University of Liège
Aerospace & Mechanical Engineering

Aircraft Structures
Instabilities

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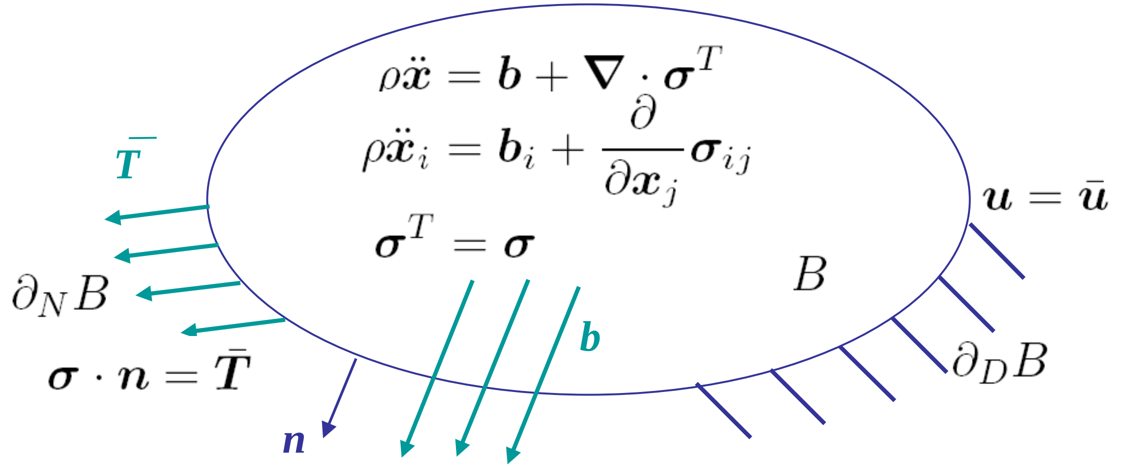
- Balance of body B
  - Momenta balance
    - Linear
    - Angular
  - Boundary conditions
    - Neumann
    - Dirichlet



- Small deformations with linear elastic, homogeneous & isotropic material
  - (Small) Strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{\nabla}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{\nabla}\right)$, or $\begin{cases} \boldsymbol{\varepsilon}_{ij} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial \boldsymbol{x}_i}\boldsymbol{u}_j + \dfrac{\partial}{\partial \boldsymbol{x}_j}\boldsymbol{u}_i\right) \\ \boldsymbol{\varepsilon}_{ij} = \dfrac{1}{2}\left(\boldsymbol{u}_{j,i} + \boldsymbol{u}_{i,j}\right) \end{cases}$
  - Hooke's law $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon}$, or $\boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl}\boldsymbol{\varepsilon}_{kl}$

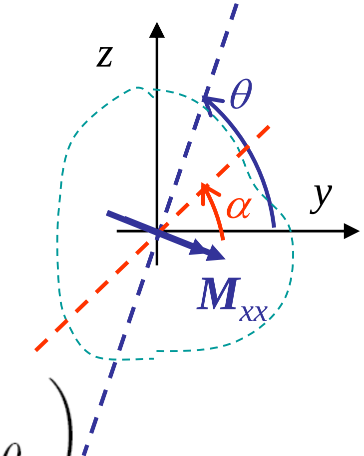
    with $$\mathcal{H}_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{1+\nu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$$

    $\lambda = K - 2\mu/3$ $\qquad$ 2μ
  - Inverse law $\boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma}$

    with $$\mathcal{G}_{ijkl} = \frac{1+\nu}{E}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right) - \frac{\nu}{E}\delta_{ij}\delta_{kl}$$

- General expression for unsymmetrical beams
  - Stress $\boldsymbol{\sigma}_{xx} = \kappa E z \cos\alpha - \kappa E y \sin\alpha$

    With $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{\kappa E} \begin{pmatrix} I_{yy} & -I_{yz} \\ -I_{yz} & I_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$

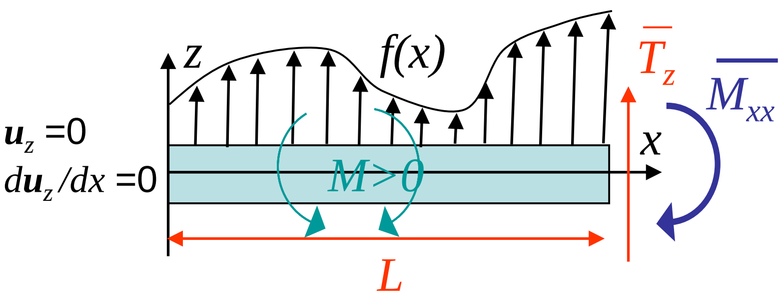


  - Curvature

$$\begin{pmatrix} -\boldsymbol{u}_{z,xx} \\ \boldsymbol{u}_{y,xx} \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{E\left(I_{yy}I_{zz} - I_{yz}I_{yz}\right)} \begin{pmatrix} I_{zz} & I_{yz} \\ I_{yz} & I_{yy} \end{pmatrix} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

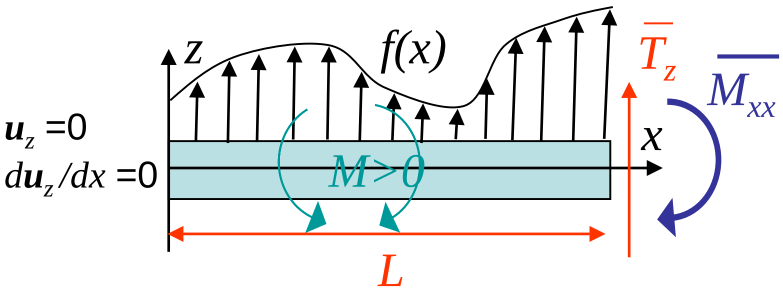
  - In the principal axes $I_{yz} = 0$

- Euler-Bernoulli equation in the principal axis
  - $\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f(x)$ for x in [0 L]
  - BCs $\begin{cases} -\frac{\partial}{\partial x}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\Big|_{0,\,L} = \bar{T}_z\big|_{0,\,L} \\ -EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\Big|_{0,\,L} = \bar{M}_{xx}\big|_{0,\,L} \end{cases}$
  - Similar equations for $\boldsymbol{u}_y$

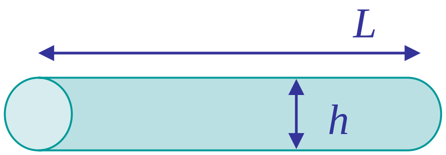


Beam shearing: linear elasticity summary

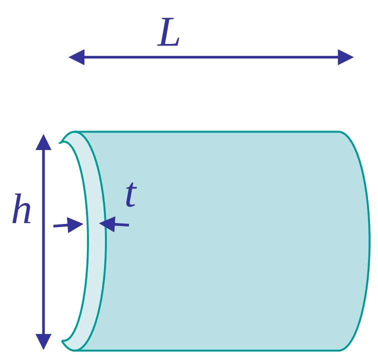
- General relationships
  - $$\begin{cases} f_z(x) = -\partial_x T_z = -\partial_{xx} M_y \\ f_y(x) = -\partial_x T_y = \partial_{xx} M_z \end{cases}$$



- Two problems considered
  - Thick symmetrical section

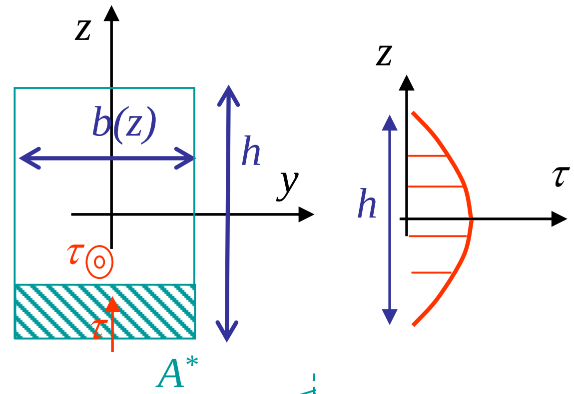


    - Shear stresses are small compared to bending stresses if $h/L << 1$
  - Thin-walled (unsymmetrical) sections

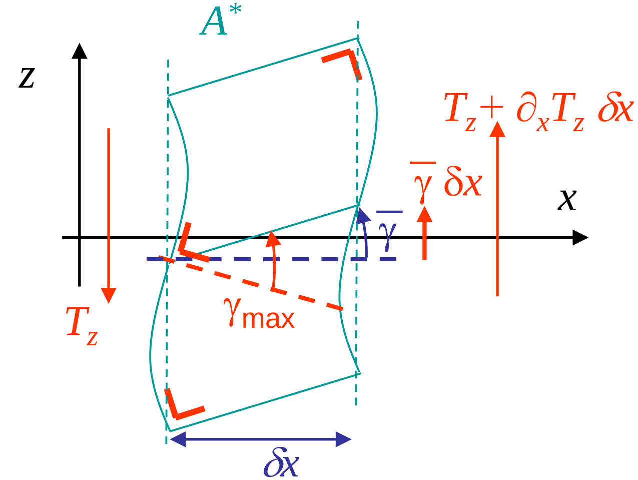


    - Shear stresses are not small compared to bending stresses
    - Deflection mainly results from bending stresses
    - 2 cases
      - Open thin-walled sections
        - » Shear = shearing through the shear center + torque
      - Closed thin-walled sections
        - » Twist due to shear has the same expression as torsion

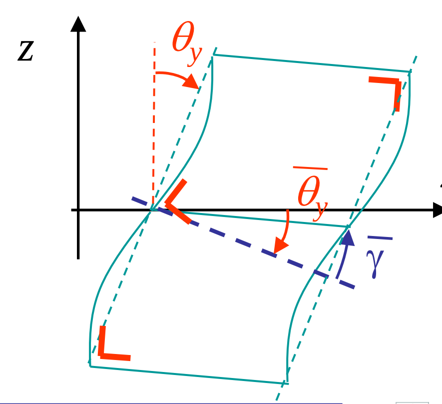
- Shearing of symmetrical thick-section beams
  - Stress $\boldsymbol{\sigma}_{zx} = -\frac{T_z S_n(z)}{I_{yy} b(z)}$
    - With $S_n(z) = \int_{A^*} z\,dA$
    - Accurate only if $h > b$



  - Energetically consistent averaged shear strain
    - $\bar{\gamma} = \frac{T_z}{A'\mu}$ with $A' = \frac{1}{\int_A \frac{S_n^2}{I_{yy}^2 b^2} dA}$
    - Shear center on symmetry axes



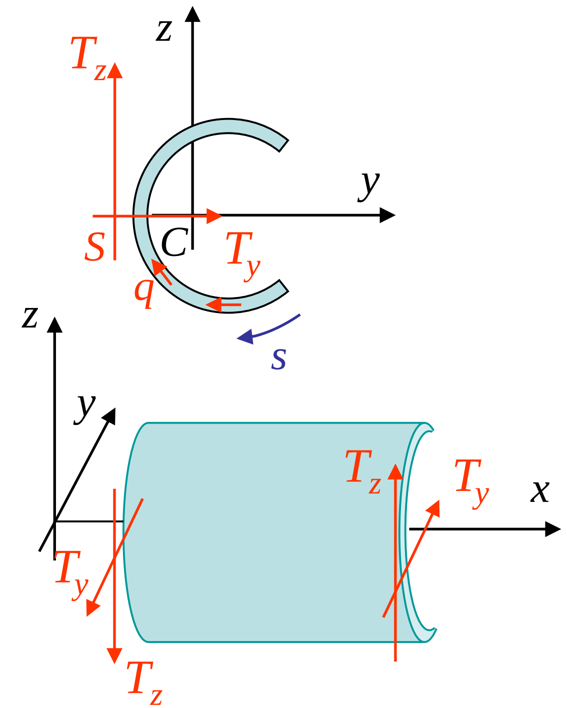
  - Timoshenko equations
    - $\bar{\gamma} = 2\bar{\boldsymbol{\varepsilon}}_{xz} = \frac{\partial \boldsymbol{u}_x}{\partial z} + \frac{\partial \boldsymbol{u}_z}{\partial x} = \theta_y + \partial_x \boldsymbol{u}_z$ & $\kappa = \frac{\partial \theta_y}{\partial x}$
    - On [0 L]: $\begin{cases} \frac{\partial}{\partial_x}\left(EI\frac{\partial \theta_y}{\partial x}\right) - \mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right) = 0 \\ \frac{\partial}{\partial x}\left(\mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right)\right) = -f \end{cases}$

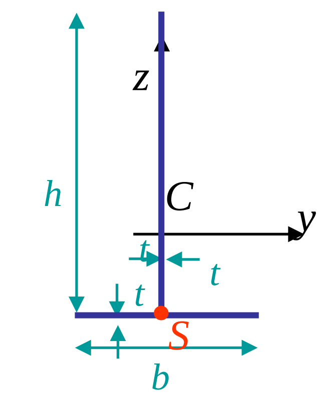


Beam shearing: linear elasticity summary

- Shearing of open thin-walled section beams
  - Shear flow $q = t\tau$
    - $$q\left(s\right) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s tz ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s ty ds'$$
    - In the principal axes

      $$q\left(s\right) = -\frac{T_z}{I_{yy}}\int_0^s tz ds' - \frac{T_y}{I_{zz}}\int_0^s ty ds'$$
  - Shear center S
    - On symmetry axes
    - At walls intersection
    - Determined by momentum balance
  - Shear loads correspond to
    - Shear loads passing through the shear center &
    - Torque





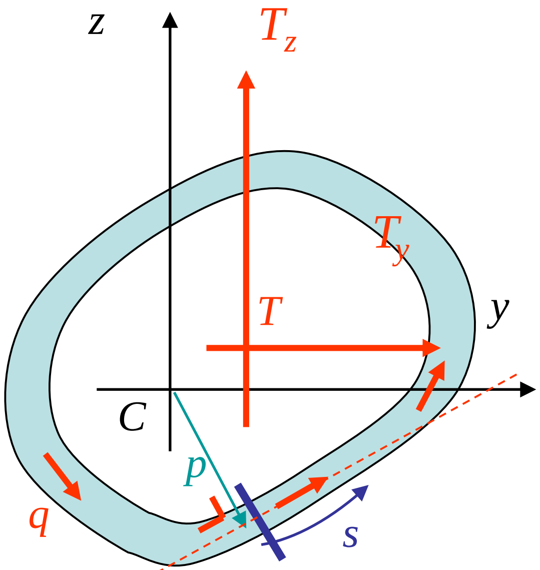
- Shearing of closed thin-walled section beams
  - Shear flow $q = t\tau$
    - $q(s) = q_o(s) + q(0)$
    - Open part (for anticlockwise of q, s)

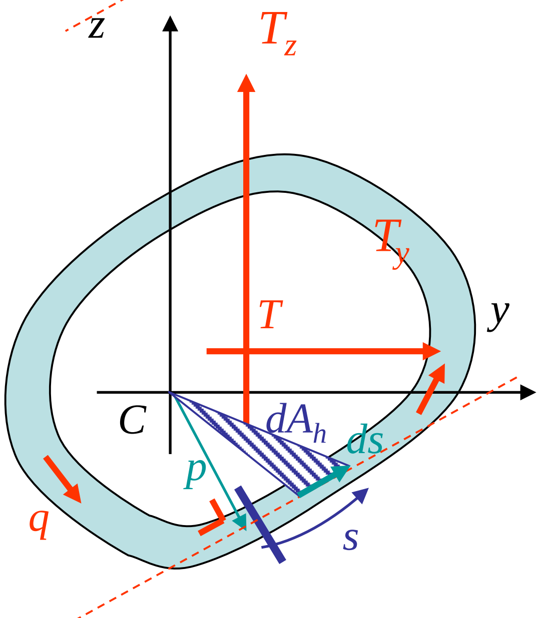
$$q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s t(s')\,z(s')\,ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s t(s')\,y(s')\,ds'$$

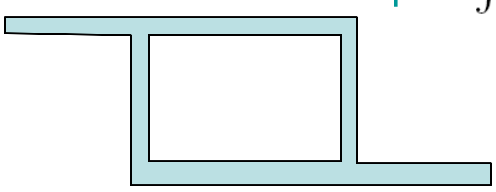
    - Constant twist part

$$q(s=0) = \frac{y_T T_z - z_T T_y - \oint p(s)\,q_o(s)\,ds}{2A_h}$$

    - The $q(0)$ is related to the closed part of the section, but there is a $q_o(s)$ in the open part which should be considered for the shear torque $\oint p(s)\,q_o(s)\,ds$



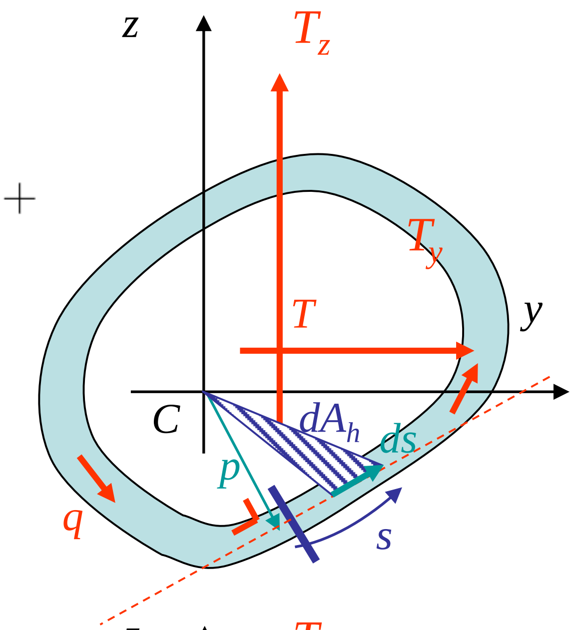


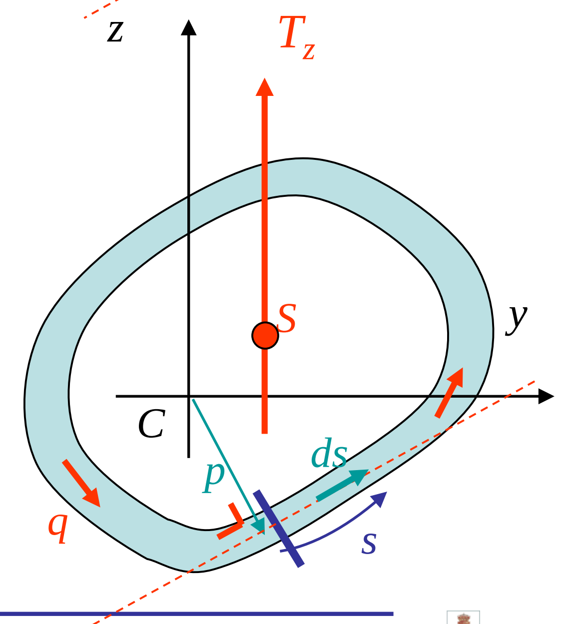


Beam shearing: linear elasticity summary

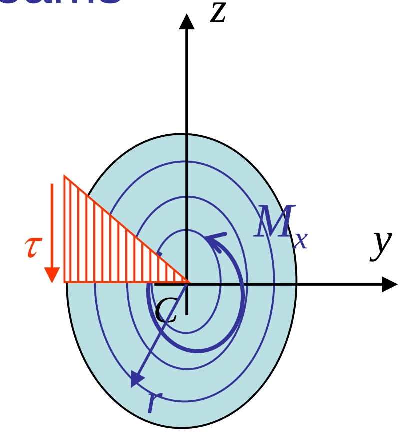
- Shearing of closed thin-walled section beams
  - Warping
    - $$\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \int_0^s \frac{q}{\mu t} ds - \frac{1}{A_h} \oint \frac{q}{\mu t} ds \left\{ A_{Cp}(s) + \frac{z_R\left[y(s) - y(0)\right] - y_R\left[z(s) - z(0)\right]}{2} \right\}$$
    - With $\boldsymbol{u}_x(0) = \frac{\oint t \boldsymbol{u}_x(s)\, ds}{\oint t(s)\, ds}$
      - $\boldsymbol{u}_x(0)=0$ for symmetrical section if origin on the symmetry axis
  - Shear center S
    - Compute q for shear passing thought S
    - Use

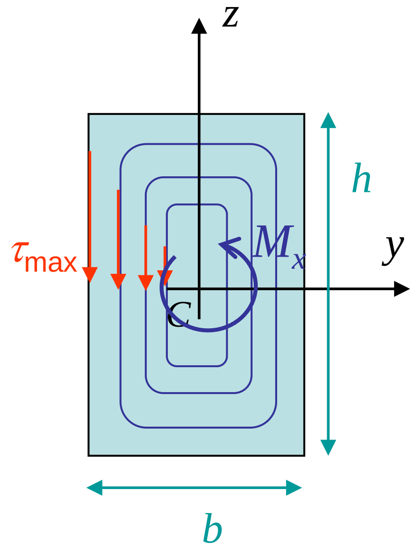
$$q(s=0) = \frac{y_T T_z - z_T T_y - \oint p(s)\, q_o(s)\, ds}{2A_h}$$





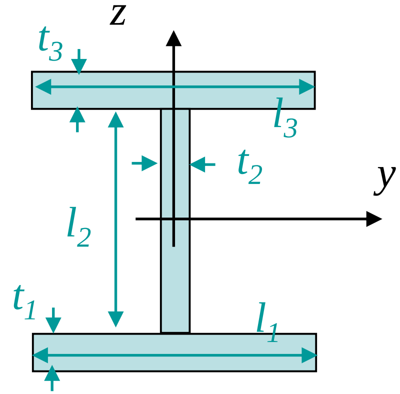
- Torsion of symmetrical thick-section beams
  - Circular section
    - $\tau = \mu\gamma = r\mu\theta_{,x}$
    - $C = \frac{M_x}{\theta_{,x}} = \int_A \mu r^2 dA$
  - Rectangular section
    - $\tau_{\max} = \frac{M_x}{\alpha h b^2}$
    - $C = \frac{M_x}{\theta_{,x}} = \beta h b^3 \mu$
    - If $h >> b$
      - $\tau_{xy} = 0$ & $\tau_{xz} = 2\mu y \theta_{,x}$
      - $\tau_{\max} = \frac{3M_x}{hb^2}$
      - $C = \frac{M_x}{\theta_{,x}} = \frac{hb^3\mu}{3}$

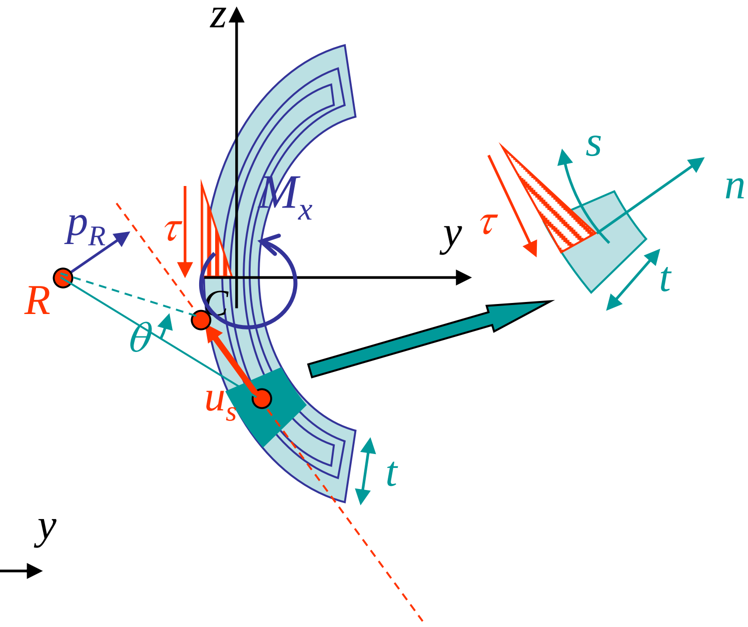




h/b	1	1.5	2	4	∞
α	0.208	0.231	0.246	0.282	1/3
β	0.141	0.196	0.229	0.281	1/3

- Torsion of open thin-walled section beams
  - Approximated solution for twist rate
    - Thin curved section
      - $\tau_{xs} = 2\mu n \theta_{,x}$
      - $C = \dfrac{M_x}{\theta_{,x}} = \dfrac{1}{3}\int \mu t^3 ds$
    - Rectangles
      - $\tau_{\max_i} = \mu t_i \theta_{,x}$
      - $\dfrac{M_x}{\theta_{,x}} = \sum_i \dfrac{l_i t_i^3 \mu}{3}$

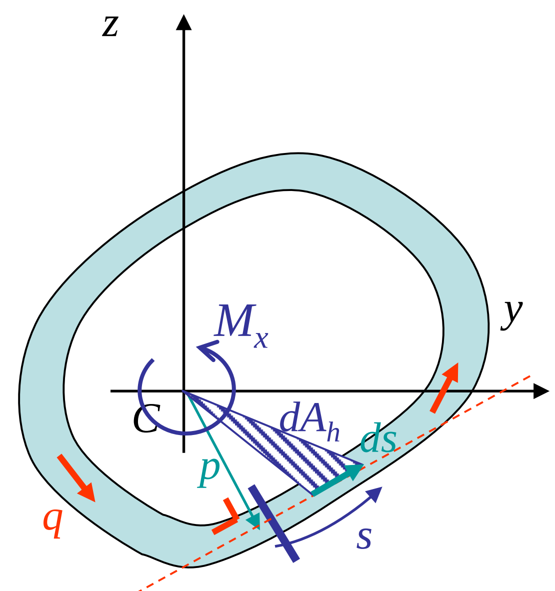




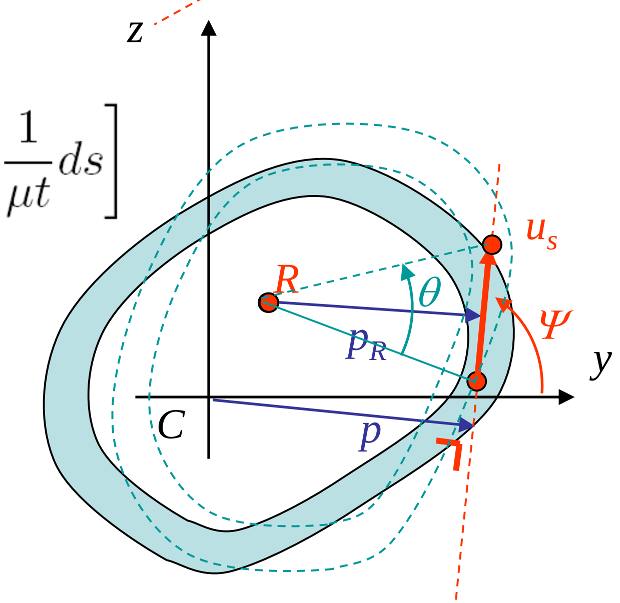
  - Warping of s-axis
    - $\boldsymbol{u}_x^s(s) = \boldsymbol{u}_x^s(0) - \theta_{,x}\int_0^s p_R ds' = \boldsymbol{u}_x^s(0) - 2A_{R_p}(s)\,\theta_{,x}$

- Torsion of closed thin-walled section beams
  - Shear flow due to torsion $M_x = 2A_h q$
  - Rate of twist
    - $$\theta_{,x} = \frac{M_x}{4A_h^2}\oint \frac{1}{\mu t} ds$$
    - Torsion rigidity for constant μ

$$I_T = \frac{4A_h^2}{\oint \frac{1}{t} ds} \leq I_p = \int_A r^2 dA$$

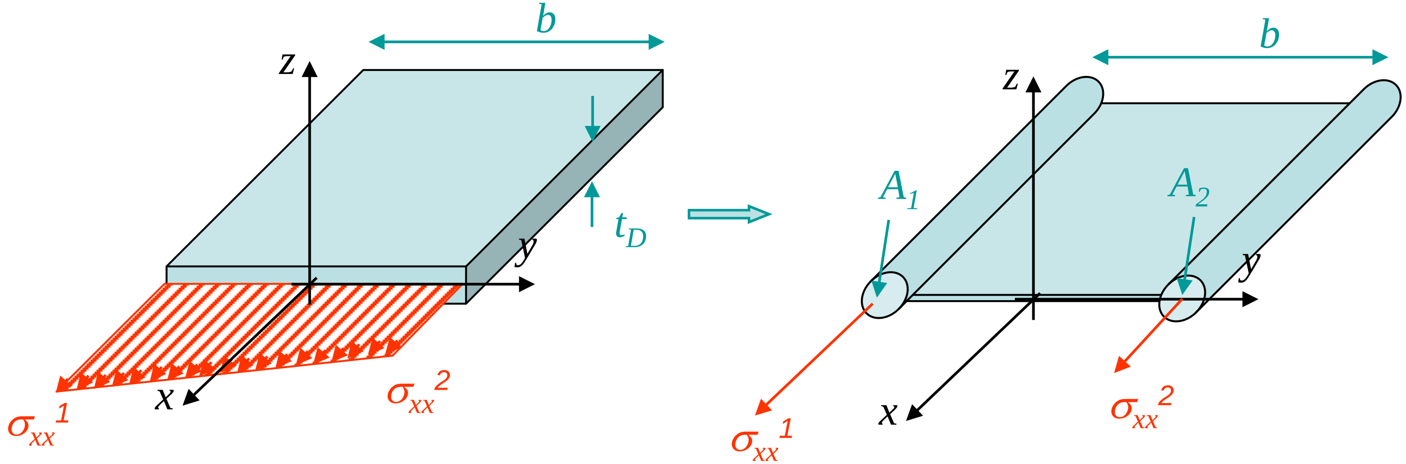


  - Warping due to torsion
    - $$\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \frac{M_x}{2A_h}\left[\int_0^s \frac{1}{\mu t} ds - \frac{A_{R_p}(s)}{A_h}\oint \frac{1}{\mu t} ds\right]$$
    - A_{Rp} from twist center



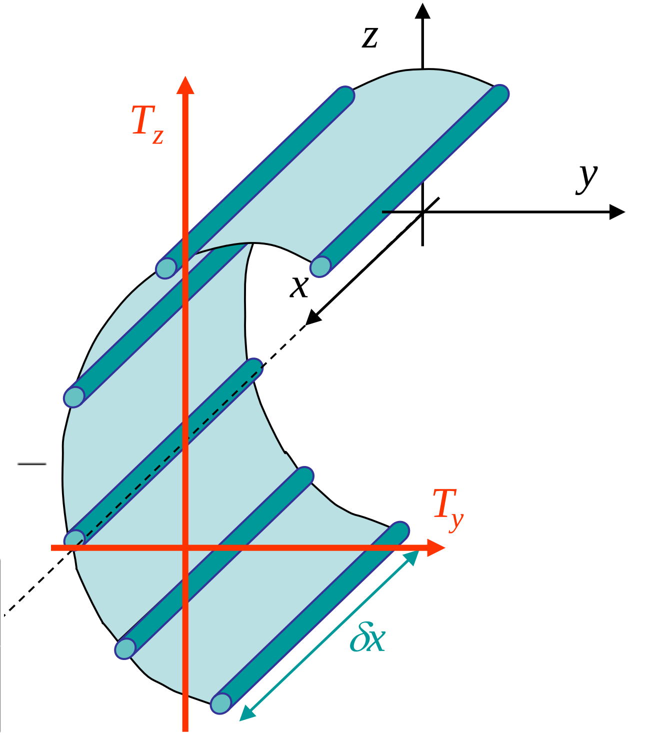
- Panel idealization
  - Booms' area **depending on loading**
    - For linear direct stress distribution

$$\begin{cases} A_1 = \dfrac{t_D b}{6}\left(2 + \dfrac{\boldsymbol{\sigma}_{xx}^2}{\boldsymbol{\sigma}_{xx}^1}\right) \\ A_2 = \dfrac{t_D b}{6}\left(2 + \dfrac{\boldsymbol{\sigma}_{xx}^1}{\boldsymbol{\sigma}_{xx}^2}\right) \end{cases}$$



- Consequence on bending
  - If Direct stress due to bending is carried by booms only
    - The position of the neutral axis, and thus the second moments of area
      - Refer to the direct stress carrying area only
      - Depend on the loading case only
- Consequence on shearing
  - Open part of the shear flux
    - Shear flux for open sections

$$q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\sigma} z ds + \sum_{i:\, s_i \leq s} z_i A_i\right] - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\sigma} y ds + \sum_{i:\, s_i \leq s} y_i A_i\right]$$

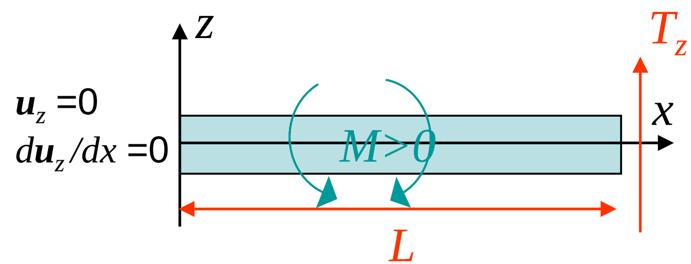


- Consequence on torsion
  - If no axial constraint
    - Torsion analysis does not involve axial stress
    - So torsion is unaffected by the structural idealization

- Virtual displacement
  - In linear elasticity the general formula of virtual displacement reads

$$\int_0^L \int_A \boldsymbol{\sigma}^{(1)} : \boldsymbol{\varepsilon} dA dx = P^{(1)} \Delta_P$$

    - $\boldsymbol{\sigma}^{(1)}$ is the stress distribution corresponding to a (unit) load $P^{(1)}$
    - Δ_P is the energetically conjugated displacement to P in the direction of $P^{(1)}$ that corresponds to the strain distribution $\boldsymbol{\varepsilon}$
  - Example bending of semi cantilever beam



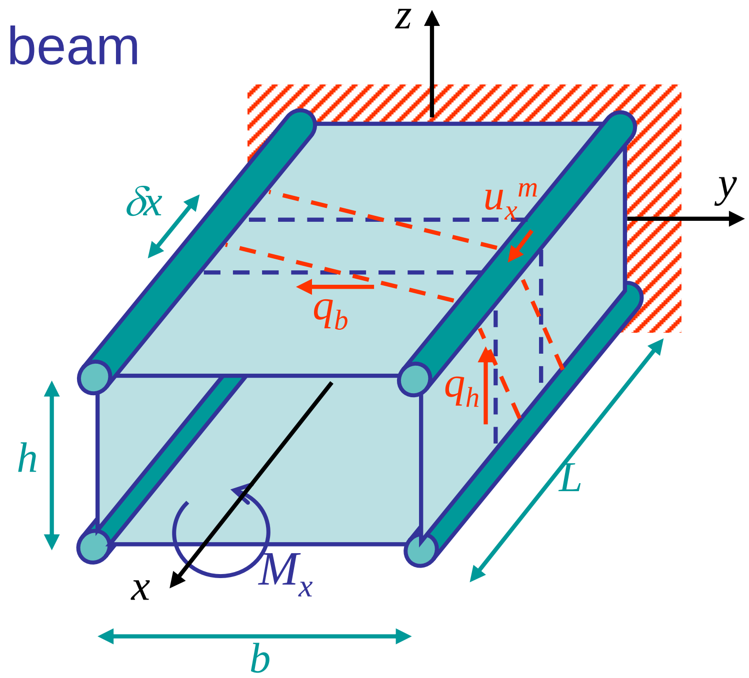
    - $$\int_0^L \int_A \boldsymbol{\sigma}_{xx}^{(1)} \boldsymbol{\varepsilon}_{xx} dA dx = \Delta_P u$$
      - In the principal axes

$$\Delta_P u = \frac{1}{E I_{yy} I_{zz}} \int_0^L \left\{ I_{zz} M_y^{(1)} M_y + I_{yy} M_z^{(1)} M_z \right\} dx$$

  - Example shearing of semi-cantilever beam
    - $$\int_0^L \int_s q^{(1)} \frac{q}{\mu t} ds dx = \boldsymbol{T}^{(1)} \Delta \bar{\boldsymbol{u}} = \Delta_T u$$

Structural discontinuities summary

- Torsion of a built-in end closed-section beam
  - If warping is constrained (built-in end)
    - Direct stresses are introduced
    - Different shear stress distribution



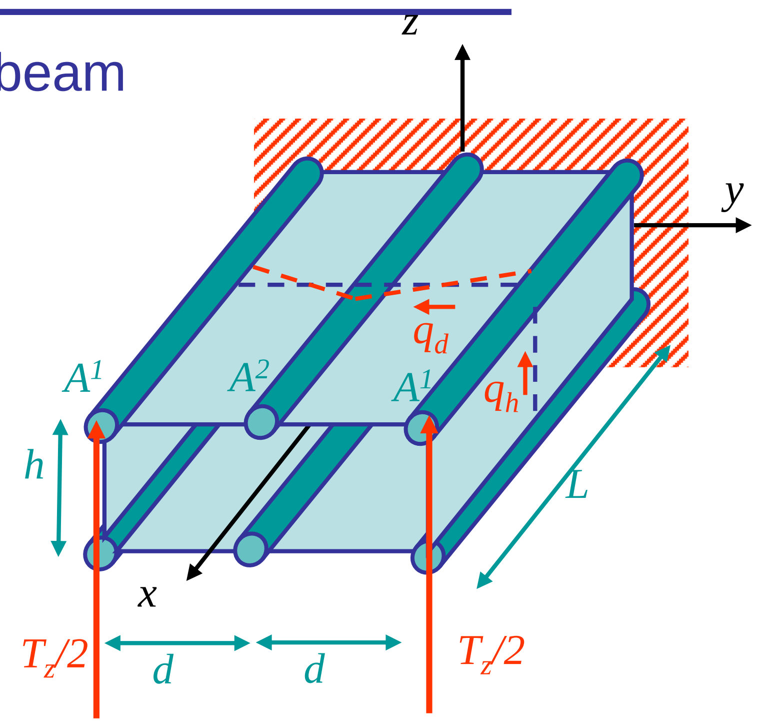
  - Example: square idealized section
    - Warping

$$\boldsymbol{u}_x^m(x) = \frac{M_x}{8\mu h b}\frac{t_b h - t_h b}{t_h t_b}\left(1 - \frac{\cosh(wL - wx)}{\cosh wL}\right)$$

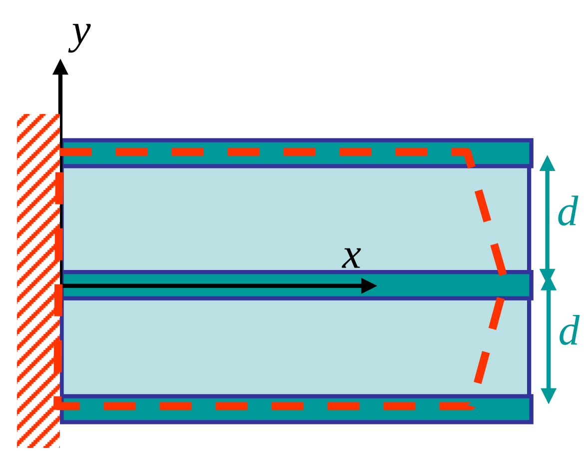
    - Shear stress

$$\begin{cases} \tau_h = \dfrac{q_h}{t_h} = \dfrac{M_x}{2hbt_h}\left(1 + \dfrac{t_h b - t_b h}{t_h b + t_b h}\dfrac{\cosh(wL - wx)}{\cosh wL}\right) \\ \tau_b = \dfrac{q_b}{t_b} = \dfrac{M_x}{2hbt_b}\left(1 + \dfrac{t_b h - t_h b}{t_h b + t_b h}\dfrac{\cosh(wL - wx)}{\cosh wL}\right) \end{cases}$$

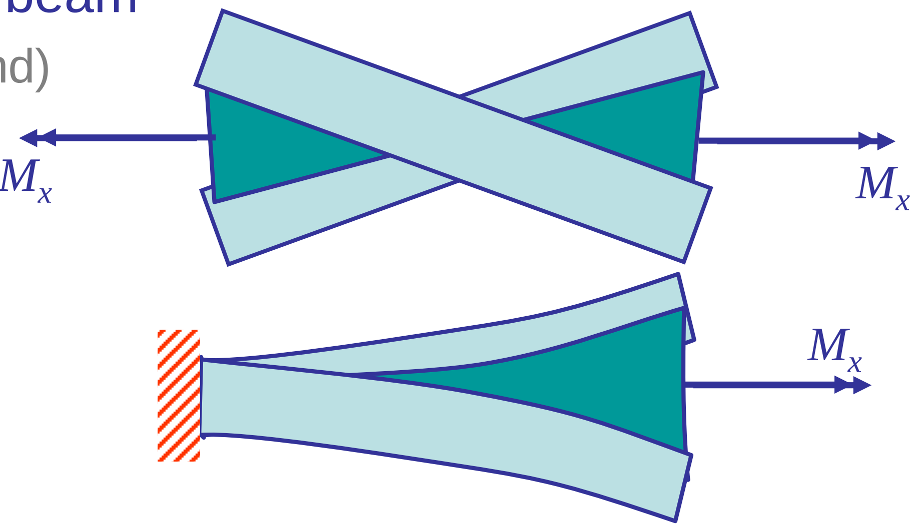
- Shear lag of a built-in end closed-section beam
  - Beam shearing
    - Shear strain in cross-section
    - Deformation of cross-section
    - Elementary theory of bending
      - For pure bending
      - Not valid anymore because of the cross section deformation
  - Example
    - 6-boom wing
    - Deformation of top cover



$$\gamma_{xy} = -\frac{q_d}{\mu t}$$
$$= -\frac{T_z}{2h\mu t\left(\frac{2A^1}{A^2}+1\right)}\left(1-\frac{\cosh w\,(L-x)}{\cosh wL}\right)$$



- Torsion of a built-end open-section beam
  - If warping is constrained (built-in end)
    - Direct stresses are introduced
    - There is a bending contribution to the torque



$$M_x = C\theta_{,x} - C^{\Gamma}\theta_{,xxx}$$

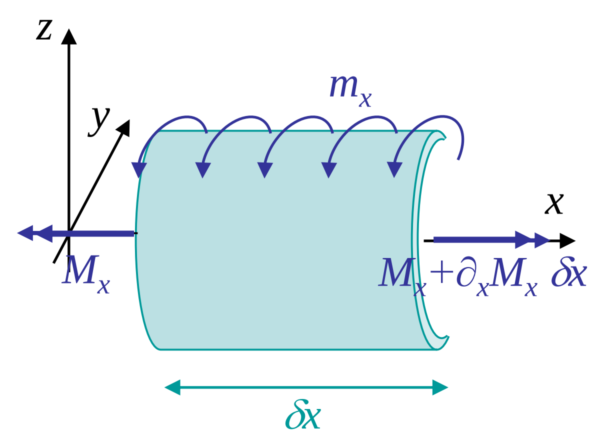
$$C^{\Gamma} = \int_C 4A_{R_p}^2 Etds - \frac{\left(\int_C Et2A_{R_p}(s)\,ds\right)^2}{\int_C Etds}$$

  - Examples
    - Equation for pure torque

$$\theta_{,xxx} - w^2\theta_{,x} = -\frac{w^2}{C}M_x \quad \text{with} \quad w^2 = \frac{C}{C^{\Gamma}}$$

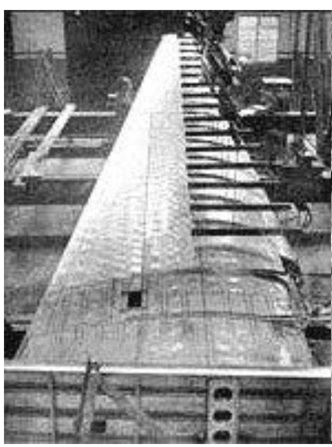
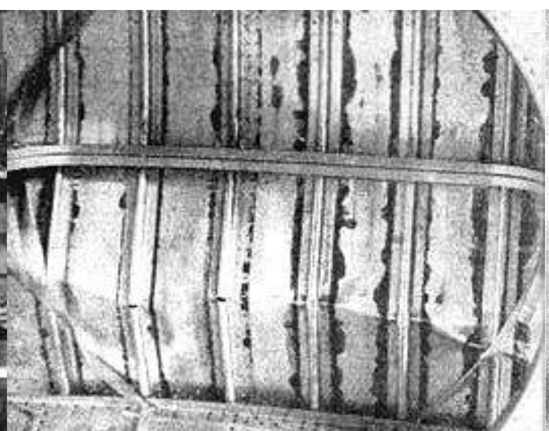
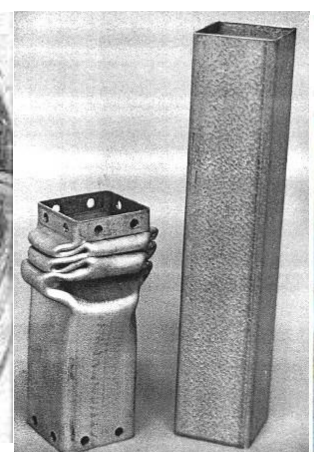
    - Equation for distributed torque

$$\partial_x\left(C^{\Gamma}\theta_{,xxx} - C\theta_{,x}\right) = m_x(x)$$



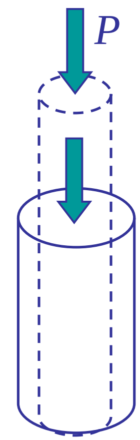
Column instabilities

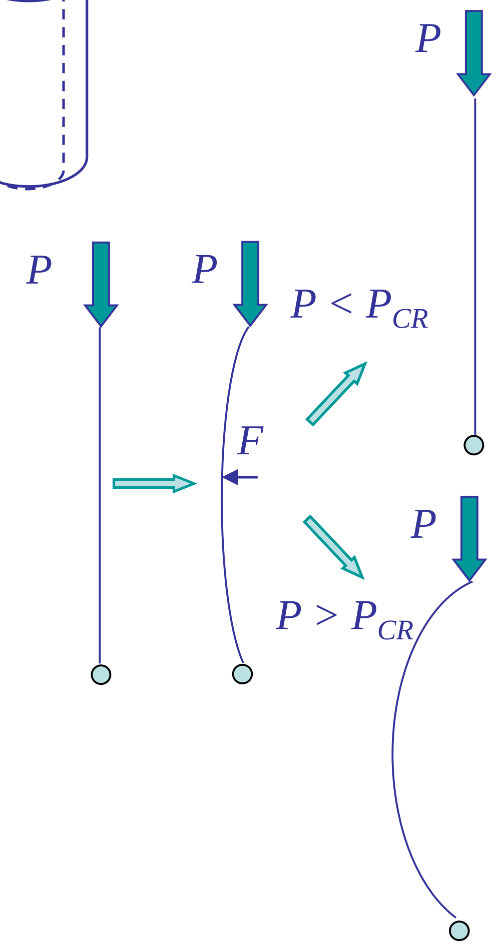
- 2 kinds of buckling
  - Primary buckling
    - No changes in cross-section
    - Wavelength of buckle ~ length of element
    - Solid & thick-walled column
  - Secondary (local) buckling
    - Changes in cross-section
    - Wavelength of buckle ~ cross-sectional dimensions
    - Thin-walled column & stiffened panels
  - Pictures:
    - D.H. Dove wing (max loading test)
    - Automotive beam
    - Local buckling

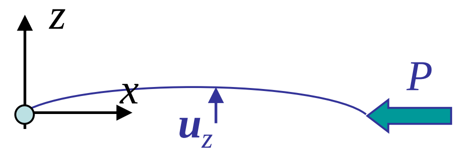

Euler buckling

- Assumptions
  - Perfectly symmetrical column (no imperfection)
  - Axial load perfectly aligned along centroidal axis
  - Linear elasticity
- Theoretically
  - Deformed structure should remain symmetrical
  - Solution is then a shortening of the column
  - Buckling load P_{CR} is defined as P such that if a small lateral displacement is enforced by a lateral force, once this force is removed
    - If $P = P_{CR}$, the lateral deformation is constant (neutral stability)
    - If $P > P_{CR}$, this lateral displacement increases & the column is unstable
    - If $P < P_{CR}$, this lateral displacement disappears & the column is stable
- Practically
  - The initial lateral displacement is due to imperfections (geometrical or material)



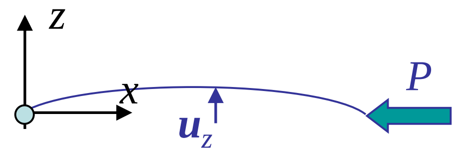


- Euler critical axial load
  - Bending theory
    - $-EI_{yy}\boldsymbol{u}_{z,xx} = M_{xx} = P_{CR}\boldsymbol{u}_z$



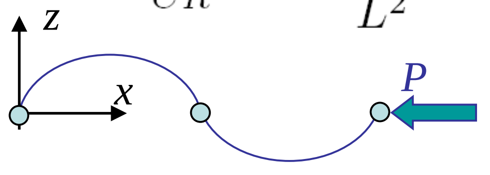
  - Solution
    - General form: $\boldsymbol{u}_z = C_1 \cos\omega x + C_2 \sin\omega x$ with $\omega = \sqrt{\frac{P_{CR}}{EI_{yy}}}$
    - BCs at $x = 0$ & $x = L$ imply $C_1 = 0$ & $C_2 \sin\omega L = 0$
    - Non trivial solution $\omega L = k\pi$ with $k = 1, 2, 3, \ldots$
    - In that case C_2 is undetermined and can $\rightarrow \infty$
  - Euler critical load for pinned-pinned BCs
    - $P_{CR} = \frac{k^2\pi^2 EI_{yy}}{L^2}$ with $k = 1, 2, 3, \ldots$

- Euler critical axial load (2)
  - Euler critical load for pinned-pinned BCs (2)

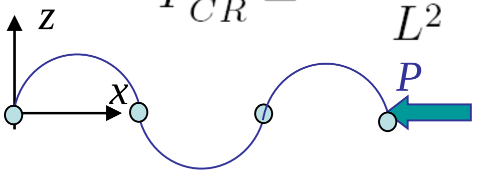


    - $P_{CR} = \frac{k^2 \pi^2 E I_{yy}}{L^2}$ with k = 1, 2, 3, …
    - Buckling will occur for lowest P_{CR}
      - In the plane of lowest I
      - For the lowest k $\Longrightarrow$ $k = 1$ $\Longrightarrow$ $P_{CR}^{(1)} = \frac{\pi^2 E I_{\min}}{L^2}$
  - In case modes 1, .. k-1 are prevented, critical load becomes the load k

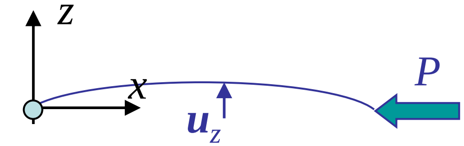
$$P_{CR}^{(2)} = \frac{4\pi^2 E I_{\min}}{L^2}$$



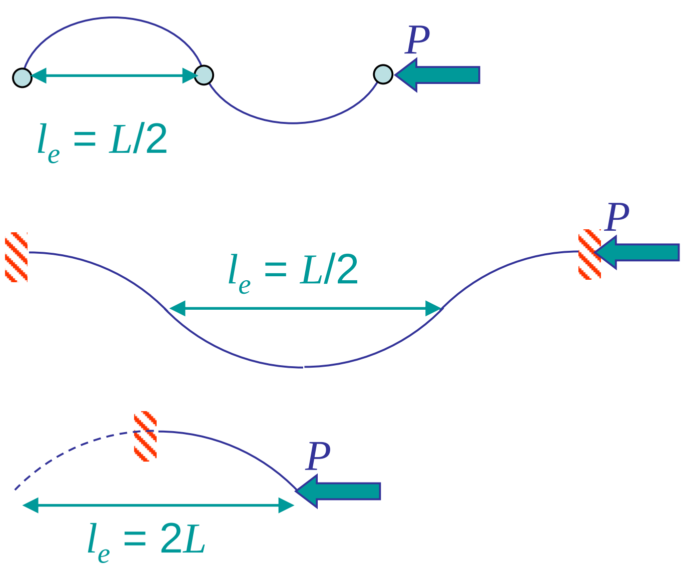
$$P_{CR}^{(3)} = \frac{9\pi^2 E I_{\min}}{L^2}$$



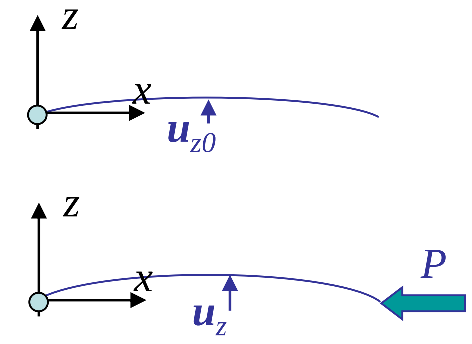
- Euler critical axial load (3)
  - For pinned-pinned BCs



    - $P_{CR} = \frac{\pi^2 E I_{\min}}{L^2}$
    - $\sigma_{CR} = -\pi^2 E \frac{r^2}{L^2}$ (compression) with gyration radius $r = \sqrt{\frac{I_{\min}}{A}}$
  - General case
    - Euler critical loads $P_{CR,E} = \frac{\pi^2 E I_{\min}}{l_e^2}$, $\sigma_{CR,E} = -\pi^2 E \frac{r^2}{l_e^2}$ (compressive)
    - With l_e the effective length



- Practical case: initial imperfection
  - Let us assume an initial small curvature of the beam
    - $\kappa_0 = -\frac{\partial^2 \boldsymbol{u}_{z0}}{\partial x^2}$



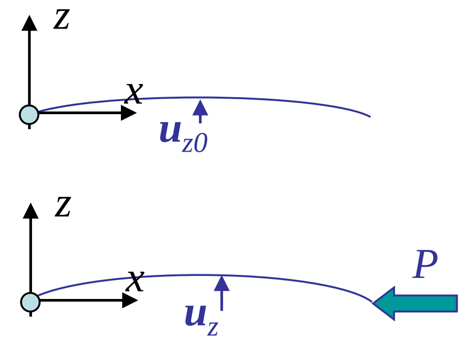
  - As this curvature is small the equation of bending for straight beam can still be used, but with the change of curvature being considered for the strain
    - $-EI_{yy}\left(\boldsymbol{u}_{z,xx} - \boldsymbol{u}_{z0,xx}\right) = M_{xx} = P\boldsymbol{u}_z$
    - The general form of the initial deflection satisfying the BCs is

$$\boldsymbol{u}_{z0} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$ ⟹ the deflection equation becomes

$$EI_{yy}\boldsymbol{u}_{z,xx} + P\boldsymbol{u}_z = EI_{yy}\boldsymbol{u}_{z0,xx} = -\frac{EI_{yy}\pi^2}{L^2}\sum_{n=1}^{\infty} A_n n^2 \sin\frac{n\pi x}{L}$$

  - Solution $\boldsymbol{u}_z = C_1 \cos\omega x + C_2 \sin\omega x + \sum_{n=1}^{\infty} \frac{A_n n^2}{n^2 - \frac{\omega^2 L^2}{\pi^2}} \sin\frac{n\pi x}{L}$
    - With $\omega = \sqrt{\frac{P}{EI_{yy}}}$

- Practical case: initial imperfection (2)
  - Solution for an initial small curvature of the beam
    - $\boldsymbol{u}_z = C_1 \cos \omega x + C_2 \sin \omega x + \sum_{n=1}^{\infty} \frac{A_n n^2}{n^2 - \frac{\omega^2 L^2}{\pi^2}} \sin \frac{n\pi x}{L}$
    - With $\omega = \sqrt{\frac{P}{EI_{yy}}}$
    - BCs at $x = 0$ & $x = L$ imply $C_1 = C_2 = 0$, & as $P_{CR}^{(1)} = \frac{\pi^2 EI_{\min}}{L^2}$



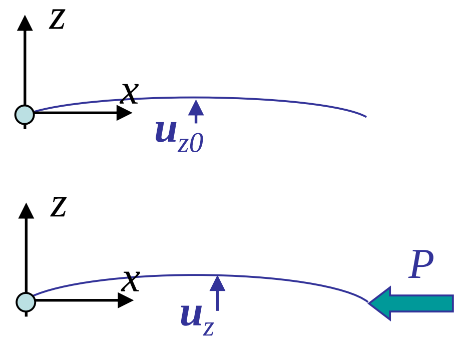
$$\Rightarrow \boldsymbol{u}_z = \sum_{n=1}^{\infty} \frac{A_n n^2}{n^2 - \frac{\omega^2 L^2}{\pi^2}} \sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} \frac{A_n n^2}{n^2 - \frac{P}{P_{CR}}} \sin \frac{n\pi x}{L}$$

- Clearly near buckling, so $P \rightarrow P_{CR}$, and the dominant term of the solution is for $n = 1$

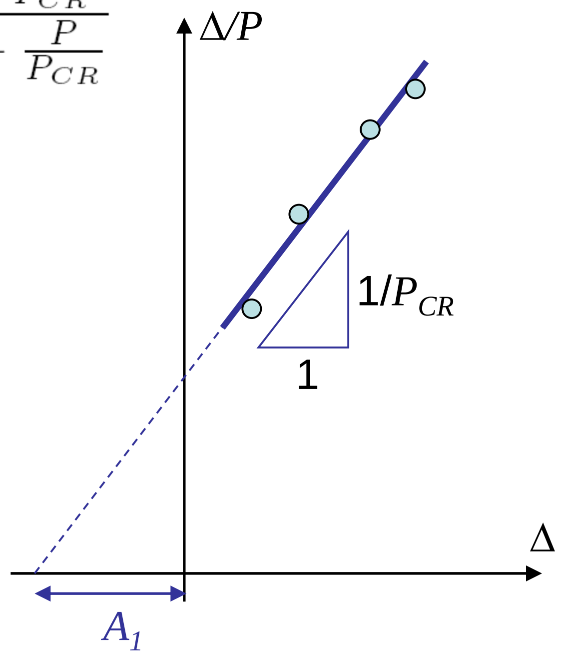
$$\Rightarrow \boldsymbol{u}_z \simeq \frac{A_1}{1 - \frac{P}{P_{CR}}} \sin \frac{\pi x}{L}$$

Initial imperfection

- Practical case: initial imperfection (3)
  - Solution for an initial small curvature of the beam
    - $\boldsymbol{u}_z \simeq \frac{A_1}{1-\frac{P}{P_{CR}}} \sin \frac{\pi x}{L}$ near buckling



  - If central deflection is measured *vs* axial load
    - As $\boldsymbol{u}_{z0}(L/2) \sim A_1$
    - $\Delta = \boldsymbol{u}_z\left(\frac{L}{2}\right) - A_1 = \frac{A_1}{1-\frac{P}{P_{CR}}} - A_1 = \frac{A_1 \frac{P}{P_{CR}}}{1-\frac{P}{P_{CR}}}$
  - Southwell diagram
    - $\Delta = P_{CR}\frac{\Delta}{P} - A_1$
    - Allows measuring buckling loads without breaking the columns



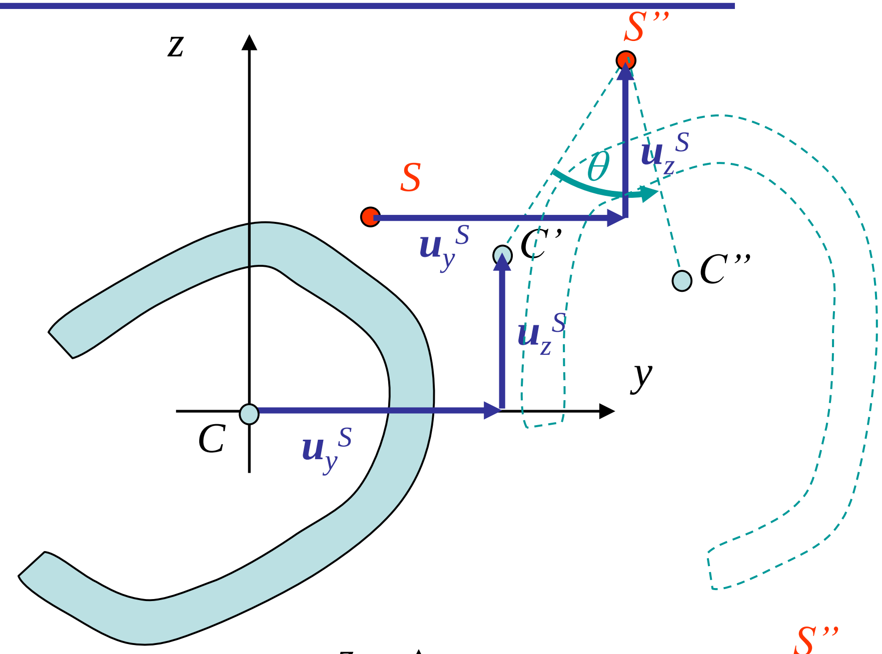
- Remark
  - Critical Euler loading depends on BCs

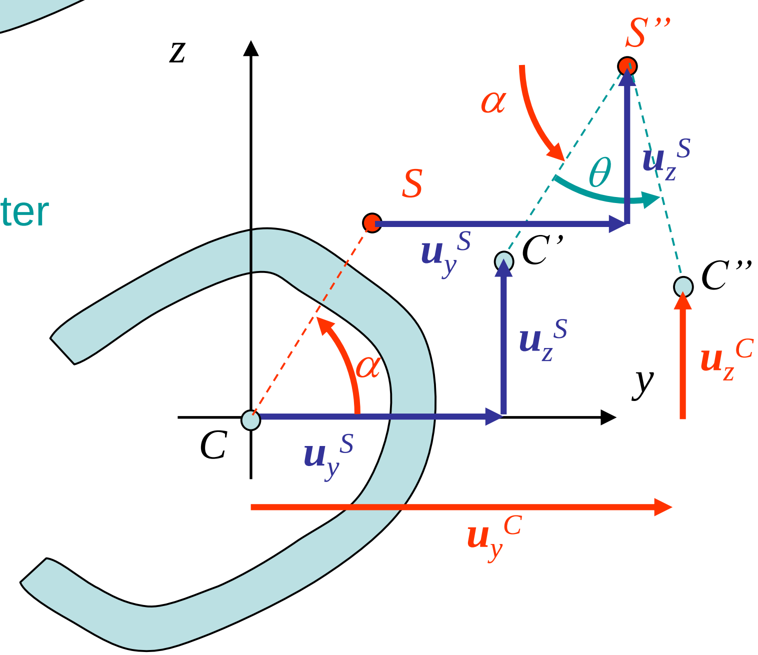
Flexural-torsional buckling of open thin-walled columns

- Thin-walled column under critical flexural loads
  - Can twist without bending or
  - Can twist and bend simultaneously

Flexural-torsional buckling of open thin-walled columns

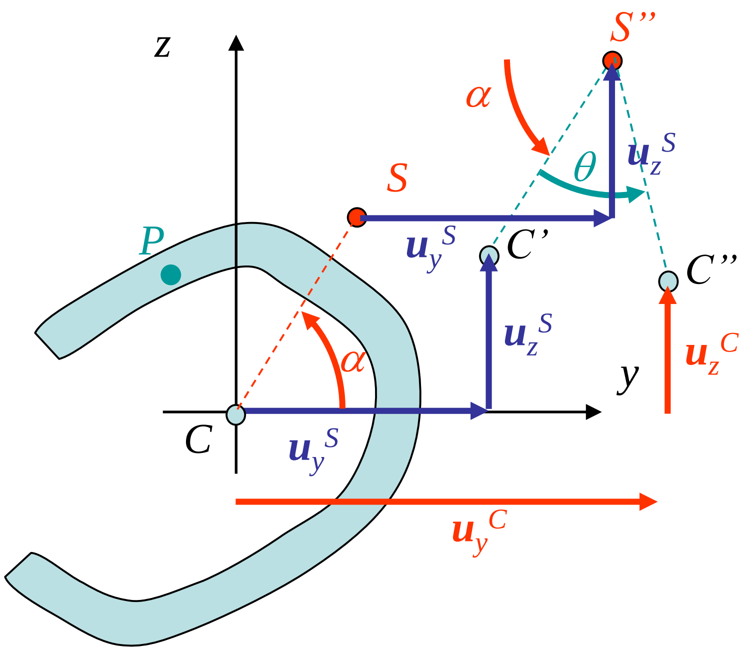
- Kinematics
  - Consider
    - A thin-walled section
    - Centroid C
    - Cyz principal axes
    - Shear center S
  - Section motion (CSRD)
    - Translation
      - Shear center is moved
      - By u_y^S & u_z^S
      - To S''
    - Rotation around shear (twist) center
      - We assume shear center=twist center
      - By θ
    - Centroid motion
      - To C' after section translation
      - To C'' after rotation
      - Resulting displacements u_y^C & u_z^C
      - Same decomposition for other points of the section





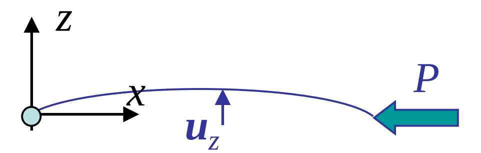
- Kinematics (2)
  - Relations
    - Centroid $\begin{cases} \delta \boldsymbol{u}_y^C = \delta \boldsymbol{u}_y^S + z_S \delta\theta \\ \delta \boldsymbol{u}_z^C = \delta \boldsymbol{u}_z^S - y_S \delta\theta \end{cases}$
    - Other points P of the section

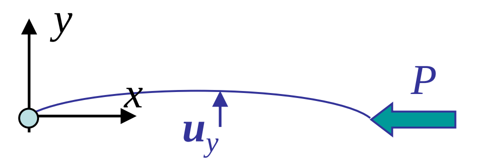
$$\begin{cases} \delta \boldsymbol{u}_y^P = \delta \boldsymbol{u}_y^S + (z_S - z_P)\,\delta\theta \\ \delta \boldsymbol{u}_z^P = \delta \boldsymbol{u}_z^S - (y_S - y_P)\,\delta\theta \end{cases}$$



- Considering axial loading
  - If θ remains small, the induced momentums are

$$\begin{cases} M_y = P\boldsymbol{u}_z^C = P\boldsymbol{u}_z^S - P y_S \theta \\ M_z = -P\boldsymbol{u}_y^C = -P\boldsymbol{u}_y^S - P z_S \theta \end{cases}$$





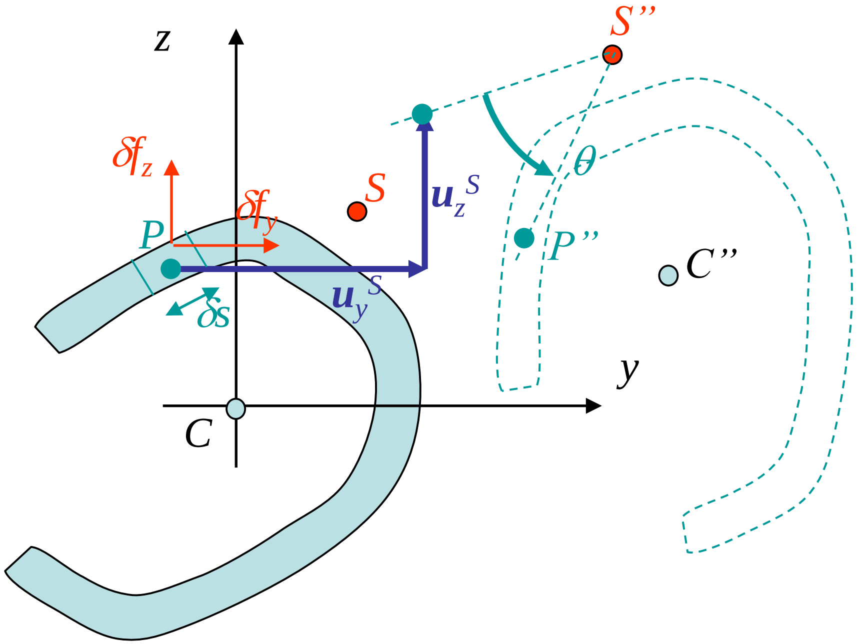
  - As we are in the principal axes (I_{yz}=0), and as motion resulting from bending is $\boldsymbol{u}^S$

$$\begin{cases} EI_{zz}\boldsymbol{u}_{y,xx}^S = M_z = -P\boldsymbol{u}_y^S - P z_S \theta \\ EI_{yy}\boldsymbol{u}_{z,xx}^S = -M_y = -P\boldsymbol{u}_z^S + P y_S \theta \end{cases}$$

- Torsion
  - Any point P of the section

$$\begin{cases} \delta \boldsymbol{u}_y^P = \delta \boldsymbol{u}_y^S + (z_S - z_P)\,\delta\theta \\ \delta \boldsymbol{u}_z^P = \delta \boldsymbol{u}_z^S - (y_S - y_P)\,\delta\theta \end{cases}$$

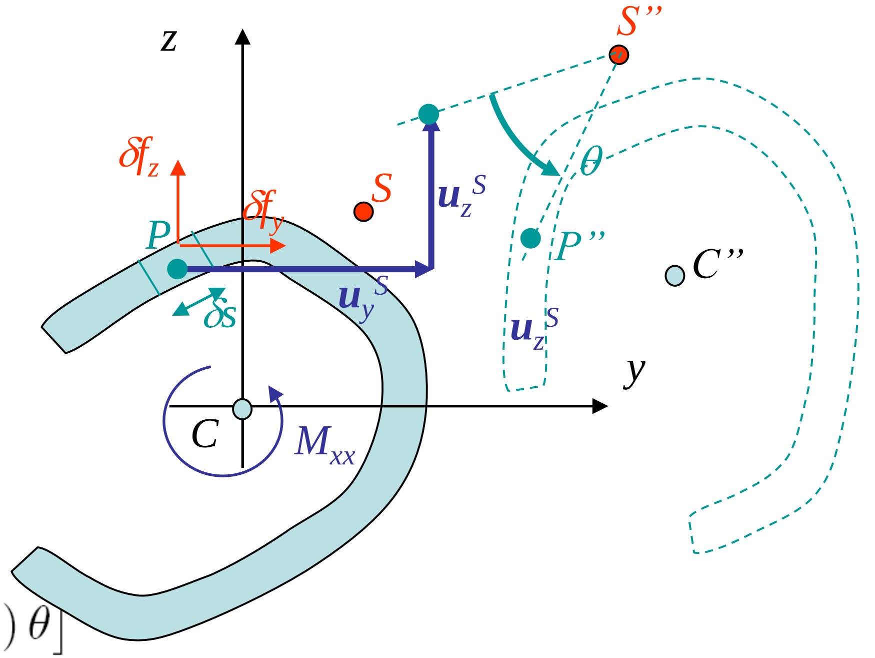
  - As torsion results from axial loading, this corresponds to a torque with warping constraint
    - See previous lecture
  - Analogy between beam bending/pin-ended column



  - As $\begin{cases} f_z(x) = -\partial_x T_z = -\partial_{xx} M_y \\ f_y(x) = -\partial_x T_y = \partial_{xx} M_z \end{cases}$
  - The momentum at point P can be substituted by lateral loading
  - Contributions on δs $\begin{cases} \delta f_z = -\partial_{xx}\delta M_y = -\delta P \partial_{xx}\left[\boldsymbol{u}_z^S - (y_S - y_P)\,\theta\right] \\ \delta f_y = \partial_{xx}\delta M_z = -\delta P \partial_{xx}\left[\boldsymbol{u}_y^S + (z_S - z_P)\,\theta\right] \end{cases}$

- Torsion (2)
  - Lateral loading analogy
    - Contributions on δs

$$\begin{cases} \delta f_y = -\delta P \partial_{xx} \left[\boldsymbol{u}_y^S + (z_S - z_P)\,\theta \right] \\ \delta f_z = -\delta P \partial_{xx} \left[\boldsymbol{u}_z^S - (y_S - y_P)\,\theta \right] \end{cases}$$



    - As axial load leads to uniform compressive stress on section of area A

$$\begin{cases} df_z = -\dfrac{P}{A} t ds \partial_{xx} \left[\boldsymbol{u}_z^S - (y_S - y_P)\,\theta \right] \\ df_y = -\dfrac{P}{A} t ds \partial_{xx} \left[\boldsymbol{u}_y^S + (z_S - z_P)\,\theta \right] \end{cases}$$

    - Resulting distributed torque (per unit length) on ds
      - $dm_x = (y_P - y_S)\, df_z - (z_P - z_S)\, df_y$

$$\Longrightarrow dm_x = \frac{P}{A} t \left\{ - (y_P - y_S)\, \partial_{xx} \left[\boldsymbol{u}_z^S - (y_S - y_P)\,\theta \right] + (z_P - z_S)\, \partial_{xx} \left[\boldsymbol{u}_y^S + (z_S - z_P)\,\theta \right] \right\} ds$$

- Distributed torque
  - As $dm_x = \frac{P}{A} t \left\{ -\left(y_P - y_S\right) \partial_{xx} \left[\boldsymbol{u}_z^S - \left(y_S - y_P\right) \theta \right] + \left(z_P - z_S\right) \partial_{xx} \left[\boldsymbol{u}_y^S + \left(z_S - z_P\right) \theta \right] \right\} ds$

$$\Rightarrow m_x = -\frac{P}{A} \int_s \left(y_P - y_S\right) \boldsymbol{u}^S_{z,xx} t ds - \frac{P}{A} \int_s \left(y_P - y_S\right)^2 \theta_{,xx} t ds + \frac{P}{A} \int_s \left(z_P - z_S\right) \boldsymbol{u}^S_{y,xx} t ds - \frac{P}{A} \int_s \left(z_P - z_S\right)^2 \theta_{,xx} t ds$$

$$\Rightarrow m_x = \frac{P}{A} y_S \boldsymbol{u}^S_{z,xx} \int_s t ds - \frac{P}{A} z_S \boldsymbol{u}^S_{y,xx} \int_s t ds - \frac{P}{A} \boldsymbol{u}^S_{z,xx} \int_s y_P t ds + \frac{P}{A} \boldsymbol{u}^S_{y,xx} \int_s z_P t ds - \frac{P}{A} y_S^2 \theta_{,xx} \int_s t ds + 2 \frac{P}{A} y_S \theta_{,xx} \int_s y_P t ds - \theta_{,xx} \frac{P}{A} \int_s y_P^2 t ds - \frac{P}{A} z_S^2 \theta_{,xx} \int_s t ds + 2 \frac{P}{A} z_S \theta_{,xx} \int_s z_P t ds - \theta_{,xx} \frac{P}{A} \int_s z_P^2 t ds$$

  - As C is the centroid

$$\Rightarrow m_x = P y_S \boldsymbol{u}^S_{z,xx} - P z_S \boldsymbol{u}^S_{y,xx} - P y_S^2 \theta_{,xx} - \theta_{,xx} \frac{P}{A} I_{yy} - P z_S^2 \theta_{,xx} - \theta_{,xx} \frac{P}{A} I_{zz}$$

- Distributed torque (2)
  - The analogous torque by unit length resulting from the bending reads

$$m_x \quad = \quad P y_S \boldsymbol{u}^S_{z,xx} - P z_S \boldsymbol{u}^S_{y,xx} - P y_S^2 \theta_{,xx} - \theta_{,xx} \frac{P}{A} I_{yy} - P z_S^2 \theta_{,xx} - \theta_{,xx} \frac{P}{A} I_{zz}$$

  - Polar second moment of area around S:

$$I_p^S = I_{yy} + I_{zz} + A\left(y_S^2 + z_S^2\right)$$

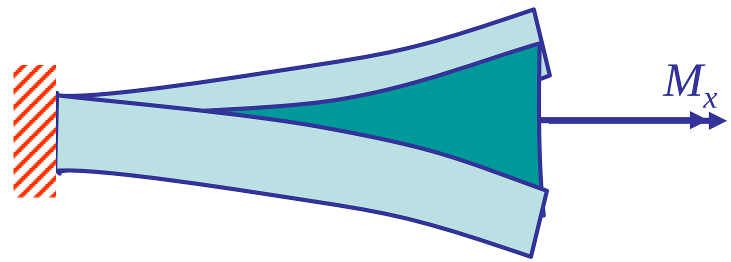
$$\Longrightarrow m_x = P\left(y_S \boldsymbol{u}^S_{z,xx} - z_S \boldsymbol{u}^S_{y,xx}\right) - \frac{P}{A} I_p^S \theta_{,xx}$$

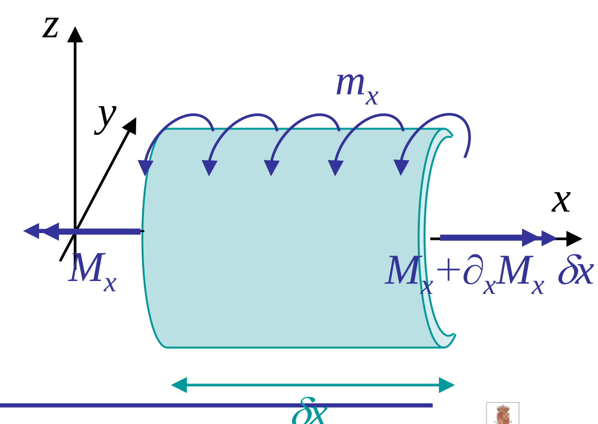
  - For a built-in end open-section beam
    - Warping is constrained
      - Bending contribution to the torque
    - New equation

$$\partial_x \left(C^\Gamma \theta_{,xxx} - C\theta_{,x}\right) = m_x(x)$$

    - For a constant section

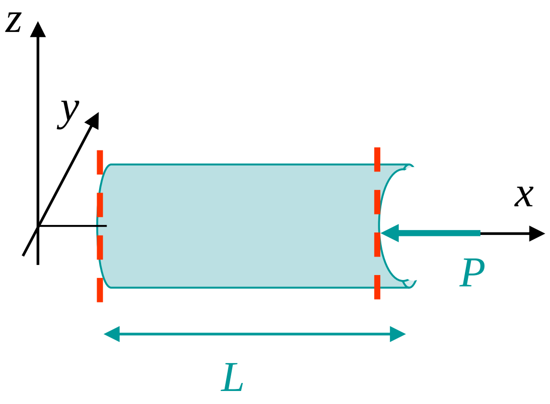
$$C^\Gamma \theta_{,xxxx} + \left(\frac{P}{A} I_p^S - C\right)\theta_{,xx} -$$
$$P y_S \boldsymbol{u}^S_{z,xx} + P z_S \boldsymbol{u}^S_{y,xx} = 0$$





- Equations
  - In the principal axes
    - $EI_{zz}\boldsymbol{u}^S_{y,xx} = M_z = -P\boldsymbol{u}^S_y - Pz_S\theta$
    - $EI_{yy}\boldsymbol{u}^S_{z,xx} = -M_y = -P\boldsymbol{u}^S_z + Py_S\theta$
    - $C^\Gamma\theta_{,xxxx} + \left(\frac{P}{A}I^S_p - C\right)\theta_{,xx} - Py_S\boldsymbol{u}^S_{z,xx} + Pz_S\boldsymbol{u}^S_{y,xx} = 0$

- Example
  - Column with
    - Deflection and rotation around x constrained at both ends
      - $\boldsymbol{u}_y(0) = \boldsymbol{u}_y(L) = 0$ & $\boldsymbol{u}_z(0) = \boldsymbol{u}_z(L) = 0$
      - $\theta(0) = 0$ & $\theta(L) = 0$

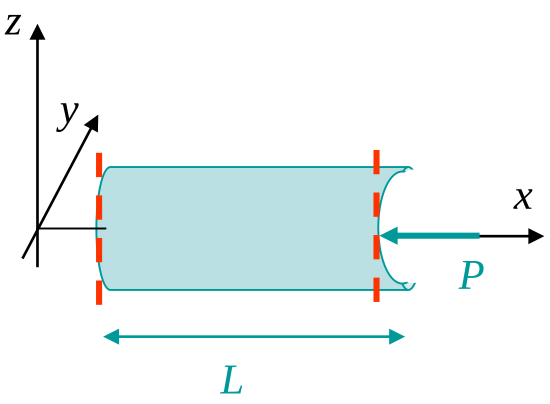


    - Warping and rotation around y & z allowed at both ends
      - Twist center = shear center
      - $M_y(0) = M_y(L) = 0 \Longrightarrow \boldsymbol{u}_{y,xx}(0) = \boldsymbol{u}_{y,xx}(L) = 0$
      - $M_z(0) = M_z(L) = 0 \Longrightarrow \boldsymbol{u}_{z,xx}(0) = \boldsymbol{u}_{z,xx}(L) = 0$
      - As warping is allowed

$$\left\{\begin{array}{l} \boldsymbol{\sigma}^{\Gamma}_{xx}(s) = E\boldsymbol{u}^{s}_{x,x}(0) - 2EA_{R_p}(s)\,\theta_{,xx} \\ \boldsymbol{u}^{s}_{x,x}(0) = \dfrac{\theta_{,xx}\int_C Et2A_{R_p}(s)\,ds}{\int_C Et\,ds} \end{array}\right\} \text{ are equal to zero}$$

$$\Longrightarrow \theta_{,xx}(0) = 0 \;\&\; \theta_{,xx}(L) = 0$$

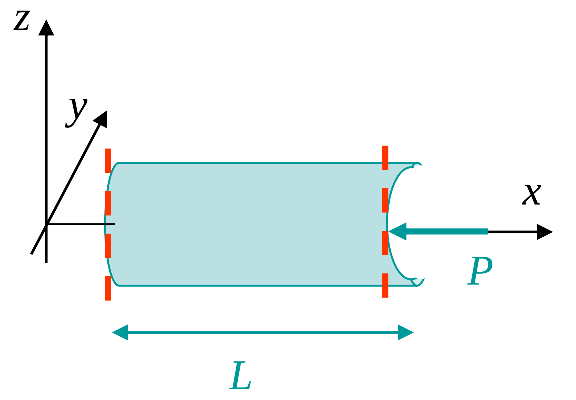
- Resolution
  - Assuming the following fields satisfying the BCs
    - $\boldsymbol{u}_y^S = C_y \sin\frac{\pi x}{L}$
    - $\boldsymbol{u}_z^S = C_z \sin\frac{\pi x}{L}$
    - $\theta = C_\theta \sin\frac{\pi x}{L}$
  - The system of equations



$$\begin{cases} EI_{zz}\boldsymbol{u}^S_{y,xx} = M_z = -P\boldsymbol{u}^S_y - Pz_S\theta \\ EI_{yy}\boldsymbol{u}^S_{z,xx} = -M_y = -P\boldsymbol{u}^S_z + Py_S\theta \\ C^\Gamma\theta_{,xxxx} + \left(\frac{P}{A}I_p^S - C\right)\theta_{,xx} - Py_S\boldsymbol{u}^S_{z,xx} + Pz_S\boldsymbol{u}^S_{y,xx} = 0 \end{cases}$$

$$\begin{cases} \left(P - \frac{\pi^2 EI_{zz}}{L^2}\right)C_y + Pz_S C_\theta = 0 \\ \left(\frac{\pi^2 EI_{yy}}{L^2} - P\right)C_z + Py_S C_\theta = 0 \\ \left(\frac{C^\Gamma\pi^2}{L^2} + C - \frac{P}{A}I_p^S\right)C_\theta + Py_S C_z - Pz_S C_y = 0 \end{cases}$$

- Resolution (2)
  - Non trivial solution leads to buckling load P

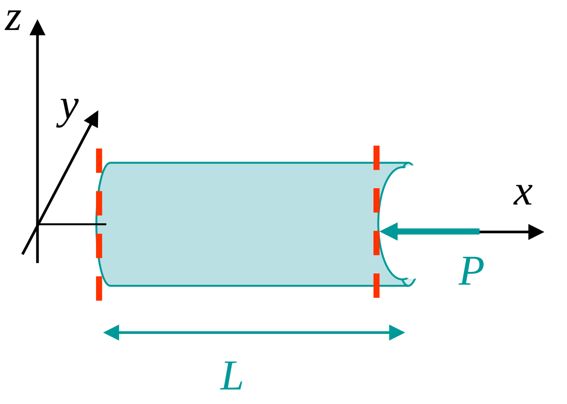


$$\begin{cases} \left(P - \frac{\pi^2 E I_{zz}}{L^2}\right) C_y + P z_S C_\theta = 0 \\ \left(\frac{\pi^2 E I_{yy}}{L^2} - P\right) C_z + P y_S C_\theta = 0 \\ \left(\frac{C^\Gamma \pi^2}{L^2} + C - \frac{P}{A} I_p^S\right) C_\theta + P y_S C_z - P z_S C_y = 0 \end{cases}$$

$$\Rightarrow \begin{vmatrix} \left(P_{CR} - \frac{\pi^2 E I_{zz}}{L^2}\right) & 0 & P_{CR} z_S \\ 0 & \left(\frac{\pi^2 E I_{yy}}{L^2} - P_{CR}\right) & P_{CR} y_S \\ -P_{CR} z_S & P_{CR} y_S & \left(\frac{C^\Gamma \pi^2}{L^2} + C - \frac{P_{CR}}{A} I_p^S\right) \end{vmatrix} = 0$$

- Buckling load is the minimum root

- If shear center and centroid coincide
  - System becomes



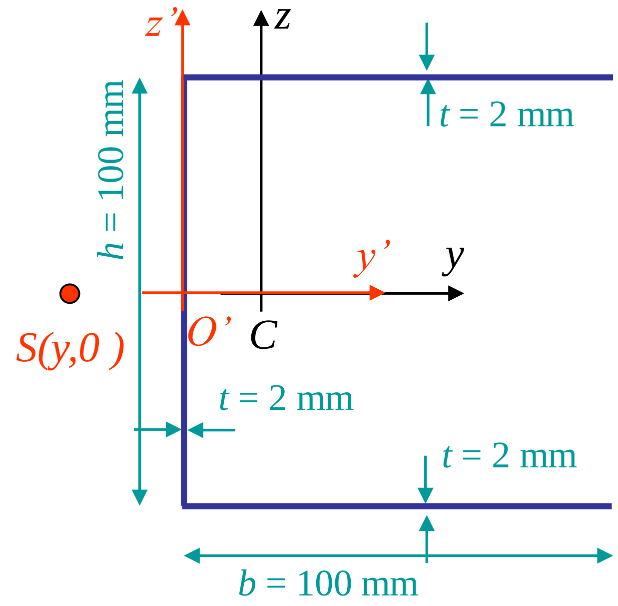
$$\begin{cases} \left(P - \dfrac{\pi^2 E I_{zz}}{L^2}\right) C_y = 0 \\ \left(\dfrac{\pi^2 E I_{yy}}{L^2} - P\right) C_z = 0 \\ \left(\dfrac{C^\Gamma \pi^2}{L^2} + C - \dfrac{P}{A} I_p^S\right) C_\theta = 0 \end{cases}$$

  - This system is uncoupled and leads to 3 critical loads

$$\begin{cases} P_{CR}^y = \dfrac{\pi^2 E I_{yy}}{L^2} \\ P_{CR}^z = \dfrac{\pi^2 E I_{zz}}{L^2} \\ P_{CR}^\theta = \dfrac{A}{I_p^S} \left(\dfrac{C^\Gamma \pi^2}{L^2} + C\right) \end{cases}$$

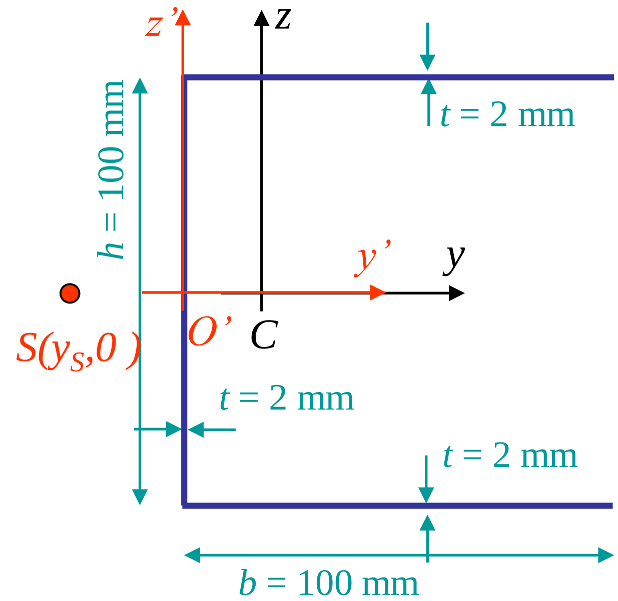
  - Buckling load is the minimum value

- Example
  - Column
    - Length: L = 1 m
    - Young: E = 70 GPa
    - Shear modulus: μ = 30 GPa
  - Buckling load?
    - Deflection and rotation around x constrained at both ends
      - $\boldsymbol{u}_y(0) = \boldsymbol{u}_y(L) = 0$ & $\boldsymbol{u}_z(0) = \boldsymbol{u}_z(L) = 0$
      - $\theta(0) = 0$ & $\theta(L) = 0$
    - Warping and rotation around y & z allowed at both ends



- Centroid position
  - $y'_C = \dfrac{2bt\frac{b}{2}}{2bt + ht} = \dfrac{0.1^2\ 0.002}{0.002\ 0.3} = 0.033 \quad \text{m}$

    $\Longrightarrow y_{O'} = -y'_C = -0.033 \quad \text{m}$
  - By symmetry on *Oy*



- Second moment of area
  - $I_{yy} = \dfrac{th^3}{12} + 2bt\left(\dfrac{h}{2}\right)^2 = \dfrac{th^3}{12}\left(1 + \dfrac{6b}{h}\right)$

    $\Longrightarrow I_{yy} = \dfrac{0.002\ 0.1^3}{12}(1+6) = 1.17\ 10^{-6}\ \text{m}^4$
  - $I_{zz} = 2\dfrac{tb^3}{12} + 2tb\left(\dfrac{h}{2} + y_{O'}\right)^2 + hty_{O'}^2$

    $\Longrightarrow I_{zz} = 2\dfrac{0.002\ 0.1^3}{12} + 2\ 0.002\ 0.1\ (0.05 - 0.033)^2 + 0.1\ 0.002\ 0.033^2$

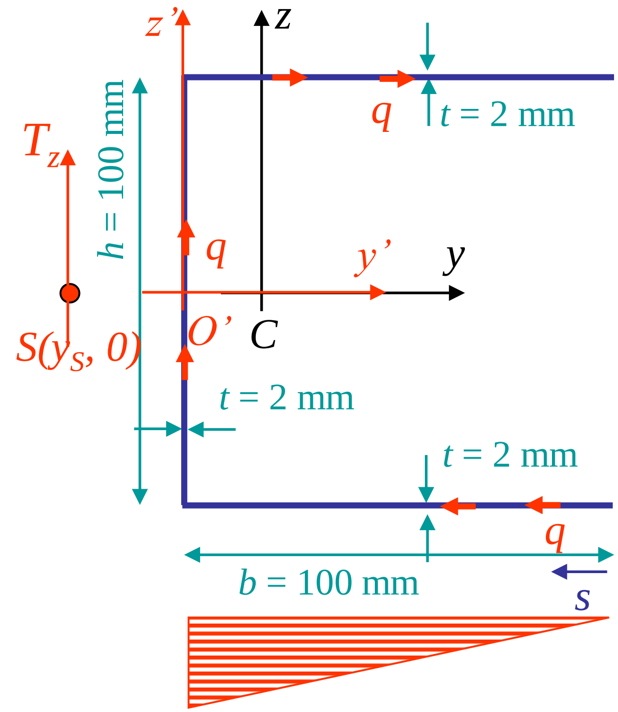
    $\Longrightarrow I_{zz} = 0.667\ 10^{-6}\ \text{m}^4$
  - By symmetry

    $I_{xy} = 0$

- Shear center
  - On C_y by symmetry
  - Consider shear force T_z
    - As I_{yz} = 0 $\Longrightarrow$ $q(s) = -\frac{T_z}{I_{yy}} \int_0^s tz ds'$
    - Lower flange, considering frame $O'y'z'$

$$q(y') = \frac{T_z}{I_{yy}} \frac{th}{2} (b - y')$$

$$\Longrightarrow q(y') = \frac{T_z}{1.17\ 10^{-6}} 0.05\ 0.002\ (0.1 - y')$$
$$= T_z \left(8.547 \text{ m}^{-1} - 85.47 \text{ m}^{-2} y'\right)$$



    - Upper flange by symmetry
    - As Tz passes through the shear center: no torsional flux

$$-y'_S T_Z = 2 \int_b^0 \frac{h}{2} q(y')(-dy') = 0$$

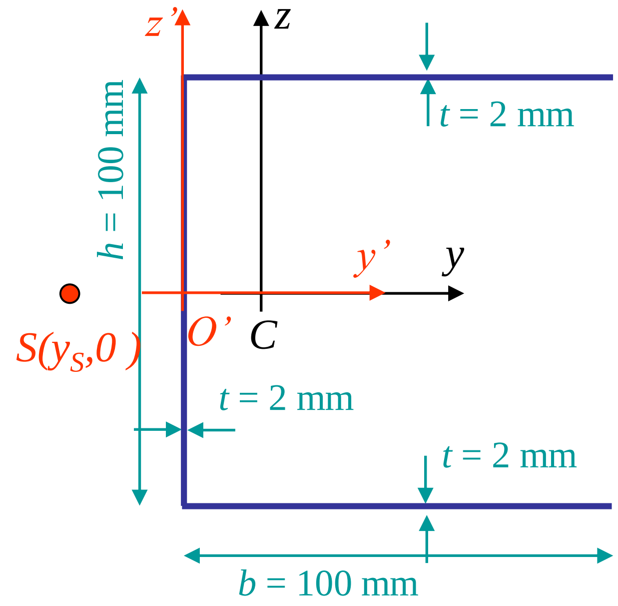
$M_{O'}$

$$\Longrightarrow y'_S = -\frac{h}{T_z} \int_0^b q(y')dy' = -0.1 \left(8.547b - \frac{85.47}{2} b^2\right) = -0.0427 \quad \text{m}$$

$$\Longrightarrow y(S) = y'_S - y'_C = -0.0427 - 0.033 = -0.076 \quad \text{m}$$

- Uncoupled critical loads
  - Using following definitions

$$\begin{cases} P^y_{CR} = \dfrac{\pi^2 EI_{yy}}{L^2} \\ P^z_{CR} = \dfrac{\pi^2 EI_{zz}}{L^2} \\ P^\theta_{CR} = \dfrac{A}{I_p^S}\left(\dfrac{C^\Gamma \pi^2}{L^2} + C\right) \end{cases}$$



- These values would be the critical loads for an uncoupled system (if $C = S$)

$$P^y_{CR} = \frac{\pi^2 EI_{yy}}{L^2} = \pi^2\ 70\ 10^9\ 1.17\ 10^{-6} = 808 \text{ kN}$$

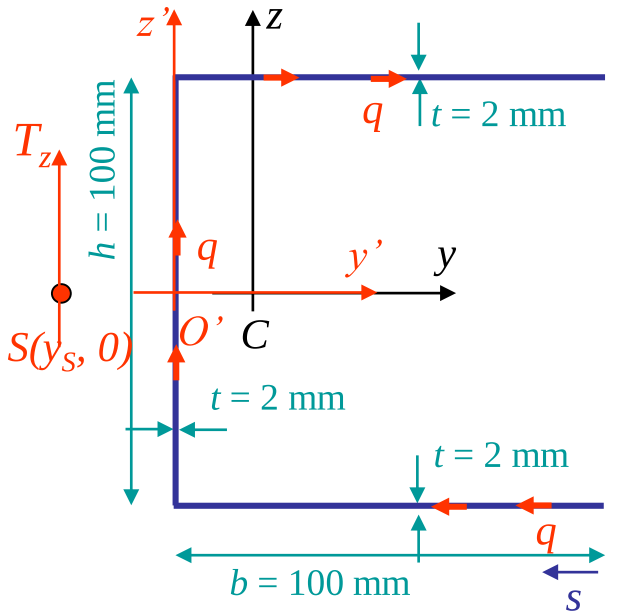
$$P^z_{CR} = \frac{\pi^2 EI_{zz}}{L^2} = \pi^2\ 70\ 10^9\ 0.667\ 10^{-6} = 461 \text{ kN}$$

P^θ_{CR} ?

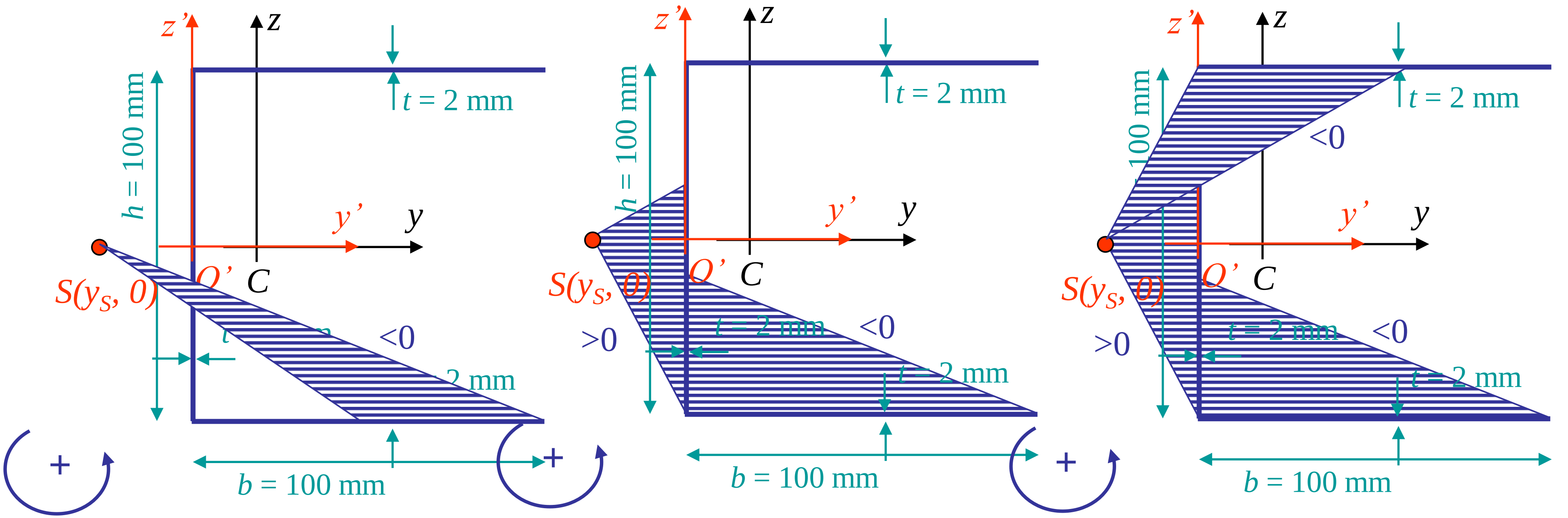
Some values are missing

- Uncoupled critical loads (2)
  - $P^{\theta}_{CR} = \frac{A}{I_p^S}\left(\frac{C^{\Gamma}\pi^2}{L^2} + C\right)$
    - $C = \sum \frac{l_i t_i^3 \mu}{3} = \frac{\mu t^3}{3}(2b+h)$
      $= \frac{30\ 10^9\ 0.002^3\ 0.3}{3} = 24 \quad \mathrm{N}\cdot\mathrm{m}^2$
    - $A = 2bt + ht = 0.002\ 0.3 = 0.6\ 10^{-3}\ \mathrm{m}^2$
    - $I_p^S = I_{yy} + I_{zz} + A\left(y_S^2 + z_S^2\right)$

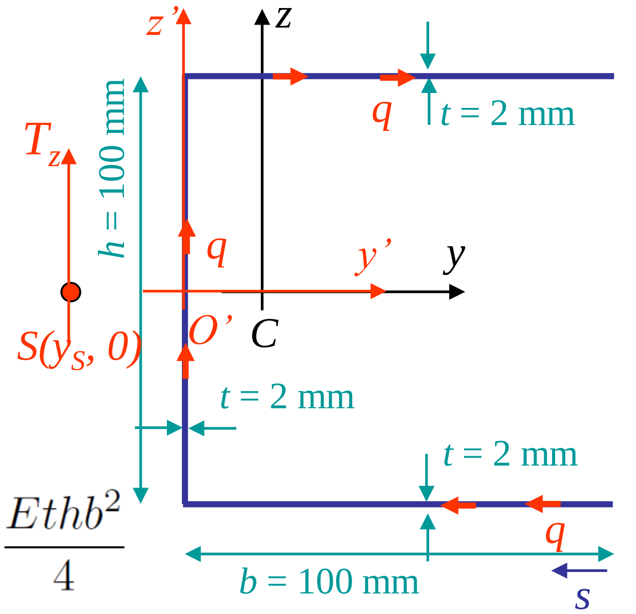
      $\Longrightarrow I_p^S = 1.17\ 10^{-6} + 0.667\ 10^{-6} + 0.6\ 10^{-3} 0.076^2 = 5.3\ 10^{-6} \quad \mathrm{m}^4$
    - $C^{\Gamma} = \int_C 4A_{R_p}^2 Etds - \frac{\left(\int_C Et2A_{R_p}(s)\,ds\right)^2}{\int_C Etds}$
  - Requires $A_{Rp}(s)$



- Uncoupled critical loads (3)
  - Evaluation of the $A_{Rp}(s)$
    - Lower flange $A_{R_p}(y') = -\frac{(b-y')h}{4}$
    - Web $A_{R_p}(z') = -\frac{bh}{4} + \frac{\left(z' + \frac{h}{2}\right)(-y'_S)}{2}$
    - Upper flange $A_{R_p}(y') = -\frac{bh}{4} - \frac{hy'_S}{2} - \frac{y'h}{4}$



- Uncoupled critical loads (4)
  - $$C^{\Gamma} = \int_C 4A_{R_p}^2 Etds - \frac{\left(\int_C Et2A_{R_p}(s)\,ds\right)^2}{\int_C Etds}$$



  - Lower flange:
    - $$A_{R_p}(y') = -\frac{(b-y')h}{4}$$
    - $$\int_f 2A_{R_p}Etds = \int_b^0 2Eth\frac{y'-b}{4}(-dy') = -\frac{Ethb^2}{4}$$

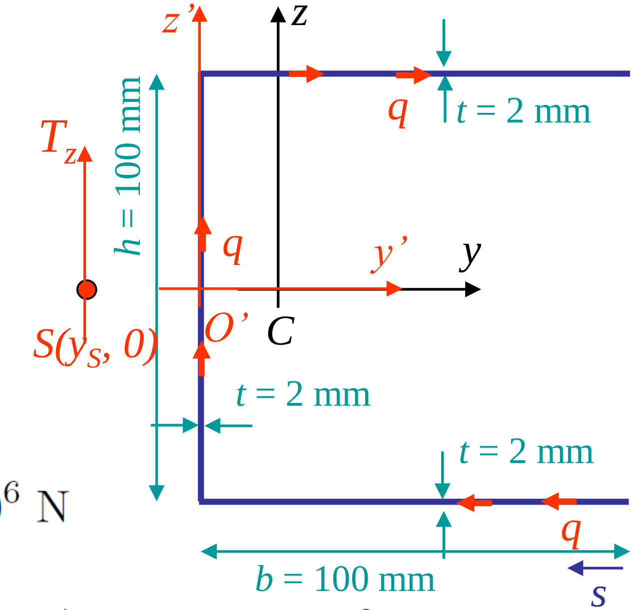
$$\Longrightarrow \int_f 2A_{R_p}Etds = -\frac{70\;10^9\;0.002\;0.1^3}{4} = -35\ \ \text{kN}\cdot m^2$$

    - $$\int_f Etds = \int_b^0 Et(-dy') = Etb = 70\;10^9\;0.002\;0.1 = 14\,10^6\ \text{N}$$
    - $$\int_f 4A_{R_p}^2 Etds = \int_b^0 4Et\left(\frac{y'-b}{4}h\right)^2(-dy') = \frac{Eth^2b^3}{12}$$

$$\Longrightarrow \int_f 4A_{R_p}^2 Etds = \frac{70\;10^9\;0.002\;0.1^5}{12} = 116.7\ \ \text{N}\cdot m^4$$

Flexural-torsional buckling of open thin-walled columns

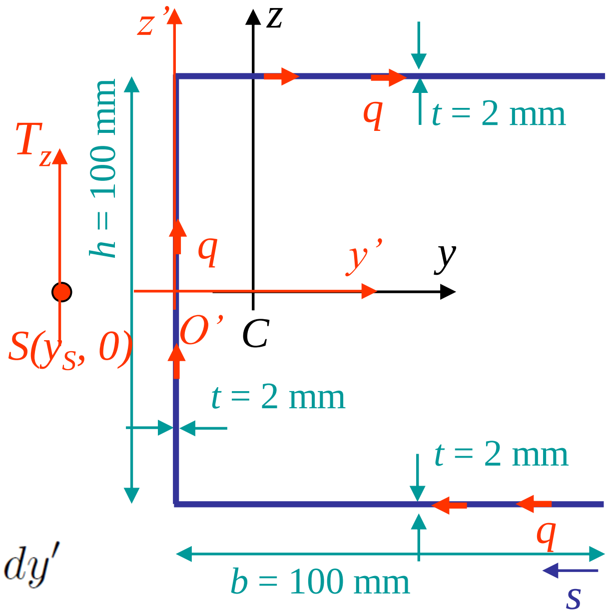
- Uncoupled critical loads (5)



  - $$C^{\Gamma} = \int_C 4A_{R_p}^2 Etds - \frac{\left(\int_C Et2A_{R_p}(s)\,ds\right)^2}{\int_C Etds}$$

    - Web:
      - $$A_{R_p}(z') = -\frac{bh}{4} + \frac{\left(z' + \frac{h}{2}\right)(-y'_S)}{2}$$
      - $$\int_w Etds = Eth = 70\ 10^9\ 0.002\ 0.1 = 14\ 10^6\ \text{N}$$
      - $$\int_w 2A_{R_p}Etds = -\int_{-\frac{h}{2}}^{\frac{h}{2}} 2Et\left[\frac{h}{4}(b+y'_S) + \cancel{\frac{z'y'_S}{2}}\right]dz = -\frac{Eth^2(b+y'_S)}{2}$$
        $$= -\frac{70\ 10^9\ 0.002\ 0.1^2\,(0.1-0.0427)}{2} = -40.1\ \ \text{kN}\cdot m^2$$
      - $$\int_w 4A_{R_p}^2 Etds = \int_{-\frac{h}{2}}^{\frac{h}{2}} 4Et\left[\frac{h}{4}(b+y'_S) + \frac{z'y'_S}{2}\right]^2 dz$$
        $$= \frac{Eth^3(b+y'_S)^2}{4} + \frac{Ety'^{\,2}_S h^3}{12} = \frac{70\ 10^9\ 0.002\ 0.1^3\,(0.1-0.0427)^2}{4} +$$
        $$\frac{70\ 10^9\ 0.002\ 0.1^3\,(-0.0427)^2}{12} = 136.2\ \ \text{N}\cdot m^4$$

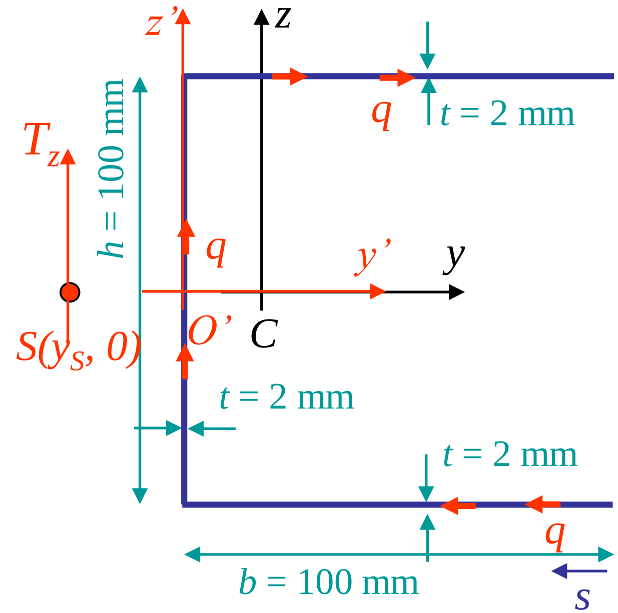
- Uncoupled critical loads (6)
  - $$C^{\Gamma} = \int_C 4A_{R_p}^2 Etds - \frac{\left(\int_C Et2A_{R_p}(s)\,ds\right)^2}{\int_C Etds}$$
    - Upper flange:
      - $$A_{R_p}(y') = -\frac{bh}{4} - \frac{hy'_S}{2} - \frac{y'h}{4}$$



$$\int_f 2A_{R_p}Etds = -\int_0^b 2Et\left[\frac{h}{4}(b+2y'_S) + \frac{y'h}{4}\right]dy'$$
$$= -Et\left[bh\frac{b+2y'_S}{2} + \frac{b^2h}{4}\right]$$
$$= -70\ 10^9\ 0.002\ \left(0.1^2\frac{0.1-2\ 0.0427}{2} + \frac{0.1^3}{4}\right) = -45.2\ \ \mathrm{kN}\cdot m^2$$

$$\int_f Etds = Etb = 70\ 10^9\ 0.002\ 0.1 = 14\ 10^6\ \mathrm{N}$$

- Uncoupled critical loads (7)
  - $$C^{\Gamma} = \int_C 4A_{R_p}^2 Etds - \frac{\left(\int_C Et2A_{R_p}(s)\,ds\right)^2}{\int_C Etds}$$
    - Upper flange (2):
      - $$A_{R_p}(y') = -\frac{bh}{4} - \frac{hy'_S}{2} - \frac{y'h}{4}$$



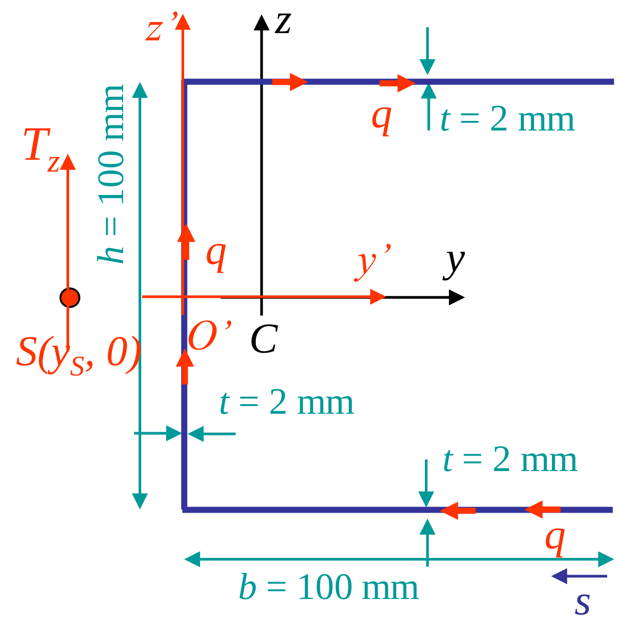
- $$\int_f 4A_{R_p}^2 Etds = \int_0^b 4Et\left[\frac{h^2}{16}\left(b+2y'_S\right)^2 + \frac{h^2y'\left(b+2y'_S\right)}{8} + \frac{y'^2h^2}{16}\right]dy'$$
$$= Et\left[bh^2\frac{\left(b+2y'_S\right)^2}{4} + \frac{h^2b^2\left(b+2y'_S\right)}{4} + \frac{b^3h^2}{12}\right]$$
$$= 70\;10^9\;0.002\;\left(0.1^3\frac{(0.1-2\;0.0427)^2}{4} + \frac{0.1^4\,(0.1-2\;0.0427)}{4} + \frac{0.1^5}{12}\right) = 175.2\;\;\mathrm{N}\cdot m^4$$

- Uncoupled critical loads (8)
  - $$C^{\Gamma} = \int_C 4A_{R_p}^2 Etds - \frac{\left(\int_C Et2A_{R_p}(s)\,ds\right)^2}{\int_C Etds}$$
    - All contributions
      - $$\int_C 4A_{R_p}^2 Etds = 116.7 + 136.2 + 175.2 = 428 \ \ \mathrm{N}\cdot m^4$$
      - $$\int_C 2A_{R_p} Etds = -35000 - 40100 - 45200 = -120.3 \ \ \mathrm{kN}\cdot m^2$$
      - $$\int_C Etds = 3\,14\,10^6 = 42\,10^6 \ \mathrm{N}$$

$$\Longrightarrow C^{\Gamma} = 428 - \frac{120300^2}{42\,10^6} = 83.4 \ \ \mathrm{N}\cdot m^4$$

- Uncoupled critical loads (9)
  - $P^{\theta}_{CR} = \frac{A}{I_p^S}\left(\frac{C^{\Gamma}\pi^2}{L^2} + C\right)$
    - $C = \sum_i \frac{l_i t_i^3 \mu}{3} = \frac{\mu t^3}{3}(2b+h)$
      $= \frac{30\ 10^9\ 0.002^3\ 0.3}{3} = 24\ \ \mathrm{N}\cdot\mathrm{m}^2$
    - $A = 2bt + ht = 0.002\ 0.3 = 0.6\ 10^{-3}\ \mathrm{m}^2$
    - $I_p^S = I_{yy} + I_{zz} + A\left(y_S^2 + z_S^2\right)$

      $\Longrightarrow I_p^S = 1.17\ 10^{-6} + 0.667\ 10^{-6} + 0.6\ 10^{-3} 0.076^2 = 5.3\ 10^{-6}\ \ \mathrm{m}^4$
    - $C^{\Gamma} = 428 - \frac{120300^2}{42\ 10^6} = 83.4\ \ \mathrm{N}\cdot m^4$

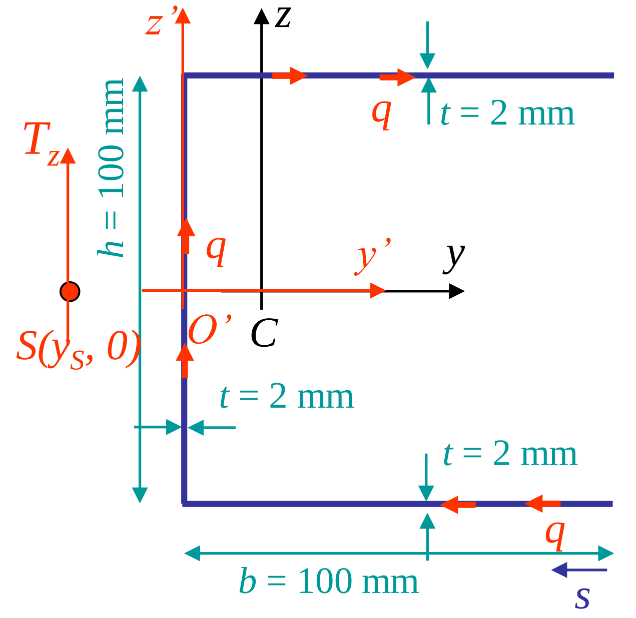


$\Longrightarrow P^{\theta}_{CR} = \frac{0.6\,10^{-3}}{5.3\,10^{-6}}\left(83.4\pi^2 + 24\right) = 95.9\ \ \mathrm{kN}$

- Critical load
  - As the uncoupled critical loads read

$$P_{CR}^{y} = \frac{\pi^2 E I_{yy}}{L^2} \quad , \quad P_{CR}^{z} = \frac{\pi^2 E I_{zz}}{L^2} \ \&$$

$$P_{CR}^{\theta} = \frac{A}{I_p^S}\left(\frac{C^{\Gamma}\pi^2}{L^2} + C\right)$$



& as $z_S = 0$, the coupled system is rewritten

$$\begin{vmatrix} \left(P_{CR} - \frac{\pi^2 E I_{zz}}{L^2}\right) & 0 & P_{CR} z_S \\ 0 & \left(\frac{\pi^2 E I_{yy}}{L^2} - P_{CR}\right) & P_{CR} y_S \\ -P_{CR} z_S & P_{CR} y_S & \left(\frac{C^{\Gamma}\pi^2}{L^2} + C - \frac{P_{CR}}{A} I_p^S\right) \end{vmatrix} = 0$$

$$\begin{vmatrix} (P_{CR} - P_{CR}^{z}) & 0 & 0 \\ 0 & (P_{CR}^{y} - P_{CR}) & P_{CR} y_S \\ 0 & P_{CR} y_S & (P_{CR}^{\theta} - P_{CR})\frac{I_p^S}{A} \end{vmatrix} = 0$$

- Critical load (2)
  - Resolution

$$\begin{vmatrix} (P_{CR} - P^z_{CR}) & 0 & 0 \\ 0 & (P^y_{CR} - P_{CR}) & P_{CR} y_S \\ 0 & P_{CR} y_S & (P^\theta_{CR} - P_{CR}) \frac{I_p^S}{A} \end{vmatrix} = 0$$

$$\frac{I_p^S}{A} (P_{CR} - P^z_{CR}) \left[P^2_{CR} \left(\frac{A}{I_p^S} y_S^2 - 1 \right) + P_{CR} \left(P^\theta_{CR} + P^y_{CR} \right) - P^\theta_{CR} P^y_{CR} \right] = 0$$

$$\left\{ \begin{array}{l} P^z_{CR} = 461 \text{ kN} \\ P^y_{CR} = 808 \text{ kN} \\ P^\theta_{CR} = 95.9 \text{ kN} \\ I_p^S = 5.3\ 10^{-6} \text{ m}^4 \\ A = 0.6\ 10^{-3} \text{ m}^2 \\ y(S) = -0.076 \text{ m} \end{array} \right.$$

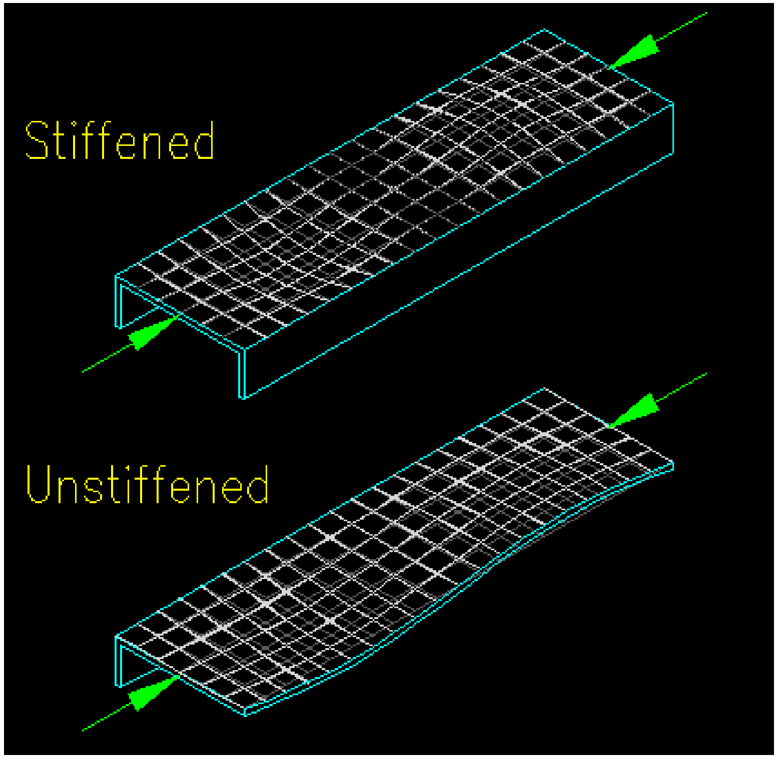
$$\left\{ \begin{array}{l} P^{(1)}_{CR} = P^z_{CR} = 461 \text{ kN} \\ -0.34 P^2_{CR} + 903.9\, 10^3 P_{CR} - 77.5\, 10^9 = 0 \end{array} \right.$$

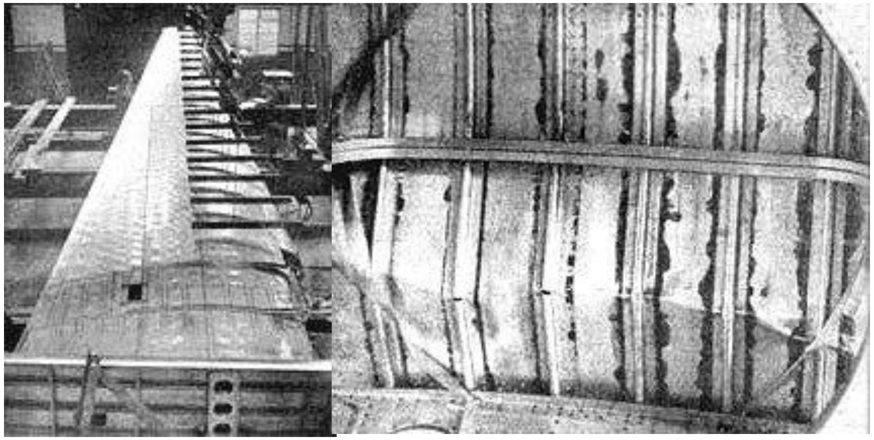
$$\left\{ \begin{array}{l} P^{(1)}_{CR} = P^z_{CR} = 461 \text{ kN} \\ P^2_{CR} - 2.62\, 10^6 P_{CR} + 224.5\, 10^9 = 0 \end{array} \right.$$

>

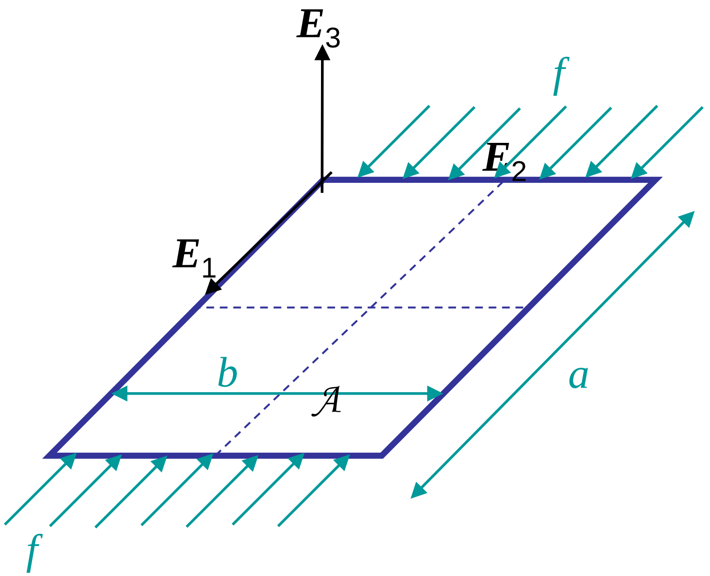
$$\left\{ \begin{array}{l} P^{(1)}_{CR} = P^z_{CR} = 461 \text{ kN} \\ \boxed{P^{(2)}_{CR} = 88.7 \text{ kN}} \\ P^{(3)}_{CR} = 2530 \text{ kN} \end{array} \right.$$

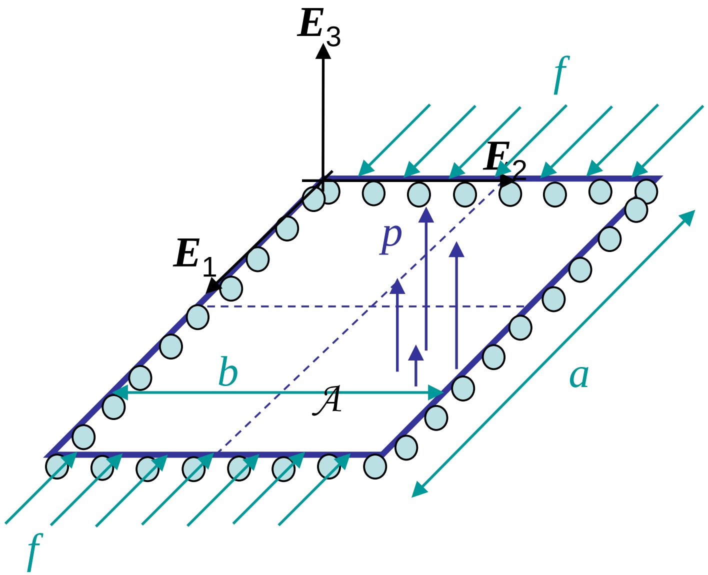
- Thin plates
  - Are subject to primary buckling
    - Wavelength of buckle ~ length of element
  - So they are stiffened

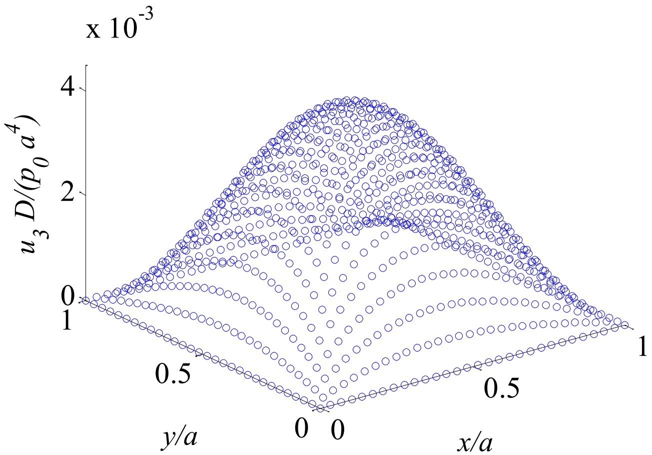




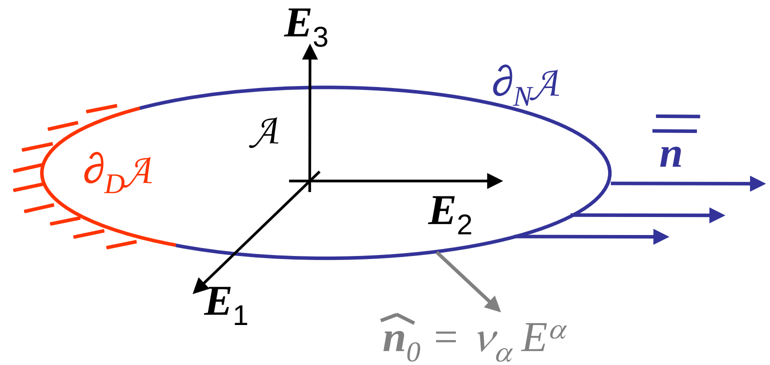
- Primary buckling of thin plates
  - Plates without support
    - Similar to column buckling
      - Same shape
      - Use D instead of EI_{zz}
  - Supported plates
    - Other displacement buckling shapes
    - Depend on BCs







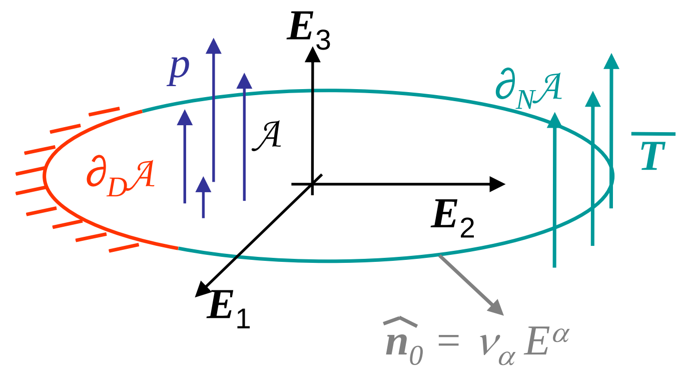
- Kirchhoff-Love membrane mode
  - On $\mathcal{A}$: $\mathcal{H}_n^{\alpha\beta\gamma\delta}\frac{\boldsymbol{u}_{\gamma,\delta\alpha}+\boldsymbol{u}_{\delta,\gamma\alpha}}{2}+\bar{\boldsymbol{n}}_\beta=\bar{\rho}\ddot{\boldsymbol{u}}_\beta$
    - With $\mathcal{H}_n^{\alpha\beta\gamma\delta}=\frac{h_0 E}{1-\nu^2}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta}+\frac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta}+\delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$

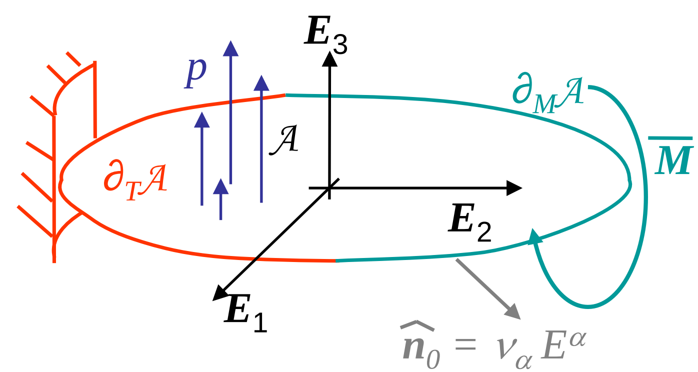


  - Completed by appropriate BCs
    - Dirichlet $\boldsymbol{u}_\alpha=\bar{\boldsymbol{u}}_\alpha$
    - Neumann $\boldsymbol{n}_\beta^\alpha\nu_\alpha=\mathcal{H}_n^{\alpha\beta\gamma\delta}\frac{\boldsymbol{u}_{\gamma,\delta\alpha}+\boldsymbol{u}_{\delta,\gamma\alpha}}{2}\nu_\alpha=\bar{\bar{\boldsymbol{n}}}_\beta$

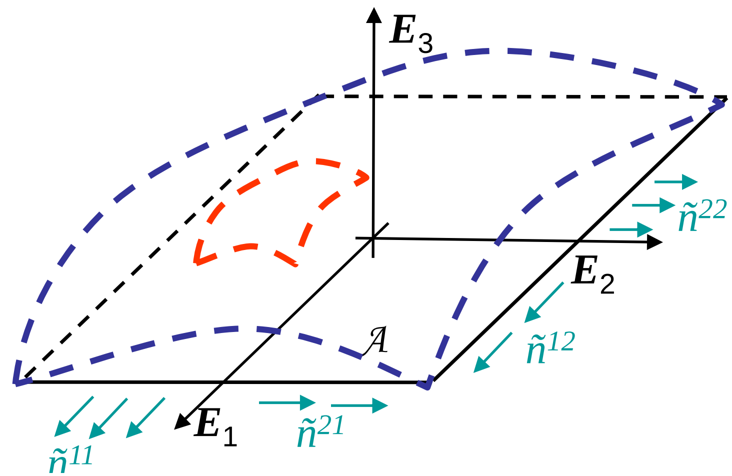
- Kirchhoff-Love bending mode
  – On $\mathcal{A}$: $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\alpha\beta} = p$
    - With $\mathcal{H}_m^{\alpha\beta\gamma\delta} = \underbrace{\frac{h_0^3 E}{12\left(1-\nu^2\right)}}_{D}\left[\nu\delta^{\alpha\beta}\delta^{\gamma\delta} + \frac{1-\nu}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}\right)\right]$
  – Completed by appropriate BCs
    - Low order
      – On $\partial_N\mathcal{A}$: $-\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)_{,\beta}\nu_\alpha = \bar{T}$
      – On $\partial_D\mathcal{A}$: $\boldsymbol{u}_3 = \bar{\boldsymbol{u}}_3$
    - High order
      – On $\partial_T\mathcal{A}$: $\boldsymbol{\Delta t} = \bar{\boldsymbol{\Delta t}}$

        with $\boldsymbol{\Delta t} = -\boldsymbol{u}_{3,\alpha}\boldsymbol{E}_\alpha$
      – On $\partial_M\mathcal{A}$: $-\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta}\right)\nu_\beta = \bar{M}\nu_\alpha$



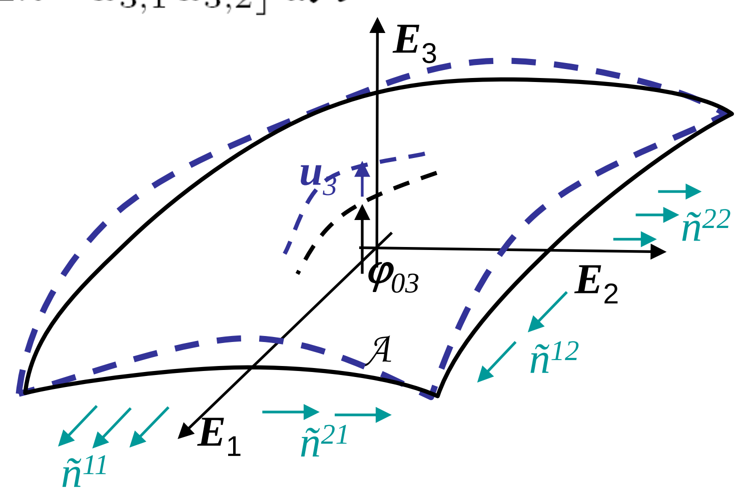


- Membrane-bending coupling
  - The first order theory is uncoupled
  - For second order theory
    - On $\mathcal{A}$: $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta} \boldsymbol{u}_{3,\gamma\delta} - \tilde{n}^{\alpha\beta} \boldsymbol{u}_3\right)_{,\alpha\beta} = p$
    - Tension increases the bending stiffness of the plate
    - Internal energy



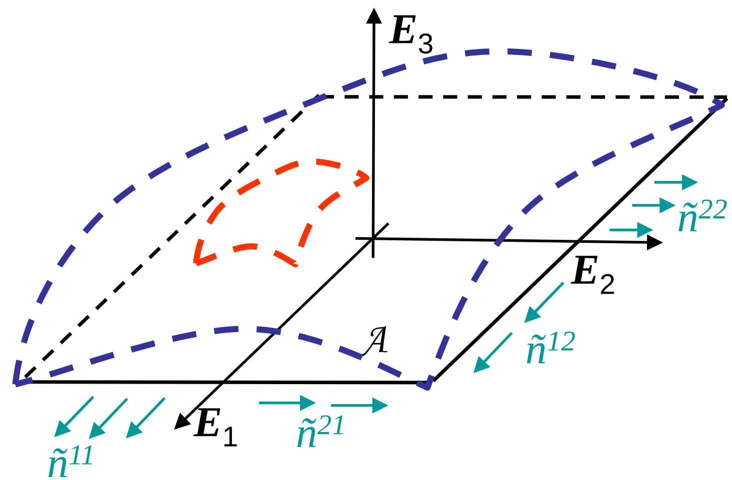
$$E_{\text{int}} = \int_{\mathcal{A}} \frac{D}{2} \left[\boldsymbol{u}_{3,11}^2 + \boldsymbol{u}_{3,22}^2 + 2(1-\nu)\boldsymbol{u}_{3,12}^2 + 2\nu \boldsymbol{u}_{3,11}\boldsymbol{u}_{3,22}\right] d\mathcal{A} + \frac{1}{2}\int_{\mathcal{A}} \left[\tilde{n}^{11}\boldsymbol{u}_{3,1}{}^2 + \tilde{n}^{22}\boldsymbol{u}_{3,2}{}^2 + 2\tilde{n}^{12}\boldsymbol{u}_{3,1}\boldsymbol{u}_{3,2}\right] d\mathcal{A}$$

  - In case of small initial curvature (κ >>)
    - On $\mathcal{A}$:
      $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta} \boldsymbol{u}_{3,\gamma\delta} - \tilde{n}^{\alpha\beta} \boldsymbol{u}_3 - \tilde{n}^{\alpha\beta} \boldsymbol{\varphi}_{03}\right)_{,\alpha\beta} = p$
    - Tension induces bending effect



- Primary buckling theory of thin plates
  - Second order theory
    - On $\mathcal{A}$: $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} - \tilde{n}^{\alpha\beta}\boldsymbol{u}_3\right)_{,\alpha\beta} = p$
  - Simply supported plate with arbitrary pressure
    - Pressure is written in a Fourier series

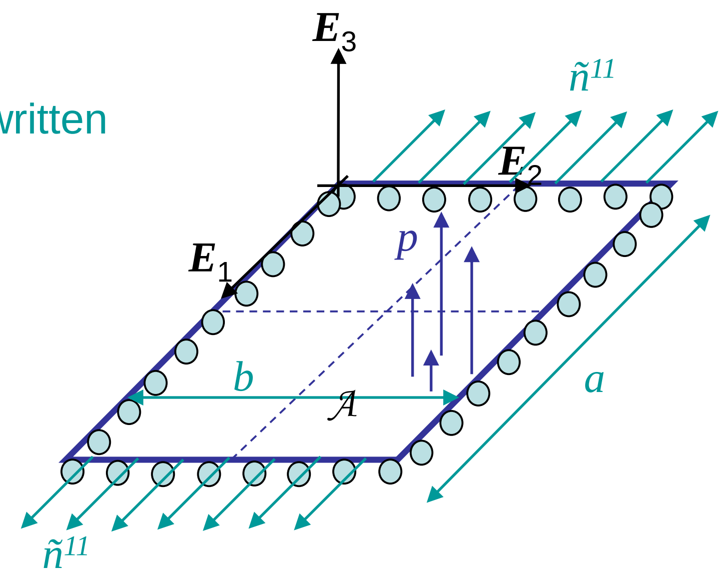
$$p\left(x,\,y\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$



- Displacements with these BCs can also be written

$$\boldsymbol{u}_3 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

with $A_{mn} = \dfrac{a_{mn}}{\pi^4 D\left[\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + \frac{\tilde{n}^{11}m^2}{\pi^2 D a^2}\right]}$



- There is a buckling load $\tilde{n}^{11}$ leading to infinite displacements for every couple (m, n)
  - Lowest one?

- Primary buckling theory of thin plates (2)
  - Simply supported plate
    - Displacements in terms of

$$A_{mn} = \frac{a_{mn}}{\pi^4 D \left[\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 + \frac{\tilde{n}^{11} m^2}{\pi^2 D a^2} \right]}$$

    - Buckling load $\tilde{n}^{11}$
      - Minimal (in absolute value) for n=1

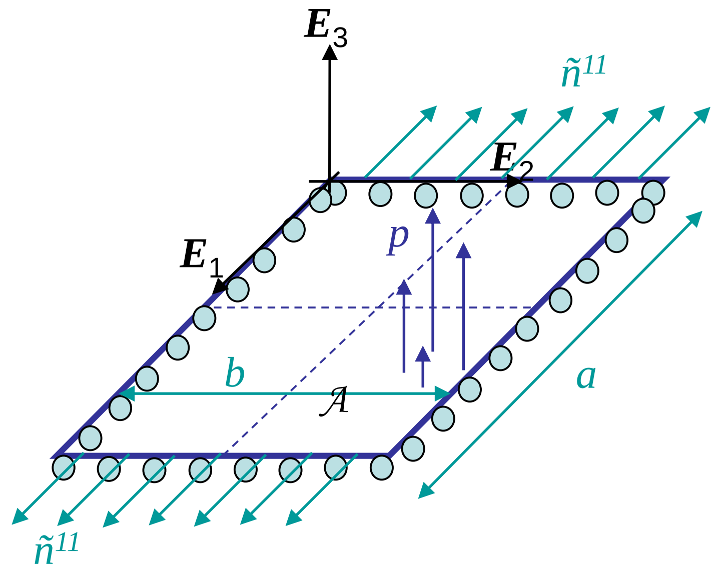
$$\Rightarrow \tilde{n}^{11}_{CR} = -\frac{\pi^2 a^2 D}{m^2} \left(\frac{m^2}{a^2} + \frac{1}{b^2} \right)^2$$

      - Or again $\tilde{n}^{11}_{CR} = -k \frac{\pi^2 D}{b^2}$

        with the buckling coefficient k

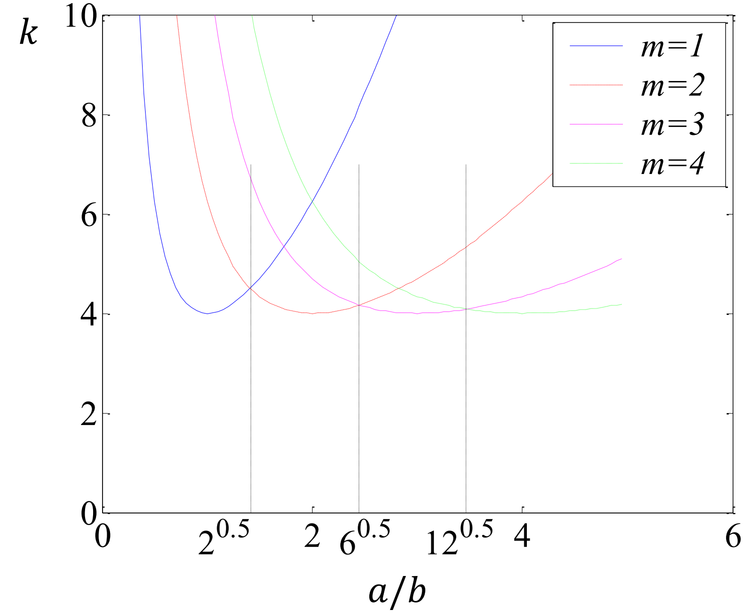
$$k = \left(\frac{mb}{a} + \frac{a}{mb} \right)^2$$

      - Depends on ration a/b



- Primary buckling theory of thin plates (3)
  - Simply supported plate (2)
    - Buckling coefficient k

$$k = \left(\frac{mb}{a} + \frac{a}{mb}\right)^2$$

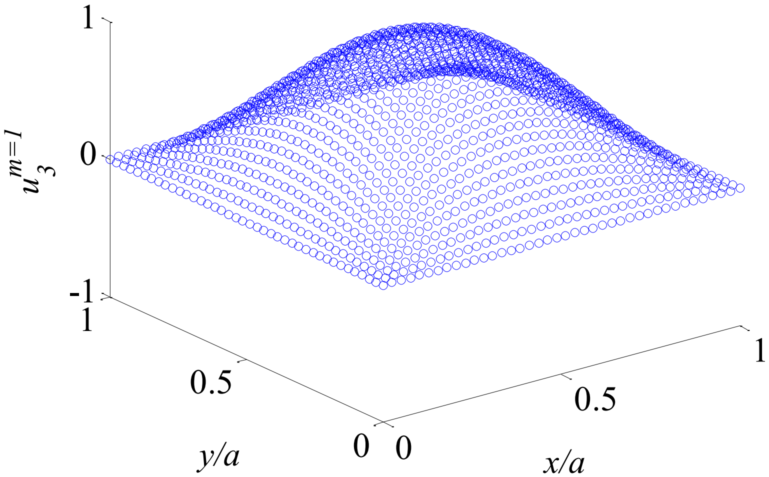


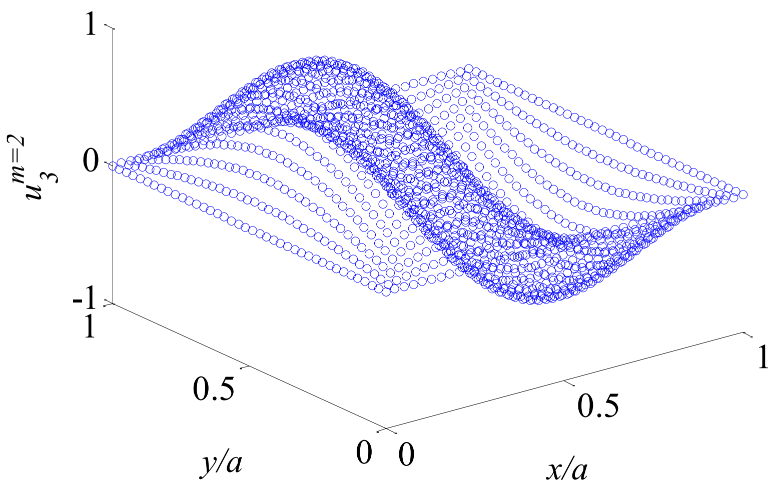
    - Mode of buckling depends on a/b
    - k is minimal (=4) for a/b = 1, 2, 3, …
    - Mode transition for

$$\frac{a}{b} = \sqrt{m\,(m+1)}$$

    - For a/b > 3: k ~ 4
  - This analysis depends on the BCs, but same behaviors for
    - Other BCs
    - Other loadings (bending, shearing) instead of compression
    - Only the value of k is changing (tables)

- Primary buckling theory of thin plates (4)
  - Shape of the modes for
    - Simply supported plate
    - In compression
    - *n*=1



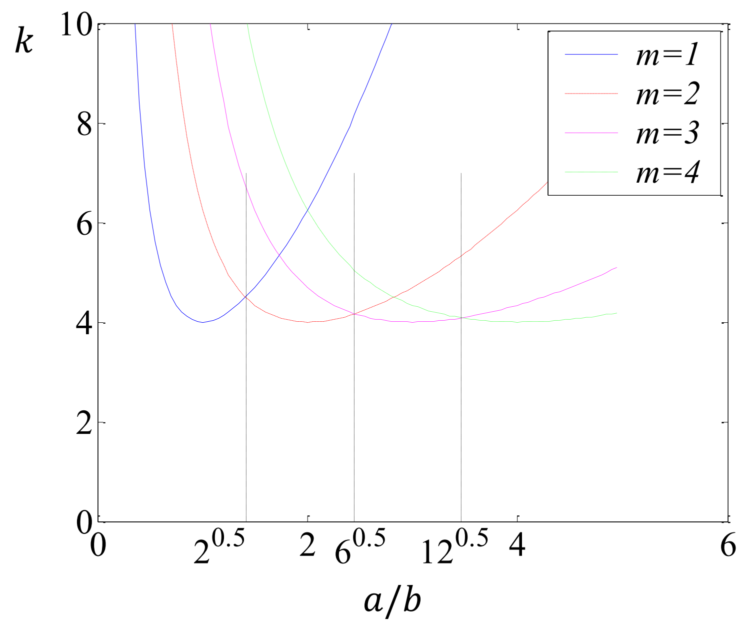


- Primary buckling theory of thin plates (5)
  - Critical stress
    - We found $\tilde{n}^{11}_{CR} = -k\dfrac{\pi^2 D}{b^2}$

      $\Longrightarrow \sigma_{CR} = \tilde{n}^{11}_{CR}/h_0$
    - Or again

$$\sigma_{CR} = -k\frac{\pi^2 E h_0^2}{12\left(1-\nu^2\right)b^2}$$

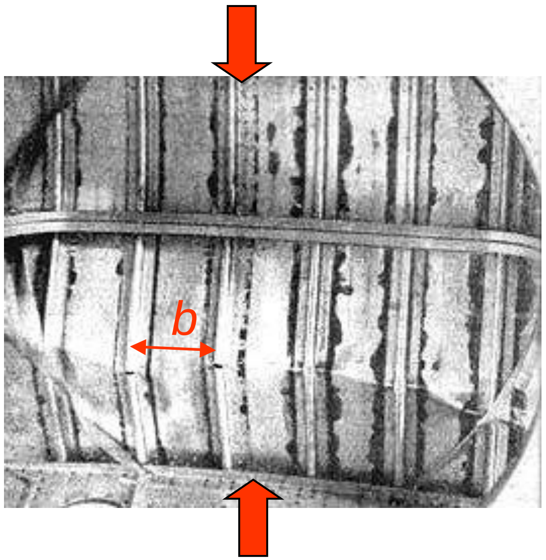
    - This can be generalized to other loading cases with k depending on the problem
      - Picture for simply supported plate in compression



  - As
    - $\sigma_{CR} = -k\dfrac{\pi^2 E h_0^2}{12\left(1-\nu^2\right)b^2}$
    - $k \sim$ cst for $a/b > 3$

    We use stiffeners to reduce b

    $\Longrightarrow$ to increase σ_{CR} of the skin
    - As long as $\sigma_{CR} < \sigma_p^0$



- Primary buckling theory of thin plates (6)
  - What happens for other BCs?
    - We cannot say $\boldsymbol{u}_3 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$ anymore
    - But buckling corresponds to a stationary point of the internal energy (neutral equilibrium)
    - So we can plug any Fourier series or displacement approximations in the form

$$E_{\text{int}} = \int_{\mathcal{A}} \frac{D}{2} \left[\boldsymbol{u}_{3,11}^2 + \boldsymbol{u}_{3,22}^2 + 2(1-\nu)\, \boldsymbol{u}_{3,12}^2 + 2\nu \boldsymbol{u}_{3,11} \boldsymbol{u}_{3,22} \right] d\mathcal{A} + \frac{1}{2} \int_{\mathcal{A}} \left[\tilde{n}^{11} {\boldsymbol{u}_{3,1}}^2 + \tilde{n}^{22} {\boldsymbol{u}_{3,2}}^2 + 2\tilde{n}^{12} \boldsymbol{u}_{3,1} \boldsymbol{u}_{3,2} \right] d\mathcal{A}$$

and find the stationary point

- Primary buckling theory of thin plates (7)
  - Energy method
    - Let us analyze the simply supported plate

$$\Rightarrow \boldsymbol{u}_3 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

    - Internal energy

$$\begin{aligned} E_{\text{int}} &= \int_{\mathcal{A}} \frac{D}{2}\left[\boldsymbol{u}_{3,11}^2 + \boldsymbol{u}_{3,22}^2 + 2(1-\nu)\,\boldsymbol{u}_{3,12}^2 + 2\nu\boldsymbol{u}_{3,11}\boldsymbol{u}_{3,22}\right] d\mathcal{A} + \\ & \frac{1}{2}\int_{\mathcal{A}} \left[\tilde{n}^{11}{\boldsymbol{u}_{3,1}}^2 + \cancel{\tilde{n}^{22}{\boldsymbol{u}_{3,2}}^2} + \cancel{2\tilde{n}^{12}\boldsymbol{u}_{3,1}\boldsymbol{u}_{3,2}}\right] d\mathcal{A} \end{aligned}$$

    - First term

$$\int_0^a\int_0^b \boldsymbol{u}_{3,11}^2 dydx = \int_0^a\int_0^b \left(\sum_m\sum_n -A_{mn}\frac{m^2\pi^2}{a^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right)^2 dydx$$

      - As $\int_0^a \sin\frac{m\pi x}{a}\sin\frac{m'\pi x}{a}dx = \frac{a}{2}\delta_{mm'}$ the cross-terms vanish

$$\Rightarrow \int_0^a\int_0^b \boldsymbol{u}_{3,11}^2 dydx = \sum_m\sum_n\int_0^a\int_0^b \left(A_{mn}\frac{m^2\pi^2}{a^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right)^2 dydx$$

- Primary buckling theory of thin plates (8)
  - Energy method (2)
    - Internal energy

$$E_{\text{int}} = \int_{\mathcal{A}} \frac{D}{2}\left[\boldsymbol{u}_{3,11}^2 + \boldsymbol{u}_{3,22}^2 + 2(1-\nu)\,\boldsymbol{u}_{3,12}^2 + 2\nu \boldsymbol{u}_{3,11}\boldsymbol{u}_{3,22}\right] d\mathcal{A} + \frac{1}{2}\int_{\mathcal{A}}\left[\tilde{n}^{11}\boldsymbol{u}_{3,1}{}^2 + \cancel{\tilde{n}^{22}\boldsymbol{u}_{3,2}{}^2} + \cancel{2\tilde{n}^{12}\boldsymbol{u}_{3,1}\boldsymbol{u}_{3,2}}\right] d\mathcal{A}$$

  - As $\boldsymbol{u}_3 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$
  - And as cross-terms vanish

$$E_{\text{int}} = \frac{D}{2}\sum_m\sum_n \int_0^a\int_0^b A_{mn}^2 \left[\frac{m^4\pi^4}{a^4}\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b} + \frac{n^4\pi^4}{b^4}\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b} + 2(1-\nu)\frac{m^2n^2\pi^4}{a^2b^2}\cos^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b} + 2\nu\frac{m^2n^2\pi^4}{a^2b^2}\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b} + \frac{\tilde{n}^{11}}{D}\frac{m^2\pi^2}{a^2}\cos^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}\right] dx\,dy$$

- Primary buckling theory of thin plates (9)
  - Energy method (3)
    - As $\int_0^a \sin^2 \frac{m\pi x}{a} dx = \int_0^a \cos^2 \frac{m\pi x}{a} dx = \frac{a}{2}$

$$
\begin{aligned}
E_{\text{int}} \quad = \quad & \frac{D}{2} \sum_m \sum_n \int_0^a \int_0^b A_{mn}^2 \left[\frac{m^4\pi^4}{a^4} \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} + \frac{n^4\pi^4}{b^4} \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} + \right. \\
& 2\left(1-\nu\right) \frac{m^2n^2\pi^4}{a^2b^2} \cos^2 \frac{m\pi x}{a} \cos^2 \frac{n\pi y}{b} + 2\nu \frac{m^2n^2\pi^4}{a^2b^2} \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} + \\
& \left. \frac{\tilde{n}^{11}}{D} \frac{m^2\pi^2}{a^2} \cos^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} \right] dx dy
\end{aligned}
$$

$$
\begin{aligned}
E_{\text{int}} \quad = \quad & \frac{D}{2} \sum_m \sum_n A_{mn}^2 \left[\frac{m^4\pi^4 b}{4a^3} + \frac{n^4\pi^4 a}{4b^3} + \right. \\
& \left. 2\left(1-\cancel{\nu}\right) \frac{m^2n^2\pi^4}{4ab} + 2\nu \cancel{\frac{m^2n^2\pi^4}{4ab}} + \frac{\tilde{n}^{11}}{D} \frac{m^2\pi^2 b}{4a} \right]
\end{aligned}
$$

$$
\Longrightarrow \quad E_{\text{int}} \quad = \quad \frac{D}{2ab} \sum_m \sum_n A_{mn}^2 \left[\left(\frac{m^2\pi^2 b}{2a} + \frac{n^2\pi^2 a}{2b} \right)^2 + \frac{\tilde{n}^{11}}{D} \frac{m^2\pi^2 b^2}{4} \right]
$$

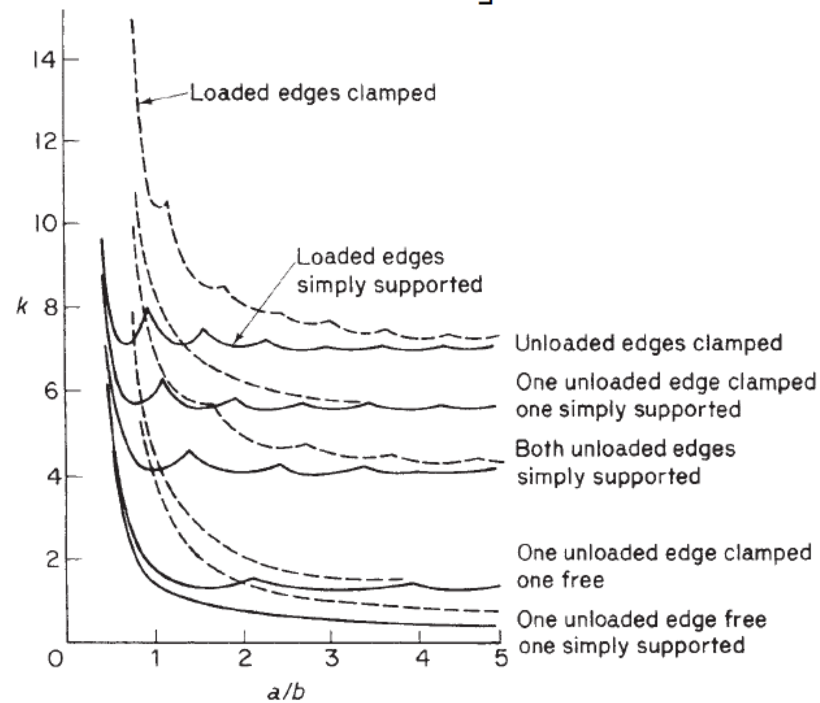
- Primary buckling theory of thin plates (10)
  - Energy method (4)
    - As $E_{\text{int}} \;=\; \frac{D}{2ab}\sum_m\sum_n A_{mn}^2\left[\left(\frac{m^2\pi^2 b}{2a}+\frac{n^2\pi^2 a}{2b}\right)^2+\frac{\tilde{n}^{11}}{D}\frac{m^2\pi^2 b^2}{4}\right]$
    - At buckling we have at least for one couple (m, n)

$$0=\partial_{A_{mn}}E_{\text{int}}=\frac{D\pi^2}{4ab}A_{mn}\left[\left(\frac{m^2\pi b}{a}+\frac{n^2\pi a}{b}\right)^2+\frac{m^2 b^2\tilde{n}^{11}_{CR}}{D}\right]$$

    - Most critical value for n=1

$$\Longrightarrow \tilde{n}^{11}_{CR}=-\frac{\pi^2 a^2 D}{m^2}\left(\frac{m^2}{a^2}+\frac{1}{b^2}\right)^2$$

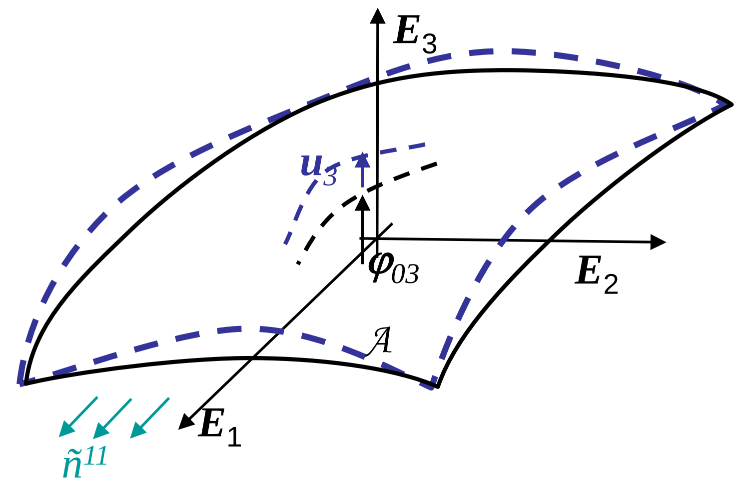
    - In general $\tilde{n}^{11}_{CR}=-k\frac{\pi^2 D}{b^2}$



- Experimental determination of critical load
  - Avoid buckling $\Longrightarrow$ Southwell diagram
  - Plate with small initial curvature
    - $\left(\mathcal{H}_m^{\alpha\beta\gamma\delta}\boldsymbol{u}_{3,\gamma\delta} - \tilde{n}^{\alpha\beta}\boldsymbol{u}_3 - \tilde{n}^{\alpha\beta}\boldsymbol{\varphi}_{03}\right)_{,\alpha\beta} = p$
  - Particular case of $p = 0$, tension $\tilde{n}^{11}$, simply supported edges
    - For $\boldsymbol{\varphi}_{03}(x,\, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

      $\Longrightarrow$ $\boldsymbol{u}_3 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

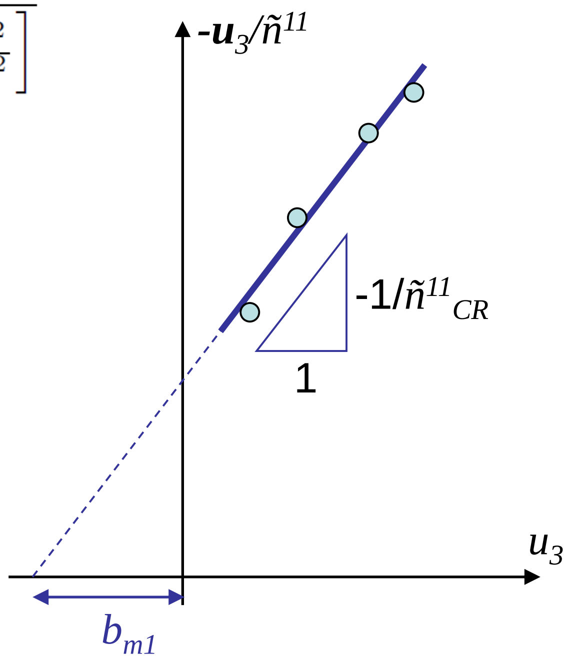
      with $A_{mn} = -\frac{b_{mn}m^2\tilde{n}^{11}}{a^2\pi^2 D\left[\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + \frac{\tilde{n}^{11}m^2}{\pi^2 D a^2}\right]}$
    - When $\tilde{n}^{11} \rightarrow \tilde{n}^{11}_{CR} = -\frac{\pi^2 a^2 D}{m^2}\left(\frac{m^2}{a^2}+\frac{1}{b^2}\right)^2$
      - Term b_{m1} is the dominant one in the solution
      - Displacement takes the shape of buckling mode m ($n=1$)



- Experimental determination of critical load (2)
  - Particular case of p=0, tension $\tilde{n}^{11}$, simply supported edges (2)
    - When $\tilde{n}^{11} \rightarrow$ $\tilde{n}^{11}_{CR} = -\frac{\pi^2 a^2 D}{m^2}\left(\frac{m^2}{a^2}+\frac{1}{b^2}\right)^2$
      - Term b_{m1} is the dominant one in the solution
      - As $\boldsymbol{u}_3 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

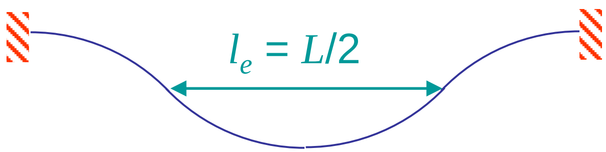
        with $A_{mn} = -\frac{b_{mn} m^2 \tilde{n}^{11}}{a^2\pi^2 D\left[\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+\frac{\tilde{n}^{11}m^2}{\pi^2 D a^2}\right]}$

        $\Longrightarrow$ $\boldsymbol{u}_3 \rightarrow \frac{b_{m1}\tilde{n}^{11}}{\tilde{n}^{11}_{CR}-\tilde{n}^{11}}$
    - Rearranging: $\boldsymbol{u}_3 \rightarrow \tilde{n}^{11}_{CR}\frac{\boldsymbol{u}_3}{\tilde{n}^{11}} - b_{m1}$
      - m depends on ratio a/b



Secondary buckling of columns

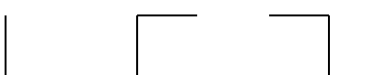
- Primary to secondary buckling of columns
  - Slenderness ratio l_e/r with
    - l_e: effective length of the column
      - Depends on BCs and mode



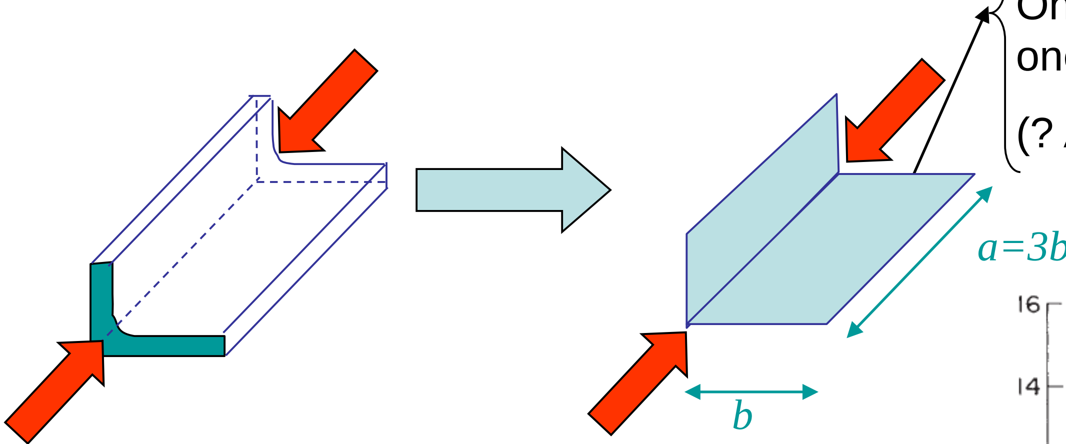
    - r: radius of gyration
  - High slenderness (l_e/r >80)
    - Primary buckling
  - Low slenderness (l_e/r <20)
    - Secondary (local) buckling
    - Usually in flanges
  - In between slenderness
    - Combination

- Example of secondary buckling
  - Composite beam
  - Design such that
    - Load of primary buckling > limit load > web local buckling load
  - Final year project
    - Alice Salmon
    - coexpair Composites Expertise for Aircraft

- How to determine secondary buckling?
  - Easy cases: particular sections



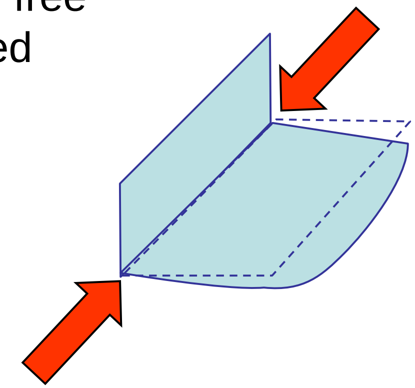
- Secondary buckling of a L-section
  - Represent the beam as plates



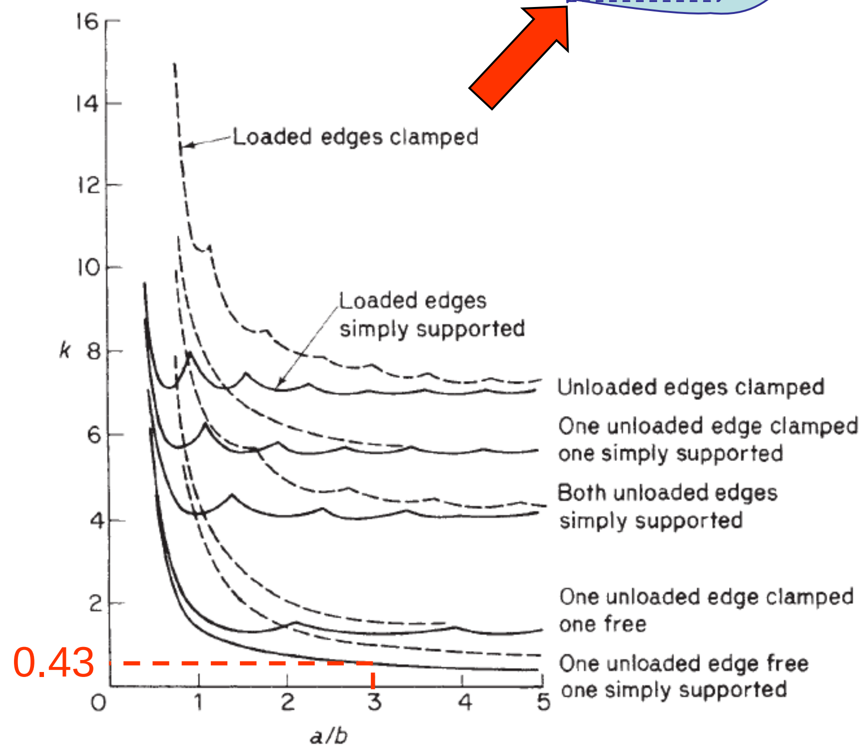
Loaded edges simply supported

One unloaded edge free one simply supported

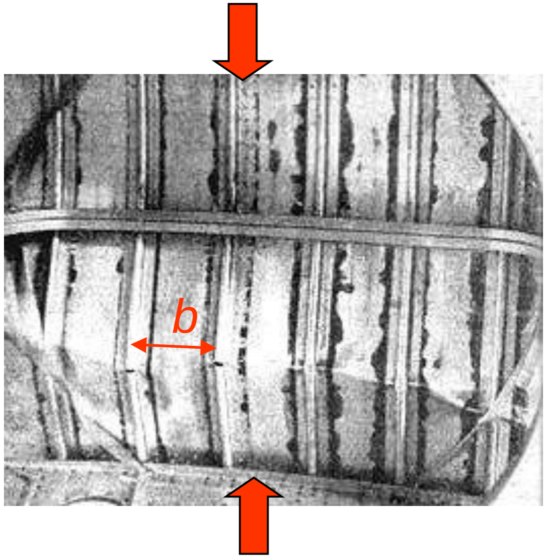
(? Assumption)

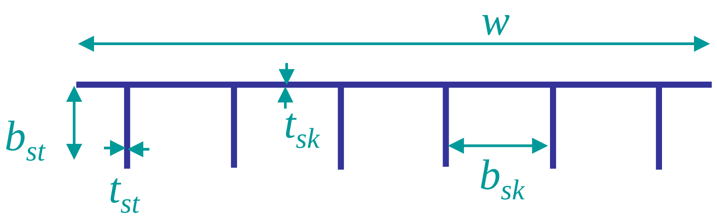


  - Take critical plate and evaluate k from plate analysis
    - k = 0.43, mode m=1
  - Deduce buckling load
    - $\sigma_{CR} = -k \dfrac{\pi^2 E h_0^2}{12\left(1-\nu^2\right) b^2}$
    - Check if lower than σ_p^0
  - This method is an approximation
    - Experimental determination

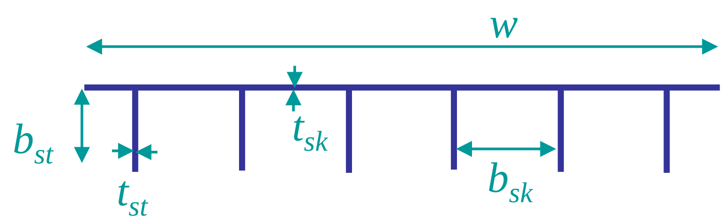


- Primary buckling of thin plates
  - We found
    - $\sigma_{CR} = -k\frac{\pi^2 E h_0^2}{12\left(1-\nu^2\right)b^2}$
    - With k ~ constant for a/b >3
  - In order to increase the buckling stress
    - Increase h_0/b ratio, or
    - Use stiffeners to reduce effective b of skin



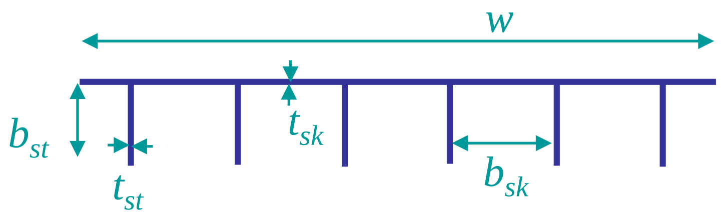


- Buckling modes of stiffened panels
  - Consider the section



  - Different buckling possibilities
    - High slenderness
      - Euler column (primary) buckling with cross-section depicted
    - Low slenderness and stiffeners with high degree of strength compared to skin
      - Structure can be assumed to be flat plates
        - » Of width b_{sk}
        - » Simply supported by the (rigid) stringers
      - Structure too heavy
    - More efficient structure if buckling occurs in stiffeners and skin at the same time
      - Closely spaced stiffeners of comparable thickness to the skin
      - Warning: both buckling modes could interact and reduces critical load
      - Section should be considered as a whole unit
      - Prediction of critical load relies on assumptions and semi-empirical methods
  - Skin can also buckled between the rivets

- A simple method to determine buckling



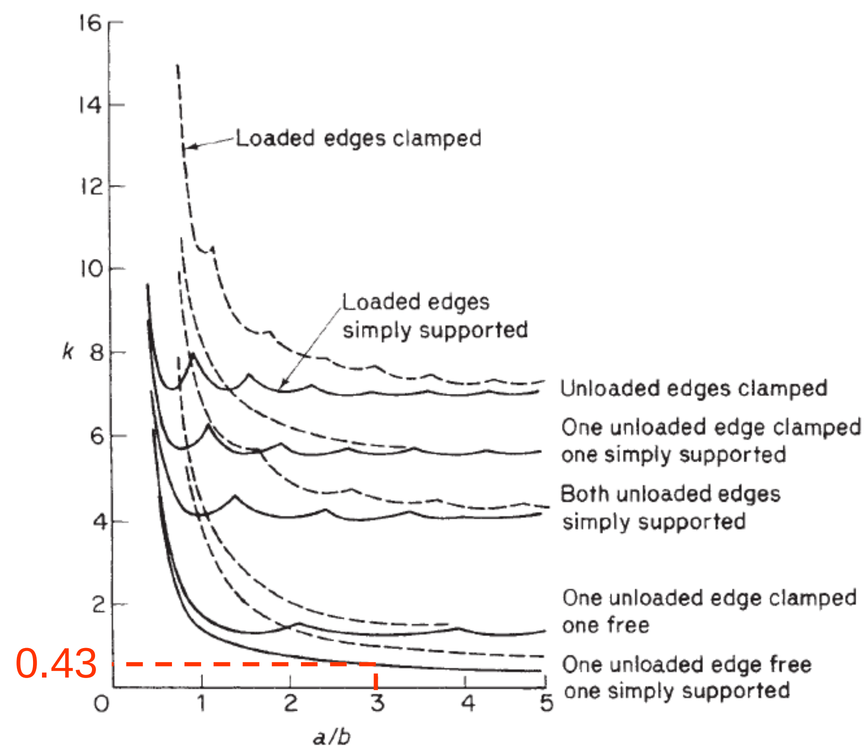
  - First check Euler primary buckling: $\sigma_{CR,E} = -\pi^2 E \frac{r^2}{l_e^2}$
  - Buckling of a skin panel
    - Simply supported on 4 edges
    - Assumed to remain elastic

$$\sigma_{CR,sk} = -4 \frac{\pi^2 E t_{sk}^2}{12\left(1-\nu^2\right) b_{sk}^2}$$

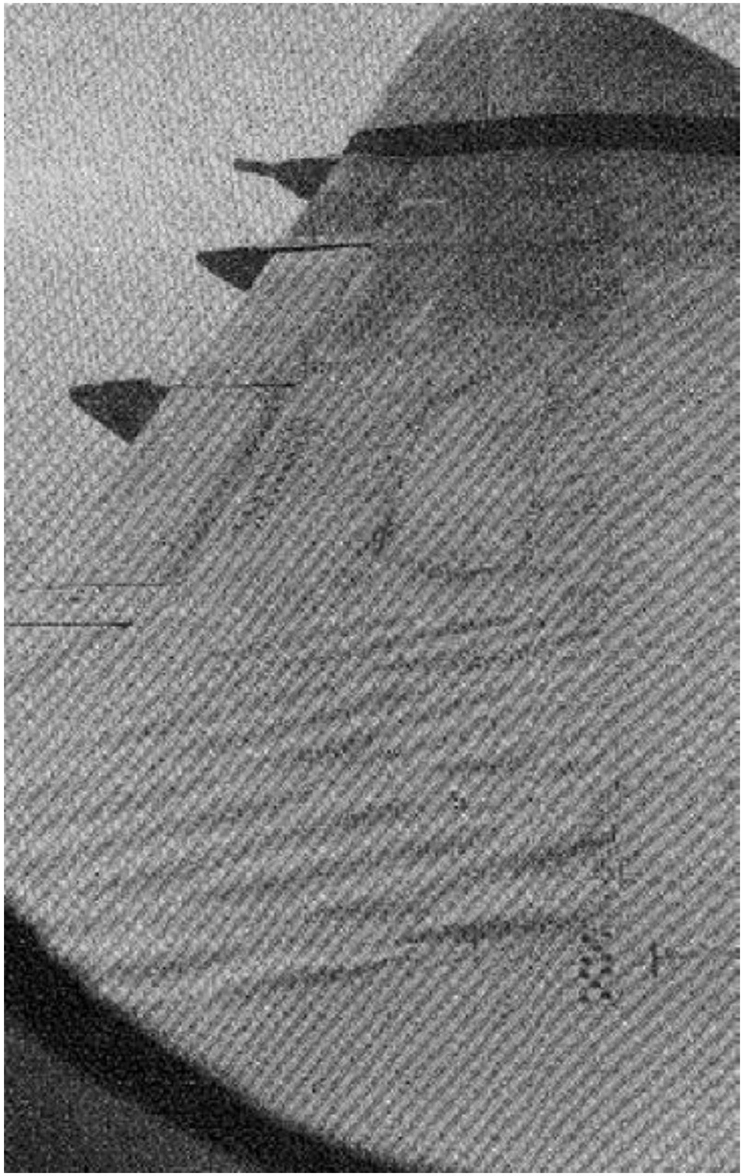
  - Buckling of a stiffener
    - Simply supported on 3 edges & 1 edge free
    - Assumed to remain elastic

$$\sigma_{CR,st} = -0.43 \frac{\pi^2 E t_{st}^2}{12\left(1-\nu^2\right) b_{st}^2}$$

  - Take lowest one (in absolute value)



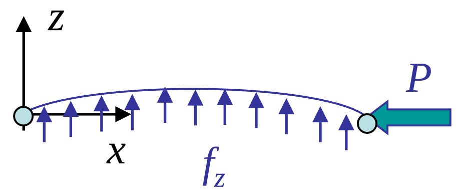
- Shearing instability
  - Shearing
    - Produces compression in the skin
    - Leads to wrinkles
  - The structure keeps some stiffness
  - Picture: Wing of a Boeing stratocruiser



References

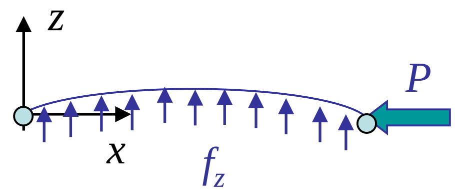
- Lecture notes
  - Aircraft Structures for engineering students, T. H. G. Megson, Butterworth-Heinemann, An imprint of Elsevier Science, 2003, ISBN 0 340 70588 4
- Other references
  - Books
    - Mécanique des matériaux, C. Massonet & S. Cescotto, De boek Université, 1994, ISBN 2-8041-2021-X

- Example
  - Uniform transverse load f_z
  - Pinned-pinned BCs
  - Maximum deflection?
  - Maximum momentum?



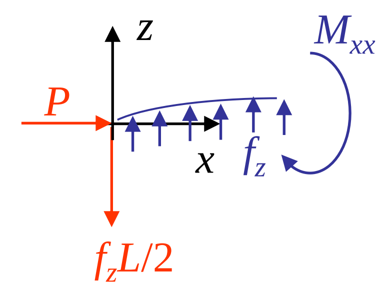
Annex 1: Transversely loaded columns

- Equation
  - Euler-Bernouilli



$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f\left(x\right)$$

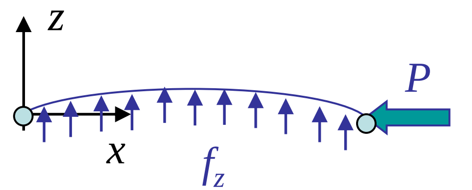
    - This assumes deformed configuration ~ initial configuration
    - But near buckling, due to the deflection, P is exerting a moment
    - So we cannot apply superposition principle as the axial loading also produces a deflection
  - Going back to bending equation



    - $$-EI_{yy}\boldsymbol{u}_{z,xx} = M_{xx} = P\boldsymbol{u}_z - \int_0^x x' f_z dx' + x\frac{f_z L}{2}$$
$$= P\boldsymbol{u}_z - f_z\frac{x^2 - Lx}{2}$$

$$\Longrightarrow \boldsymbol{u}_{z,xx} + \frac{P}{EI_{yy}}\boldsymbol{u}_z = \frac{f_z}{2EI_{yy}}\left(x^2 - Lx\right)$$

- Solution
  - Going back to bending equation
    - $\boldsymbol{u}_{z,xx} + \frac{P}{EI_{yy}}\boldsymbol{u}_z = \frac{f_z}{2EI_{yy}}\left(x^2 - Lx\right)$



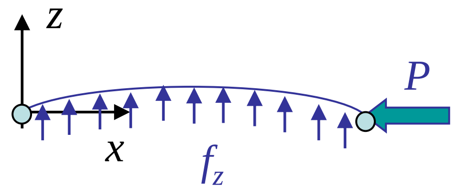
  - General solution
    - $\boldsymbol{u}_z = C_1 \cos\omega x + C_2 \sin\omega x + \frac{f_z}{2P}\left(x^2 - Lx - \frac{2}{\omega^2}\right)$ with $\omega = \sqrt{\frac{P}{EI_{yy}}}$
    - BC at $x = 0$: $C_1 = \frac{f_z}{\omega^2 P}$
    - BC at $x = L$: $0 = \frac{f_z}{\omega^2 P}\cos\omega L + C_2 \sin\omega L - \frac{f_z}{P\omega^2}$

$$\Longrightarrow C_2 = \frac{f_z}{\omega^2 P \sin\omega L}\left(1 - \cos\omega L\right)$$

  - Deflection $\boldsymbol{u}_z = \frac{f_z}{\omega^2 P}\left[\cos\omega x + \frac{1-\cos\omega L}{\sin\omega L}\sin\omega x\right] + \frac{f_z}{2P}\left(x^2 - Lx - \frac{2}{\omega^2}\right)$
    - Deflection and momentum are maximum at $x = L/2$

Annex 1: Transversely loaded columns

- Maximum deflection
  - Deflection is maximum at $x = L/2$



$$\boldsymbol{u}_z = \frac{f_z}{\omega^2 P}\left[\cos\omega x + \frac{1-\cos\omega L}{\sin\omega L}\sin\omega x\right] + \frac{f_z}{2P}\left(x^2 - Lx - \frac{2}{\omega^2}\right)$$

$$\Rightarrow \max_x \boldsymbol{u}_z = \frac{f_z}{\omega^2 P}\left[\cos\frac{\omega L}{2} + \frac{1-\cos\omega L}{\sin\omega L}\sin\frac{\omega L}{2}\right] + \frac{f_z}{2P}\left(-\frac{L^2}{4} - \frac{2}{\omega^2}\right)$$

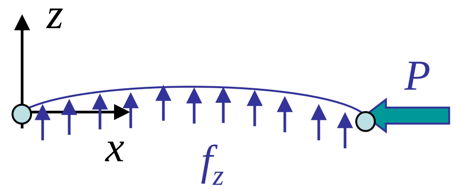
$$\Rightarrow \max_x \boldsymbol{u}_z = \frac{f_z}{\omega^2 P}\left[\cos\frac{\omega L}{2} + \frac{\cancel{2}\,\sin^{\cancel{2}}\frac{\omega L}{2}}{\cancel{2}\,\cancel{\sin\frac{\omega L}{2}}\cos\frac{\omega L}{2}}\sin\cancel{\frac{\omega L}{2}} - 1\right] - \frac{f_z L^2}{8P}$$

$$\Rightarrow \max_x \boldsymbol{u}_z = \frac{f_z}{\omega^2 P}\left[\frac{1}{\cos\frac{\omega L}{2}} - 1\right] - \frac{f_z L^2}{8P}$$

$$\left\{\begin{array}{l} P_{CR} = \dfrac{\pi^2 E I_{\min}}{L^2} \\ \omega = \sqrt{\dfrac{P}{E I_{yy}}} \end{array}\right. \Rightarrow \max_x \boldsymbol{u}_z = \frac{f_z L^2 P_{CR}}{\pi^2 P^2}\left[\frac{1}{\cos\frac{\sqrt{\frac{P}{P_{CR}}}\pi}{2}} - 1\right] - \frac{f_z L^2}{8P}$$

$$\Rightarrow \max_x \boldsymbol{u}_z = \frac{f_z L^4 P_{CR}^2}{\pi^4 E I_{yy} P^2}\left[\frac{1}{\cos\frac{\sqrt{\frac{P}{P_{CR}}}\pi}{2}} - 1\right] - \frac{f_z L^4 P_{CR}}{8\pi^2 E I_{yy} P}$$

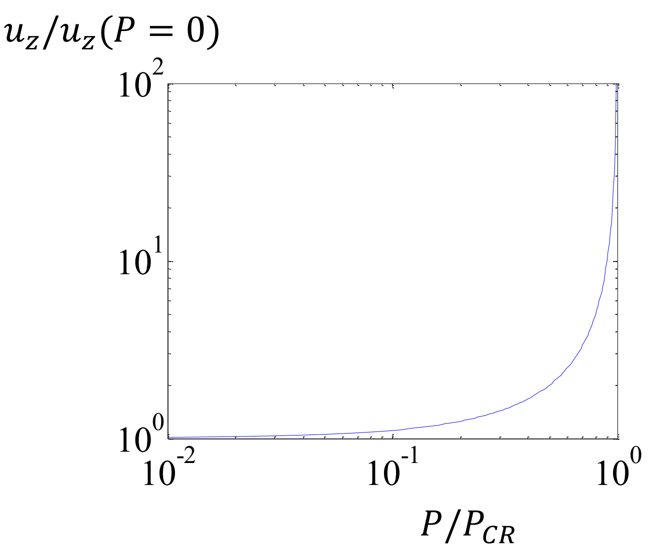
- Maximum deflection (2)
  - Deflection is maximum at $x = L/2$ (2)



$$\max_x \boldsymbol{u}_z = \frac{f_z L^4 P_{CR}^2}{\pi^4 E I_{yy} P^2}\left[\frac{1}{\cos\frac{\sqrt{\frac{P}{P_{CR}}}\pi}{2}} - 1\right] - \frac{f_z L^4 P_{CR}}{8\pi^2 E I_{yy} P}$$

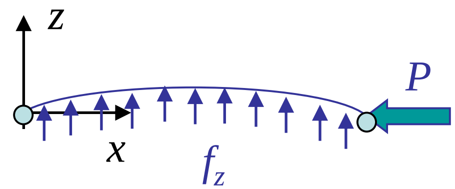
- From Euler-Bernoulli theory $\max_x \boldsymbol{u}_z\,(P=0) = \frac{5 f_z L^4}{384 E I_{yy}}$

$$\Rightarrow \frac{\max_x \boldsymbol{u}_z}{\max_x \boldsymbol{u}_z\,(P=0)} = \frac{384 P_{CR}^2}{5\pi^4 P^2}\left[\frac{1}{\cos\frac{\sqrt{\frac{P}{P_{CR}}}\pi}{2}} - 1\right] - \frac{384 P_{CR}}{40\pi^2 P}$$



- As for plates, compression induces bending due to the deflection (second order theory)

- Maximum moment
  - Maximum moment at $x = L/2$
    - $\max_x M_{xx} = P\boldsymbol{u}_z\left(\frac{L}{2}\right) + f_z\frac{L^2}{8}$
    - With $\max_x \boldsymbol{u}_z = \frac{f_z}{\omega^2 P}\left[\frac{1}{\cos\frac{\omega L}{2}} - 1\right] - \frac{f_z L^2}{8P}$



$$\Rightarrow \max_x M_{xx} = \frac{f_z}{\omega^2}\left[\frac{1}{\cos\frac{\omega L}{2}} - 1\right]$$

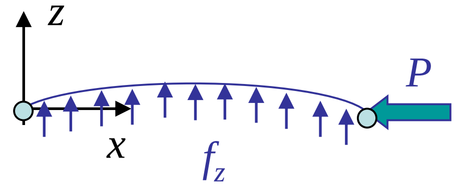
$$\begin{cases} P_{CR} = \frac{\pi^2 E I_{\min}}{L^2} \\ \omega = \sqrt{\frac{P}{EI_{yy}}} \end{cases}$$

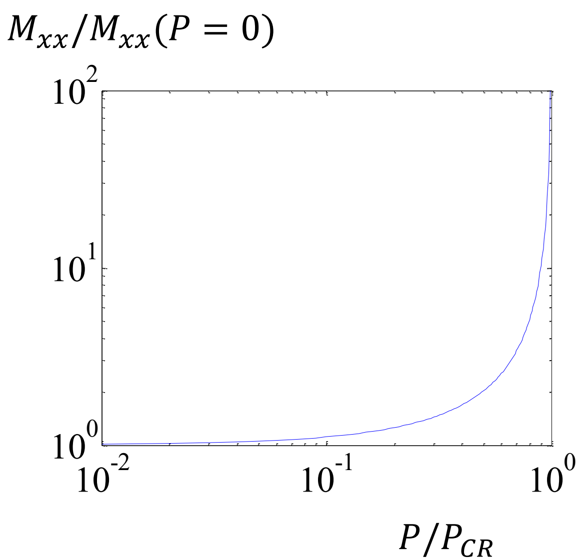
$$\Rightarrow \max_x M_{xx} = \frac{f_z P_{CR} L^2}{\pi^2 P}\left[\frac{1}{\cos\frac{\sqrt{\frac{P}{P_{CR}}}\pi}{2}} - 1\right]$$

$$\Rightarrow \frac{\max_x M_{xx}}{\max_x M_{xx}\,(P=0)} = \frac{8P_{CR}}{\pi^2 P}\left[\frac{1}{\cos\frac{\sqrt{\frac{P}{P_{CR}}}\pi}{2}} - 1\right]$$

- Maximum moment (3)
  - Maximum moment at $x = L/2$ (2)

$$\frac{\max_x M_{xx}}{\max_x M_{xx}\,(P=0)} = \frac{8P_{CR}}{\pi^2 P}\left[\frac{1}{\cos\frac{\sqrt{\frac{P}{P_{CR}}}\pi}{2}} - 1\right]$$



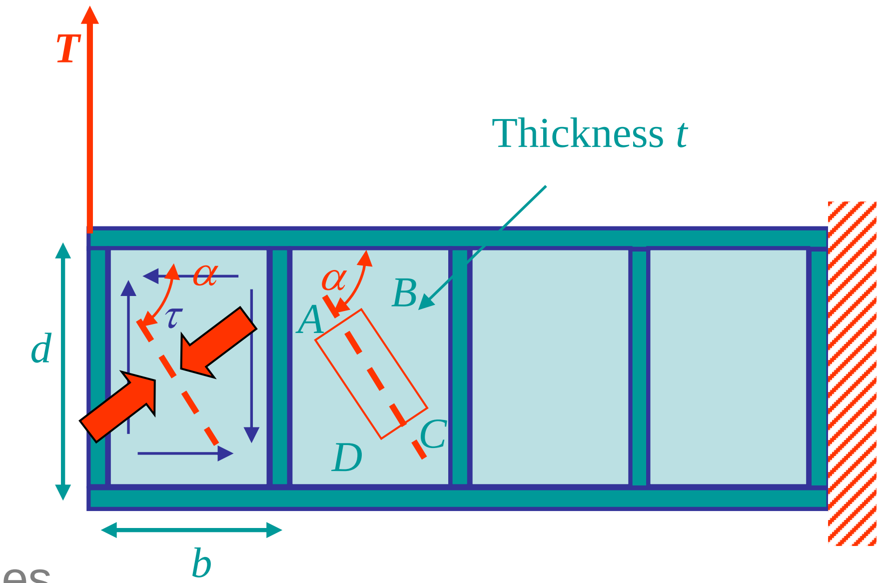


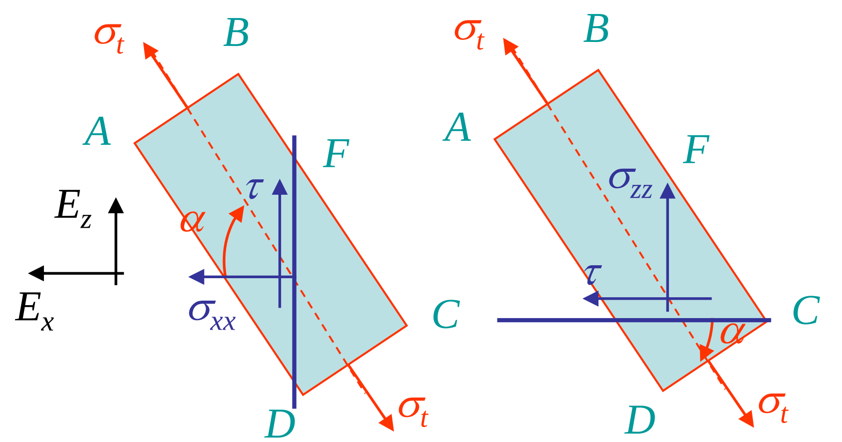
- Remark: for large deflections the bending equation which assumes linearity is no longer correct as curvature becomes $\kappa = \frac{\boldsymbol{u}_{z,xx}}{\sqrt{1+\boldsymbol{u}_{z,x}^2}^{\,3}}$

- Spar of wings
  - Usually not a simple beam
  - Assumptions before buckling:
    - Flanges resist direct stress only
    - Uniform shear stress in each web

$$\tau = \frac{T}{td}$$

  - The shearing produces compression in the web leading to α-inclined wrinkles
  - Assumptions during buckling
    - Due to the buckles the web can only carry a tensile stress σ_t in the wrinkle direction
    - This leads to a new distribution of stress in the web
      - σ_{xx} & σ_{zz}
      - Shearing τ





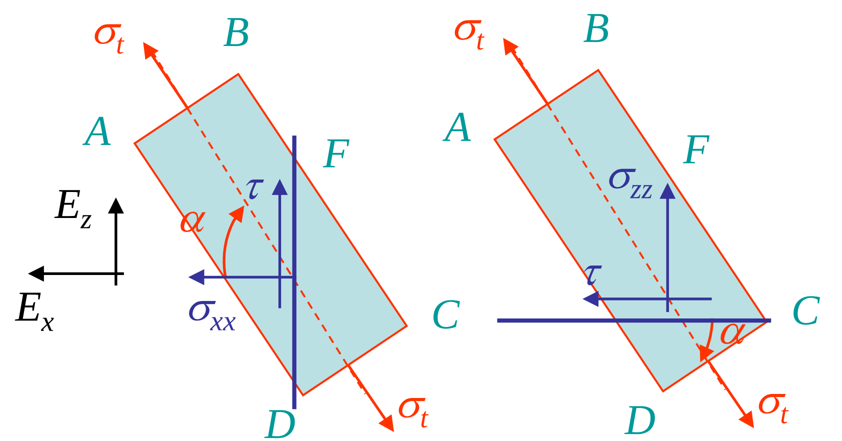
- New stress distribution
  - Use rotation tensor to compute in terms of σ_t

$$\begin{pmatrix} \boldsymbol{\sigma}_{xx} & \tau \\ \tau & \boldsymbol{\sigma}_{zz} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \sigma_t & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \boldsymbol{\sigma}_{xx} & \tau \\ \tau & \boldsymbol{\sigma}_{zz} \end{pmatrix} = \sigma_t \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \boldsymbol{\sigma}_{xx} & \tau \\ \tau & \boldsymbol{\sigma}_{zz} \end{pmatrix} = \sigma_t \begin{pmatrix} \cos^2\alpha & \frac{\sin 2\alpha}{2} \\ \frac{\sin 2\alpha}{2} & \sin^2\alpha \end{pmatrix}$$

$$\Rightarrow \begin{cases} \sigma_t = \dfrac{2\tau}{\sin 2\alpha} \\ \boldsymbol{\sigma}_{xx} = \dfrac{\tau}{\tan\alpha} \\ \boldsymbol{\sigma}_{zz} = \tau \tan\alpha \end{cases}$$



- New stress distribution (2)
  - From $\begin{cases} \sigma_t = \dfrac{2\tau}{\sin 2\alpha} \\ \boldsymbol{\sigma}_{xx} = \dfrac{\tau}{\tan\alpha} \\ \boldsymbol{\sigma}_{zz} = \tau \tan\alpha \end{cases}$
  - Shearing by vertical equilibrium

$$\Longrightarrow \tau = \frac{T}{td}$$

  - Loading in flanges
    - Moment balance around bottom flange

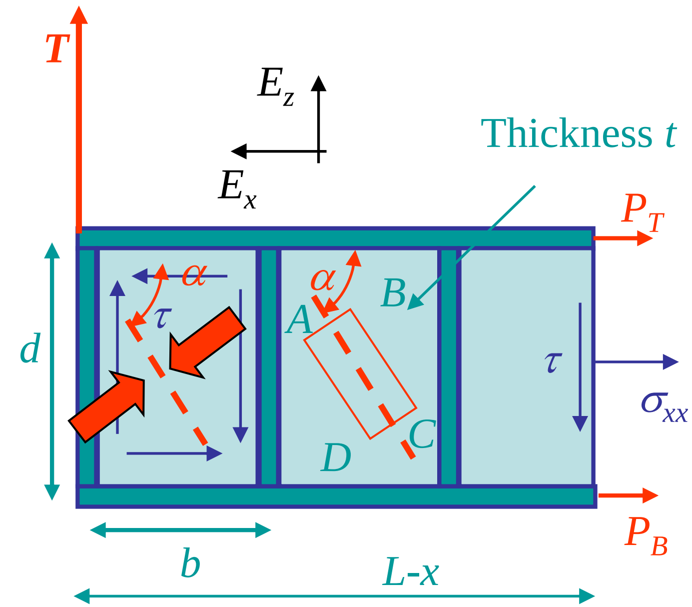
$$T\,(L-x) + P_T d + \frac{td^2}{2}\boldsymbol{\sigma}_{xx} = 0$$

$$\Longrightarrow P_T = -T\frac{(L-x)}{d} - \frac{T}{2\tan\alpha}$$

    - Horizontal equilibrium

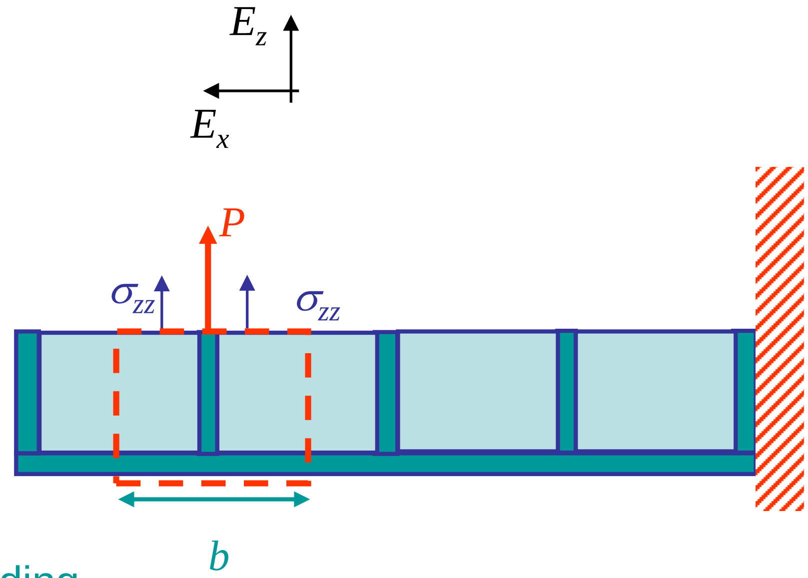
$$P_T + P_B + td\boldsymbol{\sigma}_{xx} = 0$$

$$\Longrightarrow P_B = T\frac{(L-x)}{d} - \frac{T}{2\tan\alpha}$$



- New stress distribution (3)
  - From

$$\left\{\begin{array}{l}\sigma_t = \dfrac{2\tau}{\sin 2\alpha}\\ \boldsymbol{\sigma}_{xx} = \dfrac{\tau}{\tan\alpha}\\ \boldsymbol{\sigma}_{zz} = \tau\tan\alpha\\ \tau = \dfrac{T}{td}\end{array}\right.$$



  - Loading in stiffeners
    - Assumption: each stiffener carries the loading of half of the adjacent panels

$$P = -\boldsymbol{\sigma}_{zz} bt \Longrightarrow P = -T\frac{b}{d}\tan\alpha$$

    - Stiffeners can be subject to Euler buckling if this load is too high
      - Tests show that for these particular BCs, the equivalent length reads

$$l_e = \begin{cases} \frac{d}{\sqrt{4-2\frac{b}{d}}} & \text{if } b < \frac{3}{2}d \\ d & \text{if } b > \frac{3}{2}d \end{cases}$$

Annex 2: Buckling of stiffener/web constructions

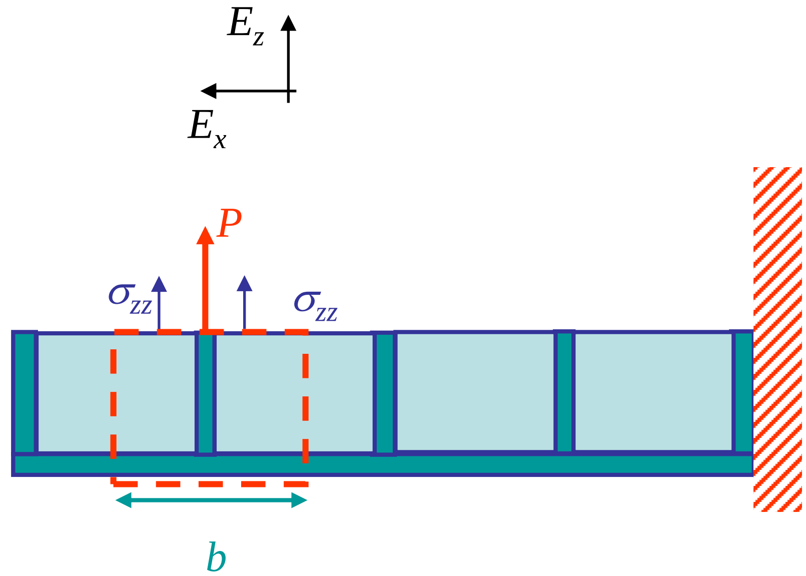
- New stress distribution (4)
  - From

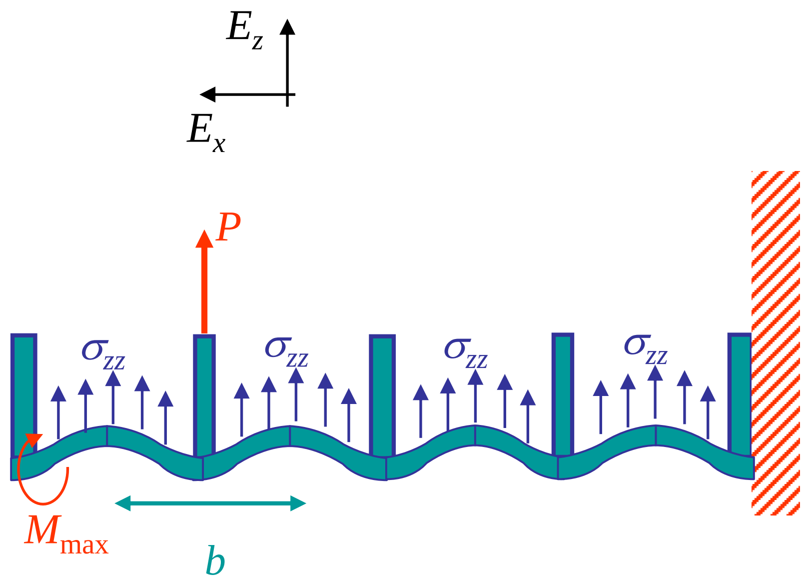
$$\boldsymbol{\sigma}_{zz} = \frac{T}{td}\tan\alpha$$

  - Bending in flanges
    - In addition to the flanges loading P_B & P_T
    - Stress σ_{zz} produces bending
      - Stiffeners constraint rotation
      - Maximum moment at stiffeners
    - Using table for double cantilever beams

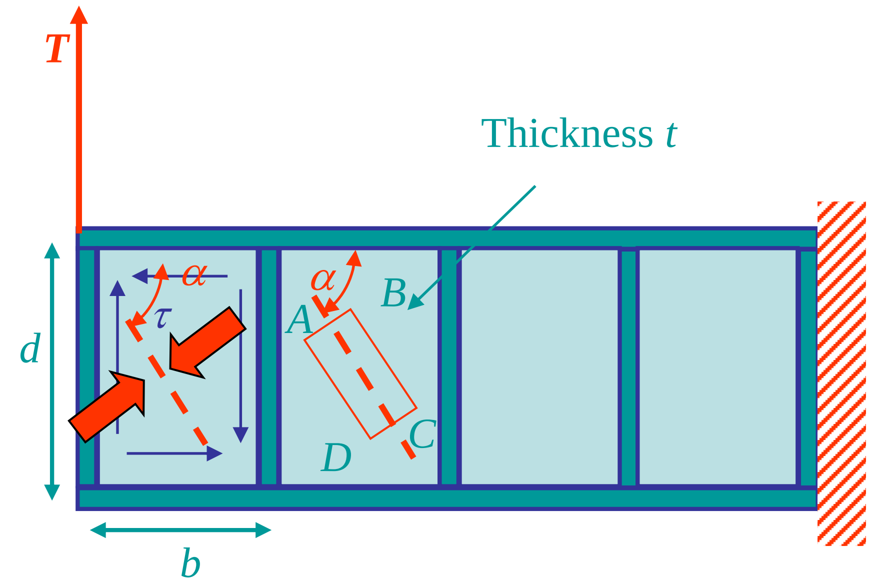
$$M_{\max} = \frac{\boldsymbol{\sigma}_{zz} t b^2}{12}$$

$$\Rightarrow M_{\max} = \frac{Tb^2}{12d}\tan\alpha$$





- Wrinkles angle
  - The angle is the one which minimizes the deformation energy of
    - Webs
    - Flanges
    - Stiffeners
  - If flanges and stiffeners are rigid
    - We should get back to $\alpha = 45°$
  - Because of the deformation of flanges and stiffeners $\alpha < 45°$
    - Empirical formula for uniform material



$$\tan^2\alpha = \frac{\sigma_t - \frac{P_F}{A_F}}{\sigma_t - \frac{P_S}{A_S}}$$

$\frac{P_F}{A_F}$ → Load in flange / Flange section

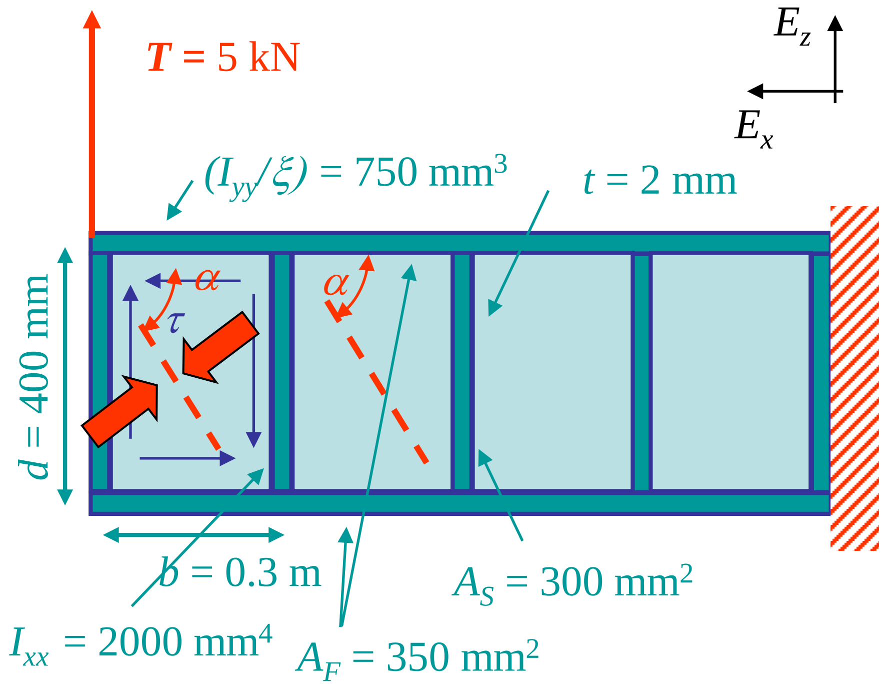
$\frac{P_S}{A_S}$ → Load in stiffener / Stiffener section

  - As $P_T = -T\dfrac{(L-x)}{d} - \dfrac{T}{2\tan\alpha}$ non constant, α non constant

    - Another empirical formula

$$\tan^4\alpha = \frac{1 + \frac{td}{2A_F}}{1 + \frac{tb}{A_S}}$$

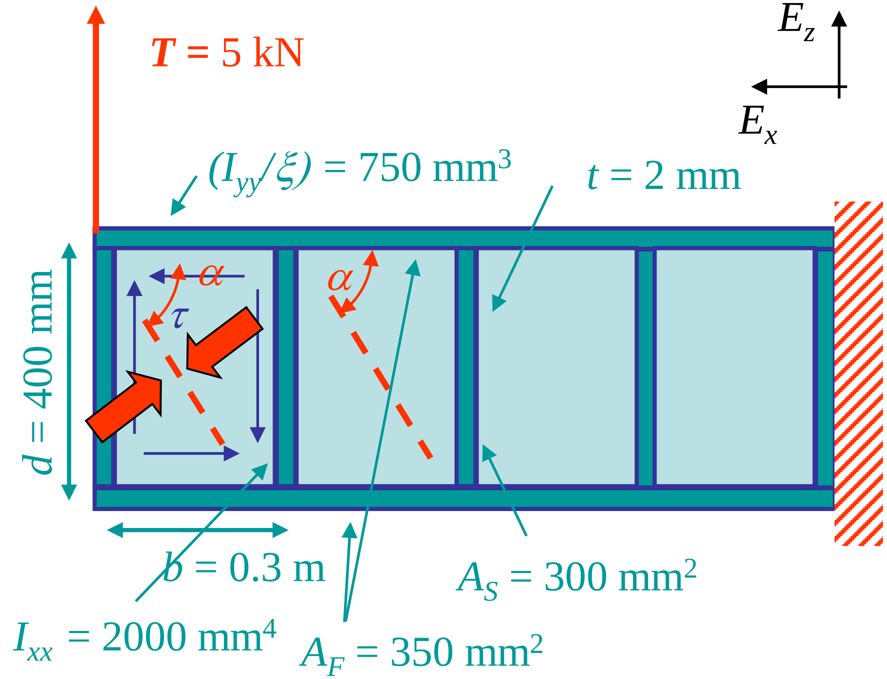
- Example
  - Web/stiffener construction
    - 2 similar flanges
    - 5 similar stiffeners
    - 4 similar webs
    - Same material
      - E = 70 GPa
  - Stress state?



- Wrinkles orientation
  - $\tan^4\alpha = \dfrac{1+\frac{td}{2A_F}}{1+\frac{tb}{A_S}}$

$$\Rightarrow \tan^4\alpha = \frac{1+\frac{0.002\ 0.4}{2\ 0.00035}}{1+\frac{0.002\ 0.3}{0.0003}} = 0.714$$

$$\Rightarrow \alpha = 42.6^o$$



- Stress in top flange (> than in bottom one)
  - 1st contribution: uniform compression

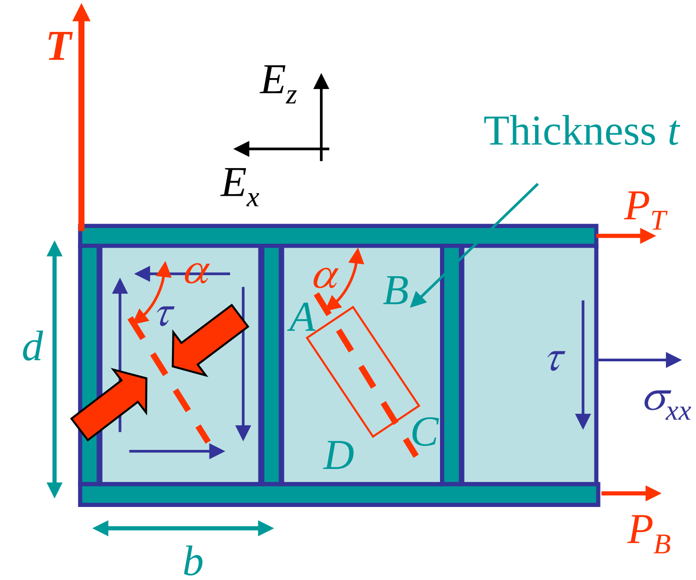
$$P_T = -T\frac{(L-x)}{d} - \frac{T}{2\tan\alpha}$$

  - Maximum at $x = 0$

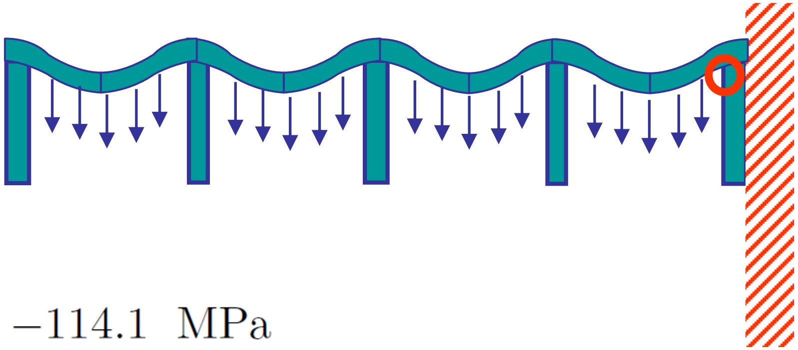
$$P_T = -T\frac{L}{d} - \frac{T}{2\tan\alpha}$$

$$P_T = -5\,10^3\frac{1.2}{0.4} - \frac{5\,10^3}{2\;0.913} = -17.7 \;\; \text{kN}$$

$$\Longrightarrow \sigma_T^{(1)} = \frac{P_T}{A_F} = \frac{17700}{0.00035} = -50.7 \;\; \text{MPa}$$



  - 2nd contribution: bending
    - Maximum at stiffener $\Longrightarrow$ at x=0



$$M_{\max} = \frac{5\,10^3 0.3^2}{12\;0.4}0.913 = 85.6 \;\; \text{N}\cdot\text{m}$$

$$\Longrightarrow \sigma_T^{(2)} = -M_{\max}\frac{\xi}{I_{yy}} = -\frac{85.6}{750}\,10^9 = -114.1 \;\; \text{MPa}$$

  - Maximum compressive stress

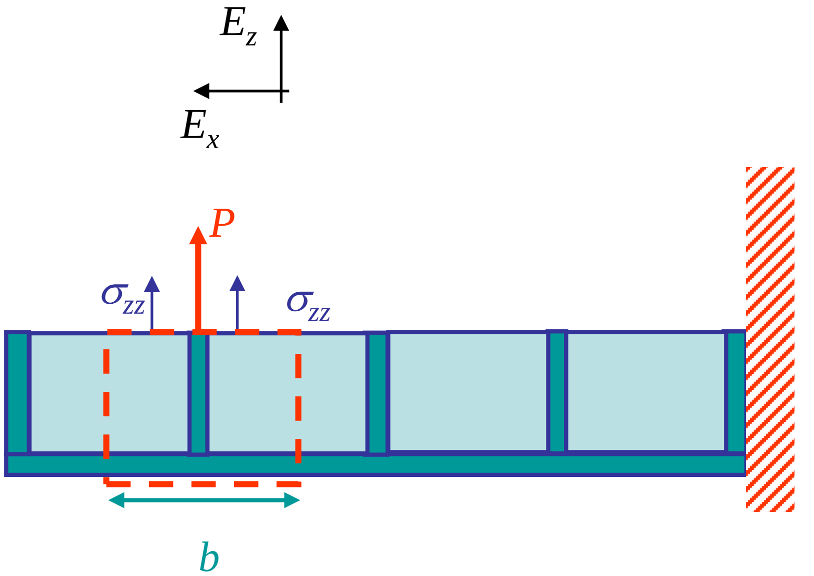
$$\sigma_{\max\ \text{compr}} = -50.7 - 114.1 = -164.8 \;\; \text{MPa}$$

- Stiffeners
  - Loading $P = -T\frac{b}{d}\tan\alpha$

    $$\Rightarrow P = -5\,10^3\frac{0.3}{0.4}0.913 = -3.4 \text{ kN}$$

  - Buckling?
    - As b < 3/2 d: $l_e = \frac{d}{\sqrt{4 - 2\frac{b}{d}}}$

$$P_{CR} = \frac{\pi^2\; 70\; 10^9\; 2\; 10^{-9}}{0.253^2} = 21.6 \text{ kN}$$



- Assuming centroid of stiffener lies in web's plane
  - We can use Euler critical load

$$P_{CR} = \frac{\pi^2 EI}{l_e^2}$$

$$\Rightarrow l_e = \frac{0.4}{\sqrt{4 - 2\frac{0.3}{0.4}}} = 0.253 \text{ m}$$

- No buckling