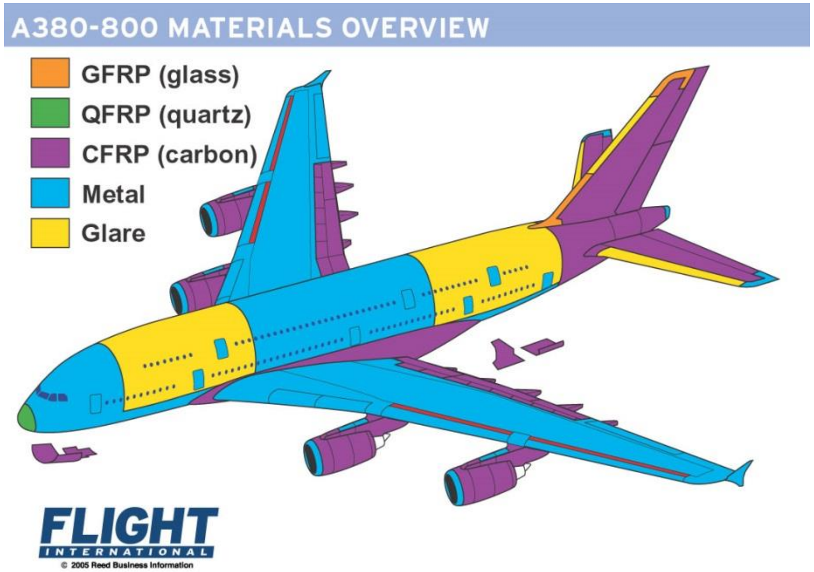
- Composite (3)
  - Drawbacks
    - "Brittle" rupture mode
    - Impact damage
    - Resin can absorb moisture
  - Glare
    - Thin layers of aluminum interspersed with Glass Fiber Reinforced Plastic
    - Improves damage resistance

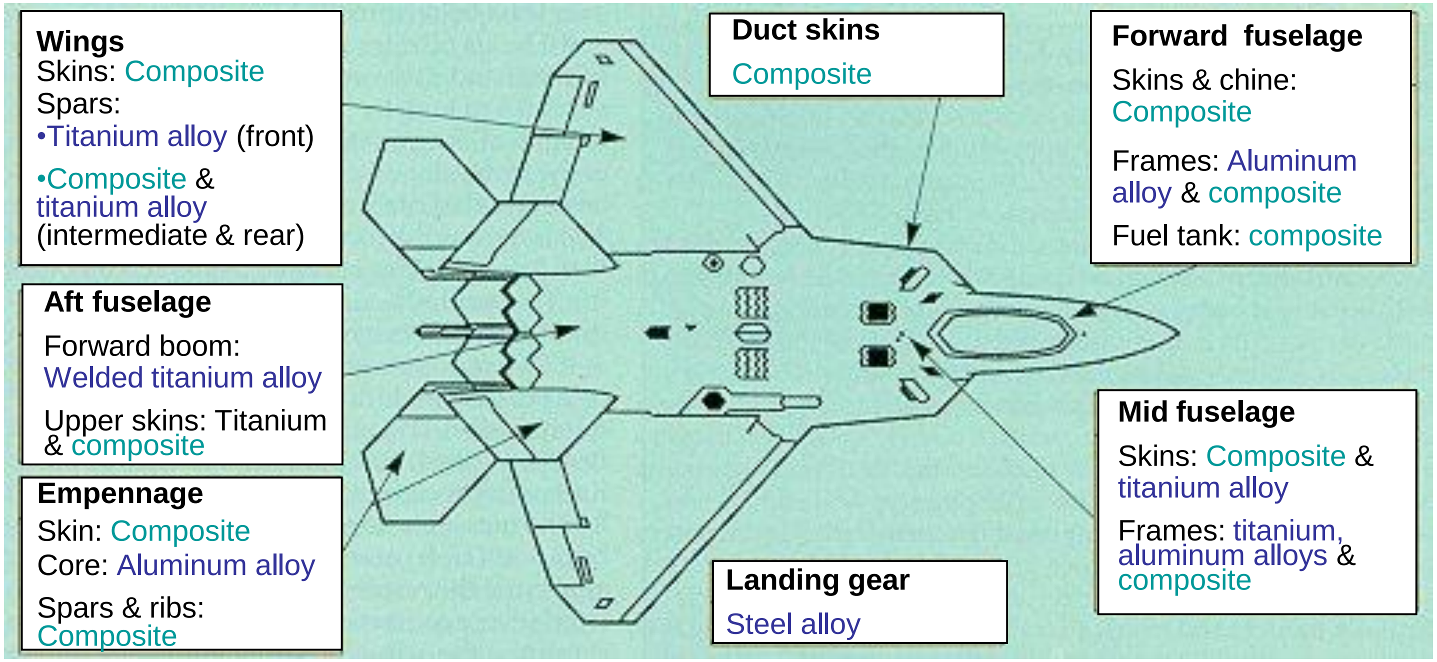




Materials

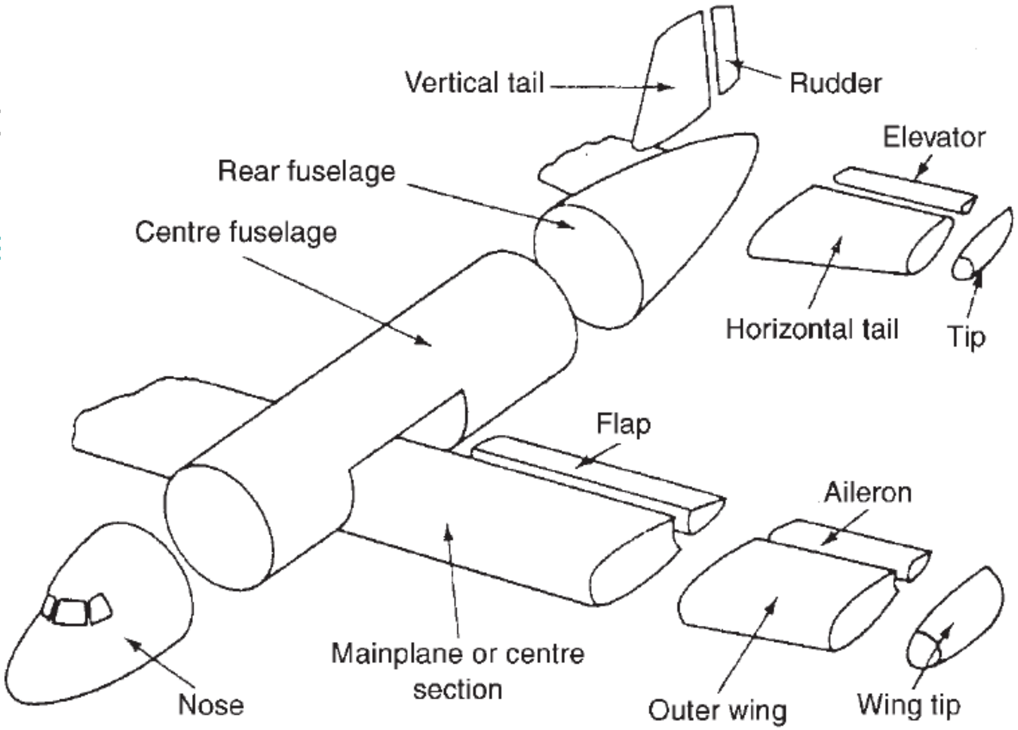
- Materials summary
  - Military aircrafts use more
    - Composite
    - Titanium alloy
  - Civil aircrafts
    - More and more composite





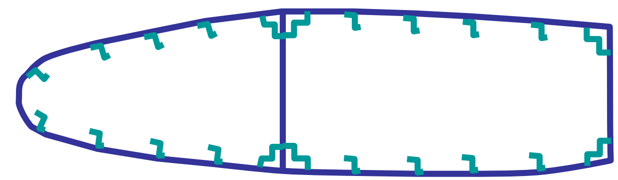
Assembly

- Sub-assembly
  - Each sub-assembly is construc
    - In specialized designed jigs
    - In different factories, countries

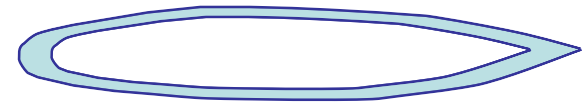


Structural idealization

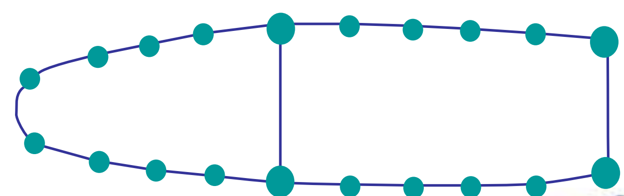
- Example 2-spar wing (one cell)
  - Stringers to stiffen thin skins
  - Angle section form spar flanges

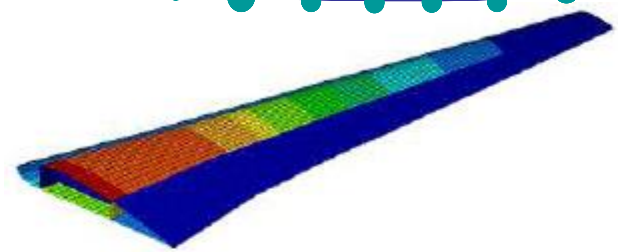


- Design stages
  - Conceptual
    - Define the plane configuration
      - Span, airfoil profile, weights, …
    - Analyses should be fast and simple
      - Formula, statistics, …

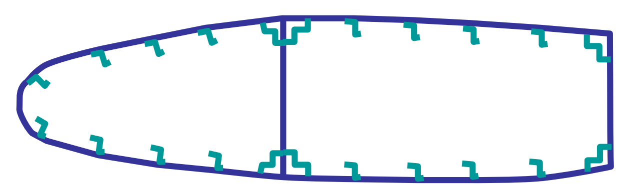


  - Preliminary design
    - Starting point: conceptual design
    - Define more variables
      - Number of stringers, stringer area, …
    - Analyses should remain fast and simple
      - Use beam idealization
        - » Part I
      - FE model of thin structures
        - » Part II



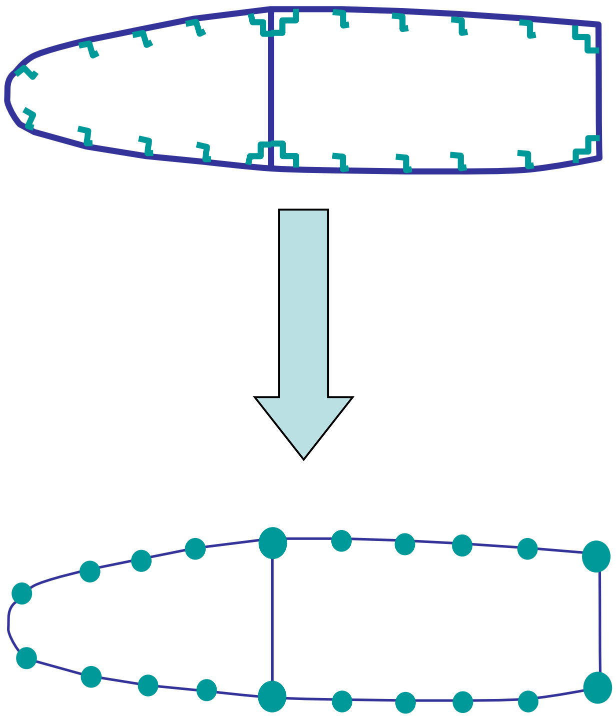


  - Detailed design
    - All details should be considered (rivets, …)
    - Most accurate analyses (3D, non-linear, FE)

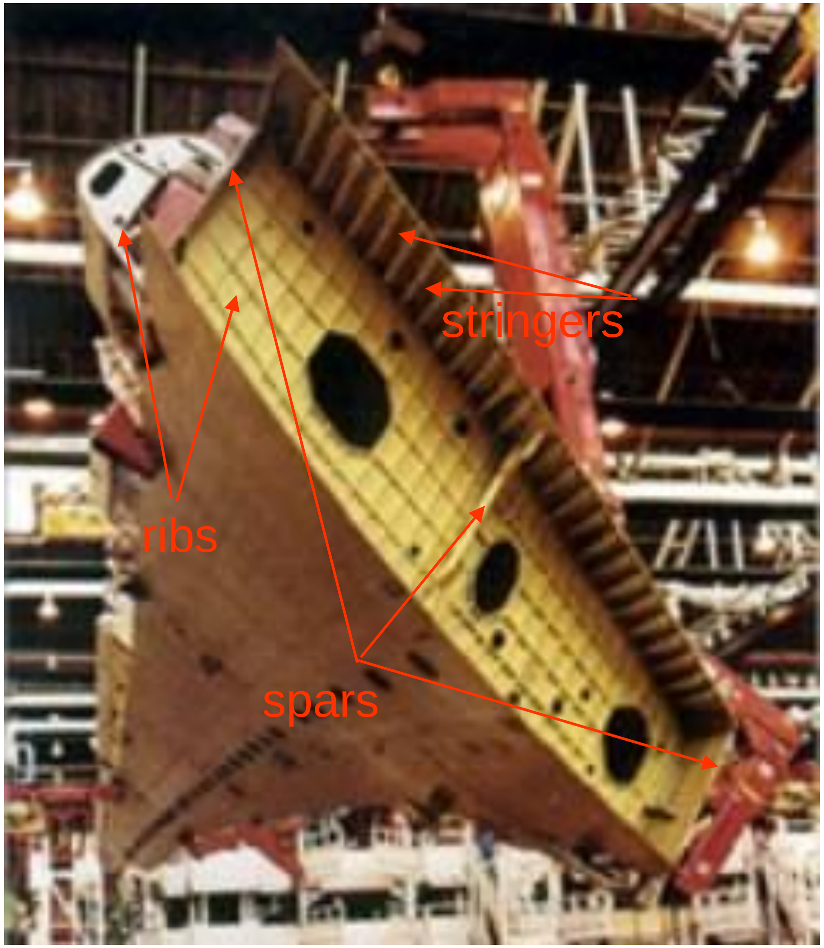


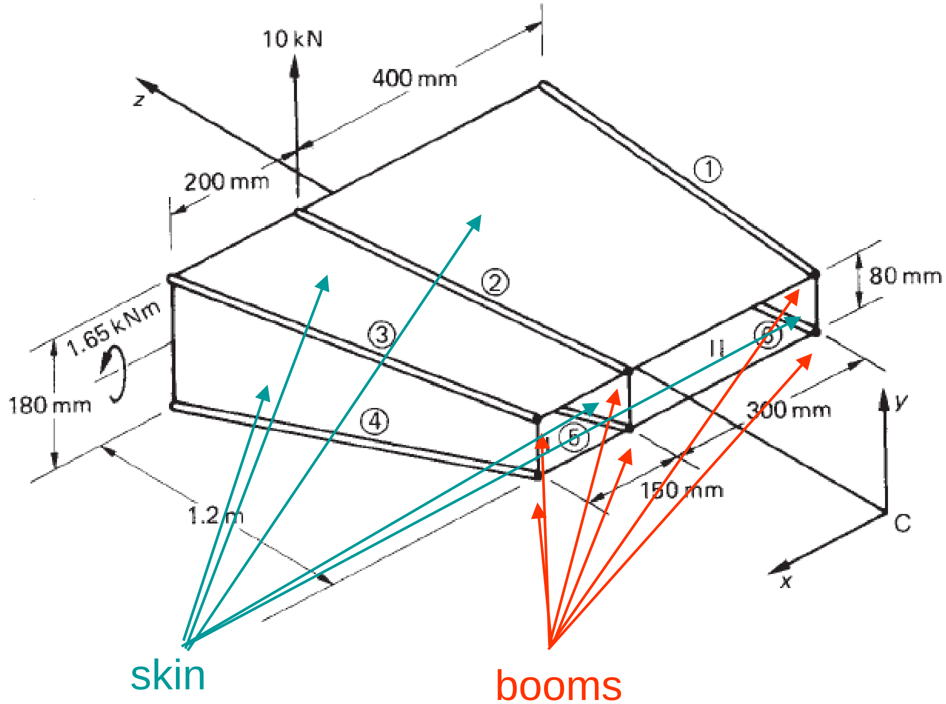
Wing section idealization

- Idealization of a 2-spar wing section
  - Booms
    - Stringers, spar flanges, …
      - Have small sections compared to airfoil
      - Direct stress due to wing bending is ~constant in each of these
      - They are replaced by concentrated area called booms
    - Booms
      - Have their centroid on the skin
      - Are carrying most direct stress due to beam bending
  - Skin
    - Skin is essentially carrying shear stress
    - It can be assumed
      - Skin is carrying only shear stress
      - If direct stress carrying capacity of skin is reported to booms by appropriate modification of their area

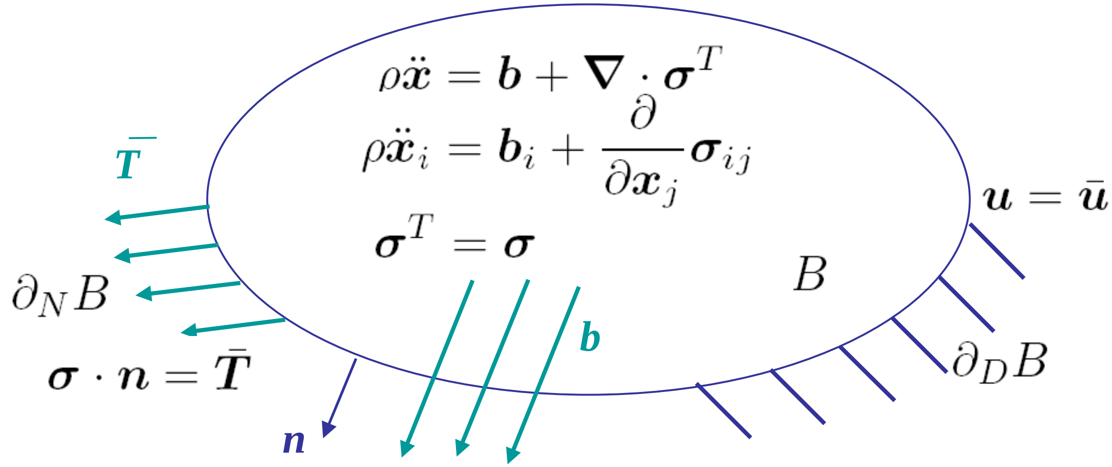


- Two-cell tapered wing





- Balance of body B
  - Momenta balance
    - Linear
    - Angular
  - Boundary conditions
    - Neumann
    - Dirichlet



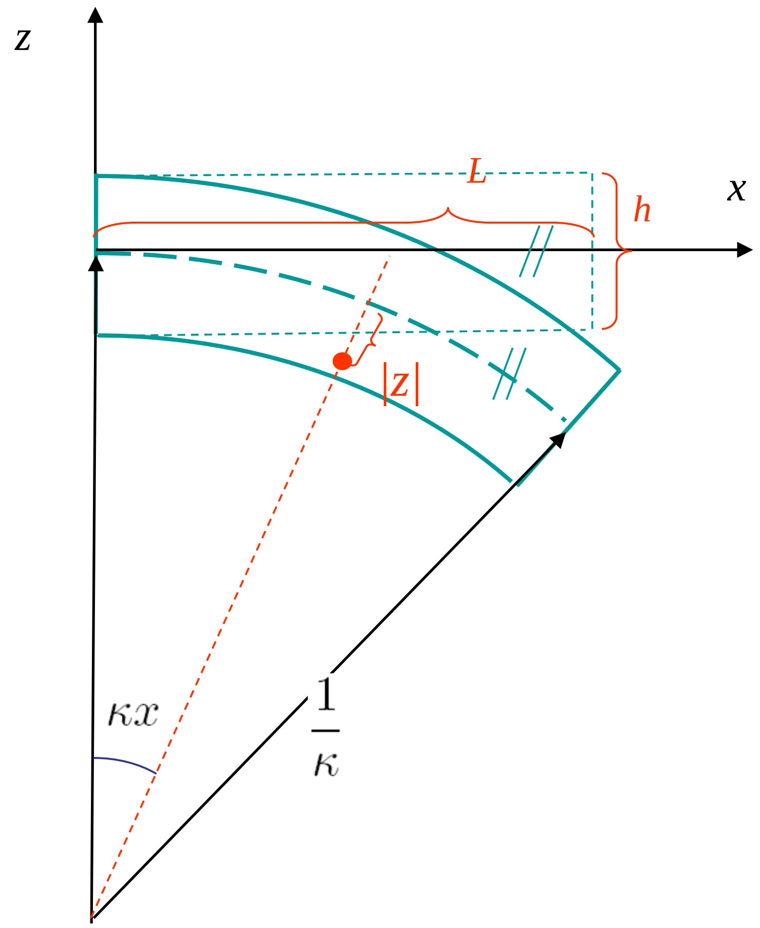
- Small deformations with linear elastic, homogeneous & isotropic material
  - (Small) Strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2} \left(\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla} \right)$, or $\begin{cases} \boldsymbol{\varepsilon}_{ij} = \frac{1}{2} \left(\frac{\partial}{\partial \boldsymbol{x}_i} \boldsymbol{u}_j + \frac{\partial}{\partial \boldsymbol{x}_j} \boldsymbol{u}_i \right) \\ \boldsymbol{\varepsilon}_{ij} = \frac{1}{2} \left(\boldsymbol{u}_{j,i} + \boldsymbol{u}_{i,j} \right) \end{cases}$
  - Hooke's law $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon}$, or $\boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl} \boldsymbol{\varepsilon}_{kl}$

    with $\mathcal{H}_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)} \delta_{ij}\delta_{kl} + \frac{E}{1+\nu} \left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk} \right)$

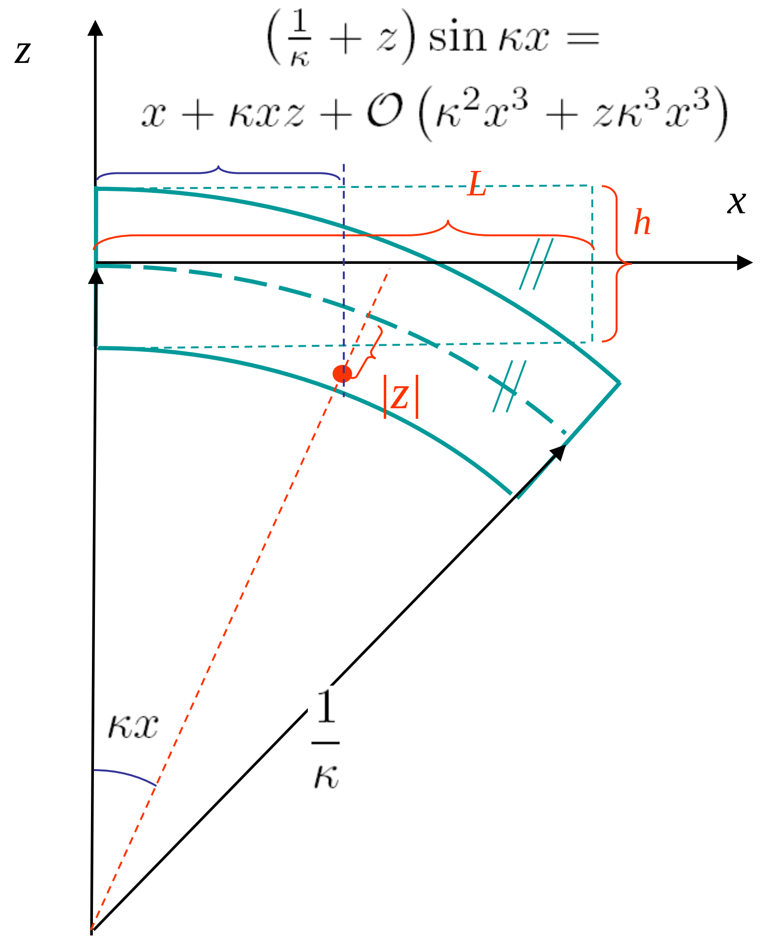
    $\lambda = K - 2\mu/3$ $\qquad$ 2μ
  - Inverse law $\boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma}$

    with $\mathcal{G}_{ijkl} = \frac{1+\nu}{E} \left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk} \right) - \frac{\nu}{E} \delta_{ij}\delta_{kl}$

- 1-D pure bending
  - Assumptions
    - Symmetrical beam
    - Filled cross section
    - Cross-section remains plane (Bernoulli or Kirchhoff-Love)
    - Only for thin structures (h/L << 1)
    - Limited bending: κL << 1
  - Curvature radius
    - $\kappa = -\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}$



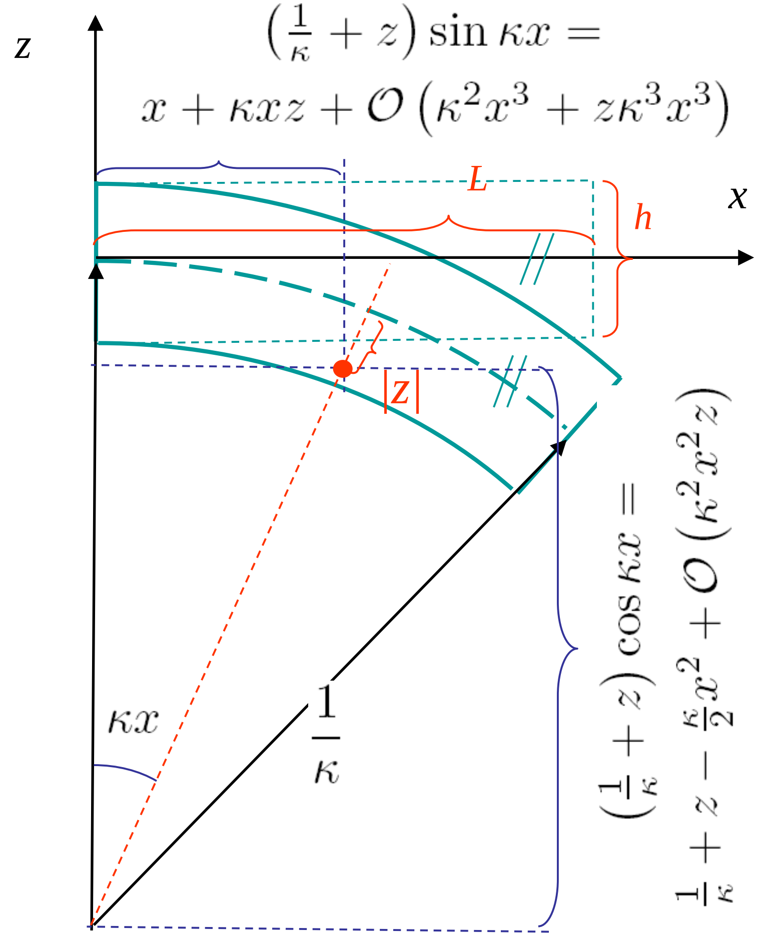
- 1-D pure bending (2)
  - Kinematics
    - $\boldsymbol{u}_x = \kappa x z$



- 1-D pure bending (3)
  - Kinematics (2)
    - $\boldsymbol{u}_x = \kappa x z$
    - $\boldsymbol{u}_z = -\frac{\kappa}{2}x^2 + ?$

Section remains plane, but the shape can change

$$\Rightarrow \begin{cases} \boldsymbol{u}_z = -\frac{\kappa}{2}x^2 + f(?) \\ \boldsymbol{u}_y = g(?) \end{cases}$$



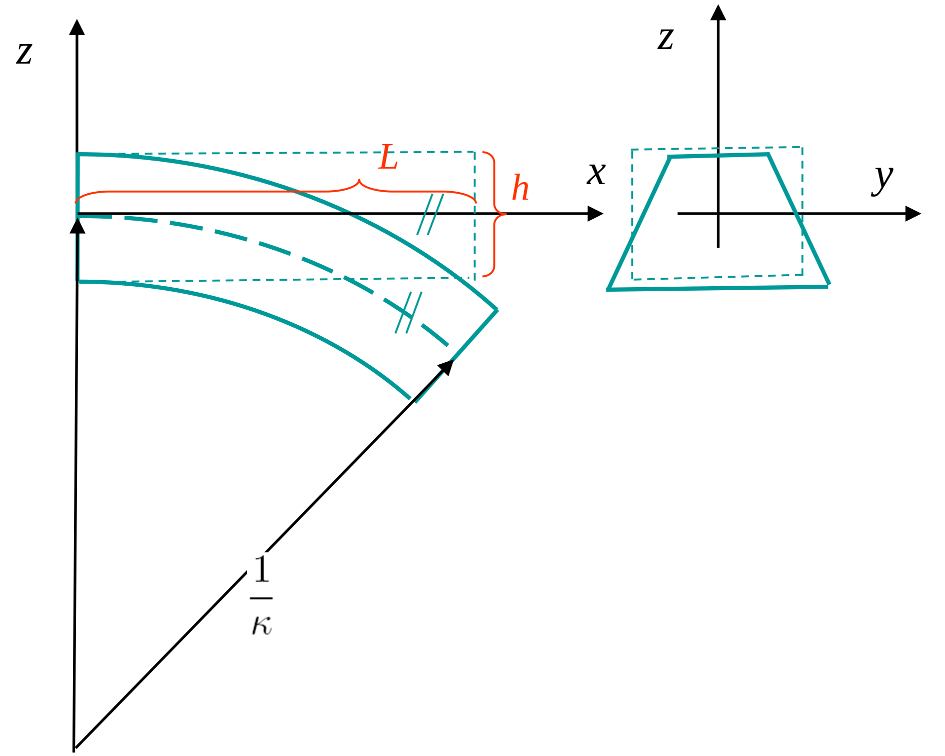
- 1-D pure bending (4)
  - Kinematics (3)
    - $\boldsymbol{u}_x = \kappa x z$
    - $\boldsymbol{u}_z = -\frac{\kappa}{2} x^2 + f\,(?)$
    - $\boldsymbol{u}_y = g\,(?)$
    - *f* & *g* should
      - Involve quadratic terms
      - Be independent of *x*

      ⟹ terms in y^2, yz, z^2
    - *f(-y)* should be equal to *f(y)*

      ⟹ terms in y^2, z^2
    - *g(-y)* should be equal to -*g(y)*

      ⟹ terms in yz
    - No shearing ⟹ $\boldsymbol{\varepsilon}_{yz} = \frac{1}{2}\left(2y\partial_{y^2} f + y\partial_{yz} g\right) = 0$ & $\boldsymbol{\varepsilon}_{xz} = 0$, $\boldsymbol{\varepsilon}_{xy} = 0$ satisfied
  - For linear elasticity, Poisson's effect induces

$$\begin{cases} \boldsymbol{u}_z = -\frac{\kappa}{2}\left[x^2 + \nu\left(\alpha z^2 - \beta y^2\right)\right] \\ \boldsymbol{u}_y = -\kappa\nu\beta y z \end{cases}$$



- 1-D pure bending (5)
  - Small deformations

$$\begin{cases} \boldsymbol{u}_x = \kappa x z \\ \boldsymbol{u}_y = -\kappa \nu \beta y z \\ \boldsymbol{u}_z = -\frac{\kappa}{2}\left[x^2 + \nu\left(\alpha z^2 - \beta y^2\right)\right] \end{cases} \Longrightarrow \begin{cases} \varepsilon_{xx} = \kappa z \\ \varepsilon_{yy} = -\kappa \beta \nu z \\ \varepsilon_{zz} = -\kappa \alpha \nu z \\ \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{xz} = 0 \end{cases}$$

  - For linear elasticity
    - $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon}$ with $\mathcal{H}_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{1+\nu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$

$$\Longrightarrow \begin{cases} \boldsymbol{\sigma}_{xx} = \frac{\kappa E z}{(1+\nu)(1-2\nu)}\left[1 - \nu - (\alpha + \beta)\nu^2\right] \\ \boldsymbol{\sigma}_{yy} = \frac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1 - \beta + \nu(\beta - \alpha)\right] \\ \boldsymbol{\sigma}_{zz} = \frac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1 - \alpha + \nu(\alpha - \beta)\right] \end{cases}$$

    - Balance equation: $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = 0$

$$\Longrightarrow \partial_z \boldsymbol{\sigma}_{zz} = 0 \Longrightarrow \alpha = \frac{1 - \beta\nu}{1 - \nu} \quad \text{or} \quad \nu = 0$$

- 1-D pure bending of beams
  - Small deformations & linear elasticity

$$\begin{cases} \varepsilon_{xx} = \kappa z \\ \varepsilon_{yy} = -\kappa\beta\nu z \\ \varepsilon_{zz} = -\kappa\alpha\nu z \\ \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{xz} = 0 \end{cases}$$

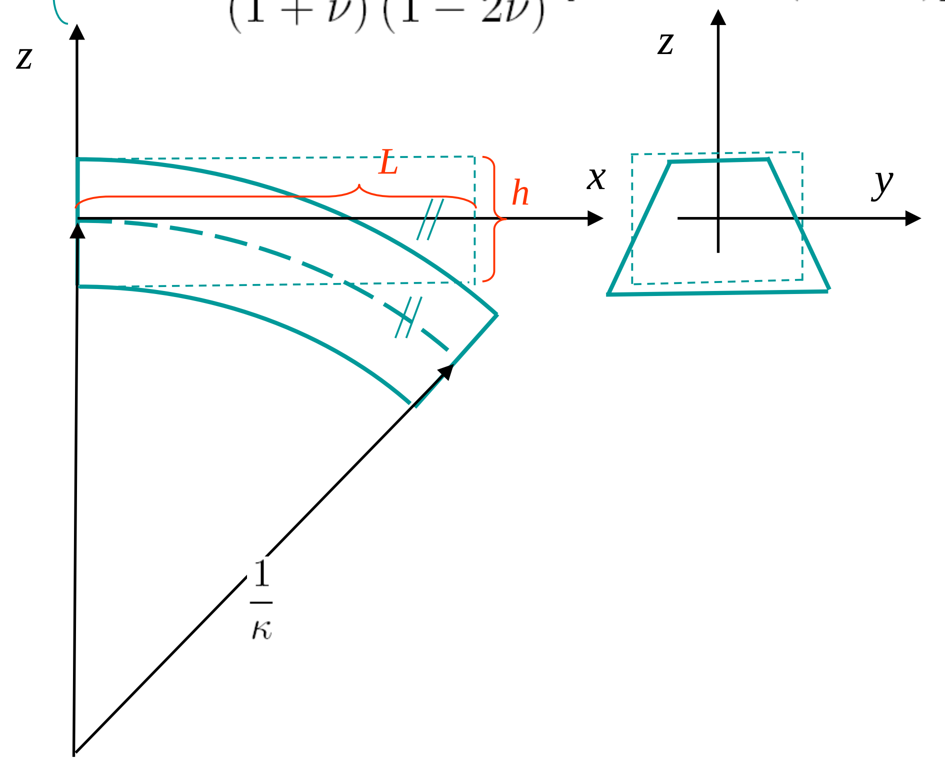
$$\begin{cases} \boldsymbol{\sigma}_{xx} = \frac{\kappa E z}{(1+\nu)(1-2\nu)}\left[1-\nu-(\alpha+\beta)\,\nu^2\right] \\ \boldsymbol{\sigma}_{yy} = \frac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1-\beta+\nu\,(\beta-\alpha)\right] \\ \boldsymbol{\sigma}_{zz} = \frac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1-\alpha+\nu\,(\alpha-\beta)\right] \end{cases}$$

  - Beam
    - Stress-free on all cross-section edges
      - $\boldsymbol{\sigma}_{yy} = \boldsymbol{\sigma}_{zz} = 0$

        ⇒ $\alpha = \beta = 1$
    - Balance equation

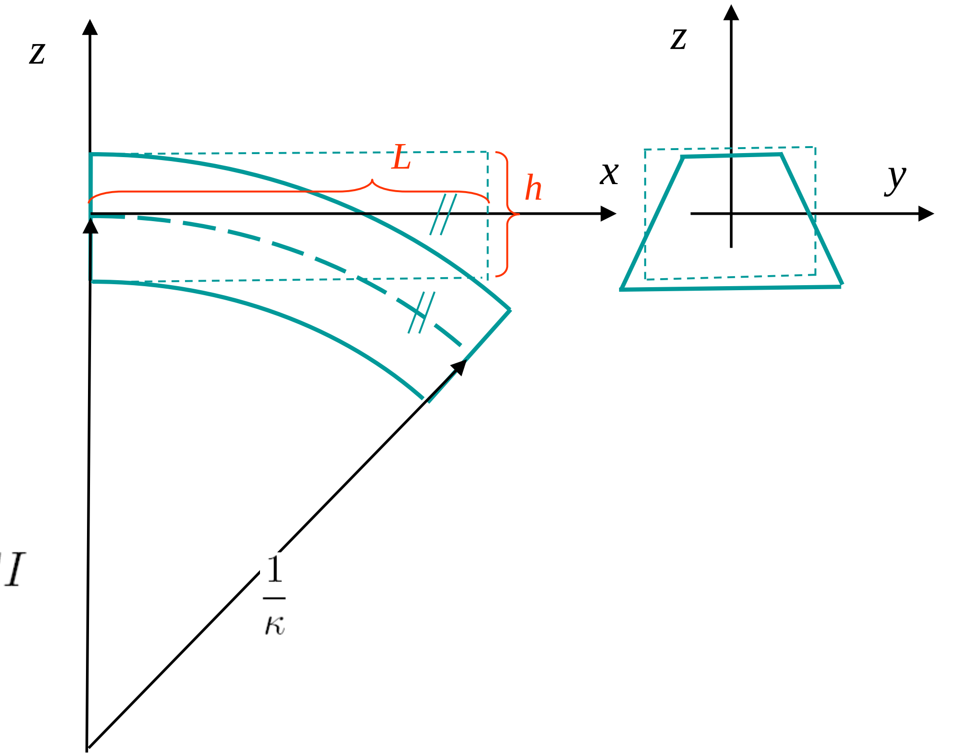
$$\alpha = \frac{1-\beta\nu}{1-\nu}$$ satisfied

⇒ $$\begin{cases} \boldsymbol{\sigma}_{xx} = \kappa E z \\ \boldsymbol{\sigma}_{yy} = \boldsymbol{\sigma}_{zz} = 0 \end{cases}$$



- 1-D pure bending of beams (2)
  - Equations

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \kappa E z \\ \boldsymbol{\sigma}_{yy} = \boldsymbol{\sigma}_{zz} = 0 \end{cases}$$



  - Momentum

$$M_{xx} = \int_A \kappa E z^2 dy dz = \kappa E I$$

    - Inertia

$$I = \int_A z^2 dy dz$$

    - For a rectangular cross-section $I = \frac{bh^3}{12}$

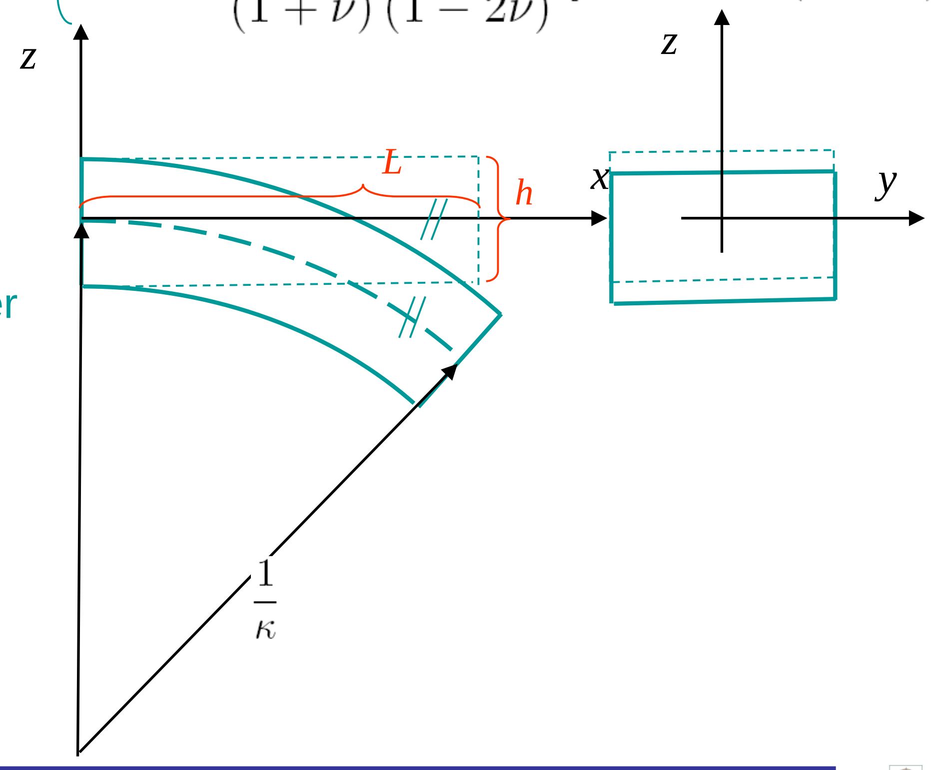
- 1-D pure bending of plates
  - Small deformations & linear elasticity

$$\begin{cases} \varepsilon_{xx} = \kappa z \\ \varepsilon_{yy} = -\kappa\beta\nu z \\ \varepsilon_{zz} = -\kappa\alpha\nu z \\ \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{xz} = 0 \end{cases}$$

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \dfrac{\kappa E z}{(1+\nu)(1-2\nu)}\left[1-\nu-(\alpha+\beta)\nu^2\right] \\ \boldsymbol{\sigma}_{yy} = \dfrac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1-\beta+\nu(\beta-\alpha)\right] \\ \boldsymbol{\sigma}_{zz} = \dfrac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1-\alpha+\nu(\alpha-\beta)\right] \end{cases}$$

  - Plate (plane - σ state)
    - No deformation along y
      ⟹ $\beta = 0$
    - Stress-free on upper and lower sides ⟹ $\boldsymbol{\sigma}_{zz} = 0$
      ⟹ $\alpha = 1 / (1-\nu)$
    - Balance equation

$$\alpha = \frac{1-\beta\nu}{1-\nu}$$ satisfied

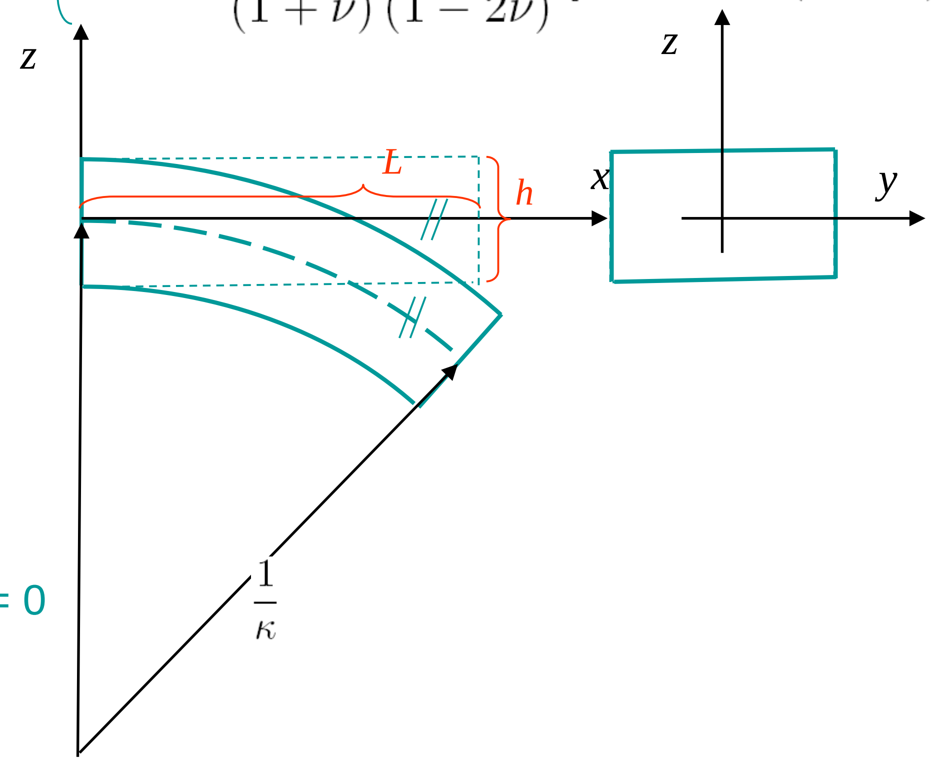


- 1-D pure bending of plates (2)
  - Small deformations & linear elasticity

$$\begin{cases} \varepsilon_{xx} = \kappa z \\ \varepsilon_{yy} = -\kappa\beta\nu z \\ \varepsilon_{zz} = -\kappa\alpha\nu z \\ \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{xz} = 0 \end{cases}$$

$$\begin{cases} \boldsymbol{\sigma}_{xx} = \frac{\kappa E z}{(1+\nu)(1-2\nu)}\left[1-\nu-(\alpha+\beta)\nu^2\right] \\ \boldsymbol{\sigma}_{yy} = \frac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1-\beta+\nu(\beta-\alpha)\right] \\ \boldsymbol{\sigma}_{zz} = \frac{\kappa E \nu z}{(1+\nu)(1-2\nu)}\left[1-\alpha+\nu(\alpha-\beta)\right] \end{cases}$$

  - Plate (plane - ε state)
    - No deformation along y
      ⇒ $\beta = 0$
    - No deformation along z
      ⇒ $\alpha = 0$
    - Balance equation
      $\alpha = \frac{1-\beta\nu}{1-\nu}$ NOT satisfied
    - This state actually requires $\nu = 0$



- 1-D pure bending of plates (3)
  - Back to plane - σ state
    - Equations

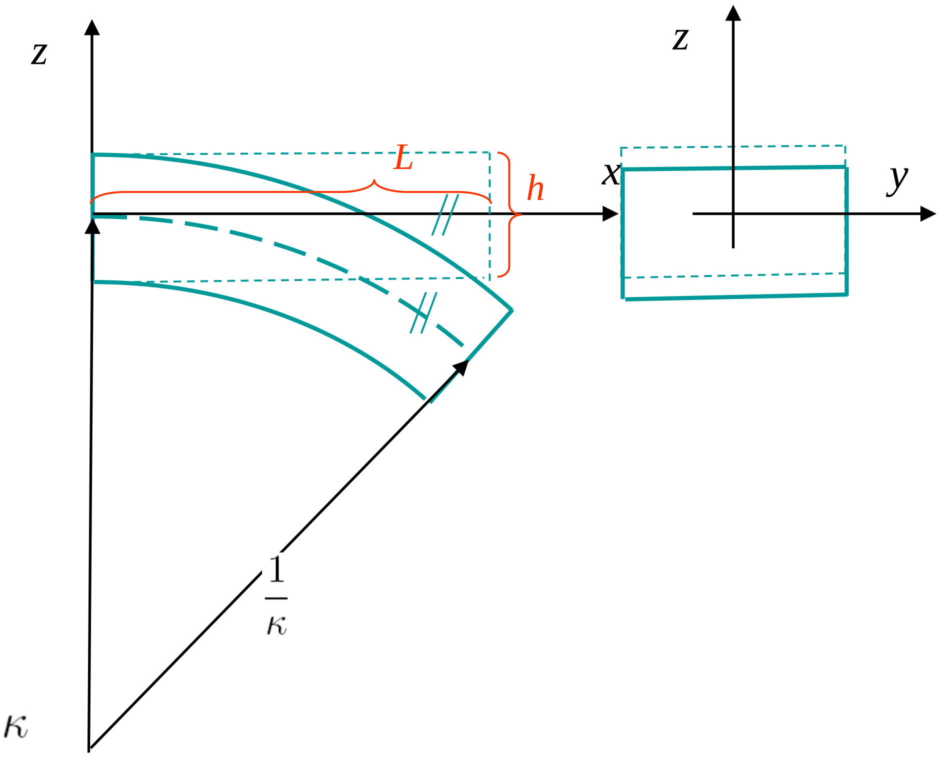
$$\begin{cases} \boldsymbol{\sigma}_{xx} = \dfrac{\kappa E z}{1-\nu^2} \\ \boldsymbol{\sigma}_{zz} = 0 \\ \boldsymbol{\sigma}_{yy} = \dfrac{\nu \kappa E z}{1-\nu^2} \end{cases}$$

    - Momentum

$$m_{xx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\kappa E}{1-\nu^2} z^2 dz = D\kappa$$

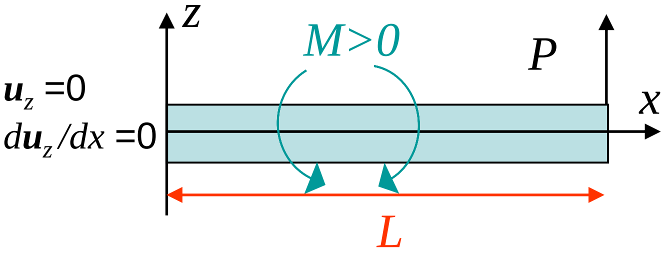
    - Flexural rigidity

$$D = \frac{Eh^3}{12\left(1-\nu^2\right)}$$



- Elastic beam
  - Equations

$$\begin{cases} M_{xx} = \int_A \kappa E z^2 dy dz = \kappa E I \\ \kappa = -\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \end{cases}$$



  - Concentrated load
    - For a uniform cross-section *hxb*: $I = \dfrac{bh^3}{12}$

$$P(L-x) = \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} EI \Longrightarrow \boldsymbol{u}_z = \frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right)$$

    - Stress

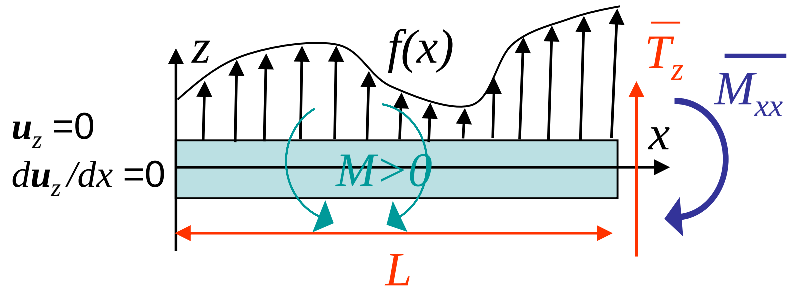
$$\boldsymbol{\sigma}_{xx}\big|_{z=-\frac{h}{2}} = -\kappa E \frac{h}{2} = \frac{Ph}{2I}(L-x) = \frac{6P}{bh^2}(L-x)$$

    - Shearing
      - There is a shearing $\boldsymbol{T}_z = P$: $\boldsymbol{T}_z = \dfrac{\partial M_{xx}}{\partial x} = \dfrac{\partial P(x-L)}{\partial x} = P$
      - Its effect on shearing stress can be neglected if h/L << as

$$\boldsymbol{\sigma}_{xy} = \mathcal{O}\left(\frac{P}{bh}\right) = \mathcal{O}\left(\frac{h}{L}\boldsymbol{\sigma}_{xx}\left(x=0,\ z=-\frac{h}{2}\right)\right)$$

- Elastic beam (2)
  - Equations

$$\begin{cases} M_{xx} = \int_A \kappa E z^2 dy dz = \kappa E I \\ \kappa = -\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \end{cases}$$



  - Non-uniform loading
    - Internal energy variation

$$\delta E_{\text{int}} = \int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \boldsymbol{\varepsilon}_{xx} dA dx = \int_0^L \int_A E \kappa \delta \kappa z^2 dA dx =$$

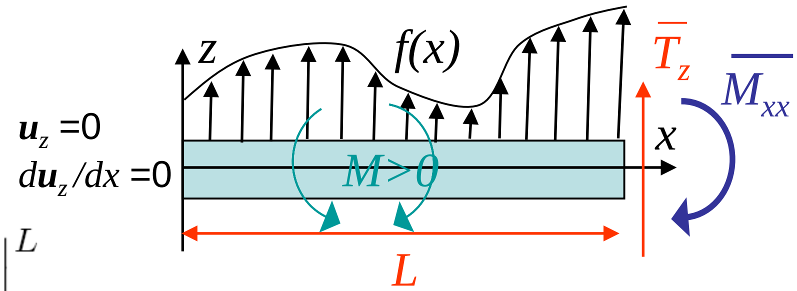
$$\int_0^L \int_A E \delta \frac{\kappa^2}{2} z^2 dA dx = \delta \int_0^L \frac{M_{xx} \kappa}{2} dx$$

    - Work variation of external forces

$$\delta W_{\text{ext}} = \int_0^L f(x) \delta \boldsymbol{u}_z dx + \bar{T}_z \delta \boldsymbol{u}_z \Big]_0^L - \bar{M}_{xx} \frac{\partial \delta \boldsymbol{u}_z}{\partial x} \Big|_0^L$$

$$\Rightarrow \int_0^L \frac{1}{2} E I \left(\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \right)^2 dx = \int_0^L f(x) \boldsymbol{u}_z dx + \bar{T}_z \boldsymbol{u}_z \Big]_0^L - \bar{M}_{xx} \frac{\partial \boldsymbol{u}_z}{\partial x} \Big|_0^L$$

- Elastic beam (3)
  - Energy conservation



$$\int_0^L \frac{1}{2} EI \left(\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)^2 dx =$$

$$\int_0^L f(x)\, \boldsymbol{u}_z dx + \bar{T}_z \boldsymbol{u}_z\Big]_0^L - \bar{M}_{xx} \frac{\partial \boldsymbol{u}_z}{\partial x}\Big|_0^L$$

- Integration by parts of the internal energy variation

$$\delta E_{\text{int}} = \int_0^L EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \frac{\partial^2 \delta \boldsymbol{u}_z}{\partial x^2} dx = \left[EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \frac{\partial \delta \boldsymbol{u}_z}{\partial x}\right]_0^L - \int_0^L \frac{\partial}{\partial x}\left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) \frac{\partial \delta \boldsymbol{u}_z}{\partial x} dx$$

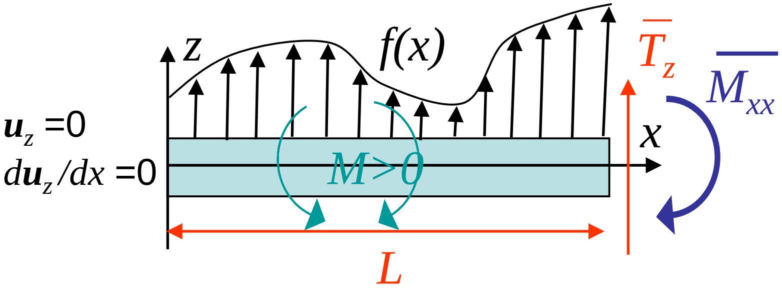
$$\delta E_{\text{int}} = \left[EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \frac{\partial \delta \boldsymbol{u}_z}{\partial x}\right]_0^L - \left[\frac{\partial}{\partial x}\left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) \delta \boldsymbol{u}_z\right]_0^L + \int_0^L \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) \delta \boldsymbol{u}_z dx$$

- Work variation of external forces

$$\delta W_{\text{ext}} = \int_0^L f(x)\, \delta \boldsymbol{u}_z dx + \bar{T}_z \delta \boldsymbol{u}_z\Big]_0^L - \bar{M}_{xx} \frac{\partial \delta \boldsymbol{u}_z}{\partial x}\Big]_0^L$$

- Elastic beam (4)
  - Energy conservation (2)
    - As $\delta \boldsymbol{u}_z$ is arbitrary:

      ⇒ Euler-Bernoulli equations



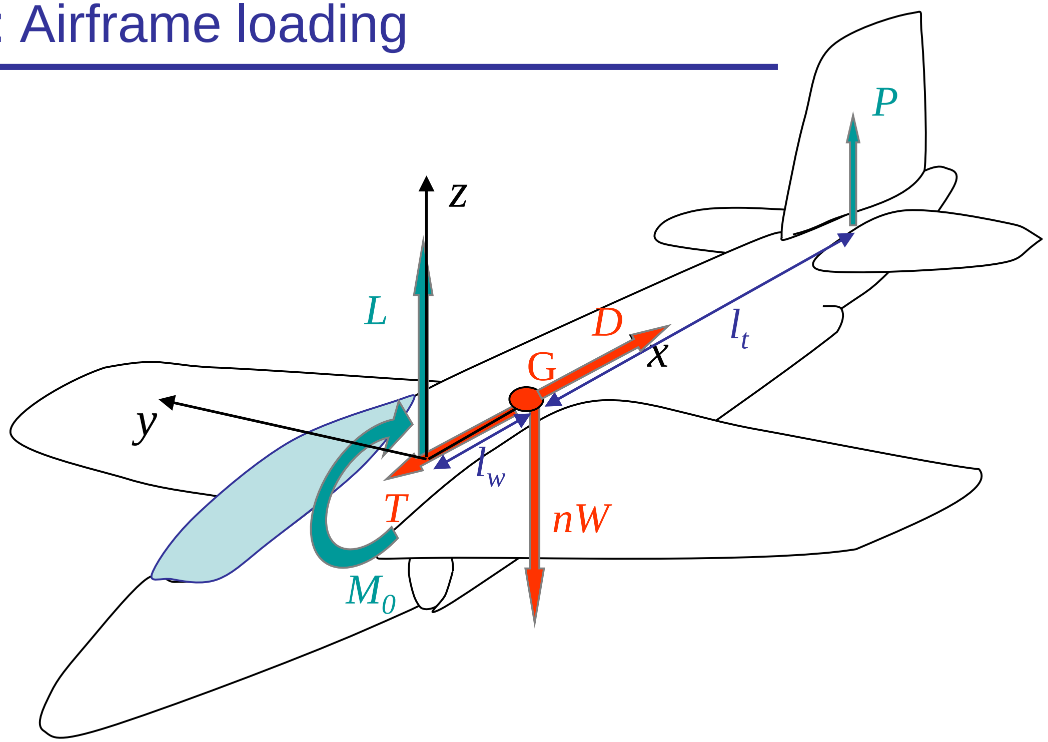
- $$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f(x)$$ on [0, L] &
- $$\begin{cases} -\frac{\partial}{\partial x}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\Big|_{0,\,L} = \bar{T}_z\big|_{0,\,L} \\ -EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\Big|_{0,\,L} = \bar{M}_{xx}\big|_{0,\,L} \end{cases}$$

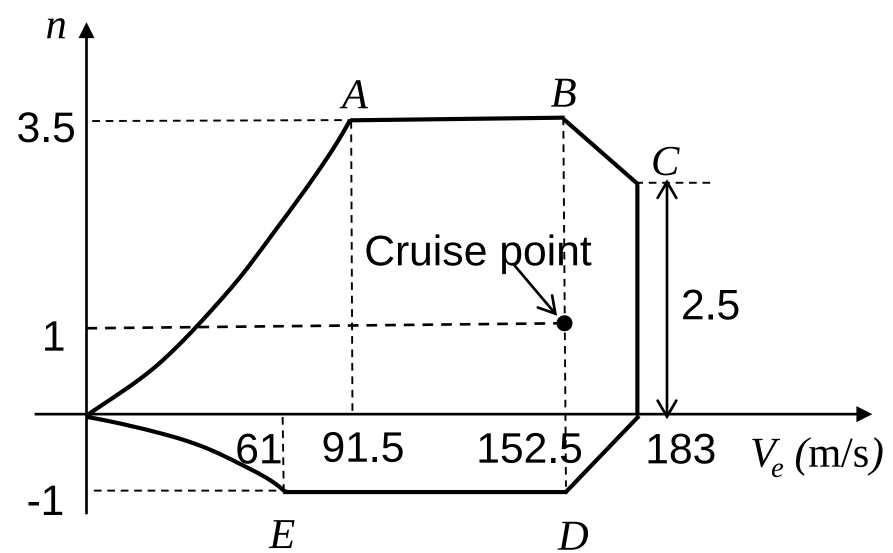
Exercise: Airframe loading

- Plane
  - Total weight: $W = 196000\,N$
  - Span: $b = 27.5\,m$
  - MAC: $\bar{c} = 3.05\,m$
  - Aerodynamic centers
    - Wing: forward of G

      $l_w = 0.915\,m$
    - Tail: aft of G

      $l_t = 16.7\,m$
  - Pitching moment
    - $C_{M,0}$ = -0.0638
    - Wing and body contributions
  - Thrust and drag
    - Supposed to be applied at G
  - Flight envelope known

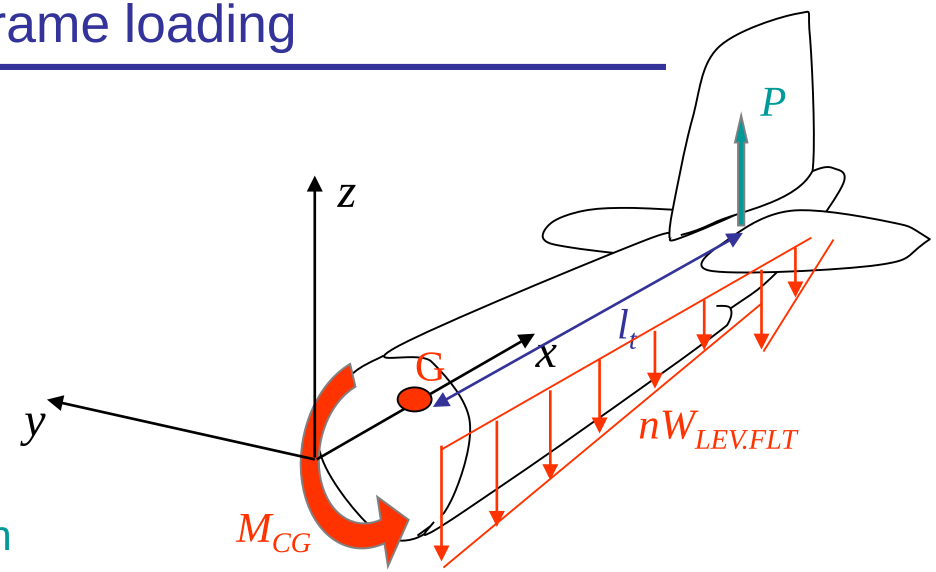


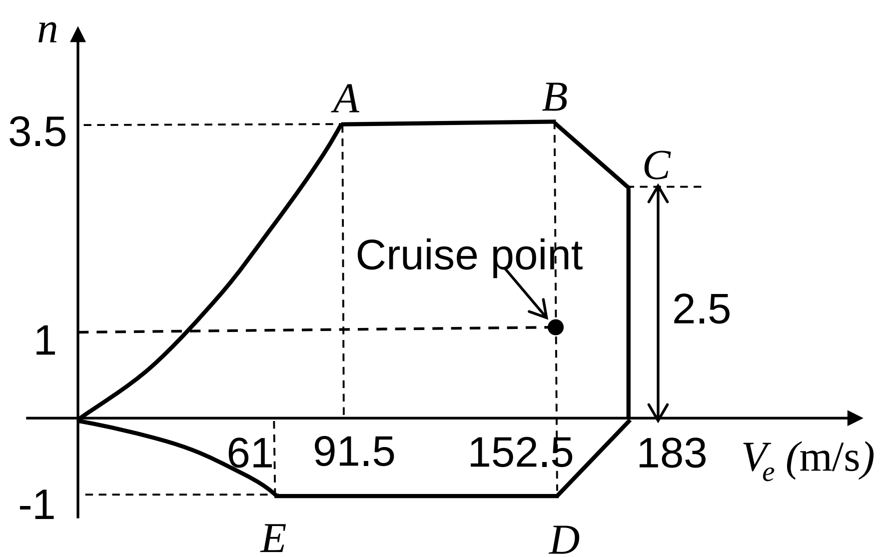


- Fuselage bending moment at G
  - Due to
    - Fuselage weight
      - For n=1, the resulting bending moment is called $M_{LEV.FLT}$
      - So, at other points of the flight envelope the contribution is $n\ M_{LEV.FLT}$
    - Tail load P
  - At cruising flight, sea-level bending moment at G is given:

$$M_{\text{CG, cruise point}} = 600000\ Nm$$

  - Maximum value of M_{CG} on the flight envelope?



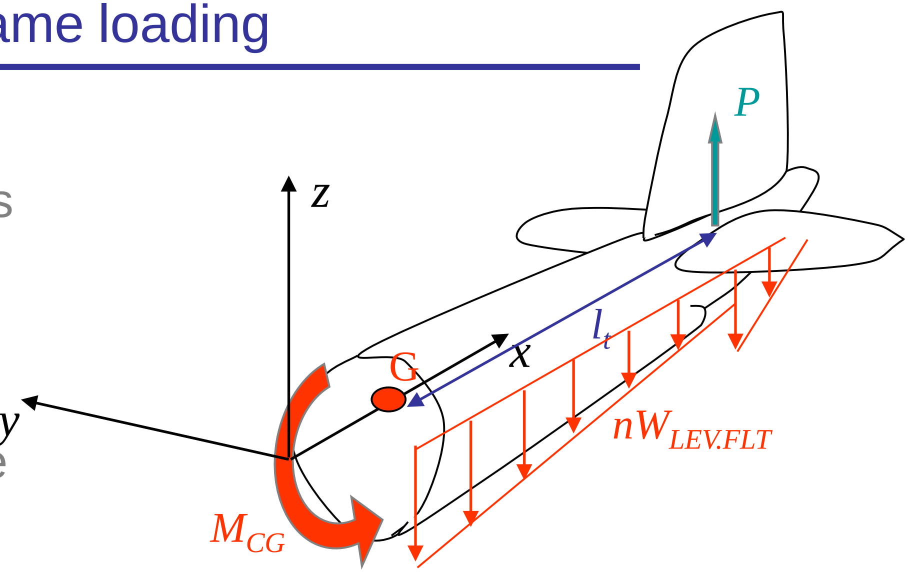


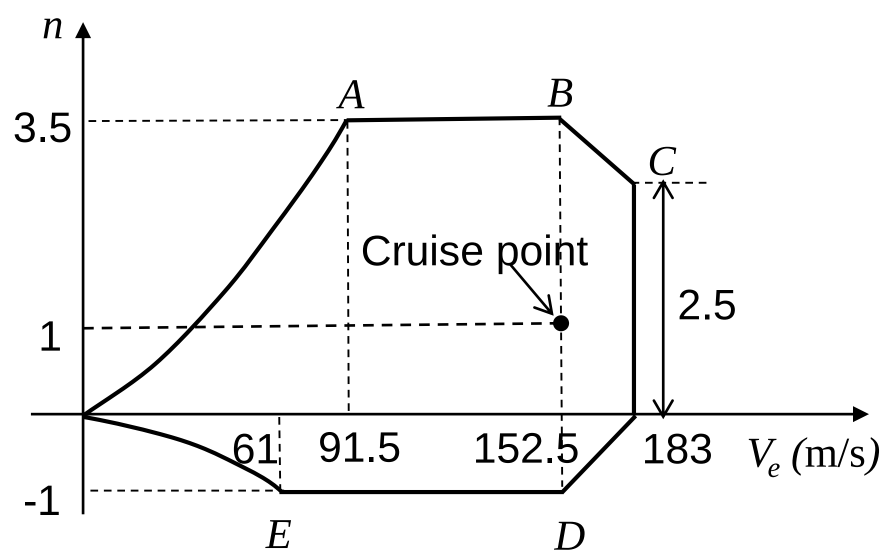
References

- Lecture notes
  - Aircraft Structures for engineering students, T. H. G. Megson, Butterworth-Heinemann, An imprint of Elsevier Science, 2003, ISBN 0 340 70588 4

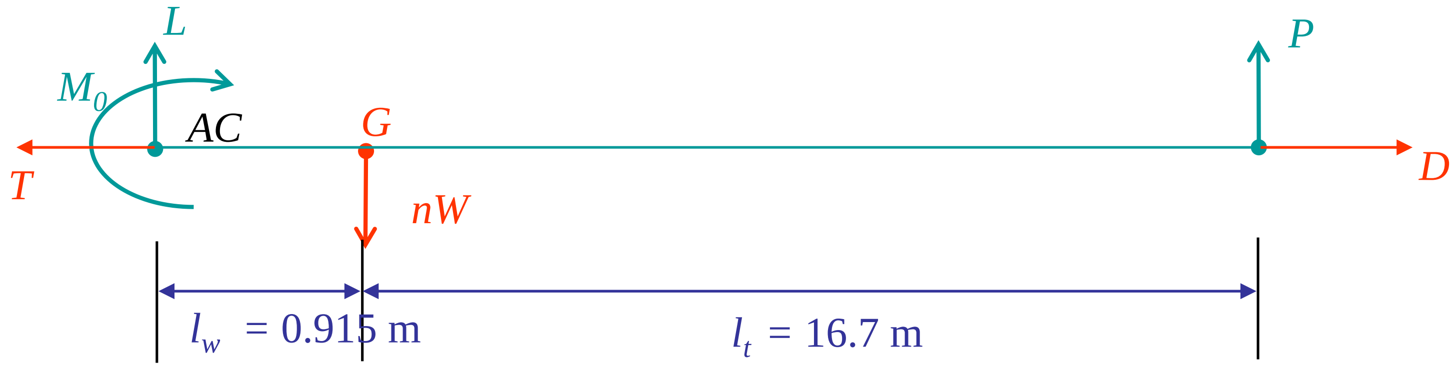
Exercise: Airframe loading

- Method
  - Write down the equilibrium equations
    - Translation and
    - Moment around the G
  - Write down an expression of the bending moment M_{CG} in the fuselage at the CG in terms of
    - The load factor n
    - The equivalent velocity V
  - Evaluate this expression in cruising conditions
    - In order to determine the extra unknowns
  - Finally, calculate this bending moment for all critical flight conditions
    - Points A, B, C, D, E
    - Bottom point of a symmetric maneuver





- Equilibrium equations
  - Bottom point of a symmetric maneuver



  - Balance equations (vertical and moment)

$$\begin{cases} 0.915L + M_0 - 16.7P = 0 \\ L + P = nW \end{cases}$$

$$\Rightarrow \begin{cases} L = \dfrac{16.7nW - M_0}{17.615} \\ P = \dfrac{0.915nW + M_0}{17.615} \end{cases}$$

- Equilibrium equations (2)
  - At sea level (or taking equivalent velocities):
    - $M_0 = \frac{1}{2}\rho V^2 S_{wing}\bar{c}C_{M_0} = \frac{1}{2} * 1.223 * 27.5 * 3.05^2 * -0.0638 * V^2$

$$\Longrightarrow M_0 = -9.98V^2$$

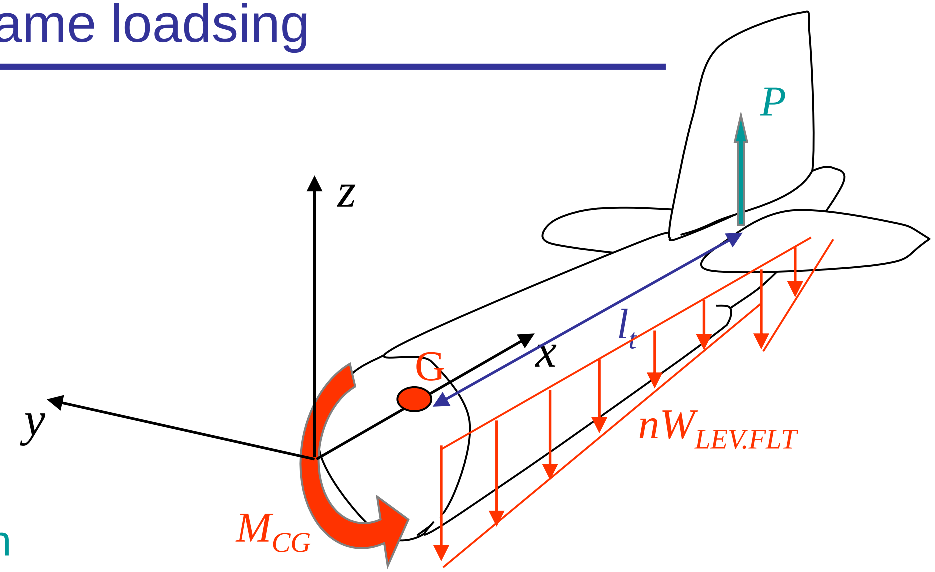
- As $P = \frac{0.915nW + M_0}{17.615}$

$$\Longrightarrow P = \frac{0.915 * 196000 * n - 9.98 * V^2}{17.615}$$

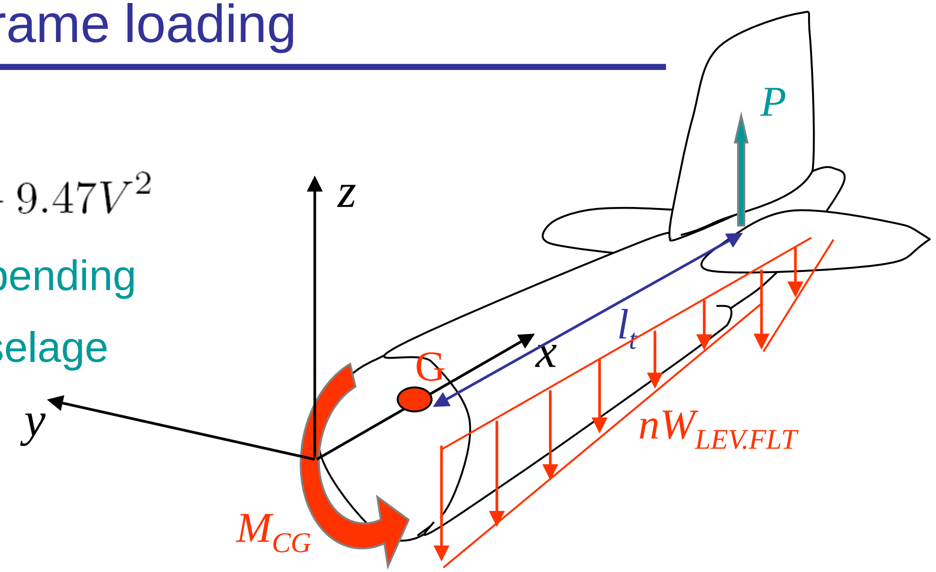
- Bending moment of fuselage at CG
  - Due to
    - Fuselage weight
      - For n=1, the resulting bending moment is called $M_{LEV.FLT}$
      - So, at other points of the flight envelope the contribution is $n\, M_{LEV.FLT}$
    - Tail load P
  - By equilibrium
    - $M_{CG} = nM_{LEV.FLT} - 16.7P$
    - As $P = \dfrac{0.915 * 196000 * n - 9.98 * V^2}{17.615}$

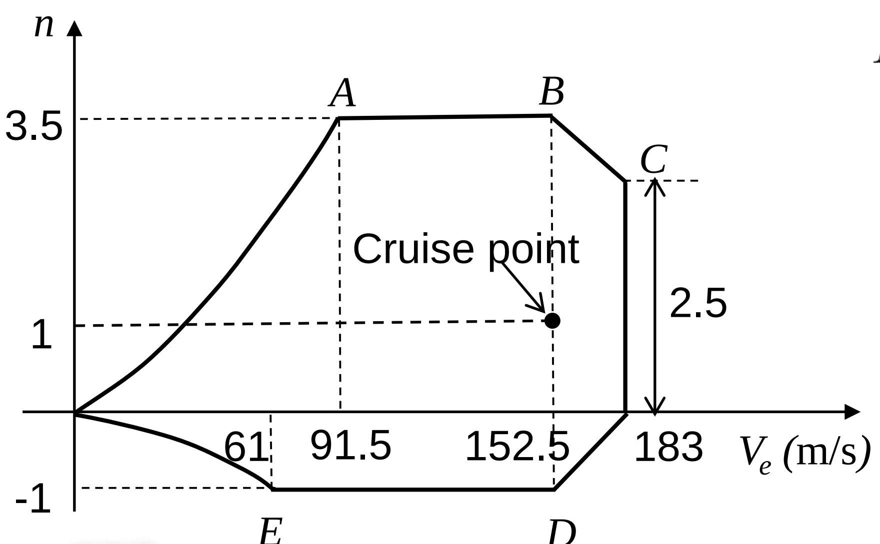
$$\Longrightarrow M_{CG} = nM_{LEV.FLT} - 16.7 * \left[\frac{0.915 * 196000 * n - 9.98V^2}{17.615}\right]$$

$$\Longrightarrow M_{CG} = n\left(M_{LEV.FLT} - 170020\right) + 9.47V^2$$



- Missing term
  - In $M_{CG} = n\left(M_{LEV.FLT} - 170020\right) + 9.47V^2$
    - We need to evaluate the resulting bending moment $M_{LEV.FLT}$ due to the rear fuselage weight at cruise condition
  - At cruise conditions
    - $n = 1$
    - $V = 152.5\ m/s$
    - $M_{\text{CG, cruise point}} = 600000\ Nm$





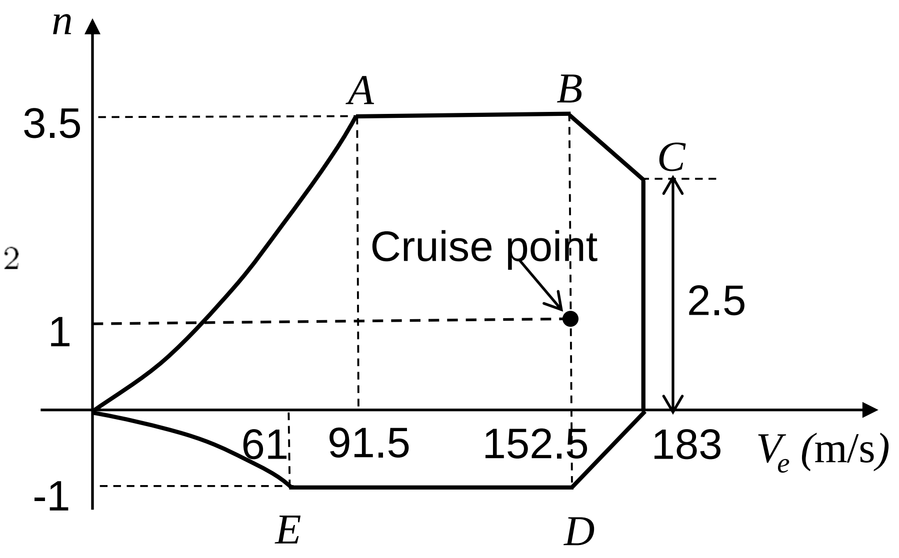
$$M_{CG} = n\left(M_{LEV.FLT} - 170020\right) + 9.47V^2$$

$$\Longrightarrow 600000 = 1 * \left(M_{LEV.FLT} - 170020\right) + 9.47 * 152.5^2$$

$$\Longrightarrow M_{LEV.FLT} = 549809\ Nm$$

- Flight envelope
  - From

$$\begin{cases} M_{CG} = n\left(M_{LEV.FLT} - 170020\right) + 9.47V^2 \\ M_{LEV.FLT} = 549809\ Nm \end{cases} \Longrightarrow M_{CG} = 379789n + 9.47V^2$$



  - Most critical cases
    - B or C
    - Point B

$$M^B_{CG} = 379789 * 3.5 + 9.47 * 152.5^2$$

$$\Longrightarrow M^B_{CG} = 1549500\ Nm$$

    - Point C

$$M^C_{CG} = 379789 * 2.5 + 9.47 * 183^2$$

$$\Longrightarrow M^C_{CG} = 1266600\ Nm$$

  - Maximum bending moment in fuselage
    - 1 549 500 Nm
    - At n = 3.5 and V = 152.5 m/s

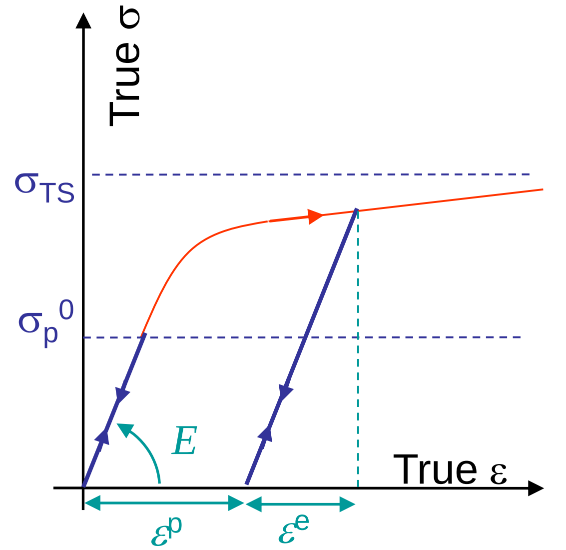
- Material law (small deformations)
  - Yield surface

$$f(\boldsymbol{\sigma}) \leq 0 \quad \begin{cases} f < 0\text{: elastic region} \\ f = 0\text{: plasticity} \end{cases}$$

  - Plastic flow
    - Assumption: deformations can be added

$$d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^{\mathrm{e}} + d\boldsymbol{\varepsilon}^{\mathrm{p}} \Longrightarrow d\boldsymbol{\sigma} = \mathcal{H} : d\boldsymbol{\varepsilon}^{\mathrm{e}}$$

    - Normal plastic flow $d\boldsymbol{\varepsilon}^{\mathrm{p}} = d\lambda \dfrac{\partial f}{\partial \boldsymbol{\sigma}}$

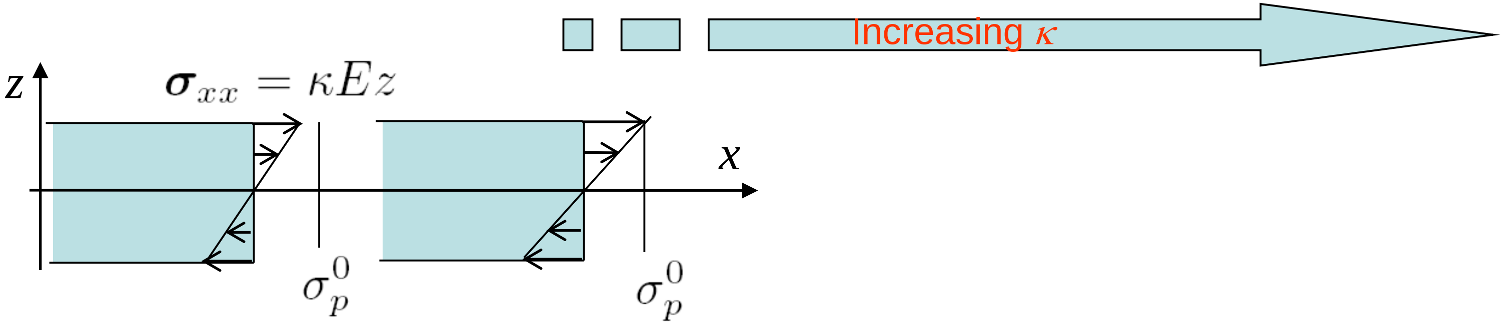


    - Von Mises surface with isotropic plastic flow (J2-plasticity)
      - Deviatoric part of the stress tensor $\mathbf{s} = \boldsymbol{\sigma} - \dfrac{\mathrm{tr}(\boldsymbol{\sigma})}{3}\mathbf{I}$
      - Yield surface $f = \sqrt{\dfrac{3}{2}\mathbf{s} : \mathbf{s}} - \sigma_p(\bar{\epsilon}^{\mathrm{p}}) \leq 0$
      - Normality: since $\dfrac{\partial\left(\boldsymbol{\sigma}_{ij} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\delta_{ij}\right)}{\partial \boldsymbol{\sigma}_{kl}} = \dfrac{1}{2}\delta_{ik}\delta_{jl} + \dfrac{1}{2}\delta_{il}\delta_{jk} - \dfrac{1}{3}\delta_{ij}\delta_{kl}$

$$\Longrightarrow \frac{\partial f}{\partial \boldsymbol{\sigma}} = \sqrt{\frac{3}{2}} \frac{\mathbf{s}}{\sqrt{\mathbf{s} : \mathbf{s}}}$$

      - Then the plastic flow becomes $d\lambda = d\bar{\varepsilon}^{\mathrm{p}} = \sqrt{\dfrac{2}{3} d\boldsymbol{\varepsilon}^{\mathrm{p}} : d\boldsymbol{\varepsilon}^{\mathrm{p}}}$
  - Path dependency (incremental equations in *d*)

- Elastoplastic beam
  - Pure bending ⟹ $\boldsymbol{\varepsilon}_{xx} = \kappa z$ is always satisfied

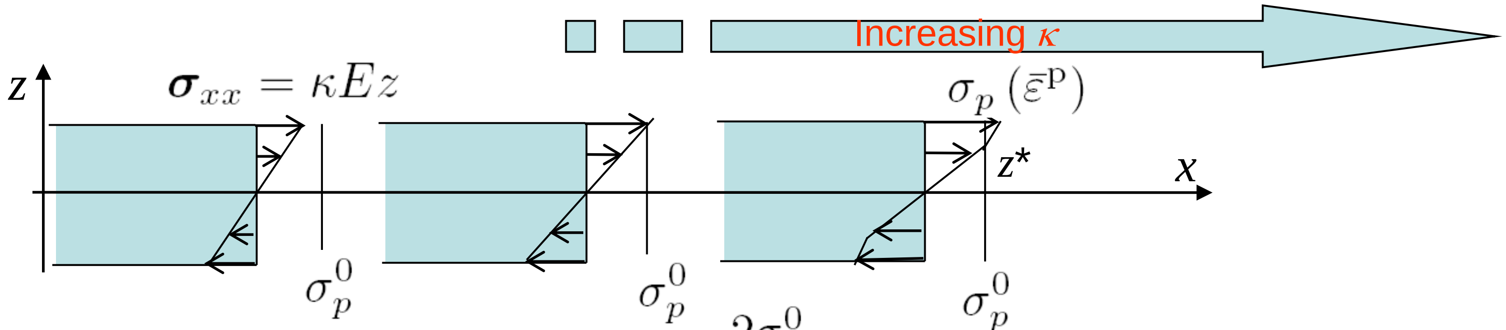


- For small κ: the beam remains elastic
- $\boldsymbol{\sigma}_{xx} = \kappa E z$ ⟹

$$\begin{cases} \boldsymbol{\sigma}_{ij} = \boldsymbol{\sigma}_{xx}\delta_{ix}\delta_{jx} = \kappa E z \delta_{ix}\delta_{jx} \\ \mathbf{s}_{ij} = \boldsymbol{\sigma}_{xx}\delta_{ix}\delta_{jx} - \frac{\boldsymbol{\sigma}_{xx}}{3}\delta_{ij} = \kappa E z \left[\delta_{ix}\delta_{jx} - \frac{1}{3}\delta_{ij}\right] \\ f = \sqrt{\frac{3}{2}\mathbf{s}_{ij}\mathbf{s}_{ij}} - \sigma_p^0 = |\kappa|\, E\, |z| - \sigma_p^0 \end{cases}$$

- Yielding for $|\kappa| = \frac{2\sigma_p^0}{hE}$

- Elastoplastic beam (2)
  – Pure bending $\Longrightarrow$ $\boldsymbol{\varepsilon}_{xx} = \kappa z$ is always satisfied (2)

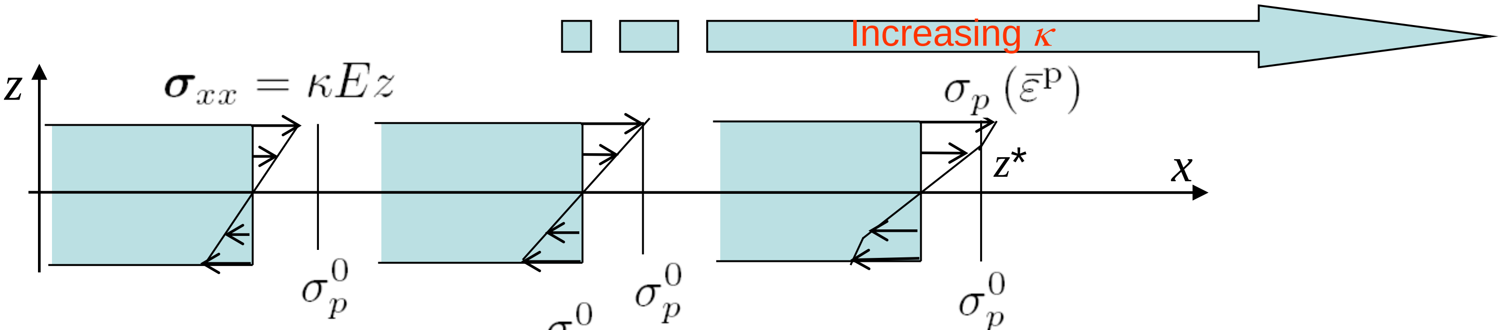


  - For κ slightly larger than $\frac{2\sigma_p^0}{hE}$: part of the beam is under plasticity
  - For |z| < $z^* = \frac{\sigma_p^0}{E\left|\kappa\right|}$: solution remains elastic $\Longrightarrow$ $\boldsymbol{\sigma}_{xx} = \kappa E z$
  - For |z| > $z^* = \frac{\sigma_p^0}{E\left|\kappa\right|}$ $\left\{\begin{array}{l} \boldsymbol{\sigma}_{ij} = \boldsymbol{\sigma}_{xx}\delta_{ix}\delta_{jx} \Longrightarrow \mathbf{s}_{ij} = \boldsymbol{\sigma}_{xx}\left[\delta_{ix}\delta_{jx} - \frac{1}{3}\delta_{ij}\right] \\ \Longrightarrow f = \sqrt{\frac{3}{2}\mathbf{s}_{ij}\mathbf{s}_{ij}} - \sigma_p\left(\bar{\varepsilon}^{\mathrm{p}}\right) = \left|\boldsymbol{\sigma}_{xx}\right| - \sigma_p\left(\bar{\varepsilon}^{\mathrm{p}}\right) = 0 \end{array}\right.$

    – Normal plastic flow:

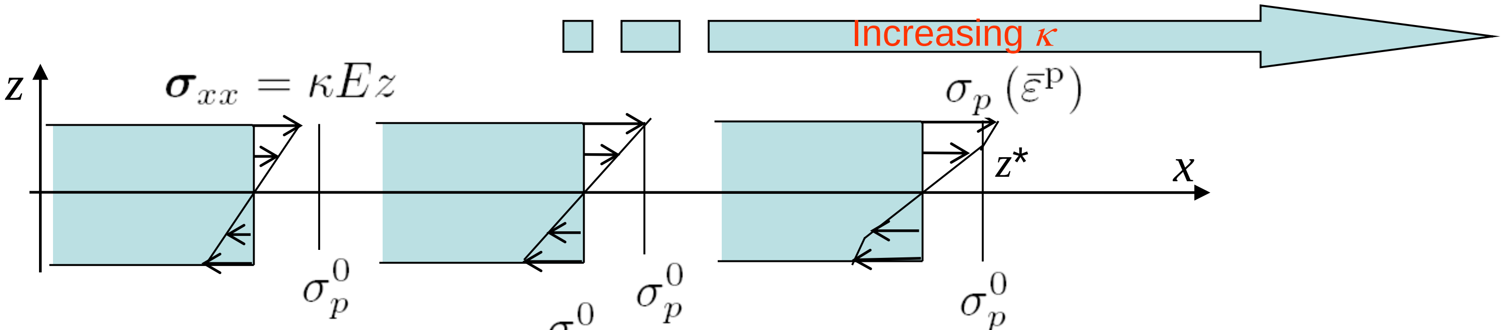
$$\boldsymbol{\varepsilon}^{\mathrm{p}} = \bar{\varepsilon}^{\mathrm{p}}\frac{\partial f}{\partial \boldsymbol{\sigma}} = \bar{\varepsilon}^{\mathrm{p}}\sqrt{\frac{3}{2}}\frac{\mathbf{s}}{\sqrt{\mathbf{s}:\mathbf{s}}} \Longrightarrow \boldsymbol{\varepsilon}_{ij}^{\mathrm{p}} = \bar{\varepsilon}^{\mathrm{p}}\left(\frac{3}{2}\delta_{ix}\delta_{jx} - \frac{1}{2}\delta_{ij}\right)$$

- Elastoplastic beam (3)
  - Pure bending ⟹ $\boldsymbol{\varepsilon}_{xx} = \kappa z$ is always satisfied (3)



- For |z| > $z^* = \dfrac{\sigma_p^0}{E\left|\kappa\right|}$
  - Additivity
$$\begin{cases} \boldsymbol{\varepsilon}^{\mathrm{e}}_{kl} = \mathcal{G}_{klij}\boldsymbol{\sigma}_{ij} = \boldsymbol{\sigma}_{xx}\left(\dfrac{1+\nu}{E}\delta_{kx}\delta_{lx} - \dfrac{\nu}{E}\delta_{kl}\right) \\ \boldsymbol{\varepsilon}^{\mathrm{p}}_{ij} = \bar{\varepsilon}^{\mathrm{p}}\left(\dfrac{3}{2}\delta_{ix}\delta_{jx} - \dfrac{1}{2}\delta_{ij}\right) \end{cases}$$
$$\Longrightarrow \kappa z = \boldsymbol{\varepsilon}^{\mathrm{e}}_{xx} + \boldsymbol{\varepsilon}^{\mathrm{p}}_{xx} = \frac{\boldsymbol{\sigma}_{xx}}{E} + \bar{\varepsilon}^{\mathrm{p}}$$
  - Linear isotropic hardening (z > 0):
$$f = \sqrt{\frac{3}{2}\mathbf{s}_{ij}\mathbf{s}_{ij}} - \sigma_p\left(\bar{\varepsilon}^{\mathrm{p}}\right) = \left|\boldsymbol{\sigma}_{xx}\right| - \sigma_p\left(\bar{\varepsilon}^{\mathrm{p}}\right) = 0 \Longrightarrow \boldsymbol{\sigma}_{xx} = \sigma_p^0 + h^p\bar{\varepsilon}^{\mathrm{p}}$$
  - Deformations tensor: $\boldsymbol{\varepsilon}_{yy} = \boldsymbol{\varepsilon}_{zz} = \boldsymbol{\varepsilon}^{\mathrm{e}}_{zz} - \dfrac{\bar{\varepsilon}^{\mathrm{p}}}{2} = -\dfrac{\nu}{E}\sigma_p^0 - \left(\dfrac{\nu h^p}{E} + \dfrac{1}{2}\right)\bar{\varepsilon}^{\mathrm{p}}$

- Elastoplastic beam (4)
  - Pure bending ⟹ $\boldsymbol{\varepsilon}_{xx} = \kappa z$ is always satisfied (4)



- For $|z|$ > $z^* = \dfrac{\sigma_p^0}{E\,|\kappa|}$
  - Solution $\left\{\begin{array}{l} \kappa z = \boldsymbol{\varepsilon}^{\mathrm{e}}_{xx} + \boldsymbol{\varepsilon}^{\mathrm{p}}_{xx} = \dfrac{\boldsymbol{\sigma}_{xx}}{E} + \bar{\varepsilon}^{\mathrm{p}} \\ \boldsymbol{\sigma}_{xx} = \sigma_p^0 + h^p \bar{\varepsilon}^{\mathrm{p}} \end{array}\right\}$ ⟹ $\bar{\varepsilon}^{\mathrm{p}} = \dfrac{E\kappa z - \sigma_p^0}{h^p + E}$

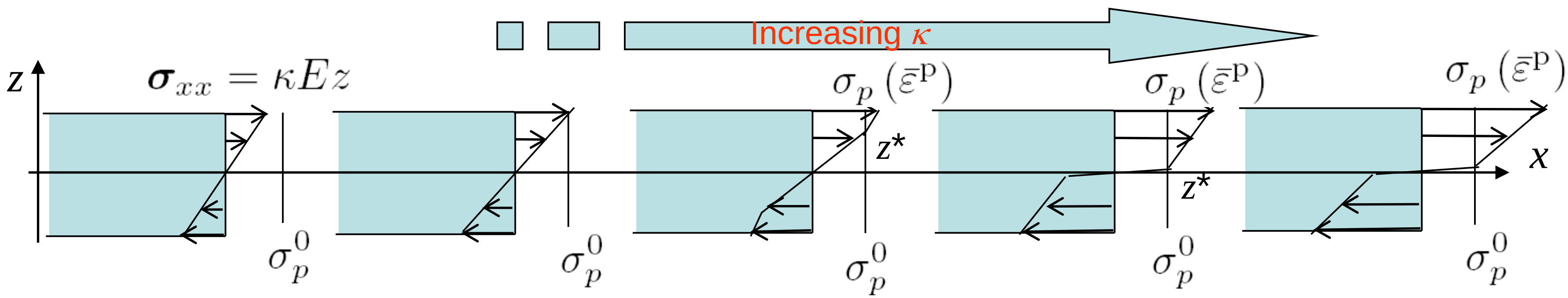
⟹ $\boldsymbol{\sigma}_{xx} = \sigma_p^0 + h^p \dfrac{E\kappa z - \sigma_p^0}{h^p + E} = \dfrac{E}{E + h^p}\left(\sigma_p^0 + h^p \kappa z\right)$

  - Bending moment:

$$M_{xx} = 2\int_0^b \left\{ \int_0^{z^*} \kappa E z^2 dz + \int_{z^*}^{\frac{h}{2}} \frac{E}{E + h^p}\left(\sigma_p^0 + h^p \kappa z\right) z dz \right\} dy$$

Annex I: Elastoplastic beam

- Elastoplastic beam (5)
  – Pure bending $\Longrightarrow$ $\boldsymbol{\varepsilon}_{xx} = \kappa z$ is always satisfied (5)



  - For $\kappa \uparrow$
    – $z^* = \dfrac{\sigma_p^0}{E\,|\kappa|} \rightarrow 0$
    – $M_{xx} \rightarrow 2\displaystyle\int_0^b \int_0^{\frac{h}{2}} \frac{E}{E+h^p}\left(\sigma_p^0 + h^p \kappa z\right) z dz dy = \frac{E\sigma_p^0 h^2 b}{4\left(E+h^p\right)} + \frac{E h^p \kappa I}{E + h^p}$
    – For perfectly plastic materials there is a plastic hinge as

$$M_{xx} \rightarrow 2\int_0^b \int_0^{\frac{h}{2}} \sigma_p^0 z dz dy = \frac{\sigma_p^0 h^2 b}{4}$$ is independent of κ