University of Liège
Aerospace & Mechanical Engineering

# Aircraft Structures
# Laminated Composites Idealization

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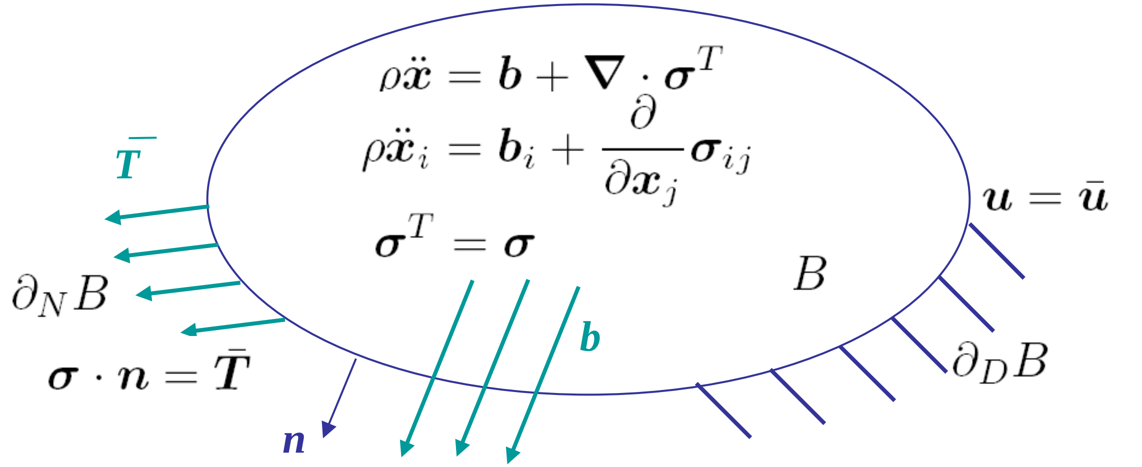
# Elasticity

- Balance of body $B$
  - Momenta balance
    - Linear
    - Angular
  - Boundary conditions
    - Neumann
    - Dirichlet




- Small deformations with linear elastic, homogeneous & isotropic material
  - (Small) Strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2} \left( \boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla} \right)$, or $\begin{cases} \boldsymbol{\varepsilon}_{ij} = \frac{1}{2} \left( \frac{\partial}{\partial \boldsymbol{x}_i} \boldsymbol{u}_j + \frac{\partial}{\partial \boldsymbol{x}_j} \boldsymbol{u}_i \right) \\ \boldsymbol{\varepsilon}_{ij} = \frac{1}{2} \left( \boldsymbol{u}_{j,i} + \boldsymbol{u}_{i,j} \right) \end{cases}$
  - Hooke's law $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon}$ , or $\boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl} \boldsymbol{\varepsilon}_{kl}$

    with $$\mathcal{H}_{ijkl} = \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda = K - 2\mu/3} \delta_{ij}\delta_{kl} + \underbrace{\frac{E}{1+\nu}}_{2\mu} \left( \frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk} \right)$$
  - Inverse law $\boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma}$

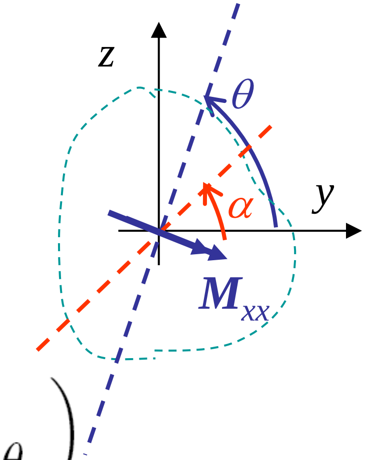
    with $$\mathcal{G}_{ijkl} = \frac{1+\nu}{E} \left( \frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk} \right) - \frac{\nu}{E}\delta_{ij}\delta_{kl}$$

- General expression for unsymmetrical beams
  - Stress $\boldsymbol{\sigma}_{xx} = \kappa E z \cos\alpha - \kappa E y \sin\alpha$

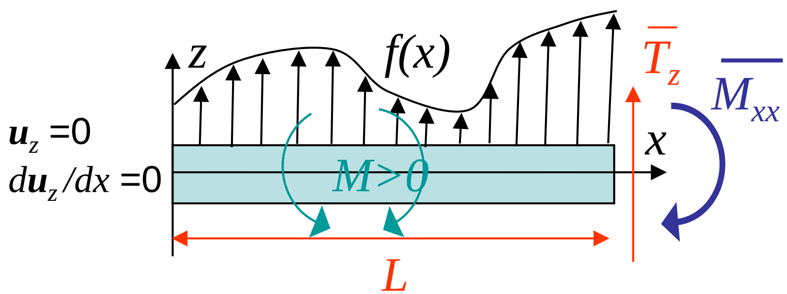
    With $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{\kappa E} \begin{pmatrix} I_{yy} & -I_{yz} \\ -I_{yz} & I_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$
  - Curvature

$$\begin{pmatrix} -\boldsymbol{u}_{z,xx} \\ \boldsymbol{u}_{y,xx} \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{E\left(I_{yy}I_{zz} - I_{yz}I_{yz}\right)} \begin{pmatrix} I_{zz} & I_{yz} \\ I_{yz} & I_{yy} \end{pmatrix} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

  - In the principal axes $I_{yz} = 0$

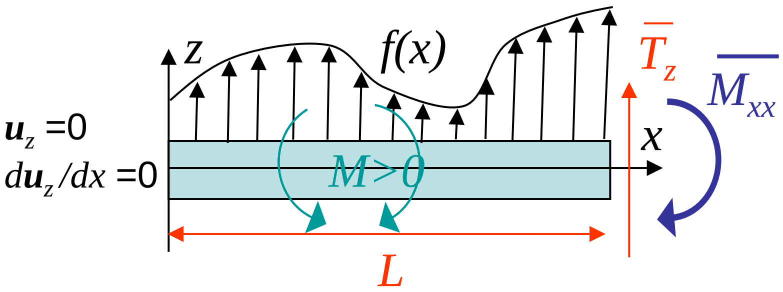



- Euler-Bernoulli equation in the principal axis
  - $\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f(x)$ for $x$ in [0 $L$]
  - BCs $\begin{cases} -\frac{\partial}{\partial x}\left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\Big|_{0,\,L} = \bar{T}_z\big|_{0,\,L} \\ -EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\Big|_{0,\,L} = \bar{M}_{xx}\big|_{0,\,L} \end{cases}$
  - Similar equations for $\boldsymbol{u}_y$

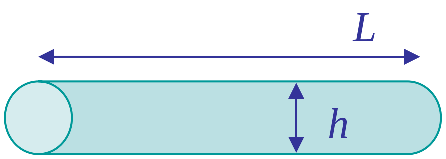


# Beam shearing: linear elasticity summary

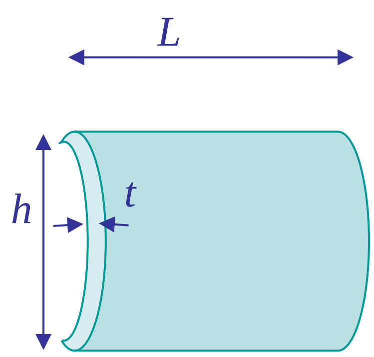
- General relationships
  - $$\begin{cases} f_z(x) = -\partial_x T_z = -\partial_{xx} M_y \\ f_y(x) = -\partial_x T_y = \partial_{xx} M_z \end{cases}$$




- Two problems considered
  - Thick symmetrical section



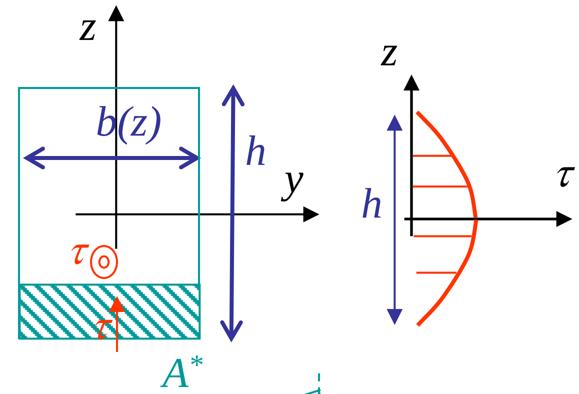

    - Shear stresses are small compared to bending stresses if $h/L \ll 1$
  - Thin-walled (unsymmetrical) sections

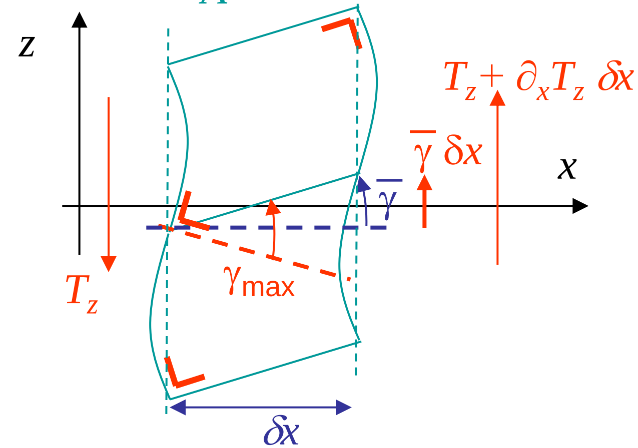


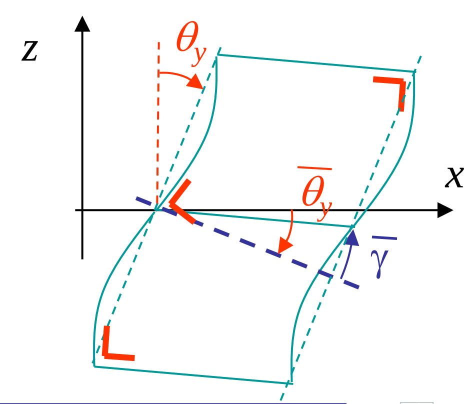

    - Shear stresses are not small compared to bending stresses
    - Deflection mainly results from bending stresses
    - 2 cases
      - Open thin-walled sections
        - » Shear = shearing through the shear center + torque
      - Closed thin-walled sections
        - » Twist due to shear has the same expression as torsion

# Beam shearing: linear elasticity summary

- Shearing of symmetrical thick-section beams
  - Stress $\boldsymbol{\sigma}_{zx} = -\frac{T_z S_n(z)}{I_{yy} b(z)}$
    - With $S_n(z) = \int_{A^*} z dA$
    - Accurate only if $h > b$
  - Energetically consistent averaged shear strain
    - $\bar{\gamma} = \frac{T_z}{A'\mu}$ with $A' = \frac{1}{\int_A \frac{S_n^2}{I_{yy}^2 b^2} dA}$
    - Shear center on symmetry axes
  - Timoshenko equations
    - $\bar{\gamma} = 2\bar{\boldsymbol{\varepsilon}}_{xz} = \frac{\partial \boldsymbol{u}_x}{\partial z} + \frac{\partial \boldsymbol{u}_z}{\partial x} = \theta_y + \partial_x \boldsymbol{u}_z$ & $\kappa = \frac{\partial \theta_y}{\partial x}$
    - On [0 $L$]: $\begin{cases} \frac{\partial}{\partial_x}\left(EI\frac{\partial \theta_y}{\partial x}\right) - \mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right) = 0 \\ \frac{\partial}{\partial x}\left(\mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right)\right) = -f \end{cases}$



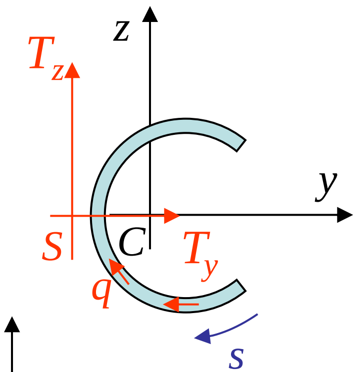



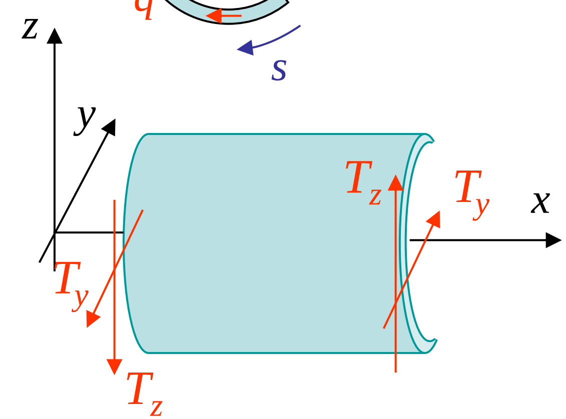
# Beam shearing: linear elasticity summary

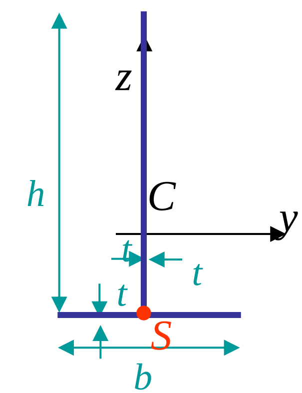
- Shearing of open thin-walled section beams
  - Shear flow $q = t\tau$
    - $$q(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s tz ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s ty ds'$$
    - In the principal axes

$$q(s) = -\frac{T_z}{I_{yy}}\int_0^s tz ds' - \frac{T_y}{I_{zz}}\int_0^s ty ds'$$

  - Shear center $S$
    - On symmetry axes
    - At walls intersection
    - Determined by momentum balance
  - Shear loads correspond to
    - Shear loads passing through the shear center &
    - Torque



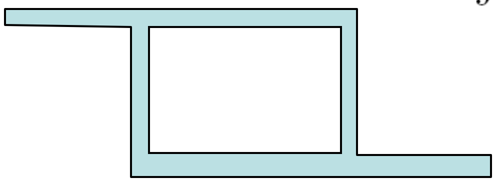
- Shearing of closed thin-walled section beams
  - Shear flow $q = t\tau$
    - $q(s) = q_o(s) + q(0)$
    - Open part (for anticlockwise of *q*, *s*)

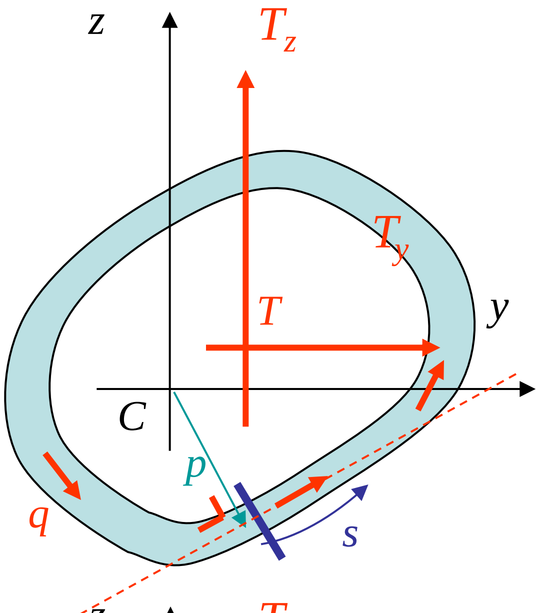
$$q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s t(s')\,z(s')\,ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s t(s')\,y(s')\,ds'$$

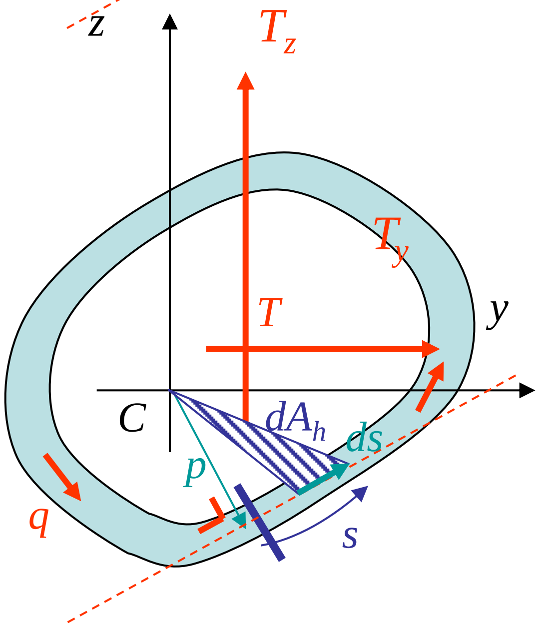
    - Constant twist part

$$q(s=0) = \frac{y_T T_z - z_T T_y - \oint p(s)\,q_o(s)\,ds}{2A_h}$$

    - The *q(0)* is related to the closed part of the section, but there is a $q_o(s)$ in the open part which should be considered for the shear torque $\oint p(s)\,q_o(s)\,ds$



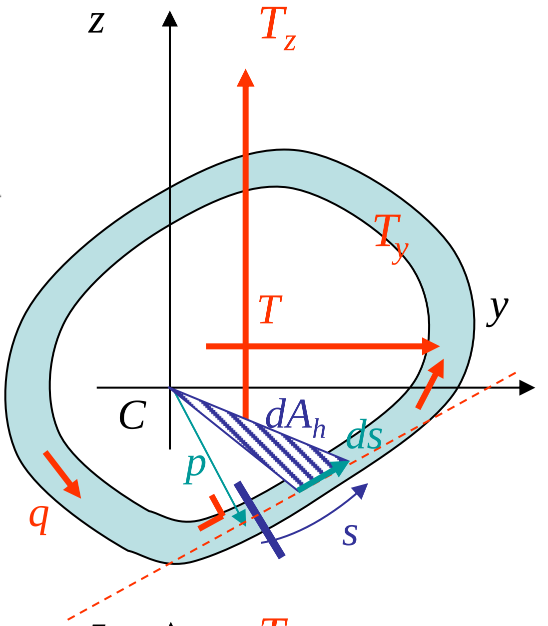


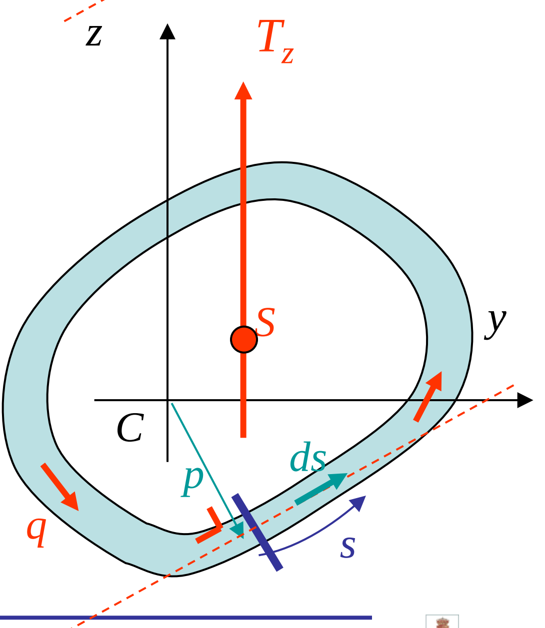



- Shearing of closed thin-walled section beams
  - Warping around twist center $R$
    - $$\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \int_0^s \frac{q}{\mu t} ds - \frac{1}{A_h} \oint \frac{q}{\mu t} ds \left\{ A_{Cp}(s) + \frac{z_R\left[y(s) - y(0)\right] - y_R\left[z(s) - z(0)\right]}{2} \right\}$$
    - With $\boldsymbol{u}_x(0) = \frac{\oint t \boldsymbol{u}_x(s)\, ds}{\oint t(s)\, ds}$
      - $\boldsymbol{u}_x(0)=0$ for symmetrical section if origin on the symmetry axis
  - Shear center $S$
    - Compute $q$ for shear passing thought $S$
    - Use

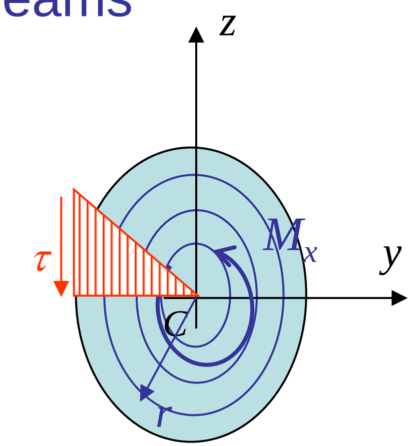
$$q(s=0) = \frac{y_T T_z - z_T T_y - \oint p(s)\, q_o(s)\, ds}{2A_h}$$

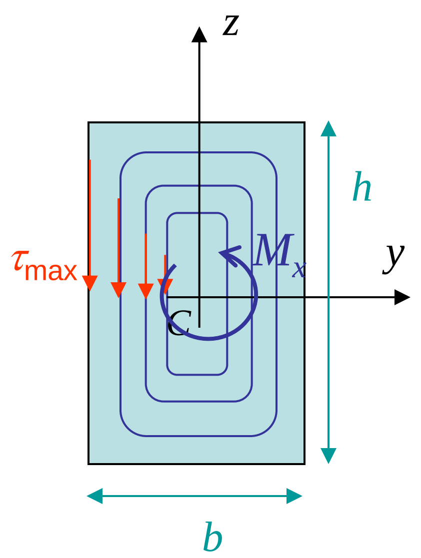
With point $S$=$T$



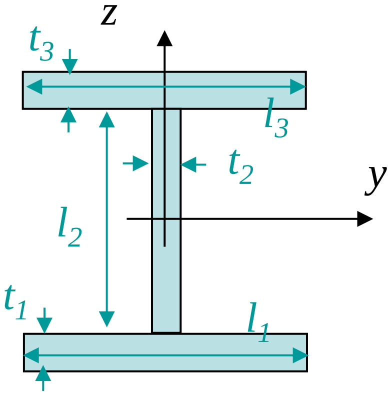
- Torsion of symmetrical thick-section beams
  - Circular section
    - $\tau = \mu\gamma = r\mu\theta_{,x}$
    - $C = \frac{M_x}{\theta_{,x}} = \int_A \mu r^2 dA$
  - Rectangular section
    - $\tau_{\max} = \frac{M_x}{\alpha h b^2}$
    - $C = \frac{M_x}{\theta_{,x}} = \beta h b^3 \mu$
    - If $h >> b$
      - $\tau_{xy} = 0$ & $\tau_{xz} = 2\mu y \theta_{,x}$
      - $\tau_{\max} = \frac{3M_x}{hb^2}$
      - $C = \frac{M_x}{\theta_{,x}} = \frac{hb^3\mu}{3}$

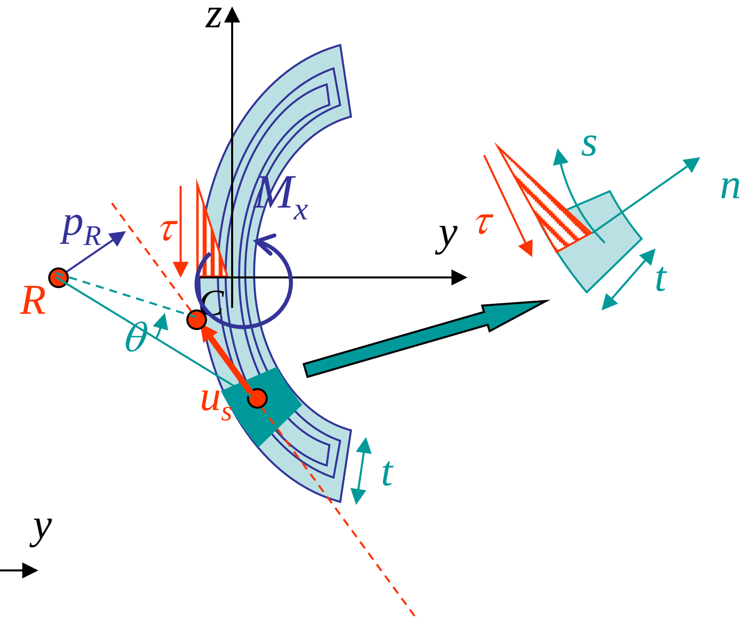



| $h/b$ | 1 | 1.5 | 2 | 4 | ∞ |
|---|---|---|---|---|---|
| $\alpha$ | 0.208 | 0.231 | 0.246 | 0.282 | 1/3 |
| $\beta$ | 0.141 | 0.196 | 0.229 | 0.281 | 1/3 |

- Torsion of open thin-walled section beams
  - Approximated solution for twist rate
    - Thin curved section
      - $\tau_{xs} = 2\mu n \theta_{,x}$
      - $C = \dfrac{M_x}{\theta_{,x}} = \dfrac{1}{3}\int \mu t^3 ds$
    - Rectangles
      - $\tau_{\max_i} = \mu t_i \theta_{,x}$
      - $\dfrac{M_x}{\theta_{,x}} = \sum_i \dfrac{l_i t_i^3 \mu}{3}$

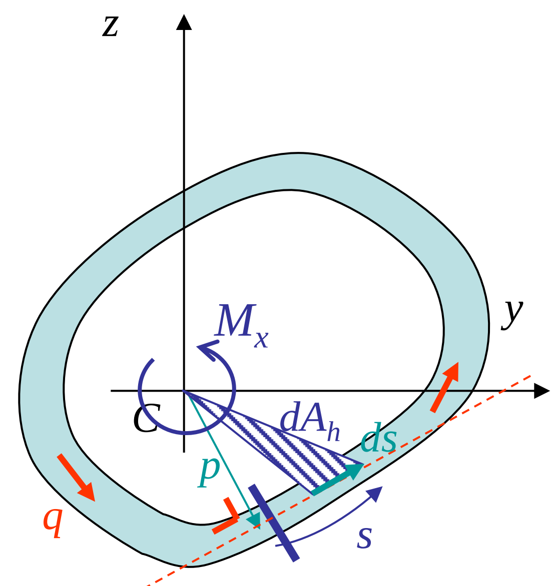


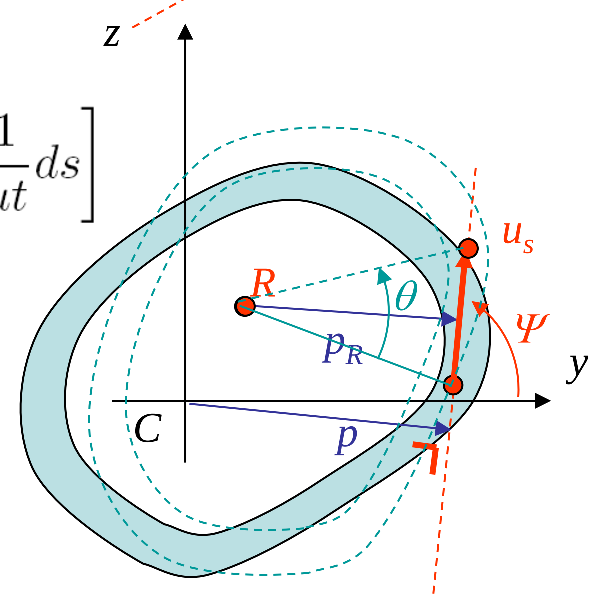

  - Warping of $s$-axis
    - $\boldsymbol{u}_x^s(s) = \boldsymbol{u}_x^s(0) - \theta_{,x}\int_0^s p_R ds' = \boldsymbol{u}_x^s(0) - 2A_{R_p}(s)\,\theta_{,x}$

- Torsion of closed thin-walled section beams
  - Shear flow due to torsion $M_x = 2A_h q$
  - Rate of twist
    - $$\theta_{,x} = \frac{M_x}{4A_h^2}\oint \frac{1}{\mu t} ds$$
    - Torsion rigidity for constant $\mu$

$$I_T = \frac{4A_h^2}{\oint \frac{1}{t} ds} \leq I_p = \int_A r^2 dA$$



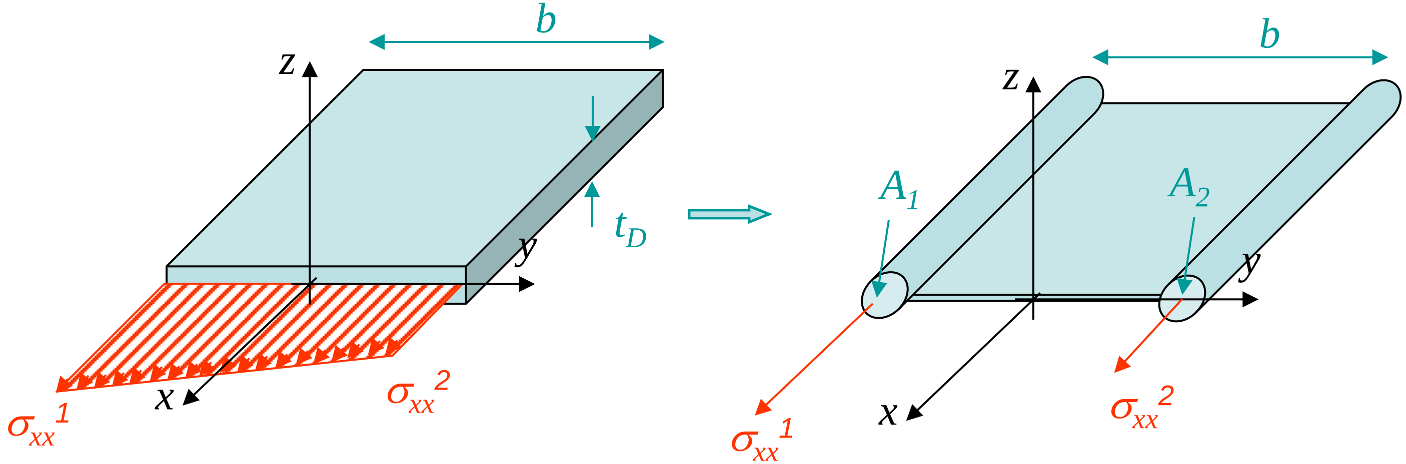

  - Warping due to torsion
    - $$\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \frac{M_x}{2A_h}\left[\int_0^s \frac{1}{\mu t} ds - \frac{A_{R_p}(s)}{A_h}\oint \frac{1}{\mu t} ds\right]$$
    - $A_{Rp}$ from twist center



# Structure idealization summary

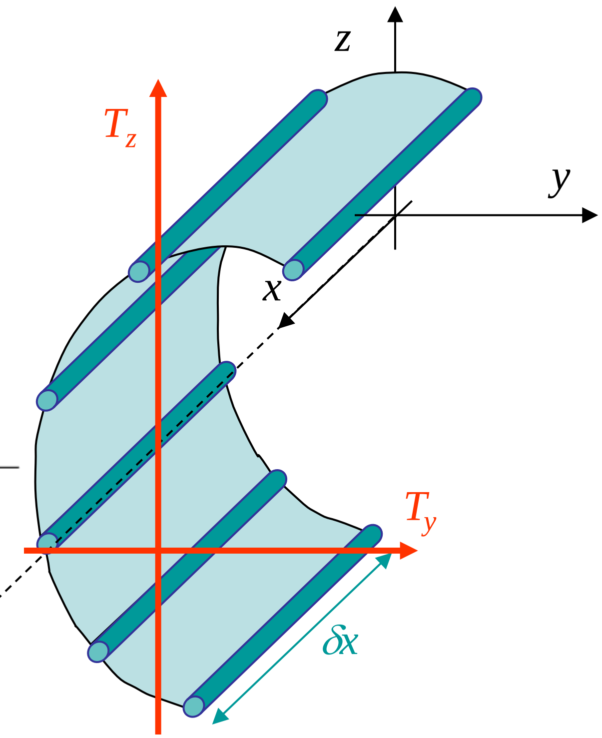
- Panel idealization
  - Booms' area **depending on loading**
    - For linear direct stress distribution

$$\begin{cases} A_1 = \dfrac{t_D b}{6}\left(2 + \dfrac{\boldsymbol{\sigma}_{xx}^2}{\boldsymbol{\sigma}_{xx}^1}\right) \\ A_2 = \dfrac{t_D b}{6}\left(2 + \dfrac{\boldsymbol{\sigma}_{xx}^1}{\boldsymbol{\sigma}_{xx}^2}\right) \end{cases}$$



- Consequence on bending
  - If Direct stress due to bending is carried by booms only
    - The position of the neutral axis, and thus the second moments of area
      - Refer to the direct stress carrying area only
      - Depend on the loading case only
- Consequence on shearing
  - Open part of the shear flux
    - Shear flux for open sections

$$q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\sigma} z ds + \sum_{i:\, s_i \leq s} z_i A_i\right] - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\sigma} y ds + \sum_{i:\, s_i \leq s} y_i A_i\right]$$



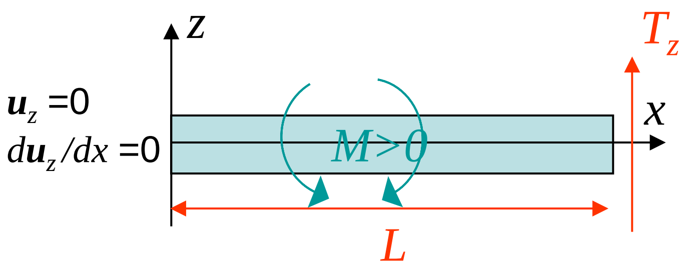

- Consequence on torsion
  - If no axial constraint
    - Torsion analysis does not involve axial stress
    - So torsion is unaffected by the structural idealization

- Virtual displacement
  - In linear elasticity the general formula of virtual displacement reads

$$\int_0^L \int_A \boldsymbol{\sigma}^{(1)} : \boldsymbol{\varepsilon} dA dx = P^{(1)} \Delta_P$$

- $\boldsymbol{\sigma}^{(1)}$ is the stress distribution corresponding to a (unit) load $P^{(1)}$
- $\Delta_P$ is the energetically conjugated displacement to $P$ in the direction of $P^{(1)}$ that corresponds to the strain distribution $\boldsymbol{\varepsilon}$

  - Example bending of semi cantilever beam




- $$\int_0^L \int_A \boldsymbol{\sigma}_{xx}^{(1)} \boldsymbol{\varepsilon}_{xx} dA dx = \Delta_P u$$

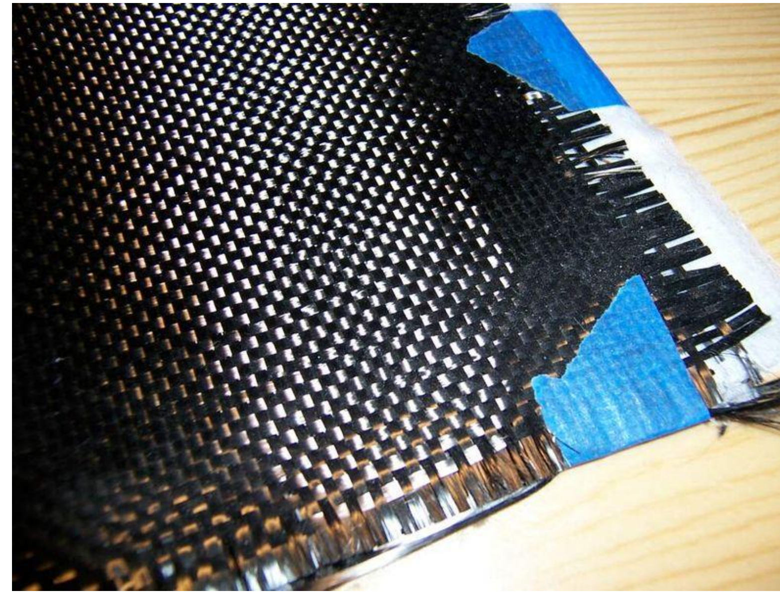
  - In the principal axes

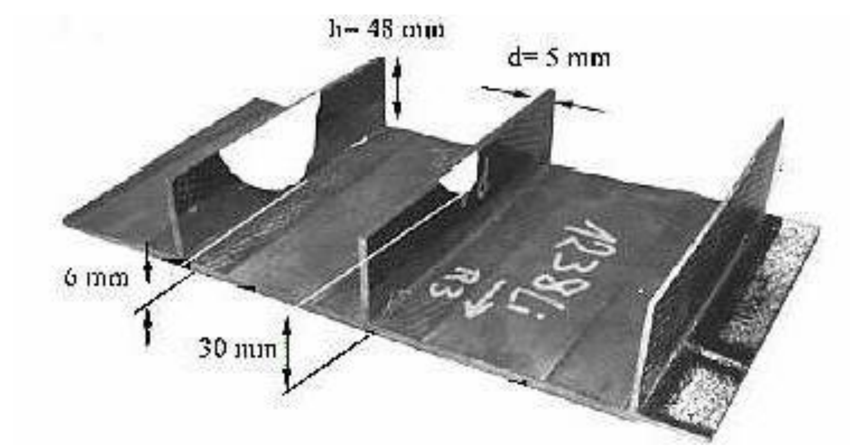
$$\Delta_P u = \frac{1}{E I_{yy} I_{zz}} \int_0^L \left\{ I_{zz} M_y^{(1)} M_y + I_{yy} M_z^{(1)} M_z \right\} dx$$

  - Example shearing of semi-cantilever beam

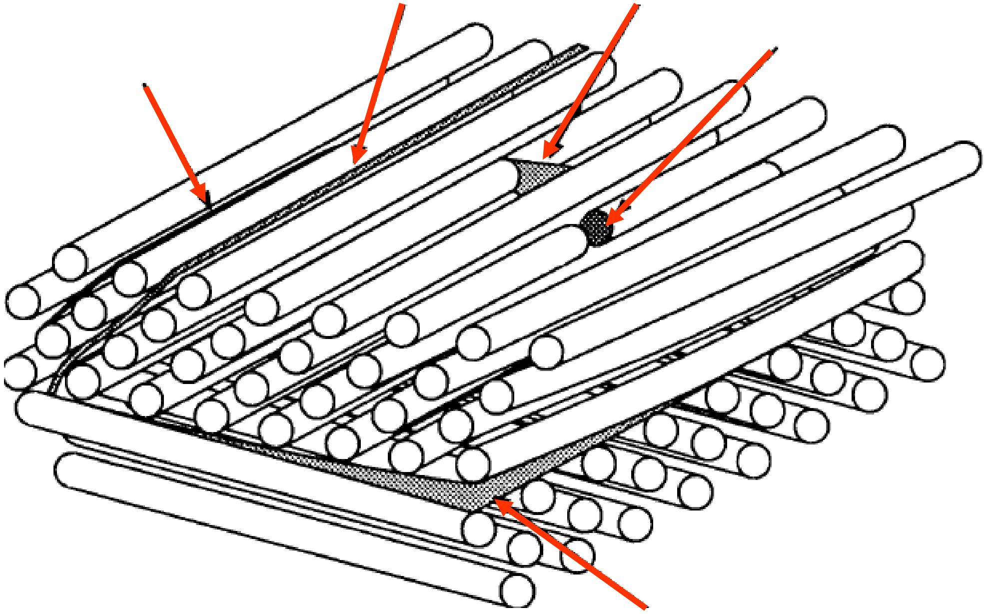
- $$\int_0^L \int_s q^{(1)} \frac{q}{\mu t} ds dx = \boldsymbol{T}^{(1)} \Delta \bar{\boldsymbol{u}} = \Delta_T u$$

- Composite
  - Fibers in a matrix
    - Fibers: polymers, metals or ceramics
    - Matrix: polymers, metals or ceramics
    - Fibers orientation: unidirectional, woven, random
  - Carbon Fiber Reinforced Plastic
    - Carbon woven fibers in epoxy resin
      - Picture: carbon fibers
    - Theoretical tensile strength: 1400 MPa
    - Density: 1800 $\mathrm{kg{\cdot}m^{-3}}$
    - A laminate is a stack of CFRP plies
      - Picture: skin with stringers



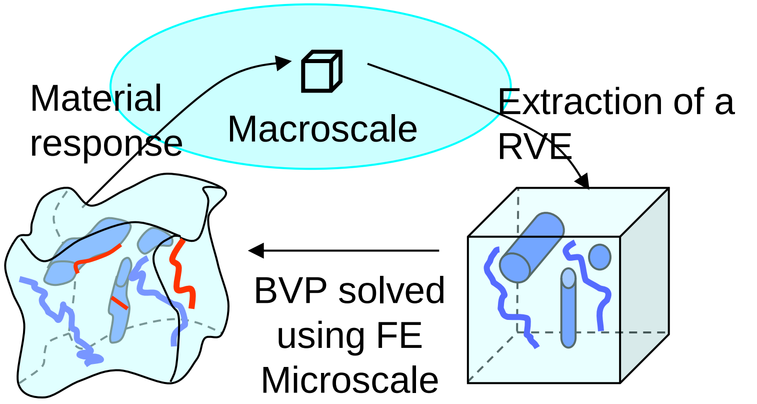


- Composite (2)
  - Drawbacks
    - "Brittle" rupture mode
    - Impact damage
    - Resin can absorb moisture
  - Complex failure modes
    - Transverse matrix fracture
    - Longitudinal matrix fracture
    - Fiber rupture
    - Fiber debonding
    - Delamination
    - Macroscopically: no plastic deformation



- Composite (3)
  - Wing, fuselage, …
  - Typhoon: CFRP
    - 70% of the skin
    - 40% of total weight
  - B787:
    - Fuselage all in CFRP

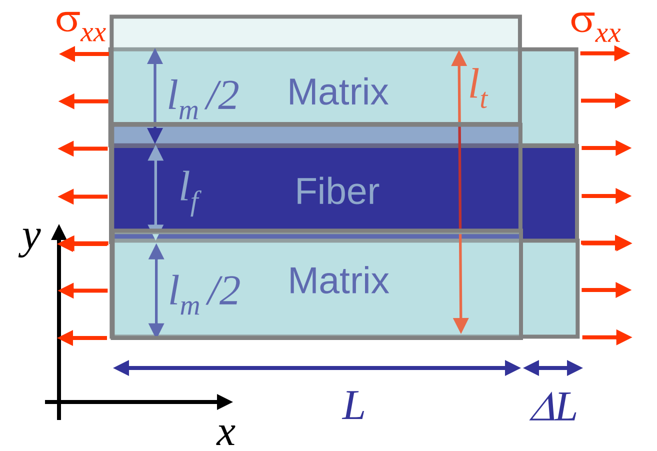
- Approaches in analyzing composite materials
  - Micromechanics
    - Composite is considered as an heterogeneous material
    - Material properties change from one point to the other
      - Resin
      - Fiber
      - Ply
    - Method used to study composite properties
  - Macromechanics
    - Composite is seen as an homogenized material
    - Material properties are constant in each direction
      - They change from one direction to the other
    - Method used in preliminary design
  - Multiscale
    - Combining both approaches



- Ply (lamina) mechanics: $E_x$
  - Symmetrical piece of lamina
    - Matrix-Fiber-Matrix
  - Constraint (small) longitudinal displacement $\Delta L$
    - Small strain $\varepsilon_{xx} = \dfrac{\Delta L}{L}$
    - Microscopic stresses
      - Fiber $\sigma_{xx\,f} = E_f \varepsilon_{xx}$
      - Matrix $\sigma_{xx\,m} = E_m \varepsilon_{xx}$
    - Resultant stress $\sigma_{xx} = E_x \varepsilon_{xx}$
    - Compatibility $\sigma_{xx} l_t = \sigma_{xx\,f} l_f + \sigma_{xx\,m} l_m$

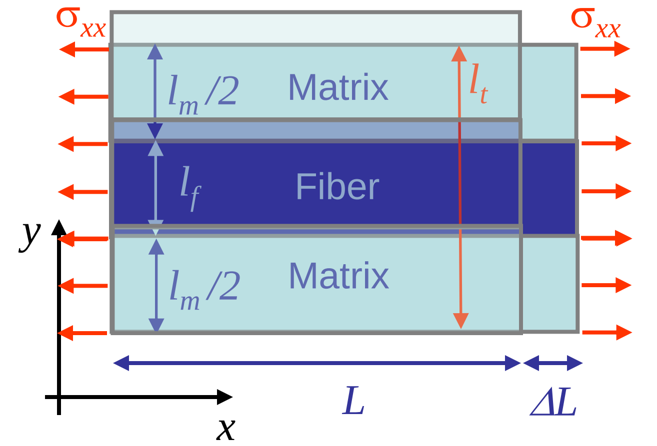
$$\Longrightarrow E_x \varepsilon_{xx} l_t = E_f \varepsilon_{xx} l_f + E_m \varepsilon_{xx} l_m$$

$$\Longrightarrow E_x = \frac{l_f}{l_t} E_f + E_m \frac{l_m}{l_t} = v_f E_f + v_m E_m$$




  - The mixture law gives the longitudinal Young modulus of a unidirectional fiber lamina from the matrix and fiber volume ratio
    - As $E_f >> E_m$, in general $E_x \sim v_f E_f$

- Ply (lamina) mechanics: $\nu_{xy}$
  - Constraint (small) longitudinal displacement $\Delta L$

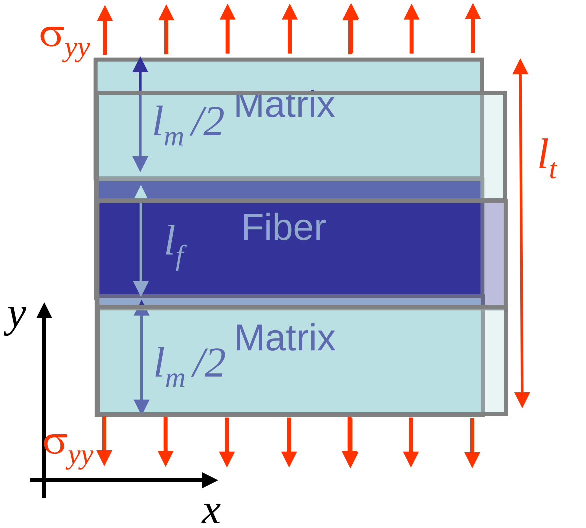



- Transverse displacement $\Delta l_t = \Delta l_f + \Delta l_m$
- Microscopic strains
  - Fiber $\Delta l_f = -\nu_f \varepsilon_{xx\,f} l_f = -\nu_f l_f \varepsilon_{xx}$
  - Matrix $\Delta l_m = -\nu_m \varepsilon_{xx\,m} l_m = -\nu_m l_m \varepsilon_{xx}$
- Resultant strain $\Delta l_t = -\nu_{xy} \varepsilon_{xx} l_t$
- Compatibility $-\nu_{xy} \varepsilon_{xx} l_t = \Delta l_t = \Delta l_f + \Delta l_m = -\nu_f \varepsilon_{xx} l_f - \nu_m \varepsilon_{xx} l_m$

$$\Longrightarrow \nu_{xy} = \nu_f \frac{l_f}{l_t} + \nu_m \frac{l_m}{l_t} = \nu_f v_f + \nu_m v_m$$

- This coefficient $\nu_{xy}$ is called major Poisson's ratio of the lamina

- Ply (lamina) mechanics: $E_y$
  - Constraint (small) transversal displacement $\Delta l$




- Total displacement $\Delta l = \Delta l_m + \Delta l_f$
- Microscopic small strains
  - Fiber $\varepsilon_{yy\,f} = \dfrac{\Delta l_f}{l_f}$
  - Matrix $\varepsilon_{yy\,m} = \dfrac{\Delta l_m}{l_m}$
- Small resultant strain
  - $\varepsilon_{yy} = \dfrac{\Delta l}{l_t} = \dfrac{\Delta l_m}{l_t} + \dfrac{\Delta l_f}{l_t} \Longrightarrow \varepsilon_{yy} = \varepsilon_{yy\,f}\dfrac{l_f}{l_t} + \varepsilon_{yy\,m}\dfrac{l_m}{l_t}$
- Resultant stresses = microscopic stresses
  - $\boldsymbol{\sigma}_{yy} = E_y\varepsilon_{yy} = E_f\varepsilon_{yy\,f} = E_m\varepsilon_{yy\,m}$
  - Relation $\varepsilon_{yy} = \varepsilon_{yy\,f}\dfrac{l_f}{l_t} + \varepsilon_{yy\,m}\dfrac{l_m}{l_t}$

$$\Longrightarrow \frac{\boldsymbol{\sigma}_{yy}}{E_y} = \frac{\boldsymbol{\sigma}_{yy}}{E_f}\frac{l_f}{l_t} + \frac{\boldsymbol{\sigma}_{yy}}{E_m}\frac{l_m}{l_t} \Longrightarrow \frac{1}{E_y} = \frac{v_f}{E_f} + \frac{v_m}{E_m}$$

- As $E_f >> E_m$, in general $E_y \sim E_m / v_m$

# Laminated composite structures

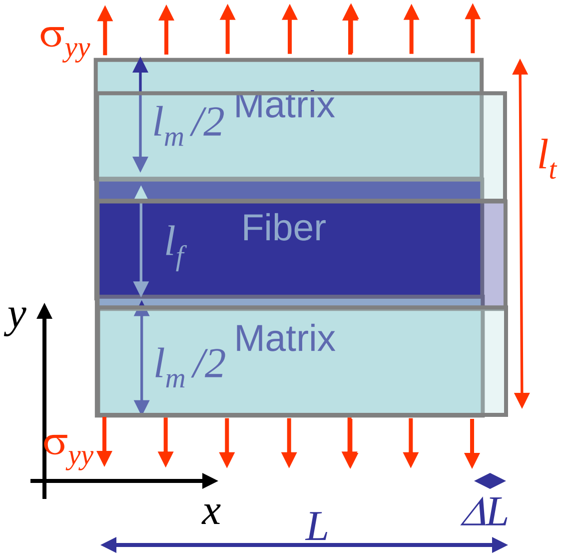
- Ply (lamina) mechanics: $\nu_{yx}$
  - Constraint (small) transversal displacement $\Delta l$ (2)
    - Longitudinal strains are equal by compatibility
      - Resultant $$\varepsilon_{xx} = \frac{\Delta L}{L} = -\nu_{yx}\varepsilon_{yy}$$
      - Fiber $$\varepsilon_{xx\,f} = -\nu_f \varepsilon_{yy\,f} = \varepsilon_{xx}$$
      - Matrix $$\varepsilon_{xx\,m} = -\nu_m \varepsilon_{yy\,m} = \varepsilon_{xx}$$
    - But from previous analysis

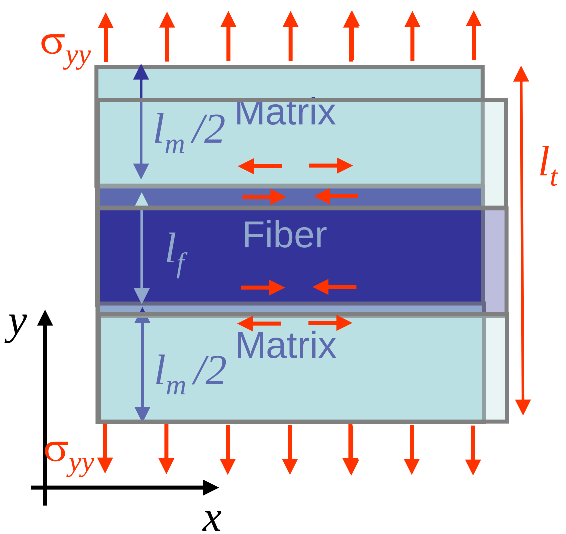
$$\varepsilon_{yy} = \varepsilon_{yy\,f} v_f + \varepsilon_{yy\,m} v_m$$

$$\Rightarrow \nu_{yx}\left(\varepsilon_{yy\,f} v_f + \varepsilon_{yy\,m} v_m\right) = \nu_f \varepsilon_{yy\,f} = \nu_m \varepsilon_{yy\,m}$$

$$\Rightarrow \nu_{yx}\varepsilon_{yy\,f}\left(v_f + \frac{v_m \nu_f}{\nu_m}\right) = \nu_f \varepsilon_{yy\,f}$$

$$\Rightarrow \nu_{yx} = \frac{\nu_f \nu_m}{v_f \nu_m + v_m \nu_f}$$




- But this is wrong as there are microscopic stresses to constrain the compatibility, so relation $\varepsilon_{xx\,f} = -\nu_f \varepsilon_{yy\,f} = \varepsilon_{xx}$ is wrong

- Ply (lamina) mechanics: $\nu_{yx}$ (2)
  - Constraint (small) transversal displacement $\Delta\, l$ (3)
    - Resultant longitudinal strain
      - $\varepsilon_{xx} = \frac{\Delta L}{L} = -\nu_{yx}\varepsilon_{yy} = -\nu_{yx}\frac{\sigma_{yy}}{E_y}$
    - Microscopic strains & compatibility
      - Fiber

$$\varepsilon_{xx\,f} = \frac{1}{E_f}\left(\sigma_{xx\,f} - \nu_f\sigma_{yy\,f}\right) = \frac{1}{E_f}\left(\sigma_{xx\,f} - \nu_f\sigma_{yy}\right)$$

      - Matrix

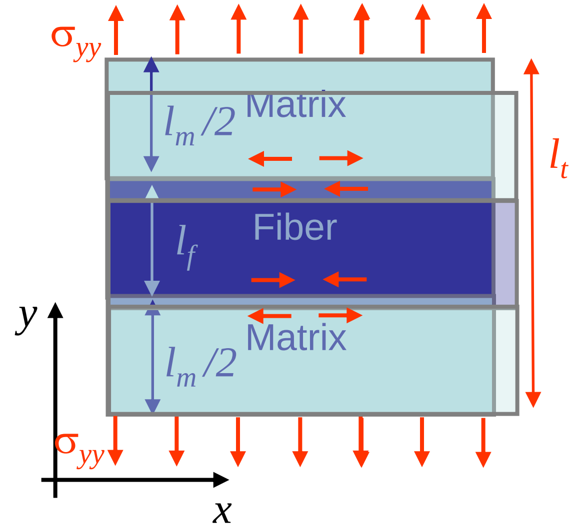
$$\varepsilon_{xx\,m} = \frac{1}{E_m}\left(\sigma_{xx\,m} - \nu_m\sigma_{yy\,m}\right) = \frac{1}{E_m}\left(\sigma_{xx\,m} - \nu_m\sigma_{yy}\right)$$

    - Resultant stress along $x$ is equal to zero

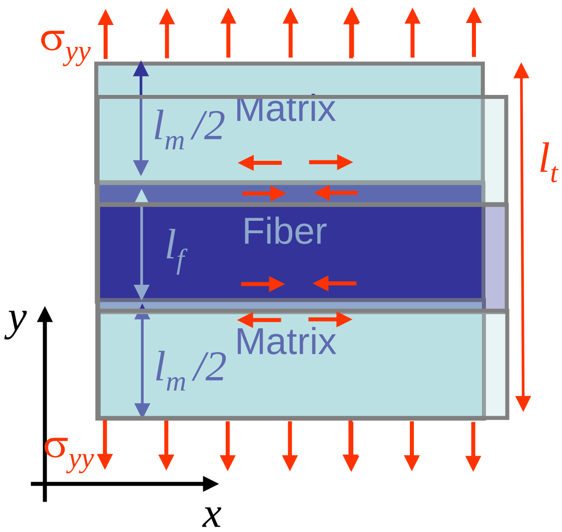
$$l_f\sigma_{xx\,f} + l_m\sigma_{xx\,m} = 0 \Longrightarrow v_f\sigma_{xx\,f} = -v_m\sigma_{xx\,m}$$

    - Using compatibility of strains

$$\frac{1}{E_m}\left(\sigma_{xx\,m} - \nu_m\sigma_{yy}\right) = \frac{1}{E_f}\left(-\frac{v_m}{v_f}\sigma_{xx\,m} - \nu_f\sigma_{yy}\right) = -\nu_{yx}\frac{\sigma_{yy}}{E_y}$$



- Ply (lamina) mechanics: $\nu_{yx}$ (3)
  - Constraint (small) transversal displacement $\Delta l$ (4)




$$\frac{1}{E_m}\left(\boldsymbol{\sigma}_{xxm} - \nu_m \boldsymbol{\sigma}_{yy}\right) = \frac{1}{E_f}\left(-\frac{v_m}{v_f}\boldsymbol{\sigma}_{xxm} - \nu_f \boldsymbol{\sigma}_{yy}\right) = -\nu_{yx}\frac{\boldsymbol{\sigma}_{yy}}{E_y}$$

$$\Rightarrow \quad v_m\left(\frac{1}{v_m E_m} + \frac{1}{E_f v_f}\right)\boldsymbol{\sigma}_{xxm} = \boldsymbol{\sigma}_{yy}\left(\frac{\nu_m}{E_m} - \frac{\nu_f}{E_f}\right)$$

$$\Rightarrow \quad \boldsymbol{\sigma}_{xxm} = \boldsymbol{\sigma}_{yy}\frac{\nu_m E_f - \nu_f E_m}{v_m E_f E_m}\,\frac{v_m E_m v_f E_f}{E_f v_f + E_m v_m} = v_f \boldsymbol{\sigma}_{yy}\frac{\nu_m E_f - \nu_f E_m}{E_f v_f + E_m v_m}$$

$$\Rightarrow \quad -\nu_{yx}\frac{\boldsymbol{\sigma}_{yy}}{E_y} = \frac{1}{E_m}\left(v_f\frac{\nu_m E_f - \nu_f E_m}{E_f v_f + E_m v_m} - \nu_m\right)\boldsymbol{\sigma}_{yy}$$

$$\Rightarrow \quad \frac{\nu_{yx}}{E_y} = \frac{1}{E_m}\,\frac{v_f \nu_f E_m + v_m \nu_m E_m}{E_f v_f + E_m v_m}$$

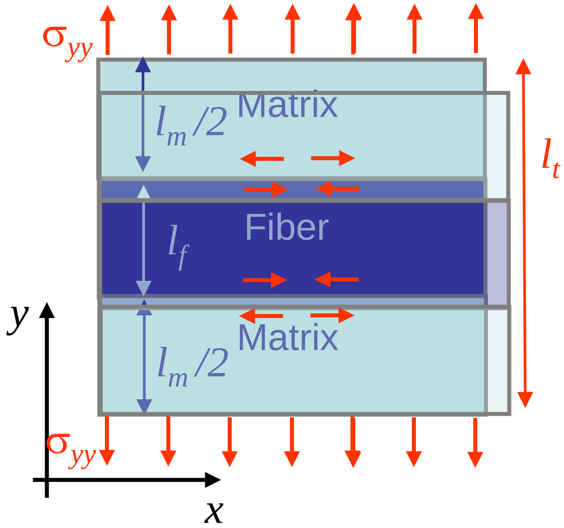
$$\Rightarrow \quad \nu_{yx} = E_y\frac{v_f \nu_f + v_m \nu_m}{E_f v_f + E_m v_m} = \frac{E_y \nu_{xy}}{E_x}$$

# Laminated composite structures

- Ply (lamina) mechanics: $\nu_{yx}$ (4)
  - Constraint (small) transversal displacement $\Delta\, l$ (5)
    - Minor Poisson coefficient

$$\nu_{yx} = E_y \frac{v_f \nu_f + v_m \nu_m}{E_f v_f + E_m v_m} = \frac{E_y \nu_{xy}}{E_x}$$

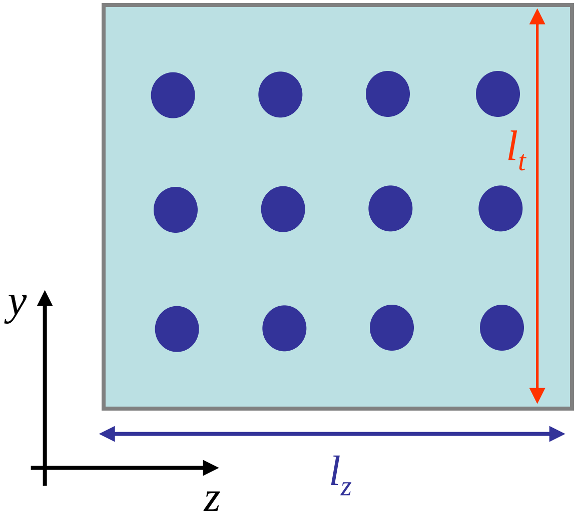
    - This is called the minor one as usually $E_m << E_f \Longrightarrow E_x >> E_y \Longrightarrow \nu_{yx} < \nu_{xy}$




- Remarks
  - The stresses in the matrix and in the fiber

$$v_f \boldsymbol{\sigma}_{xx\,f} = -v_m \boldsymbol{\sigma}_{xx\,m}$$

    can lead to fiber debonding
  - In all the previous developments we have assumed zero-stress along $z$-axis
    - This is justified as the behaviors in $z$ and $y$ directions are similar.
    - This will not be true in a stack of laminas (laminate)

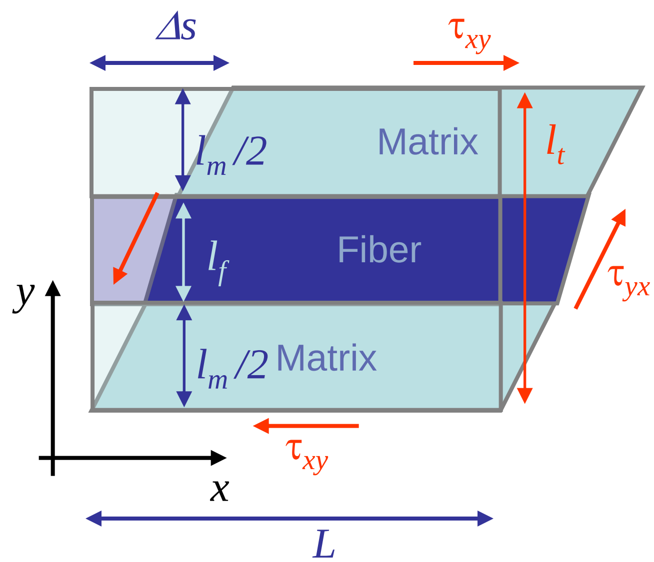


- Ply (lamina) mechanics: $\mu_{xy}$
  - Constraint (small) shearing $\gamma = \Delta s / l_t$
    - Assumption: fiber and matrix are subjected to the same shear stress $\tau_{xy} = \tau_{yx}$
    - Resultant shear sliding
      - $\mu_{xy} = \mu_{yx} = \frac{\tau_{xy} l_t}{\Delta s} = \frac{\tau_{yx} l_t}{\Delta s}$
    - Microscopic shearing
      - Fiber $\Delta s_f = \frac{\tau_{xy}}{\mu_f} l_f$
      - Matrix $\Delta s_m = \frac{\tau_{xy}}{\mu_m} l_m$
    - Compatibility

$$\Delta s = \Delta s_f + \Delta s_m \Rightarrow \frac{\tau_{xy}}{\mu_{xy}} l_t = \frac{\tau_{xy}}{\mu_f} l_f + \frac{\tau_{xy}}{\mu_m} l_m$$

$$\Rightarrow \frac{1}{\mu_{xy}} = \frac{1}{\mu_f} v_f + \frac{1}{\mu_m} v_m$$

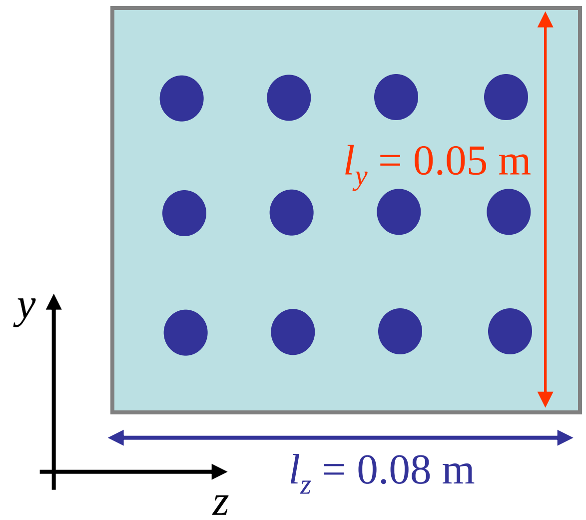
    - As $\mu_f >> \mu_m$, in general $\mu_{xy} = \mu_m / v_m$
  - Unlike isotropic materials, shear modulus is **NOT** related to $E$ and $\nu$



- Example
  - Epoxy resin matrix ($v_m$= 80%)
  - Reinforced by $x$-oriented carbon filament ($v_f$= 20%)
    - Epoxy
      - $E_m$ = 5 GPa
      - $\nu_m$ = 0.2
    - Carbon
      - $E_f$ = 200 GPa
      - $\nu_f$ = 0.3
  - What are the resultant properties (Young, Poisson, Shear modulus) ?
  - Bar of
    - Cross section 0.08 m X 0.05 m
    - Length 0.5 m
    - Subjected to an axial loading of 100 kN
  - Determine
    - Lengthening & thickness shortening?
    - Stress in components?

- Material properties
  - Section idealization
  - Young modulus
    - Longitudinal direction

$$E_x = v_f E_f + v_m E_m$$
$$= 0.2\ 200\ 10^9 + 0.8\ 5\ 10^9 = 44\ \ \mathrm{GPa}$$
$$\simeq v_f E_f = 40\ \ \mathrm{GPa}$$




    - Lateral direction

$$E_y = \left(\frac{v_f}{E_f} + \frac{v_m}{E_m}\right)^{-1} = \left(\frac{0.2}{200\ 10^9} + \frac{0.8}{5\ 10^9}\right)^{-1} = 6.2\ \ \mathrm{GPa}$$
$$\simeq \frac{E_m}{v_m} = 6.25\ \ \mathrm{GPa}$$

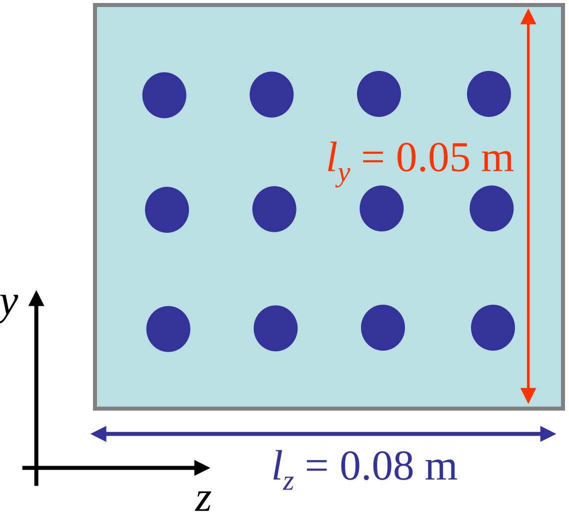
  - Poisson ratio
    - Major $\nu_{xy} = \nu_f v_f + \nu_m v_m = 0.2\ 0.3 + 0.8\ 0.2 = 0.22$
    - Minor $\nu_{yx} = \frac{E_y}{E_x}\nu_{xy} = \frac{6.2}{44}0.22 = 0.031$

- Material properties (2)
  - Shear modulus
    - Of the components
      - Fiber

$$\mu_f = \frac{E_f}{2\left(1+\nu_f\right)} = \frac{200\ 10^9}{2\left(1+0.3\right)} = 76.9 \quad \text{GPa}$$

      - Matrix

$$\mu_m = \frac{E_m}{2\left(1+\nu_m\right)} = \frac{5\ 10^9}{2\left(1+0.2\right)} = 2.08 \quad \text{GPa}$$



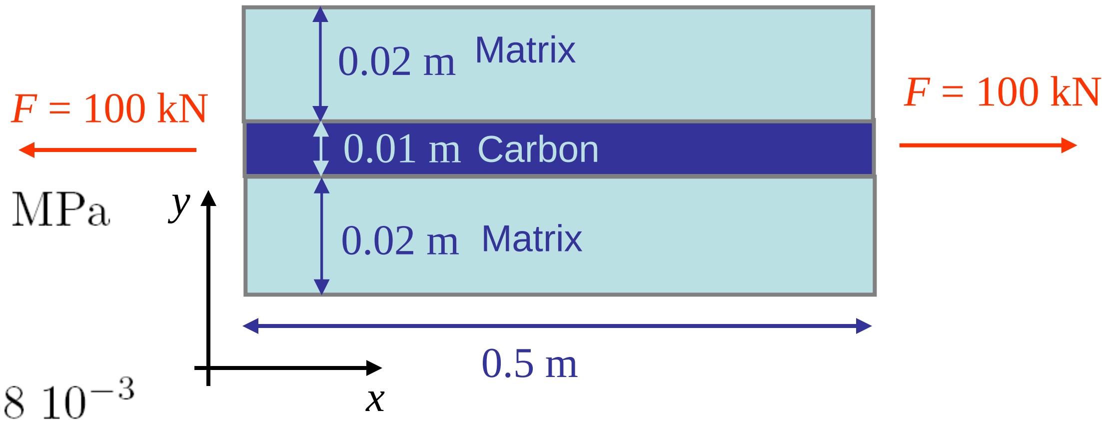

    - Resultant shear modulus

$$\mu_{xy} = \left(\frac{v_f}{\mu_f} + \frac{v_m}{\mu_m}\right)^{-1} = \left(\frac{0.2}{76.9\ 10^9} + \frac{0.8}{2.08\ 10^9}\right)^{-1}$$

$$= 2.58 \quad \text{GPa} \simeq \frac{\mu_m}{v_m} = \frac{2.08\ 10^9}{0.8} = 2.6 \quad \text{GPa}$$

- Traction
  - Resultant stress

$$\boldsymbol{\sigma}_{xx} = \frac{F}{A} = \frac{100\ 10^3}{0.08\ 0.05} = 25 \quad \text{MPa}$$




  - Lengthening

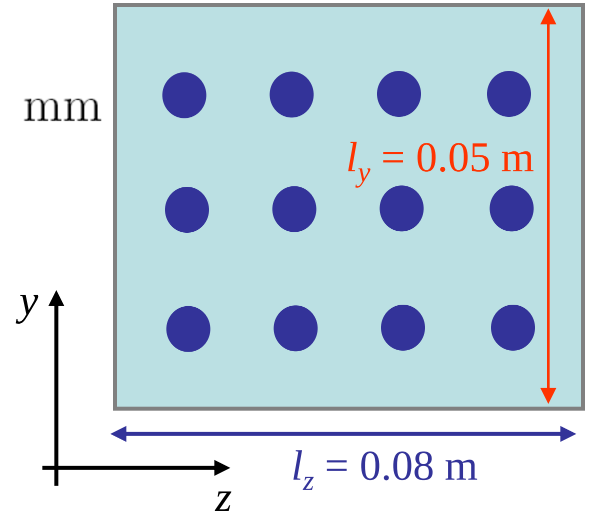
$$\boldsymbol{\varepsilon}_{xx} = \frac{\boldsymbol{\sigma}_{xx}}{E_x} = \frac{25\ 10^6}{44\ 10^9} = 0.568\ 10^{-3}$$

$$\Longrightarrow \Delta L = \boldsymbol{\varepsilon}_{xx} L = 0.568\ 10^{-3} 0.5 = 0.284 \quad \text{mm}$$

  - Thickness shortening

$$\boldsymbol{\varepsilon}_{yy} = -\nu_{xy}\boldsymbol{\varepsilon}_{xx} = -0.22\ 0.568\ 10^{-3} = -0.12496\ 10^{-3}$$

$$\Longrightarrow \Delta l_y = \boldsymbol{\varepsilon}_{yy} l_y = -0.12496\ 10^{-3}\ 0.05 = -0.0062 \quad \text{mm}$$



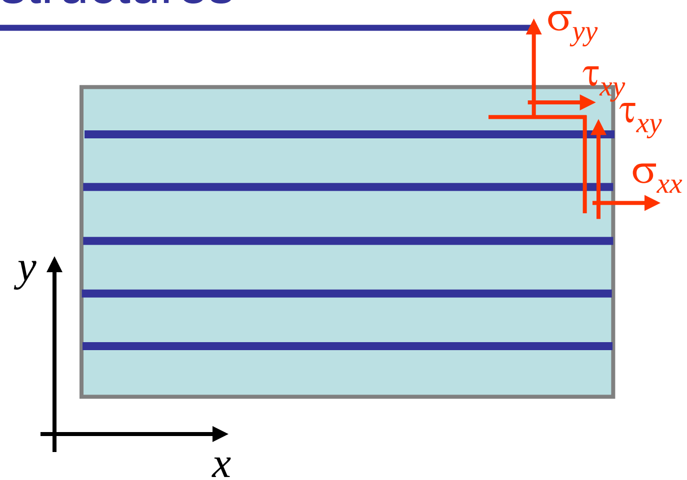

  - Microscopic stresses
    - Fiber

$$\boldsymbol{\sigma}_{xx\,f} = E_f \boldsymbol{\varepsilon}_{xx} = 200\ 10^9 \quad 0.568\ 10^{-3} = 113.6 \quad \text{MPa}$$

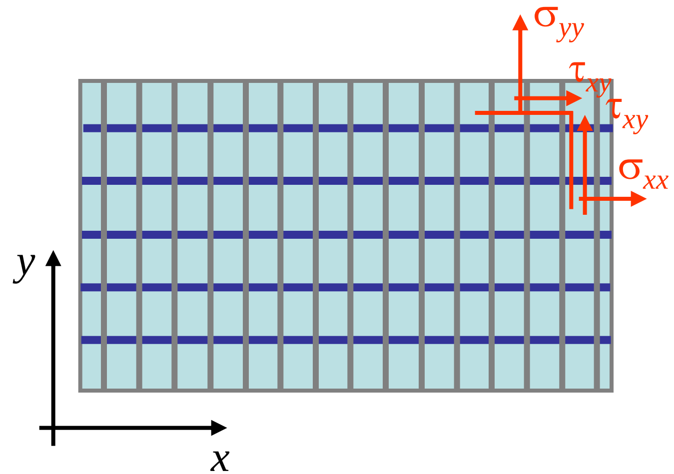
    - Matrix

$$\boldsymbol{\sigma}_{xx\,m} = E_m \boldsymbol{\varepsilon}_{xx} = 5\ 10^9 \quad 0.568\ 10^{-3} = 2.84 \quad \text{MPa}$$

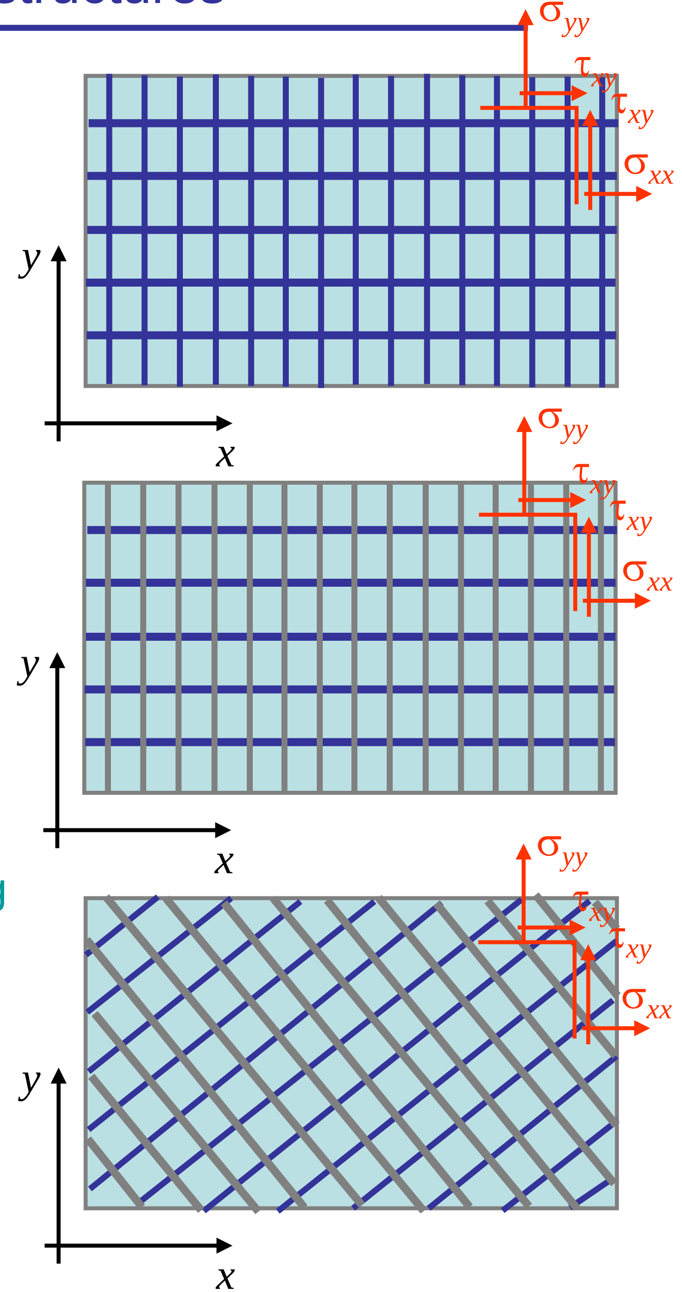
- Orthotropic ply mechanics
  - Single sheet of composite with
    - Fibers aligned in one direction: unidirectional ply or lamina




    - Fibers in perpendicular direction: woven ply



- Orthotropic ply mechanics (2)
  - Woven ply
    - Transversally isotropic
      - Fiber reinforcements the same in both directions
      - Same material properties in the 2 fiber directions
    - Orthotropic
      - Fiber reinforcements not the same in both directions
      - Different material properties in the 2 directions
      - Specially orthotropic: Applied loading in the directions of the plies
      - Generally orthotropic: Applied loading not in the directions of the plies



- Specially orthotropic ply mechanics
  - Plane stress (Plane-$\sigma$) state
    - Isotropic materials

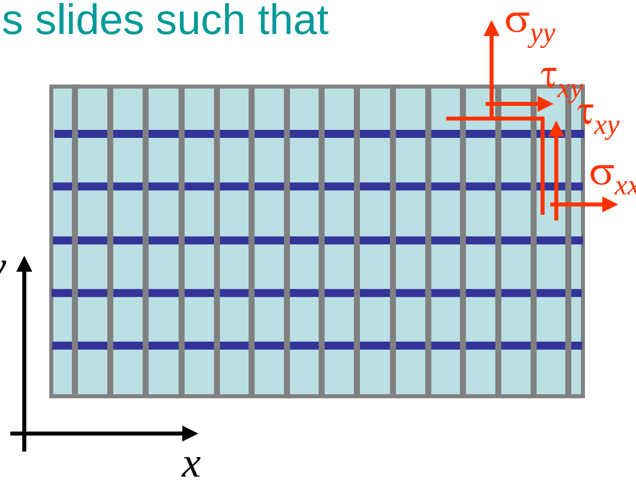
$$\begin{pmatrix} \boldsymbol{\varepsilon}_{xx} \\ \boldsymbol{\varepsilon}_{yy} \\ \boldsymbol{\varepsilon}_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{2\mu} \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{xy} \end{pmatrix}$$

- New resultant material properties defined in previous slides such that
  - For uniaxial tension along $x$

$$\boldsymbol{\varepsilon}_{xx} = \frac{\boldsymbol{\sigma}_{xx}}{E_x} \ \& \ \boldsymbol{\varepsilon}_{yy} = -\nu_{xy}\boldsymbol{\varepsilon}_{xx} = -\frac{\nu_{xy}\boldsymbol{\sigma}_{xx}}{E_x}$$

  - For uniaxial tension along $y$

$$\boldsymbol{\varepsilon}_{yy} = \frac{\boldsymbol{\sigma}_{yy}}{E_y} \ \& \ \boldsymbol{\varepsilon}_{xx} = -\nu_{yx}\boldsymbol{\varepsilon}_{yy} = -\frac{\nu_{yx}\boldsymbol{\sigma}_{yy}}{E_y}$$

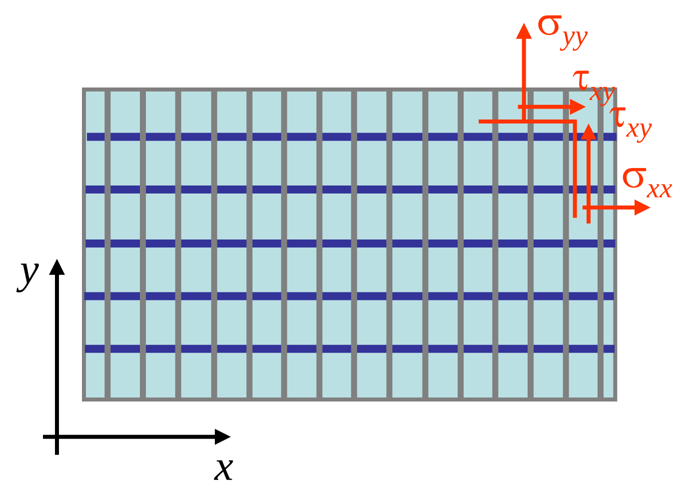



- Superposition leads to the orthotropic law

$$\begin{pmatrix} \boldsymbol{\varepsilon}_{xx} \\ \boldsymbol{\varepsilon}_{yy} \\ \boldsymbol{\varepsilon}_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & 0 \\ 0 & 0 & \frac{1}{2\mu_{xy}} \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{xy} \end{pmatrix}$$

  - Be careful $\mu_{xy}$ cannot be computed from $E_{x/y}$, $\nu_{xy/yx}$

- Specially orthotropic ply mechanics (2)
  - Plane stress (Plane-$\sigma$) state (2)
    - Reciprocal stress-strain relationship




$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & 0 \\ 0 & 0 & \frac{1}{2\mu_{xy}} \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{xy} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{xy} \end{pmatrix} = \begin{pmatrix} \frac{E_x}{1-\nu_{xy}\nu_{yx}} & \frac{\nu_{yx}E_x}{1-\nu_{xy}\nu_{yx}} & 0 \\ \frac{\nu_{xy}E_y}{1-\nu_{xy}\nu_{yx}} & \frac{E_y}{1-\nu_{xy}\nu_{yx}} & 0 \\ 0 & 0 & 2\mu_{xy} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix}$$

- To be compared to stress-strain relationship for isotropic materials

$$\begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{xy} \end{pmatrix} = \begin{pmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & 2\mu \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix}$$

- Specially orthotropic ply mechanics (3)
  - General 3D expression
    - Hooke's law $\boldsymbol{\sigma} = \mathcal{C} : \boldsymbol{\varepsilon}$ or $\boldsymbol{\sigma}_{ij} = \mathcal{C}_{ijkl}\boldsymbol{\varepsilon}_{kl}$
    - Can be rewritten under the form

$$\begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{zz} \\ \boldsymbol{\sigma}_{yz} \\ \boldsymbol{\sigma}_{zx} \\ \boldsymbol{\sigma}_{xy} \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{xxxx} & \mathcal{C}_{xxyy} & \mathcal{C}_{xxzz} & 0 & 0 & 0 \\ \mathcal{C}_{yyxx} & \mathcal{C}_{yyyy} & \mathcal{C}_{yyzz} & 0 & 0 & 0 \\ \mathcal{C}_{zzxx} & \mathcal{C}_{zzyy} & \mathcal{C}_{zzzz} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mathcal{C}_{yzyz} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mathcal{C}_{xzxz} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mathcal{C}_{xyxy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_{xx} \\ \boldsymbol{\varepsilon}_{yy} \\ \boldsymbol{\varepsilon}_{zz} \\ \boldsymbol{\varepsilon}_{yz} \\ \boldsymbol{\varepsilon}_{zx} \\ \boldsymbol{\varepsilon}_{xy} \end{pmatrix}$$

$$\boldsymbol{\sigma}_{yz} = \mathcal{C}_{yzkl}\boldsymbol{\varepsilon}_{kl} = \mathcal{C}_{yzyz}\boldsymbol{\varepsilon}_{yz} + \mathcal{C}_{yzzy}\boldsymbol{\varepsilon}_{zy} = 2\mathcal{C}_{yzyz}\boldsymbol{\varepsilon}_{yz}$$

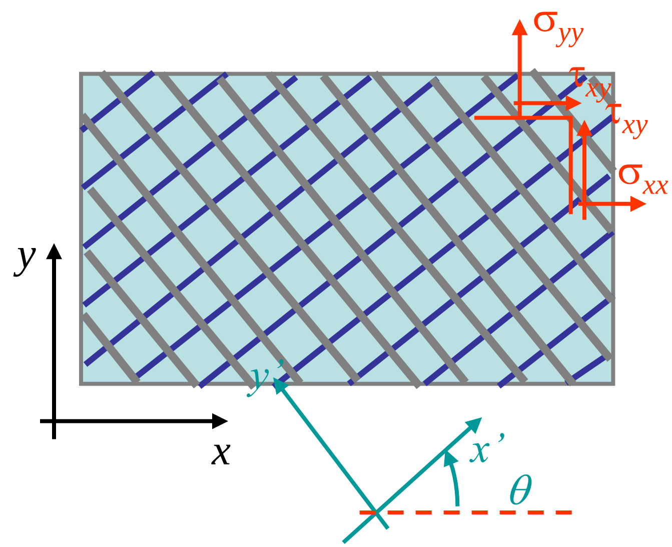
- Specially orthotropic ply mechanics (4)
  - General 3D expression (2)
    - Hooke's law $\boldsymbol{\sigma} = \mathcal{C} : \boldsymbol{\varepsilon}$ or $\boldsymbol{\sigma}_{ij} = \mathcal{C}_{ijkl}\boldsymbol{\varepsilon}_{kl}$
    - With the 21 non-zero components of the fourth-order tensor being
      - $\mathcal{C}_{xxxx} = \frac{1-\nu_{yz}\nu_{zy}}{E_y E_z D}$ , $\mathcal{C}_{xxyy} = \frac{\nu_{yx}+\nu_{zx}\nu_{yz}}{E_y E_z D}$ & $\mathcal{C}_{xxzz} = \frac{\nu_{zx}+\nu_{yx}\nu_{zy}}{E_y E_z D}$
      - $\mathcal{C}_{yyxx} = \frac{\nu_{xy}+\nu_{zy}\nu_{xz}}{E_x E_z D}$, $\mathcal{C}_{yyyy} = \frac{1-\nu_{xz}\nu_{zx}}{E_x E_z D}$ & $\mathcal{C}_{yyzz} = \frac{\nu_{zy}+\nu_{xy}\nu_{zx}}{E_x E_z D}$
      - $\mathcal{C}_{zzxx} = \frac{\nu_{xz}+\nu_{xy}\nu_{yz}}{E_y E_x D}$, $\mathcal{C}_{zzyy} = \frac{\nu_{yz}+\nu_{xz}\nu_{yx}}{E_y E_x D}$ & $\mathcal{C}_{zzzz} = \frac{1-\nu_{yx}\nu_{xy}}{E_y E_x D}$
      - $\mathcal{C}_{yzyz} = \mathcal{C}_{yzzy} = \mathcal{C}_{zyzy} = \mathcal{C}_{zyyz} = \mu_{yz}$
      - $\mathcal{C}_{xyxy} = \mathcal{C}_{xyyx} = \mathcal{C}_{yxyx} = \mathcal{C}_{yxxy} = \mu_{xy}$
      - $\mathcal{C}_{xzxz} = \mathcal{C}_{xzzx} = \mathcal{C}_{zxzx} = \mathcal{C}_{zxxz} = \mu_{xz}$
    - And the denominator $D = \frac{1-\nu_{xy}\nu_{yx}-\nu_{zy}\nu_{yz}-\nu_{xz}\nu_{zx}-2\nu_{xy}\nu_{yz}\nu_{zx}}{E_x E_y E_z}$

- Generally orthotropic ply mechanics
  - Stress-strain relationship
    - Stress-strain relationship in the axes $O'x'y'$ is known for plane-σ state

$$\begin{pmatrix} \boldsymbol{\sigma}_{x'x'} \\ \boldsymbol{\sigma}_{y'y'} \\ \boldsymbol{\sigma}_{x'y'} \end{pmatrix} = \begin{pmatrix} \frac{E_{x'}}{1-\nu_{x'y'}\nu_{y'x'}} & \frac{\nu_{y'x'}E_{x'}}{1-\nu_{x'y'}\nu_{y'x'}} & 0 \\ \frac{\nu_{x'y'}E_{y'}}{1-\nu_{x'y'}\nu_{y'x'}} & \frac{E_{y'}}{1-\nu_{x'y'}\nu_{y'x'}} & 0 \\ 0 & 0 & 2\mu_{x'y'} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_{x'x'} \\ \boldsymbol{\varepsilon}_{y'y'} \\ \boldsymbol{\varepsilon}_{x'y'} \end{pmatrix}$$

or in tensorial form $\boldsymbol{\sigma}' = \mathcal{C}' : \boldsymbol{\varepsilon}'$

    - If $\theta$ is the angle between $Ox$ & $O'x'$




– $\begin{cases} \boldsymbol{\sigma}' = \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^T \\ \boldsymbol{\varepsilon}' = \mathbf{R}\boldsymbol{\varepsilon}\mathbf{R}^T \end{cases}$

with $\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

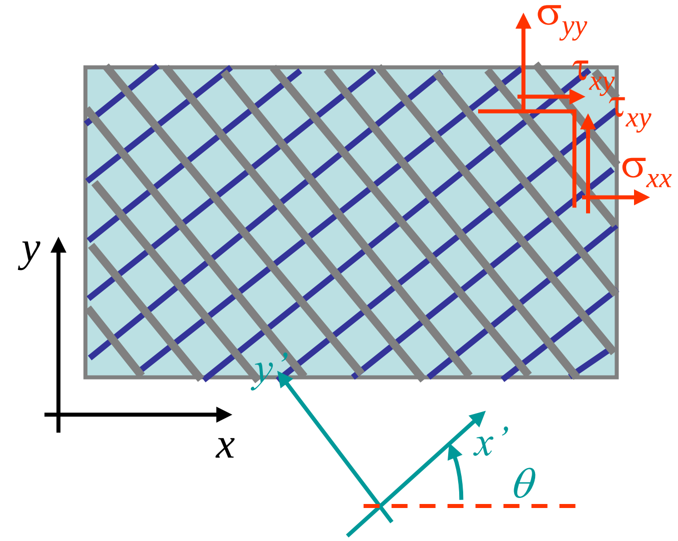
    - From there we can get $\mathcal{C}$ such that

$$\boldsymbol{\sigma} = \mathcal{C} : \boldsymbol{\varepsilon}$$

- Generally orthotropic ply mechanics (2)
  - Stress-strain relationship (2)
    - Equation $\boldsymbol{\sigma}' = \mathcal{C}' : \boldsymbol{\varepsilon}'$

      with $\boldsymbol{\sigma}' = \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^T$, $\boldsymbol{\varepsilon}' = \mathbf{R}\boldsymbol{\varepsilon}\mathbf{R}^T$

      & in 2D $\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$




    - Solution

$$\Longrightarrow \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^T = \mathcal{C}' : \mathbf{R}\boldsymbol{\varepsilon}\mathbf{R}^T$$

$$\Longrightarrow \boldsymbol{\sigma}_{ij} = \mathbf{R}_{ki}\mathcal{C}'_{klmn}\mathbf{R}_{lj}\mathbf{R}_{mp}\boldsymbol{\varepsilon}_{pq}\mathbf{R}_{nq}$$

$$\Longrightarrow \boldsymbol{\sigma}_{ij} = \mathbf{R}_{ki}\mathbf{R}_{lj}\mathcal{C}'_{klmn}\mathbf{R}_{mp}\mathbf{R}_{nq}\boldsymbol{\varepsilon}_{pq} = \mathcal{C}_{ijpq}\boldsymbol{\varepsilon}_{pq}$$

    - Or again $\boldsymbol{\sigma} = \mathcal{C} : \boldsymbol{\varepsilon}$

      with $\mathcal{C}_{ijkl} = \mathbf{R}_{mi}\mathbf{R}_{nj}\mathcal{C}'_{mnpq}\mathbf{R}_{pk}\mathbf{R}_{ql}$

- Generally orthotropic ply mechanics (3)
  - Plane σ state
    - From $$\begin{pmatrix} \boldsymbol{\sigma}_{x'x'} \\ \boldsymbol{\sigma}_{y'y'} \\ \boldsymbol{\sigma}_{x'y'} \end{pmatrix} = \begin{pmatrix} \frac{E_{x'}}{1-\nu_{x'y'}\nu_{y'x'}} & \frac{\nu_{y'x'}E_{x'}}{1-\nu_{x'y'}\nu_{y'x'}} & 0 \\ \frac{\nu_{x'y'}E_{y'}}{1-\nu_{x'y'}\nu_{y'x'}} & \frac{E_{y'}}{1-\nu_{x'y'}\nu_{y'x'}} & 0 \\ 0 & 0 & 2\mu_{x'y'} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_{x'x'} \\ \boldsymbol{\varepsilon}_{y'y'} \\ \boldsymbol{\varepsilon}_{x'y'} \end{pmatrix}$$

  - The non-zero components are
    - » $\mathcal{C}'_{x'x'x'x'} = \frac{E_{x'}}{1-\nu_{x'y'}\nu_{y'x'}}$
    - » $\mathcal{C}'_{y'y'y'y'} = \frac{E_{y'}}{1-\nu_{x'y'}\nu_{y'x'}}$
    - » $\mathcal{C}'_{x'x'y'y'} = \mathcal{C}'_{y'y'x'x'} = \frac{\nu_{y'x'}E_{x'}}{1-\nu_{x'y'}\nu_{y'x'}} = \frac{\nu_{x'y'}E_{y'}}{1-\nu_{x'y'}\nu_{y'x'}}$
    - » $\mathcal{C}'_{x'y'x'y'} = \mathcal{C}'_{x'y'y'x'} = \mathcal{C}'_{y'x'x'y'} = \mathcal{C}'_{y'x'y'x'} = \mu_{x'y'}$
  - Let $c = \cos\theta$, $s = \sin\theta$,
    - » $\mathbf{R}_{x'x} = \mathbf{R}_{y'y} = c$
    - » $\mathbf{R}_{x'y} = -\mathbf{R}_{y'x} = s$

- Generally orthotropic ply mechanics (4)
  - Plane $\sigma$ state (2)
    - Using $\mathbf{R}_{x'x} = \mathbf{R}_{y'y} = c$ & $\mathbf{R}_{x'y} = -\mathbf{R}_{y'x} = s$ expression $\mathcal{C}_{ijkl} = \mathbf{R}_{mi}\mathbf{R}_{nj}\mathcal{C}'_{mnpq}\mathbf{R}_{pk}\mathbf{R}_{ql}$ leads to

$$\begin{aligned}
\mathcal{C}_{xxxx} &= \mathbf{R}_{mx}\mathbf{R}_{nx}\mathcal{C}'_{mnpq}\mathbf{R}_{px}\mathbf{R}_{qx} \\
&= \mathbf{R}_{x'x}\mathbf{R}_{x'x}\mathcal{C}'_{x'x'x'x'}\mathbf{R}_{x'x}\mathbf{R}_{x'x} + \mathbf{R}_{x'x}\mathbf{R}_{x'x}\mathcal{C}'_{x'x'y'y'}\mathbf{R}_{y'x}\mathbf{R}_{y'x} + \\
&\quad \mathbf{R}_{x'x}\mathbf{R}_{y'x}\mathcal{C}'_{x'y'x'y'}\mathbf{R}_{x'x}\mathbf{R}_{y'x} + \mathbf{R}_{x'x}\mathbf{R}_{y'x}\mathcal{C}'_{x'y'y'x'}\mathbf{R}_{y'x}\mathbf{R}_{x'x} + \\
&\quad \mathbf{R}_{y'x}\mathbf{R}_{x'x}\mathcal{C}'_{y'x'x'y'}\mathbf{R}_{x'x}\mathbf{R}_{y'x} + \mathbf{R}_{y'x}\mathbf{R}_{x'x}\mathcal{C}'_{y'x'y'x'}\mathbf{R}_{y'x}\mathbf{R}_{x'x} + \\
&\quad \mathbf{R}_{y'x}\mathbf{R}_{y'x}\mathcal{C}'_{y'y'x'x'}\mathbf{R}_{x'x}\mathbf{R}_{x'x} + \mathbf{R}_{y'x}\mathbf{R}_{y'x}\mathcal{C}'_{y'y'y'y'}\mathbf{R}_{y'x}\mathbf{R}_{y'x}
\end{aligned}$$

$$\Rightarrow \mathcal{C}_{xxxx} = c^4\mathcal{C}'_{x'x'x'x'} + c^2s^2\left(\mathcal{C}'_{x'x'y'y'} + \mathcal{C}'_{x'y'x'y'} + \mathcal{C}'_{x'y'y'x'} + \mathcal{C}'_{y'x'x'y'} + \mathcal{C}'_{y'x'y'x'} + \mathcal{C}'_{y'y'x'x'}\right) + s^4\mathcal{C}'_{y'y'y'y'}$$

    - Eventually, using minor & major symmetry of material tensor

$$\mathcal{C}_{xxxx} = c^4\mathcal{C}'_{x'x'x'x'} + 2c^2s^2\left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'}\right) + s^4\mathcal{C}'_{y'y'y'y'}$$

- Generally orthotropic ply mechanics (5)
  - Plane σ state (3)
    - Doing the same for the other components leads to

$$\begin{cases} \mathcal{C}_{xxxx} = c^4\mathcal{C}'_{x'x'x'x'} + 2c^2s^2\left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'}\right) + s^4\mathcal{C}'_{y'y'y'y'} \\ \mathcal{C}_{yyyy} = s^4\mathcal{C}'_{x'x'x'x'} + 2c^2s^2\left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'}\right) + c^4\mathcal{C}'_{y'y'y'y'} \\ \mathcal{C}_{xxyy} = \mathcal{C}_{yyxx} = \left(c^4 + s^4\right)\mathcal{C}'_{x'x'y'y'} + c^2s^2\left(\mathcal{C}'_{x'x'x'x'} + \mathcal{C}'_{y'y'y'y'} - 4\mathcal{C}'_{x'y'x'y'}\right) \\ \mathcal{C}_{xyxy} = \mathcal{C}_{xyyx} = \mathcal{C}_{yxxy} = \mathcal{C}_{yxyx} = \\ \quad \left(c^2 - s^2\right)^2\mathcal{C}'_{x'y'x'y'} + c^2s^2\left(\mathcal{C}'_{x'x'x'x'} + \mathcal{C}'_{y'y'y'y'} - 2\mathcal{C}'_{x'x'y'y'}\right) \end{cases}$$

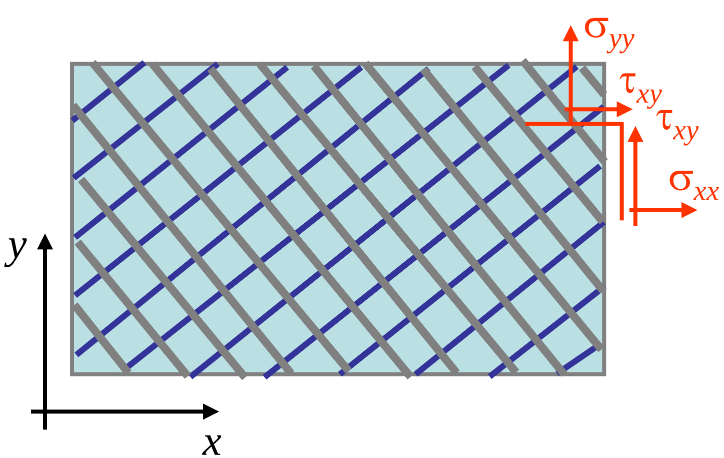
    - These are the 8 non-zero components
    - In the *O'x'y'* there were 8 non-zero components

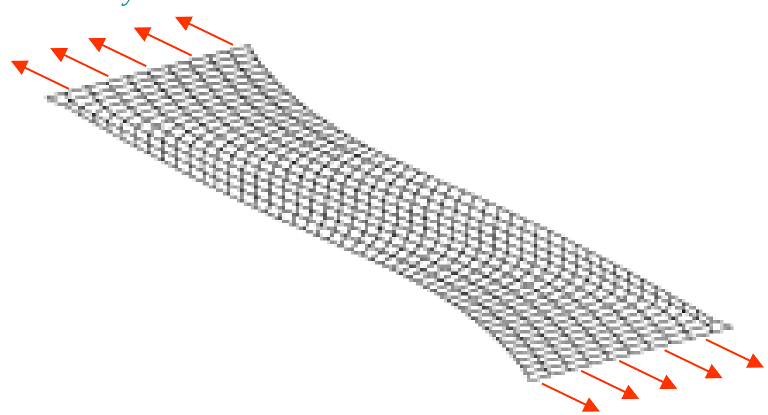
- Generally orthotropic ply mechanics (6)
  - Plane σ state (4)
    - But due to the rotation: a coupling between tension and shearing appears

$$
\begin{aligned}
\mathcal{C}_{xxxy} &= \mathbf{R}_{mx}\mathbf{R}_{nx}\mathcal{C}'_{mnpq}\mathbf{R}_{px}\mathbf{R}_{qy} \\
&= \mathbf{R}_{x'x}\mathbf{R}_{x'x}\mathcal{C}'_{x'x'x'x'}\mathbf{R}_{x'x}\mathbf{R}_{x'y} + \mathbf{R}_{x'x}\mathbf{R}_{x'x}\mathcal{C}'_{x'x'y'y'}\mathbf{R}_{y'x}\mathbf{R}_{y'y} + \\
&\quad \mathbf{R}_{x'x}\mathbf{R}_{y'x}\mathcal{C}'_{x'y'x'y'}\mathbf{R}_{x'x}\mathbf{R}_{y'y} + \mathbf{R}_{x'x}\mathbf{R}_{y'x}\mathcal{C}'_{x'y'y'x'}\mathbf{R}_{y'x}\mathbf{R}_{x'y} + \\
&\quad \mathbf{R}_{y'x}\mathbf{R}_{x'x}\mathcal{C}'_{y'x'x'y'}\mathbf{R}_{x'x}\mathbf{R}_{y'y} + \mathbf{R}_{y'x}\mathbf{R}_{x'x}\mathcal{C}'_{y'x'y'x'}\mathbf{R}_{y'x}\mathbf{R}_{x'y} + \\
&\quad \mathbf{R}_{y'x}\mathbf{R}_{y'x}\mathcal{C}'_{y'y'x'x'}\mathbf{R}_{x'x}\mathbf{R}_{x'y} + \mathbf{R}_{y'x}\mathbf{R}_{y'x}\mathcal{C}'_{y'y'y'y'}\mathbf{R}_{y'x}\mathbf{R}_{y'y}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \mathcal{C}_{xxxy} &= c^3 s\left(\mathcal{C}'_{x'x'x'x'} - \mathcal{C}'_{x'x'y'y'} - \mathcal{C}'_{x'y'x'y'} - \mathcal{C}'_{y'x'x'y'}\right) + \\
&\quad cs^3\left(\mathcal{C}'_{x'y'y'x'} + \mathcal{C}'_{y'x'y'x'} + \mathcal{C}'_{y'y'x'x'} - \mathcal{C}'_{y'y'y'y'}\right)
\end{aligned}
$$

    - A traction $\sigma_{xx}$ along $Ox$ induces a shearing $\varepsilon_{xy}$ due to the fiber orientation



- Generally orthotropic ply mechanics (7)
  - Plane σ state (5)
    - All the non-zero components are

$$
\left\{
\begin{array}{l}
\mathcal{C}_{xxxx} = c^4\mathcal{C}'_{x'x'x'x'} + 2c^2s^2\left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'}\right) + s^4\mathcal{C}'_{y'y'y'y'} \\
\mathcal{C}_{yyyy} = s^4\mathcal{C}'_{x'x'x'x'} + 2c^2s^2\left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'}\right) + c^4\mathcal{C}'_{y'y'y'y'} \\
\mathcal{C}_{xxyy} = \mathcal{C}_{yyxx} = \left(c^4 + s^4\right)\mathcal{C}'_{x'x'y'y'} + c^2s^2\left(\mathcal{C}'_{x'x'x'x'} + \mathcal{C}'_{y'y'y'y'} - 4\mathcal{C}'_{x'y'x'y'}\right) \\
\mathcal{C}_{xyxy} = \mathcal{C}_{xyyx} = \mathcal{C}_{yxxy} = \mathcal{C}_{yxyx} = \\
\quad \left(c^2 - s^2\right)^2\mathcal{C}'_{x'y'x'y'} + c^2s^2\left(\mathcal{C}'_{x'x'x'x'} + \mathcal{C}'_{y'y'y'y'} - 2\mathcal{C}'_{x'x'y'y'}\right) \\
\mathcal{C}_{xxxy} = \mathcal{C}_{xyxx} = \mathcal{C}_{xxyx} = \mathcal{C}_{yxxx} = \\
\quad c^3s\left(\mathcal{C}'_{x'x'x'x'} - \mathcal{C}'_{x'x'y'y'} - 2\mathcal{C}'_{x'y'x'y'}\right) + cs^3\left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'} - \mathcal{C}'_{y'y'y'y'}\right) \\
\mathcal{C}_{yyxy} = \mathcal{C}_{xyyy} = \mathcal{C}_{yyyx} = \mathcal{C}_{yxyy} = \\
\quad cs^3\left(\mathcal{C}'_{x'x'x'x'} - \mathcal{C}'_{x'x'y'y'} - 2\mathcal{C}'_{x'y'x'y'}\right) + c^3s\left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'} - \mathcal{C}'_{y'y'y'y'}\right)
\end{array}
\right.
$$

- Generally orthotropic ply mechanics (8)
  - Plane $\sigma$ state (6)
    - Can be rewritten under the form

$$\begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{xy} \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{xxxx} & \mathcal{C}_{xxyy} & 2\mathcal{C}_{xxxy} \\ \mathcal{C}_{yyxx} & \mathcal{C}_{yyyy} & 2\mathcal{C}_{yyxy} \\ \mathcal{C}_{xyxx} & \mathcal{C}_{xyyy} & 2\mathcal{C}_{xyxy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_{xx} \\ \boldsymbol{\varepsilon}_{yy} \\ \boldsymbol{\varepsilon}_{xy} \end{pmatrix}$$

$$\boldsymbol{\sigma}_{xx} = \mathcal{C}_{xxxx}\boldsymbol{\varepsilon}_{xx} + \mathcal{C}_{xxyy}\boldsymbol{\varepsilon}_{yy} + \mathcal{C}_{xxxy}\boldsymbol{\varepsilon}_{xy} + \mathcal{C}_{xxyx}\boldsymbol{\varepsilon}_{yx}$$

$$\boldsymbol{\sigma}_{xy} = \mathcal{C}_{xyxx}\boldsymbol{\varepsilon}_{xx} + \mathcal{C}_{xyyy}\boldsymbol{\varepsilon}_{yy} + \mathcal{C}_{xyxy}\boldsymbol{\varepsilon}_{xy} + \mathcal{C}_{xyyx}\boldsymbol{\varepsilon}_{yx}$$

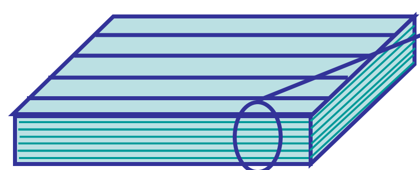
    - Remark: a symmetric matrix (not a tensor) can be recovered by using
      - The shear angle $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\,\varepsilon_{xy}$
      - $$\begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{xy} \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{xxxx} & \mathcal{C}_{xxyy} & \mathcal{C}_{xxxy} \\ \mathcal{C}_{yyxx} & \mathcal{C}_{yyyy} & \mathcal{C}_{yyxy} \\ \mathcal{C}_{xyxx} & \mathcal{C}_{xyyy} & \mathcal{C}_{xyxy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_{xx} \\ \boldsymbol{\varepsilon}_{yy} \\ \gamma_{xy} \end{pmatrix}$$

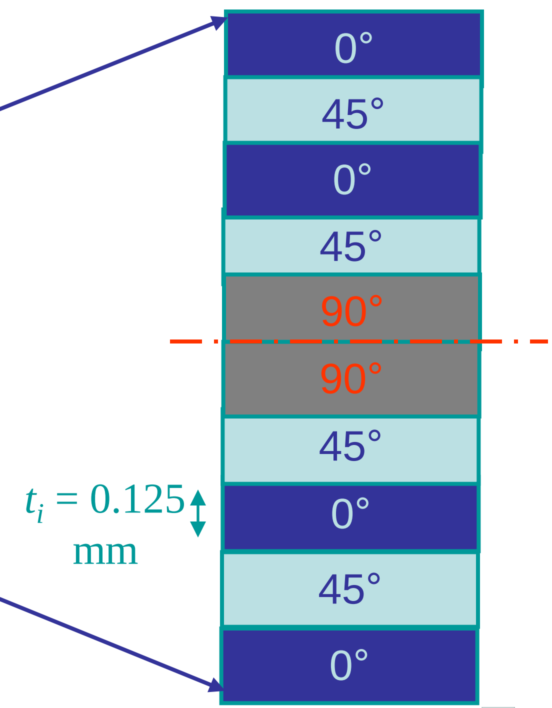
Tension/shearing coupling

- Laminated composite
  - A laminate is the superposition of different plies
    - For a ply $i$ of general orientation $\theta_i$, there is a coupling between tension and shearing

$$\begin{pmatrix} \boldsymbol{\sigma}^i_{xx} \\ \boldsymbol{\sigma}^i_{yy} \\ \boldsymbol{\sigma}^i_{xy} \end{pmatrix} = \begin{pmatrix} \mathcal{C}^i_{xxxx} & \mathcal{C}^i_{xxyy} & \mathcal{C}^i_{xxxy} \\ \mathcal{C}^i_{yyxx} & \mathcal{C}^i_{yyyy} & \mathcal{C}^i_{yyxy} \\ \mathcal{C}^i_{xyxx} & \mathcal{C}^i_{xyyy} & \mathcal{C}^i_{xyxy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}^i_{xx} \\ \boldsymbol{\varepsilon}^i_{yy} \\ \gamma^i_{xy} \end{pmatrix}$$

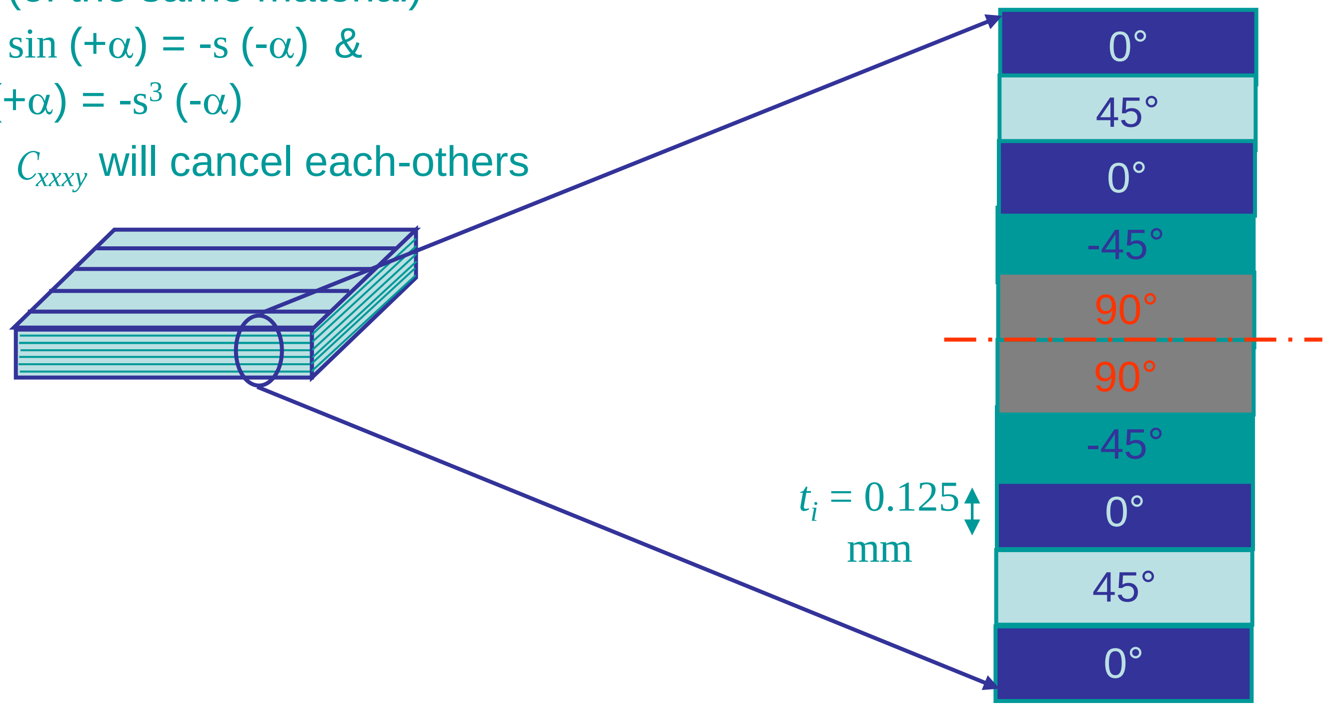
  - Symmetrical laminate
    - Symmetrical geometric and material distribution
    - Technological & coupling considerations (see later)



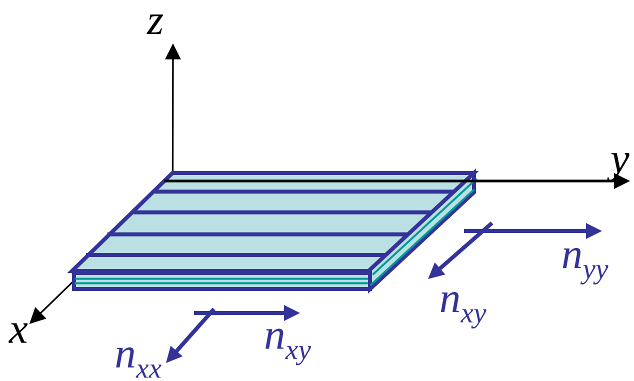



    - To ease the transfer of stress from one layer to the other
      - No more than 45° difference between plies

- Laminated composite (2)
  - Suppression of tensile/shearing coupling
    - $$\mathcal{C}_{xxxy} = c^3 s \left(\mathcal{C}'_{x'x'x'x'} - \mathcal{C}'_{x'x'y'y'} - \mathcal{C}'_{x'y'x'y'} - \mathcal{C}'_{y'x'x'y'}\right) + cs^3 \left(\mathcal{C}'_{x'y'y'x'} + \mathcal{C}'_{y'x'y'x'} + \mathcal{C}'_{y'y'x'x'} - \mathcal{C}'_{y'y'y'y'}\right)$$
    - Suppression of tensile/shearing coupling requires
      - Same proportion in $+\alpha°$ and $-\alpha°$ oriented laminas (of the same material)
    - Then $s(+\alpha) = \sin(+\alpha) = -s(-\alpha)$ & $s^3(+\alpha) = \sin^3(+\alpha) = -s^3(-\alpha)$
    - So two terms $\mathcal{C}_{xxxy}$ will cancel each-others



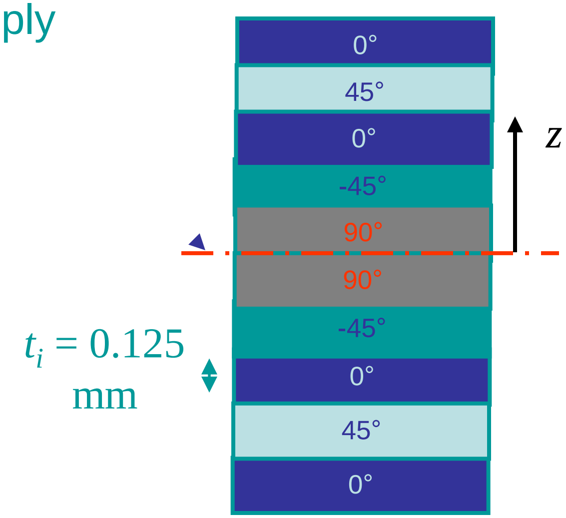
- Laminated composite (3)
  - Resulting elastic properties of a laminate can be deduced
  - Deformations of the laminate assumed to correspond to a plate
    - Membrane mode & resultant membrane stresses



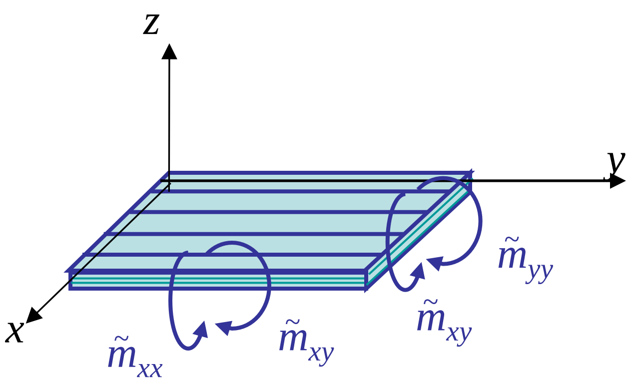

$$\begin{cases} n_{xx} = \int_h \boldsymbol{\sigma}_{xx} dz = \boldsymbol{n}_x^x = \left(\int_h \boldsymbol{\sigma} \cdot \boldsymbol{E}^x dz\right)_x \\ n_{yy} = \int_h \boldsymbol{\sigma}_{yy} dz = \boldsymbol{n}_y^y = \left(\int_h \boldsymbol{\sigma} \cdot \boldsymbol{E}^y dz\right)_y \\ n_{xy} = \int_h \boldsymbol{\sigma}_{xy} dz = \boldsymbol{n}_y^x = \boldsymbol{n}_x^y = \left(\int_h \boldsymbol{\sigma} \cdot \boldsymbol{E}^y dz\right)_x \end{cases}$$

    - For a laminate the integration is performed on each ply

$$\begin{cases} n_{xx} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{xx} dz \\ n_{yy} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{yy} dz \\ n_{xy} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{xy} dz \end{cases}$$



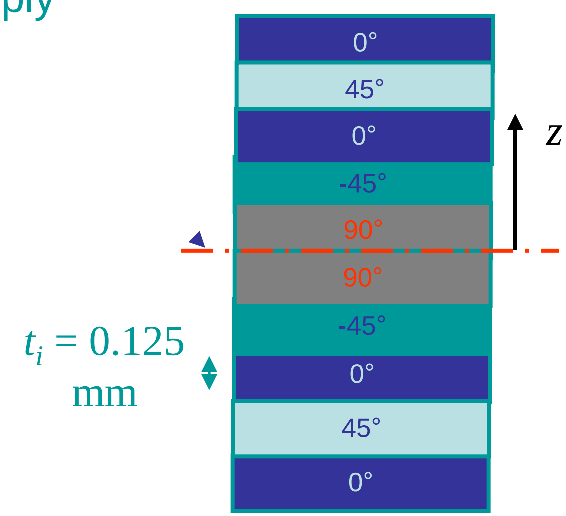
- Laminated composite (4)
  - Deformations of the laminate assumed to correspond to a plate (2)
    - Bending mode & resultant bending stresses



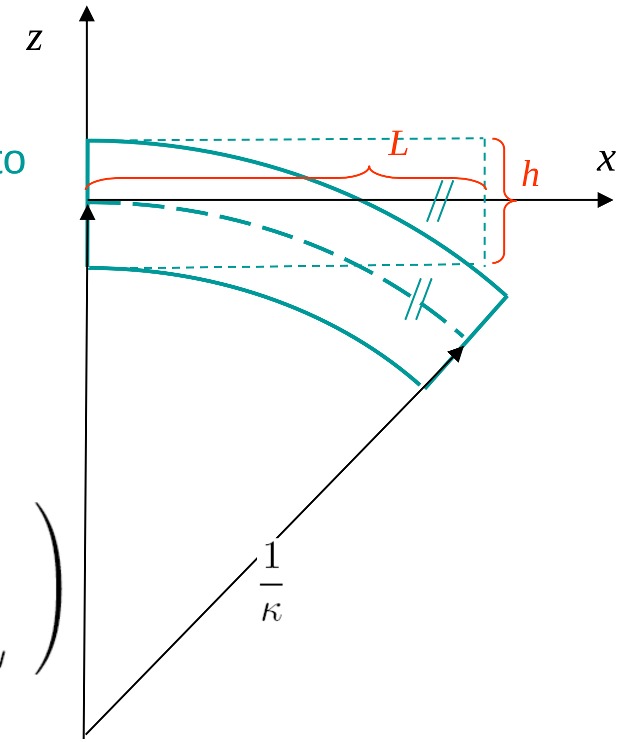

$$\begin{cases} \tilde{m}_{xx} = \int_h \boldsymbol{\sigma}_{xx} z dz = \tilde{\boldsymbol{m}}_x^x = \left( \int_h \boldsymbol{\sigma} \cdot \boldsymbol{E}^x z dz \right)_x \\ \tilde{m}_{yy} = \int_h \boldsymbol{\sigma}_{yy} z dz = \tilde{\boldsymbol{m}}_y^y = \left( \int_h \boldsymbol{\sigma} \cdot \boldsymbol{E}^y z dz \right)_y \\ \tilde{m}_{xy} = \int_h \boldsymbol{\sigma}_{xy} z dz = \tilde{\boldsymbol{m}}_y^x = \tilde{\boldsymbol{m}}_x^y = \left( \int_h \boldsymbol{\sigma} \cdot \boldsymbol{E}^y z dz \right)_x \end{cases}$$

- For a laminate the integration is performed on each ply

$$\begin{cases} \tilde{m}_{xx} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{xx} z dz \\ \tilde{m}_{yy} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{yy} z dz \\ \tilde{m}_{xy} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{xy} z dz \end{cases}$$



- Laminated composite (5)
  - Stress-strain relationship
    - Deformation of the laminate can be separated into
      - Deformation of the neutral plane
      - Deformation due to bending
        - » See picture for beam analogy
    - Strains can then be expressed as



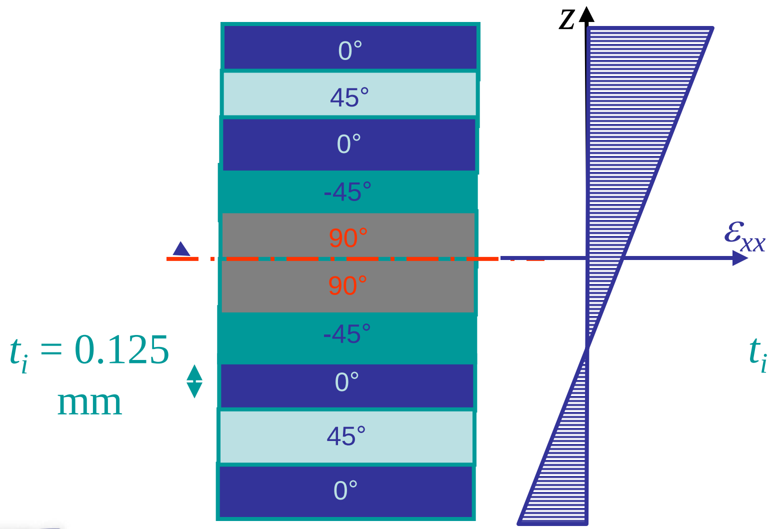

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx}\,(z=0) \\ \varepsilon_{yy}\,(z=0) \\ \gamma_{xy}\,(z=0) \end{pmatrix} + z \begin{pmatrix} -\boldsymbol{u}_{z,xx} \\ -\boldsymbol{u}_{z,yy} \\ -2\boldsymbol{u}_{z,xy} \end{pmatrix}$$

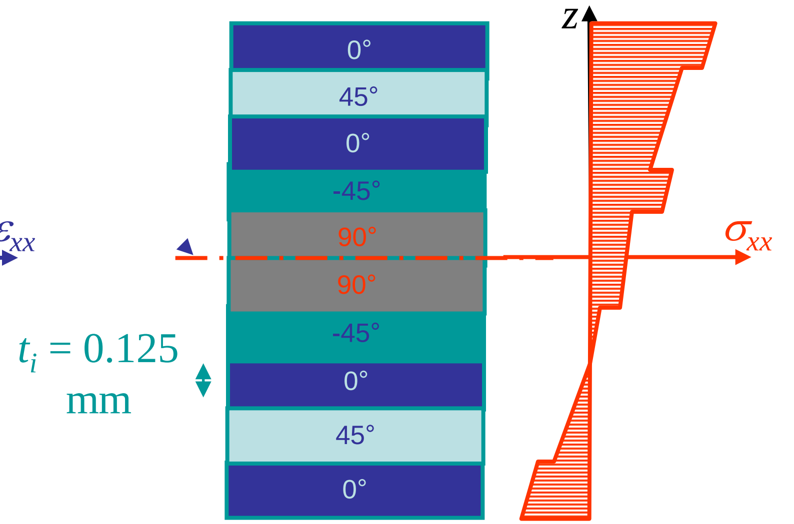
$$= \begin{pmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{pmatrix} + z \begin{pmatrix} \kappa^0_{xx} \\ \kappa^0_{yy} \\ \kappa^0_{xy} \end{pmatrix}$$

  - Exponent zero refers to neutral plane
    - » Assumed to be at $z = 0$
    - » In case of symmetric laminate it is located at the mid-plane
  - Second part of the course for rigorous demonstration

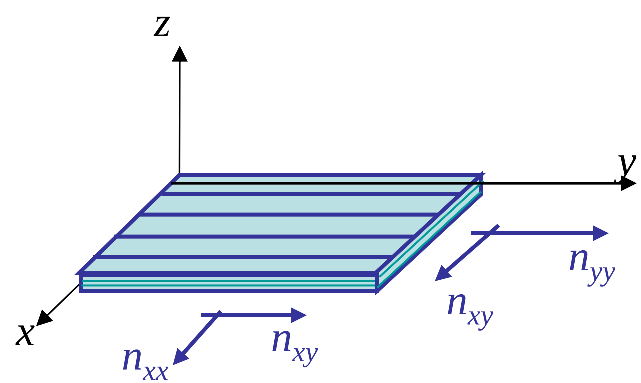
- Laminated composite (6)
  - Stress-strain relationship (2)
    - In each ply $\begin{pmatrix} \boldsymbol{\sigma}^i_{xx} \\ \boldsymbol{\sigma}^i_{yy} \\ \boldsymbol{\sigma}^i_{xy} \end{pmatrix} = \begin{pmatrix} \mathcal{C}^i_{xxxx} & \mathcal{C}^i_{xxyy} & \mathcal{C}^i_{xxxy} \\ \mathcal{C}^i_{yyxx} & \mathcal{C}^i_{yyyy} & \mathcal{C}^i_{yyxy} \\ \mathcal{C}^i_{xyxx} & \mathcal{C}^i_{xyyy} & \mathcal{C}^i_{xyxy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}^i_{xx} \\ \boldsymbol{\varepsilon}^i_{yy} \\ \gamma^i_{xy} \end{pmatrix}$

      With $\begin{pmatrix} \boldsymbol{\varepsilon}_{xx} \\ \boldsymbol{\varepsilon}_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\varepsilon}^0_{xx} \\ \boldsymbol{\varepsilon}^0_{yy} \\ \gamma^0_{xy} \end{pmatrix} + z \begin{pmatrix} \kappa^0_{xx} \\ \kappa^0_{yy} \\ \kappa^0_{xy} \end{pmatrix}$
    - So, using tensorial notation $\boldsymbol{\sigma}^i_{\alpha\beta} = \mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{\varepsilon}^0_{\gamma\delta} - z\mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{u}^0_{z,\gamma\delta}$
    - As properties change in each ply, this theoretically leads to discontinuous stress



- Laminated composite (7)
  - Stress-strain relationship (3)
    - Membrane resultant stress
      - As in each ply




$$\boldsymbol{\sigma}^i_{\alpha\beta} = \mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{\varepsilon}^0_{\gamma\delta} - z\mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{u}^0_{z,\gamma\delta}$$

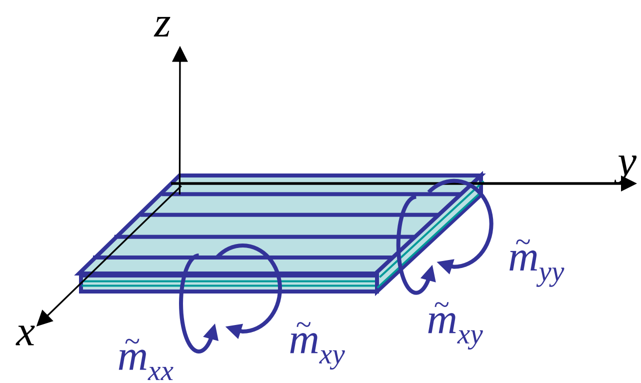
$$\Rightarrow n_{\alpha\beta} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{\alpha\beta} dz = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \int_{z_i}^{z_{i+1}} \mathcal{C}^i_{\alpha\beta\gamma\delta} dz - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \int_{z_i}^{z_{i+1}} z\mathcal{C}^i_{\alpha\beta\gamma\delta} dz$$

$$\Rightarrow n_{\alpha\beta} = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} \frac{z^2_{i+1} - z^2_i}{2}$$

$$\Rightarrow n_{\alpha\beta} = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i$$

- With the position of the neutral plane of each ply $\bar{z}_i = \dfrac{z_{i+1} + z_i}{2}$

- Laminated composite (8)
  - Stress-strain relationship (4)
    - Bending resultant stress
      - As in each ply




$$\boldsymbol{\sigma}^i_{\alpha\beta} = \mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{\varepsilon}^0_{\gamma\delta} - z\mathcal{C}^i_{\alpha\beta\gamma\delta}\boldsymbol{u}^0_{z,\gamma\delta}$$

$$\Rightarrow \tilde{m}_{\alpha\beta} = \sum_i \int_{z_i}^{z_{i+1}} \boldsymbol{\sigma}_{\alpha\beta} z dz = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \int_{z_i}^{z_{i+1}} z\mathcal{C}^i_{\alpha\beta\gamma\delta} dz - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \int_{z_i}^{z_{i+1}} z^2\mathcal{C}^i_{\alpha\beta\gamma\delta} dz$$

$$\Rightarrow \tilde{m}_{\alpha\beta} = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} \frac{z^3_{i+1} - z^3_i}{3}$$

$$\Rightarrow \tilde{m}_{\alpha\beta} = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \frac{z^2_{i+1} + z^2_i + z_i z_{i+1}}{3}$$

$$\Rightarrow \tilde{m}_{\alpha\beta} = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \left[\left(\frac{z_{i+1} + z_i}{2}\right)^2 + \frac{(z_{i+1} - z_i)^2}{12}\right]$$

$$\Rightarrow \tilde{m}_{\alpha\beta} = \boldsymbol{\varepsilon}^0_{\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i - \boldsymbol{u}^0_{z,\gamma\delta} \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} \left(t_i \bar{z}^2_i + \frac{t^3_i}{12}\right)$$

- Laminated composite (9)
  - Stress-strain relationship (5)
    - The two equations are

$$\begin{cases} n_{\alpha\beta} = \varepsilon^0_{\gamma\delta} \overbrace{\boxed{\sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i}}^{A_{\alpha\beta\gamma\delta}} - \boldsymbol{u}^0_{z,\gamma\delta} \overbrace{\boxed{\sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i}}^{B_{\alpha\beta\gamma\delta}} \\ \tilde{m}_{\alpha\beta} = \varepsilon^0_{\gamma\delta} \underbrace{\boxed{\sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i}}_{B_{\alpha\beta\gamma\delta}} - \boldsymbol{u}^0_{z,\gamma\delta} \underbrace{\boxed{\sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} \left( t_i \bar{z}_i^2 + \frac{t_i^3}{12} \right)}}_{D_{\alpha\beta\gamma\delta}} \end{cases}$$

    - Which can be rewritten under the form

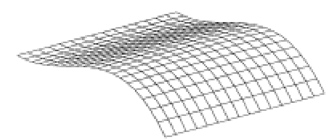
$$\begin{pmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \\ \tilde{m}_{xx} \\ \tilde{m}_{yy} \\ \tilde{m}_{xy} \end{pmatrix} = \begin{pmatrix} A_{xxxx} & A_{xxyy} & 2A_{xxxy} & B_{xxxx} & B_{xxyy} & 2B_{xxxy} \\ A_{yyxx} & A_{yyyy} & 2A_{yyxy} & B_{yyxx} & B_{yyyy} & 2B_{yyxy} \\ A_{xyxx} & A_{xyyy} & 2A_{xyxy} & B_{xyxx} & B_{xyyy} & 2B_{xyxy} \\ B_{xxxx} & B_{xxyy} & 2B_{xxxy} & D_{xxxx} & D_{xxyy} & 2D_{xxxy} \\ B_{yyxx} & B_{yyyy} & 2B_{yyxy} & D_{yyxx} & D_{yyyy} & 2D_{yyxy} \\ B_{xyxx} & B_{xyyy} & 2B_{xyxy} & D_{xyxx} & D_{xyyy} & 2D_{xyxy} \end{pmatrix} \begin{pmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \varepsilon^0_{xy} \\ -\boldsymbol{u}^0_{z,xx} \\ -\boldsymbol{u}^0_{z,yy} \\ -\boldsymbol{u}^0_{z,xy} \end{pmatrix}$$

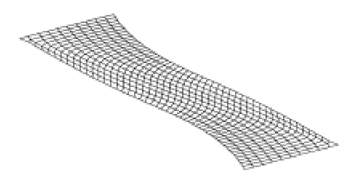
$$n_{xx} = A_{xxxx}\varepsilon^0_{xx} + A_{xxyy}\varepsilon^0_{yy} + A_{xxxy}\varepsilon^0_{xy} + A_{xxyx}\varepsilon^0_{yx} - B_{xxxx}\boldsymbol{u}^0_{z,xx} - B_{xxyy}\boldsymbol{u}^0_{z,yy} - B_{xxxy}\boldsymbol{u}^0_{z,xy} - B_{xxyx}\boldsymbol{u}^0_{z,yx}$$

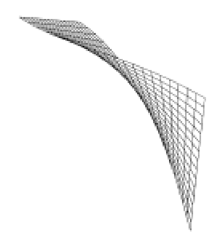
- Laminated composite (10)
  - Stress-strain relationship (6)
    - As $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx}$ & $\kappa_{xy} = -u_{z,xy} - u_{z,yx}$

$$\begin{pmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \\ \tilde{m}_{xx} \\ \tilde{m}_{yy} \\ \tilde{m}_{xy} \end{pmatrix} = \begin{pmatrix} A_{xxxx} & A_{xxyy} & A_{xxxy} & B_{xxxx} & B_{xxyy} & B_{xxxy} \\ A_{yyxx} & A_{yyyy} & A_{yyxy} & B_{yyxx} & B_{yyyy} & B_{yyxy} \\ A_{xyxx} & A_{xyyy} & A_{xyxy} & B_{xyxx} & B_{xyyy} & B_{xyxy} \\ B_{xxxx} & B_{xxyy} & B_{xxxy} & D_{xxxx} & D_{xxyy} & D_{xxxy} \\ B_{yyxx} & B_{yyyy} & B_{yyxy} & D_{yyxx} & D_{yyyy} & D_{yyxy} \\ B_{xyxx} & B_{xyyy} & B_{xyxy} & D_{xyxx} & D_{xyyy} & D_{xyxy} \end{pmatrix} \begin{pmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \\ \kappa^0_{xx} \\ \kappa^0_{yy} \\ \kappa^0_{xy} \end{pmatrix}$$

    - Terms $B$ are responsible for traction/bending coupling
      - With $B_{\alpha\beta\gamma\delta} = \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i \bar{z}_i$
      - A symmetrical stack prevents this coupling
        - » 2 identical $C^i$ at $z^i$ opposite
    - Terms $A_{xxxy}$ are responsible for tensile/shearing coupling
      - Can be avoided by using the same proportion of $+\alpha$ and $-\alpha$ plies
    - Terms $D_{xxxy}$ are responsible for torsion/bending coupling







- Symmetrical laminated composite
  - Stress-strain relationship
    - Terms $B$ vanish

$$\Rightarrow \begin{pmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{pmatrix} = \begin{pmatrix} A_{xxxx} & A_{xxyy} & A_{xxxy} \\ A_{yyxx} & A_{yyyy} & A_{yyxy} \\ A_{xyxx} & A_{xyyy} & A_{xyxy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_{xx}^{0} \\ \boldsymbol{\varepsilon}_{yy}^{0} \\ \gamma_{xy}^{0} \end{pmatrix}$$

    - If $h$ is the laminate thickness

$$\Rightarrow \begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{xy} \end{pmatrix} = \frac{1}{h} \begin{pmatrix} A_{xxxx} & A_{xxyy} & A_{xxxy} \\ A_{yyxx} & A_{yyyy} & A_{yyxy} \\ A_{xyxx} & A_{xyyy} & A_{xyxy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_{xx}^{0} \\ \boldsymbol{\varepsilon}_{yy}^{0} \\ \gamma_{xy}^{0} \end{pmatrix}$$

    - As $A_{\alpha\beta\gamma\delta} = \sum_i \mathcal{C}^i_{\alpha\beta\gamma\delta} t_i$ with

$$\mathcal{C}_{xxxy} = \mathcal{C}_{xyxx} = \mathcal{C}_{xxyx} = \mathcal{C}_{yxxx} =$$
$$c^3 s \left(\mathcal{C}'_{x'x'x'x'} - \mathcal{C}'_{x'x'y'y'} - 2\mathcal{C}'_{x'y'x'y'}\right) + cs^3 \left(\mathcal{C}'_{x'x'y'y'} + 2\mathcal{C}'_{x'y'x'y'} - \mathcal{C}'_{y'y'y'y'}\right)$$

      - Supression of tensile/shearing coupling requires same proportion in $+\alpha°$ and $-\alpha°$ oriented laminas (of the same material)
      - Then $s(+\alpha) = \sin(+\alpha) = -s(-\alpha)$ & $s^3(+\alpha) = \sin^3(+\alpha) = -s^3(-\alpha)$
      - So two terms $\mathcal{C}_{xxxy}$ will cancel each-others

- Symmetrical laminated composite without tensile/shearing coupling
  - Stress-strain relationship
    - Terms $B$ & $A_{xxxy}$ vanish

$$\Longrightarrow \begin{pmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{pmatrix} = \begin{pmatrix} A_{xxxx} & A_{xxyy} & A_{xxxy} \\ A_{yyxx} & A_{yyyy} & A_{yyxy} \\ A_{xyxx} & A_{xyyy} & A_{xyxy} \end{pmatrix} \begin{pmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{pmatrix}$$

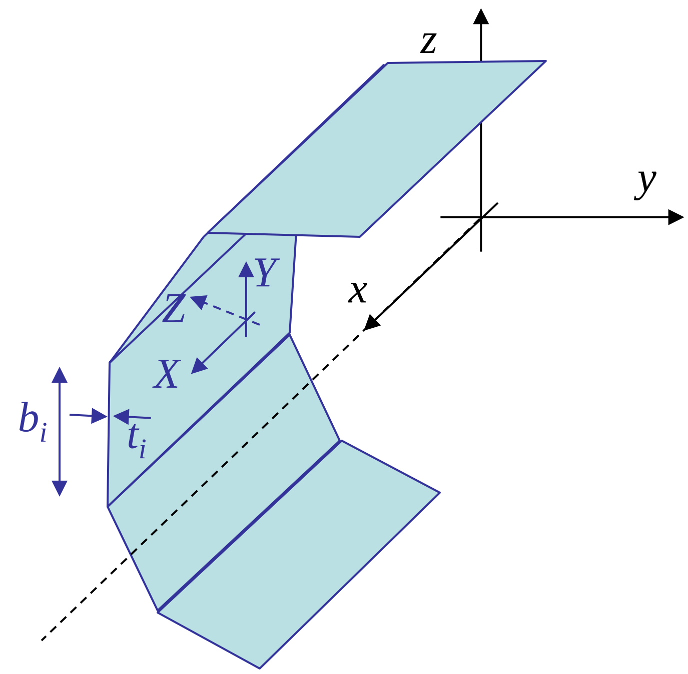
    - To be compared with an orthotropic material

$$\begin{pmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{xy} \end{pmatrix} = \begin{pmatrix} \frac{E_x}{1-\nu_{xy}\nu_{yx}} & \frac{\nu_{yx}E_x}{1-\nu_{xy}\nu_{yx}} & 0 \\ \frac{\nu_{xy}E_y}{1-\nu_{xy}\nu_{yx}} & \frac{E_y}{1-\nu_{xy}\nu_{yx}} & 0 \\ 0 & 0 & 2\mu_{xy} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_{xx} \\ \boldsymbol{\varepsilon}_{yy} \\ \boldsymbol{\varepsilon}_{xy} \end{pmatrix}$$

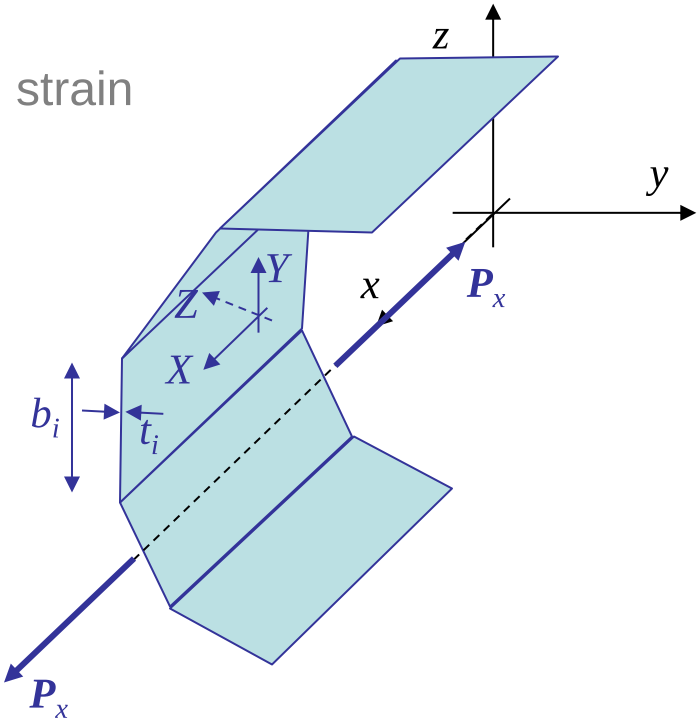
  - Homogenized orthotropic material

$$\begin{cases} E_x = \dfrac{A_{xxxx}A_{yyyy} - A^2_{xxyy}}{hA_{yyyy}} \\[2ex] E_y = \dfrac{A_{xxxx}A_{yyyy} - A^2_{xxyy}}{hA_{xxxx}} \\[2ex] \nu_{xy} = \dfrac{A_{xxyy}}{A_{yyyy}} \quad \& \quad \nu_{yx} = \dfrac{A_{xxyy}}{A_{xxxx}} \end{cases}$$

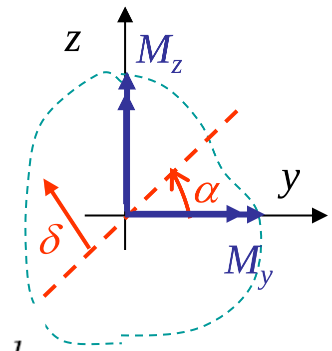
- Description
  - Consider a thin-walled beam made of laminated composite panels
  - The $i$ th panel
    - Has a thickness $t_i$
    - Has a width $b_i$
    - Has a local frame $OXYZ$
      - $OX$ correspond to the beam axis $Ox$
      - $OY$ is in the panel plane
      - $OZ$ is the out of plane direction
    - Has assumed homogenized properties
      - $E^i_X$, $E^i_Y$, $\nu^i_{XY}$ , $\nu^i_{YX}$ & $\mu^i_{XY}$
      - Rigorous for symmetrical laminates without tension/shearing coupling
      - Conceptual approximation for others
  - Panels are assumed
    - To have uniform stress on the thickness
      - Bending results from a stress distribution on the different panels
    - To carry direct and shearing stresses



- Axial loading
  - Consider that each panel has the same axial strain
    - By compatibility
    - $\boldsymbol{\varepsilon}^{i}_{XX} = \boldsymbol{\varepsilon}_{xx}$
  - Axial load carried by $i$ th panel
    - $\boldsymbol{P}_x^i = E_X^i \boldsymbol{\varepsilon}^i_{XX} t_i b_i = E_X^i t_i b_i \boldsymbol{\varepsilon}_{xx}$
  - Axial load carried by the beam
    - $\boldsymbol{P}_x = \sum_i \boldsymbol{P}_x^i = \boldsymbol{\varepsilon}_{xx} \sum_i E_X^i t_i b_i$
    - Or again $\boldsymbol{\varepsilon}_{xx} = \dfrac{\boldsymbol{P}_x}{\sum_i E_X^i t_i b_i}$



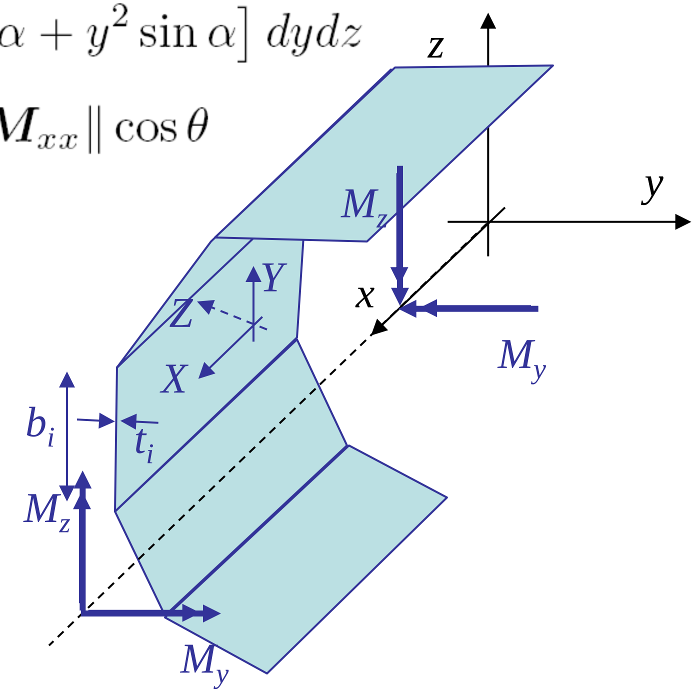
- Bending
  - For an homogeneous material, we found
    - $\boldsymbol{\sigma}_{xx} = \kappa E z \cos\alpha - \kappa E y \sin\alpha$
    - Leading to



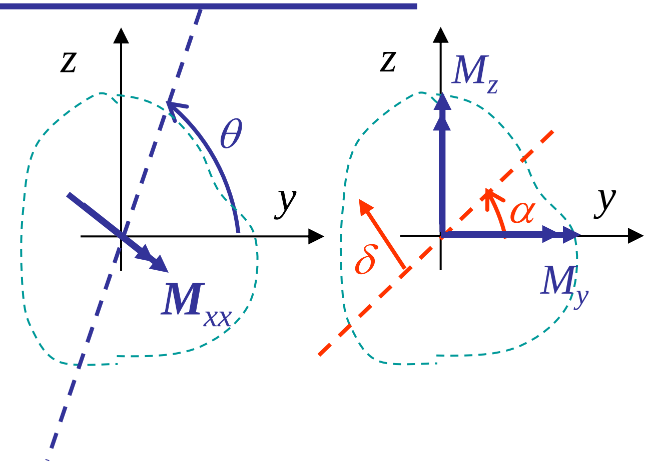

$$M_y = \int_A z\boldsymbol{\sigma}_{xx} dydz = \kappa E \int_A \left[z^2 \cos\alpha - yz\sin\alpha\right] dydz$$
$$= \kappa E I_{yy} \cos\alpha - \kappa E I_{yz} \sin\alpha = \|\boldsymbol{M}_{xx}\| \sin\theta$$
$$M_z = -\int_A y\boldsymbol{\sigma}_{xx} dydz = \kappa E \int_A \left[-zy\cos\alpha + y^2 \sin\alpha\right] dydz$$
$$= -\kappa E I_{yz} \cos\alpha + \kappa E I_{zz} \sin\alpha = -\|\boldsymbol{M}_{xx}\| \cos\theta$$

  - Here the homogenized Young modulus varies between panels
    - Curvature and neutral plane orientation are global
    - Stress in each panel becomes

$$\boldsymbol{\sigma}^i_{xx} = \kappa \left(E^i_X z\cos\alpha - E^i_X y \sin\alpha\right)$$



- Bending (2)
  - Moments
    - As $\boldsymbol{\sigma}^{i}_{xx} = \kappa \left(E^{i}_{X} z \cos\alpha - E^{i}_{X} y \sin\alpha\right)$
    - The bending moment become




– $\|\boldsymbol{M}_{xx}\| \sin\theta = M_y = \sum_i \int_{t_i \times b_i} z \boldsymbol{\sigma}^{i}_{xx} dA$

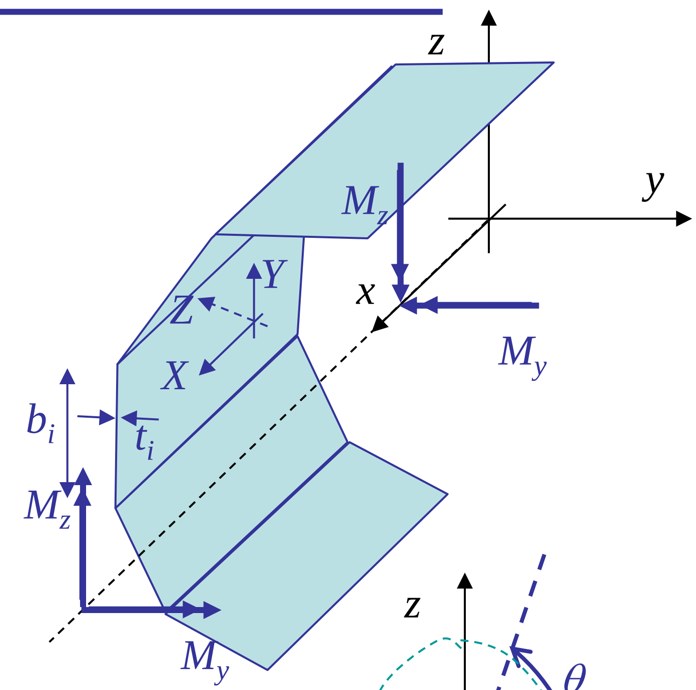
$$\Longrightarrow M_y = \kappa \cos\alpha \sum_i \int_{t_i \times b_i} E^{i}_{X} z^2 dA - \kappa \sin\alpha \sum_i \int_{t_i \times b_i} E^{i}_{X} yz dA$$

– $\|\boldsymbol{M}_{xx}\| \cos\theta = -M_z = \sum_i \int_{t_i \times b_i} y \boldsymbol{\sigma}^{i}_{xx} dA$

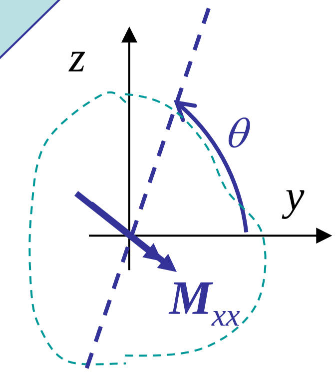
$$\Longrightarrow M_z = -\kappa \cos\alpha \sum_i \int_{t_i \times b_i} E^{i}_{X} yz dA + \kappa \sin\alpha \sum_i \int_{t_i \times b_i} E^{i}_{X} y^2 dA$$

Modified second moment of area $\bar{E}I_{zz} = \sum_i \int_{t_i \times b_i} E^{i}_{X} y^2 dA$

- Bending (3)
  - Moments (2)
    - Using the modified second moments of area
      - $\bar{E}I_{yy} = \sum_i \int_{t_i \times b_i} E_X^i z^2 dA$
      - $\bar{E}I_{zz} = \sum_i \int_{t_i \times b_i} E_X^i y^2 dA$
      - $\bar{E}I_{yz} = \sum_i \int_{t_i \times b_i} E_X^i yz dA$

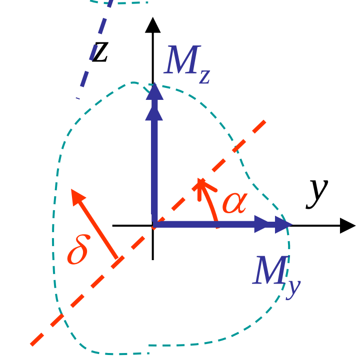



$$\Rightarrow \begin{cases} M_y = \kappa \cos\alpha \bar{E}I_{yy} - \kappa \sin\alpha \bar{E}I_{yz} = \|\boldsymbol{M}_{xx}\| \sin\theta \\ M_z = -\kappa \cos\alpha \bar{E}I_{yz} + \kappa \sin\alpha \bar{E}I_{zz} = -\|\boldsymbol{M}_{xx}\| \cos\theta \end{cases}$$




  - Orientation of the neutral axis
    - Using previous results

$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{\kappa} \begin{pmatrix} \bar{E}I_{yy} & -\bar{E}I_{yz} \\ -\bar{E}I_{yz} & \bar{E}I_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$



- Bending (4)
  - Stress
    - Starting from

$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{\kappa} \begin{pmatrix} \bar{E}I_{yy} & -\bar{E}I_{yz} \\ -\bar{E}I_{yz} & \bar{E}I_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{1}{\kappa\left(\bar{E}I_{yy}\bar{E}I_{zz} - \left(\bar{E}I_{yz}\right)^2\right)} \begin{pmatrix} \bar{E}I_{zz} & \bar{E}I_{yz} \\ \bar{E}I_{yz} & \bar{E}I_{yy} \end{pmatrix} \begin{pmatrix} M_y \\ M_z \end{pmatrix}$$

    - Using $\boldsymbol{\sigma}^i_{xx} = \kappa\left(E^i_X z\cos\alpha - E^i_X y \sin\alpha\right)$ to obtain the stress in wall number $i$

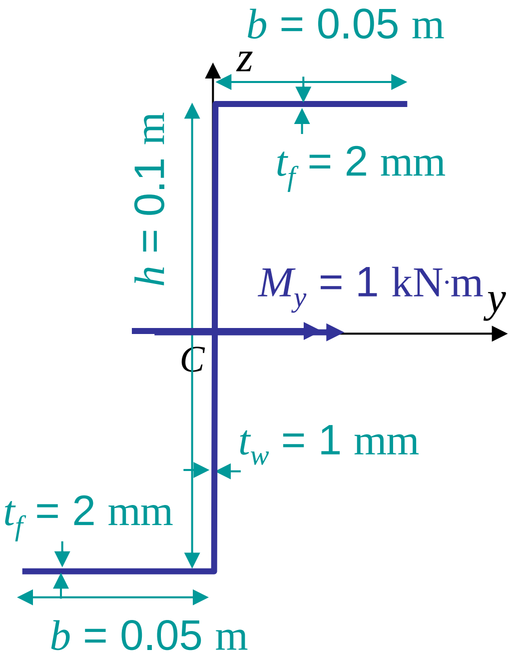
$$\Longrightarrow \boldsymbol{\sigma}^i_{xx} = \kappa E^i_X \begin{pmatrix} z & -y \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$\Longrightarrow \boldsymbol{\sigma}^i_{xx} = \frac{E^i_X}{\bar{E}I_{yy}\bar{E}I_{zz} - \left(\bar{E}I_{yz}\right)^2} \begin{pmatrix} z & -y \end{pmatrix} \begin{pmatrix} \bar{E}I_{zz} & \bar{E}I_{yz} \\ \bar{E}I_{yz} & \bar{E}I_{yy} \end{pmatrix} \begin{pmatrix} M_y \\ M_z \end{pmatrix}$$

    - At the end of the day, the stress in wall $i$ is

$$\Longrightarrow \boldsymbol{\sigma}^i_{xx} = E^i_X \frac{\left(\bar{E}I_{zz}M_y + \bar{E}I_{yz}M_z\right)z - \left(\bar{E}I_{yz}M_y + \bar{E}I_{yy}M_z\right)y}{\bar{E}I_{yy}\bar{E}I_{zz} - \left(\bar{E}I_{yz}\right)^2}$$

- Example
  - Thin-walled beam with composite cross section
    - Flange laminates
      - $E_{Xf}$ = 50 GPa
      - Thickness 2 mm
    - Web laminate
      - $E_{Xw}$ = 15 GPa
      - Thickness 1 mm
  - Bending in the vertical plane
    - $M_y$ = 1 kN·m
  - Maximum direct stress?
  - Neutral axis?



- Modified second-moments of area
  - $$\bar{E}I_{yy} = \sum_{i=1}^{3} \int_{t_i \times b_i} E^i z^2 dA \simeq 2\left(t_f b \frac{h^2}{4} E_{Xf}\right) + E_{Xw} \frac{t_w h^3}{12}$$

$$\Longrightarrow \bar{E}I_{yy} = 2\ 0.002\ 0.05 \frac{0.01}{4} 50\ 10^9 + 15\ 10^9 \frac{0.001\ 0.001}{12}$$

$$\Longrightarrow \bar{E}I_{yy} = 26.25\ 10^3\ \mathrm{N \cdot m^2}$$

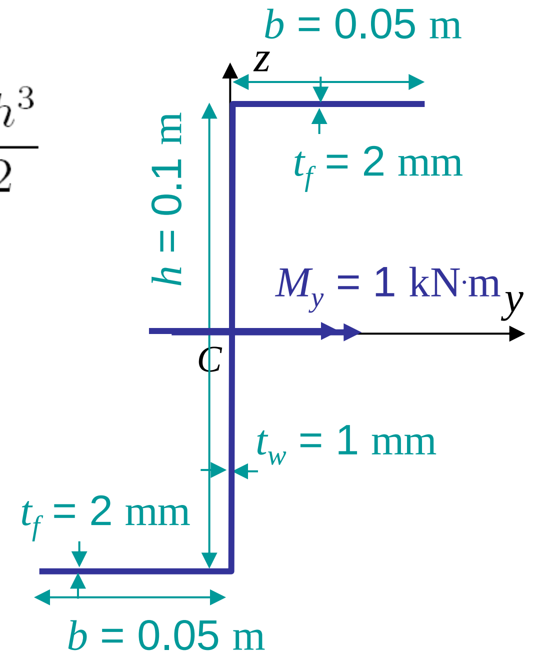
  - $$\bar{E}I_{zz} = \sum_{i=1}^{3} \int_{t_i \times b_i} E^i y^2 dA \simeq 2\left(t_f \frac{b^3}{12} + \frac{b^2}{4} t_f b\right) E_{Xf}$$

$$\Longrightarrow \bar{E}I_{zz} = 2\left(0.002 \frac{0.05^3}{12} + \frac{0.05^2}{4} 0.002\ 0.05\right) 50\ 10^9$$

$$\Longrightarrow \bar{E}I_{zz} = 8.33\ 10^3\ \mathrm{N \cdot m^2}$$

  - $$\bar{E}I_{yz} = \sum_{i=1}^{3} \int_{t_i \times b_i} E^i yz dA \simeq 2\left(t_f \frac{h}{2} \frac{b^2}{2}\right) E_{Xf}$$

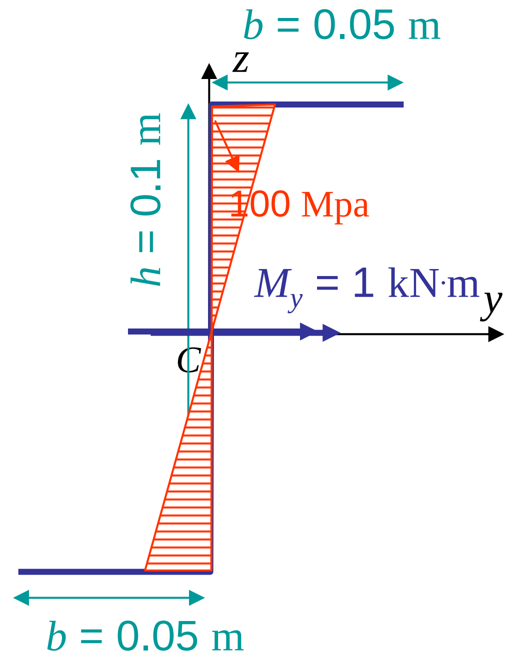
$$\Longrightarrow \bar{E}I_{yz} = 2\ 50\ 10^9\ 0.002\ 0.05 \frac{0.05^2}{2} = 12.5\ 10^3\ \mathrm{N \cdot m^2}$$



- Direct stress
  - In each wall, the stress is obtained from
    - $$\boldsymbol{\sigma}_{xx}^{i} = E_X^i \frac{\bar{E}I_{zz} M_y\, z - \bar{E}I_{yz} M_y\, y}{\bar{E}I_{yy}\bar{E}I_{zz} - \left(\bar{E}I_{yz}\right)^2}$$
    - With $D = \bar{E}I_{yy}\bar{E}I_{zz} - \left(\bar{E}I_{yz}\right)^2 = 26.25\, 8.33\, 10^6 - 12.5^2\, 10^6$
      $= 62.41\, 10^6\ \mathrm{N}^2{\cdot}\mathrm{m}^4$
  - Web
    - $$\boldsymbol{\sigma}_{xx}^{w} = E_{Xw} \frac{\bar{E}I_{zz} M_y\, z}{D}$$

$$\Rightarrow \boldsymbol{\sigma}_{xx}^{w} = 15\, 10^9 \frac{8.33\, 10^3\, 10^3}{62.41\, 10^6} z = 2\, 10^9\ z\, \mathrm{N} \cdot \mathrm{m}^{-3}$$

$$\Rightarrow \boldsymbol{\sigma}_{xx}^{w}\left(\pm\frac{h}{2}\right) = \pm 2\, 10^9\ 0.05 = \pm 100\ \mathrm{MPa}$$



- Direct stress (2)
  - In each wall, the stress is obtained from
    - $\boldsymbol{\sigma}_{xx}^{i} = E_X^i \frac{\bar{E}I_{zz} M_y\, z - \bar{E}I_{yz} M_y\, y}{\bar{E}I_{yy}\bar{E}I_{zz} - \left(\bar{E}I_{yz}\right)^2}$
    - With $D = \bar{E}I_{yy}\bar{E}I_{zz} - \left(\bar{E}I_{yz}\right)^2 = 62.41\ 10^6\ \mathrm{N}^2{\cdot}\mathrm{m}^4$
  - Flanges (+ = top, - = bottom)
    - $\boldsymbol{\sigma}_{xx}^{\pm f} = E_{Xf} \frac{\pm \bar{E}I_{zz} M_y \frac{h}{2} - \bar{E}I_{yz} M_y y}{D}$

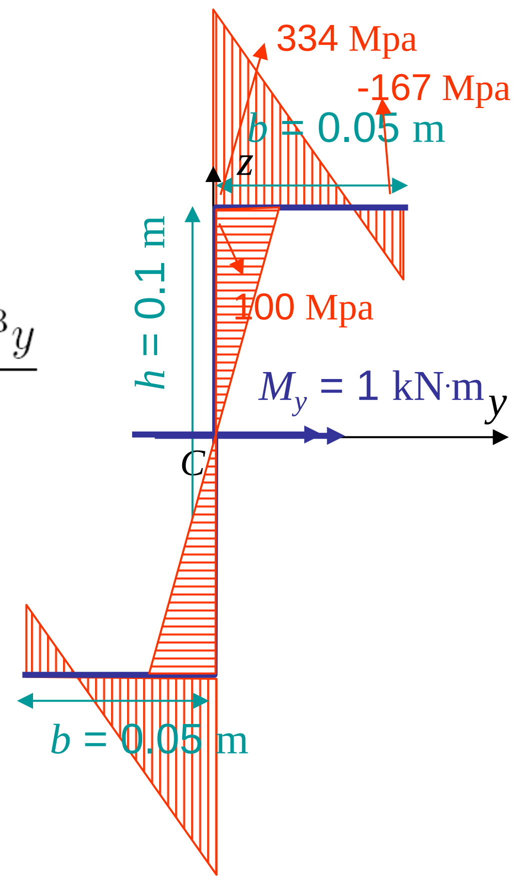
$$\Longrightarrow \boldsymbol{\sigma}_{xx}^{\pm f} = 50\ 10^9 \frac{\pm 8.33\ 10^3\ 10^3\ 0.05 - 12.5\ 10^3\ 10^3 y}{62.41\ 10^6}$$

$$\Longrightarrow \boldsymbol{\sigma}_{xx}^{\pm f} = \pm 333.7\ \mathrm{Mpa} - 10\ 10^9\ y\ \mathrm{N{\cdot}m^{-3}}$$

    - Maximal values?
      - $\boldsymbol{\sigma}_{xx}^{\pm f}(y=0) = \pm 333.7\ \mathrm{Mpa}$

        discontinuity with web

      - $\boldsymbol{\sigma}_{xx}^{\pm f}(y=\pm b) = \pm 333.7\ 10^6 \mp 10\ 10^9\ 0.05 = \mp 167\ \mathrm{Mpa}$



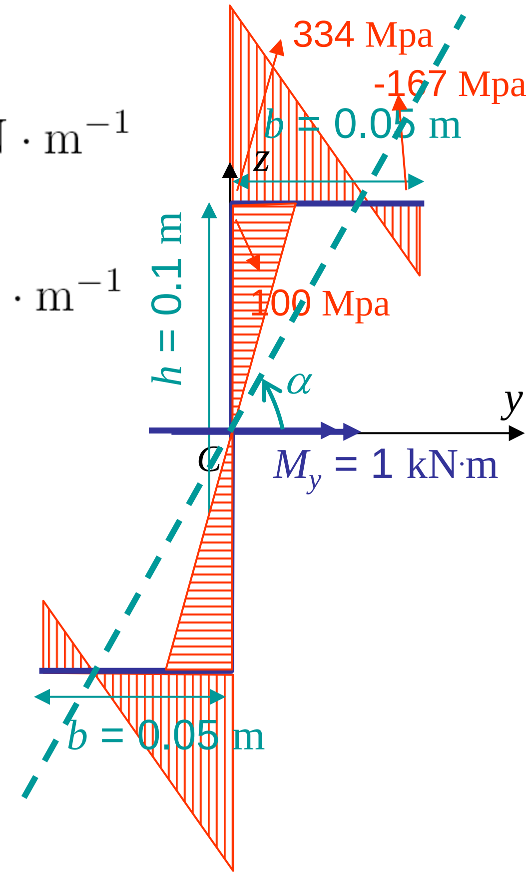
- Neutral axis
  - In each wall, the stresses satisfy
    - $\boldsymbol{\sigma}_{xx}^{i} = \kappa\, E_X^i\, z \cos\alpha - \kappa\, E_X^i\, y \sin\alpha$
    - Taking two points on top flange

$$\begin{cases} 333.7\ \mathrm{MPa} = \kappa\ 50\,10^9\ 0.05 \cos\alpha\ \mathrm{N}\cdot\mathrm{m}^{-1} \\ -167\ \mathrm{MPa} = \kappa\ 50\,10^9\ (0.05\cos\alpha - 0.05\sin\alpha)\ \mathrm{N}\cdot\mathrm{m}^{-1} \end{cases}$$

$$\Rightarrow \begin{cases} 0.133\ \mathrm{m}^{-1} = \kappa\ \cos\alpha \\ -167\ \mathrm{MPa} = 333.7\ \mathrm{MPa} - \kappa\ 50\,10^9\ 0.05\sin\alpha\ \mathrm{N}\cdot\mathrm{m}^{-1} \end{cases}$$

$$\Rightarrow \begin{cases} 0.133\ \mathrm{m}^{-1} = \kappa\ \cos\alpha \\ 0.2\ \mathrm{m}^{-1} = \kappa\ \sin\alpha \end{cases}$$

$$\Rightarrow \tan\alpha = 1.5 \Rightarrow \alpha = 56.31\,\mathrm{deg}$$

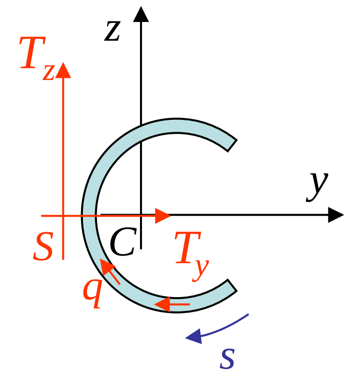



    - Curvature radius from

$$333.7\ \mathrm{MPa} = \kappa\ 50\,10^9\ 0.05\cos\alpha\ \mathrm{N}\cdot\mathrm{m}^{-1}$$

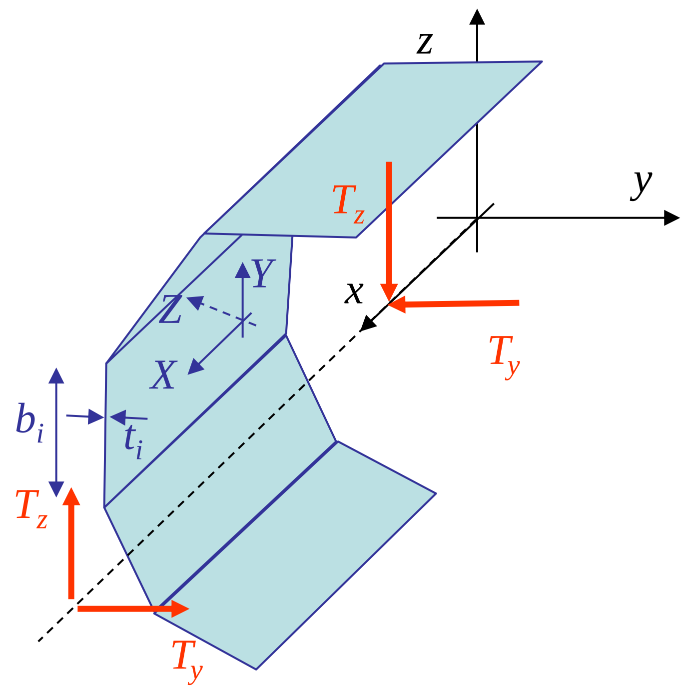
$$\Rightarrow \frac{1}{\kappa} = 13.51\ \mathrm{m}$$

- Shearing of open-section beams
  - For non-composite structures we found
    - $$q(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s tz\,ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s t(s')\,y(s')\,ds'$$

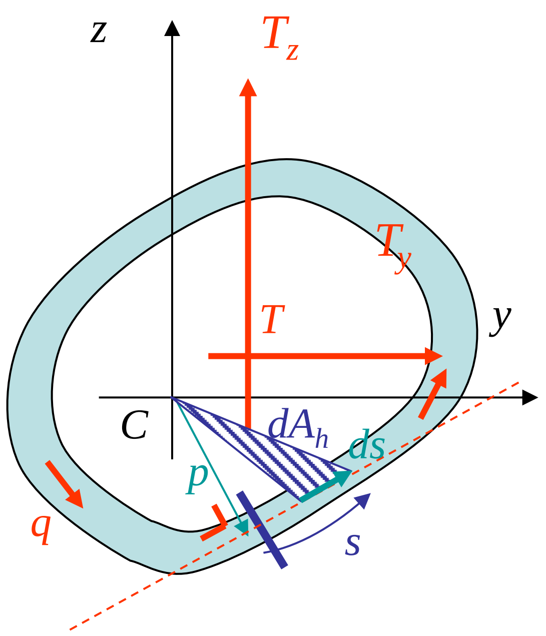



  - For composite wall, this expression becomes
    - Using direct stress expression for bending and same argumentation as thin walled beams
    - In wall $i$:

$$q^i(s) = -E_X^i\frac{\bar{E}I_{zz}T_z - \bar{E}I_{yz}T_y}{\bar{E}I_{yy}\bar{E}I_{zz} - \bar{E}I_{yz}^2}\int_0^s t_i z\,ds - E_X^i\frac{\bar{E}I_{yy}T_y - \bar{E}I_{yz}T_z}{\bar{E}I_{yy}\bar{E}I_{zz} - \bar{E}I_{yz}^2}\int_0^s t_i y\,ds$$



- Shearing of closed-section beams
  - $q(s) = q_o(s) + q(0)$
  - For non-composite structures we found
    - With $\begin{cases} q_o(s) = -\dfrac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\displaystyle\int_0^s t(s')\,z(s')\,ds' - \\ \qquad \dfrac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\displaystyle\int_0^s t(s')\,y(s')\,ds' \\ q(s=0) = \dfrac{y_T T_z - z_T T_y - \oint p(s)\,q_o(s)\,ds}{2A_h} \end{cases}$

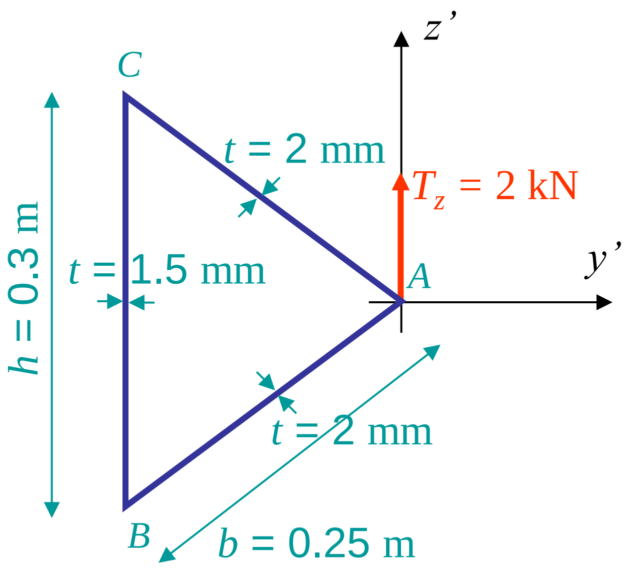



  - For composite wall, these expressions becomes

$$\begin{cases} q_o^i(s) = -E_X^i \dfrac{\bar{E}I_{zz}T_z - \bar{E}I_{yz}T_y}{\bar{E}I_{yy}\bar{E}I_{zz} - \bar{E}I_{yz}^2}\displaystyle\int_0^s t_i z ds - \\ \qquad E_X^i \dfrac{\bar{E}I_{yy}T_y - \bar{E}I_{yz}T_z}{\bar{E}I_{yy}\bar{E}I_{zz} - \bar{E}I_{yz}^2}\displaystyle\int_0^s t_i y ds \\ q(s=0) = \dfrac{y_T T_z - z_T T_y - \oint p(s)\,q_o(s)\,ds}{2A_h} \end{cases}$$

    - Anti-clockwise orientation for $q$, $s$ in this last expression

- Example
  - Simply symmetrical closed-section
    - Wall BC
      - $E_X^{BC}$ = 20 GPa
      - Thickness 1.5 mm
    - Walls AB & CA
      - $E_X^{AB} = E_X^{CA}$ = 45 GPa
      - Thickness 2 mm
  - Shear flow in the section?

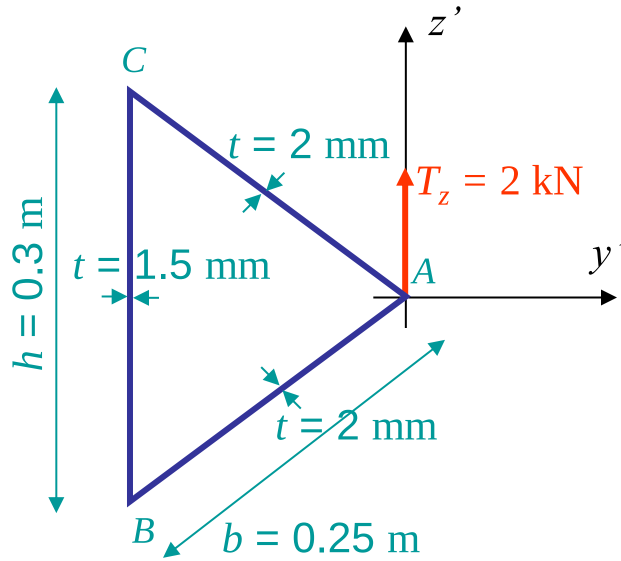


- Second moments of area
  - Simply symmetrical section

$$\Rightarrow \bar{E}I_{yz} = \sum_{i=1}^{3} \int_{t_i \times b_i} E^i yz dA = 0$$

  - So, as $T_y$=0, open shear flow reads

$$q_o^i(s) = -E_X^i \frac{T_z}{\bar{E}I_{yy}} \int_0^s t_i z ds$$




- With

$$\bar{E}I_{yy} = \bar{E}I_{y'y'} = \sum_{i=1}^{3} \int_{t_i \times b_i} E^i z'^2 dA \simeq E_X^{BC} \frac{t_{BC} h^3}{12} + 2E_X^{AC} t_{AC} \int_A^C z'^2 dl$$

$$\Rightarrow \bar{E}I_{yy} = E_X^{BC} \frac{t_{BC} h^3}{12} + 2E_X^{AC} t_{AC} \int_0^{\frac{h}{2}} z'^2 \frac{2|AB|}{h} dz' = E_X^{BC} \frac{t_{BC} h^3}{12} + \frac{4|AB| E_X^{AC} t_{AC}}{h} \frac{h^3}{24}$$

$$\Rightarrow \bar{E}I_{yy} = 20\,10^9 \frac{0.0015\ 0.3^3}{12} + \frac{4\ 0.25\ 45\ 10^9\ 0.002\ 0.3^3}{24\ 0.3}$$

$$\Rightarrow \bar{E}I_{yy} = 405\,10^3\ \mathrm{N} \cdot \mathrm{m}^2$$

- Open shear flow
  - Starting from

$$q_o^i(s) = -E_X^i \frac{T_z}{\bar{E}I_{yy}} \int_0^s t_i z ds$$

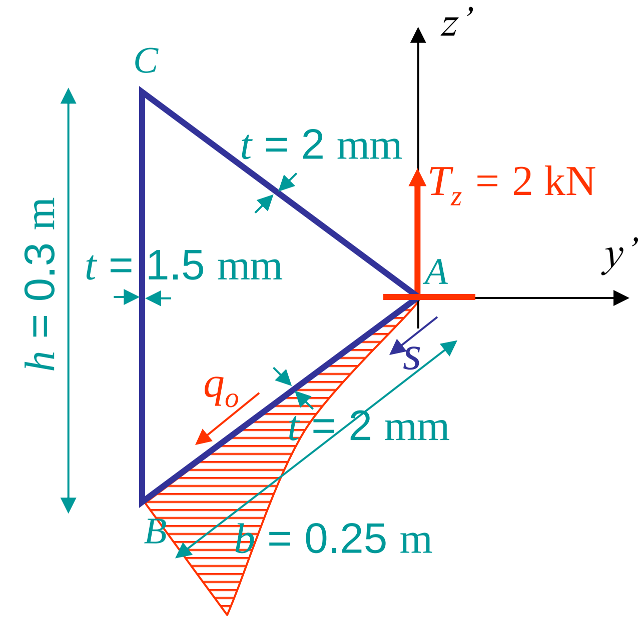
    - With a cut at the origin $A$

  - Wall $A \rightarrow B$

$$q_o^{AB}(s) = -E_X^{AB} \frac{T_z}{\bar{E}I_{yy}} \int_0^{z''} t_{AB} z' \frac{-2|AB|}{h} dz'$$
$$= E_X^{AB} \frac{2T_z|AB|}{h\bar{E}I_{yy}} t_{AB} \frac{z'^2}{2}$$

$$\Longrightarrow q_o^{AB}(z') = 45\,10^9 \frac{2\;2\,10^3\;0.250}{0.3\;405\,10^3} 0.002 \frac{z'^2}{2}$$
$$= 370\,10^3\; z'^2\; \mathrm{N \cdot m^{-3}}$$

$$\Longrightarrow q_o^{AB}(z' = -0.15) = 8.3\,10^3 \mathrm{N \cdot m^{-1}}$$



- Open shear flow (2)
  - Starting from (2)

$$q_o^i(s) = -E_X^i \frac{T_z}{\bar{E}I_{yy}} \int_0^s t_i z ds$$

  - With a cut at the origin $A$

  - Wall $A \rightarrow C$

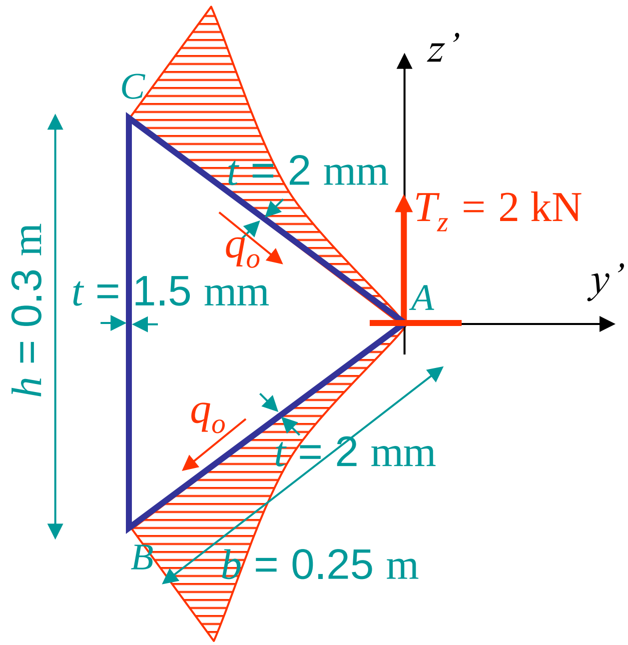
$$q_o^{AC}(s) = -E_X^{AC} \frac{T_z}{\bar{E}I_{yy}} \int_0^{z''} t_{AB} z' \frac{2|AB|}{h} dz'$$

$$= -E_X^{AC} \frac{2T_z|AB|}{h\bar{E}I_{yy}} t_{AB} \frac{z'^2}{2}$$

$$\Rightarrow q_o^{AC}(z') = -45\,10^9 \frac{2\;2\,10^3\;0.250}{0.3\;405\,10^3} 0.002 \frac{z'^2}{2}$$

$$= -370\,10^3\; z'^2\;\mathrm{N}\cdot\mathrm{m}^{-3}$$

$$\Rightarrow q_o^{AC}(z' = 0.15) = -8.3\,10^3 \mathrm{N}\cdot\mathrm{m}^{-1}$$



- Open shear flow (3)
  - Starting from (3)

$$q_o^i(s) = -E_X^i \frac{T_z}{\bar{E}I_{yy}} \int_0^s t_i z ds$$

- With a cut at the origin $A$

  - Wall $B \rightarrow C$

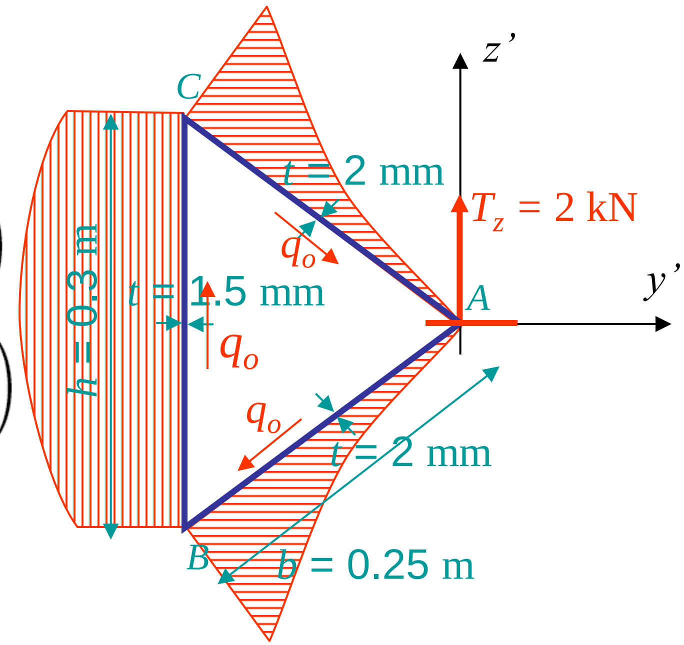
$$q_o^{BC}(s) = q_o^{AB}(z' = -0.15) - E_X^{BC} \frac{T_z}{\bar{E}I_{yy}} \int_{-\frac{h}{2}}^{z''} t_{BC} z' dz'$$

$$= q_o^{AB}(z' = -0.15) - E_X^{BC} \frac{T_z}{\bar{E}I_{yy}} t_{BC} \left(\frac{z'^2}{2} - \frac{h^2}{8}\right)$$

$$\Rightarrow q_o^{BC}(z') = 8.3\,10^3 - \frac{20\,10^9\ 2\,10^3\ 0.0015}{405\,10^3} \left(\frac{z'^2}{2} - \frac{0.3^2}{8}\right)$$

$$= 9.967\,10^3 \text{N}{\cdot}\text{m}^{-1} - 74.1\,10^3\ z'^2\ \text{N}{\cdot}\text{m}^{-3}$$

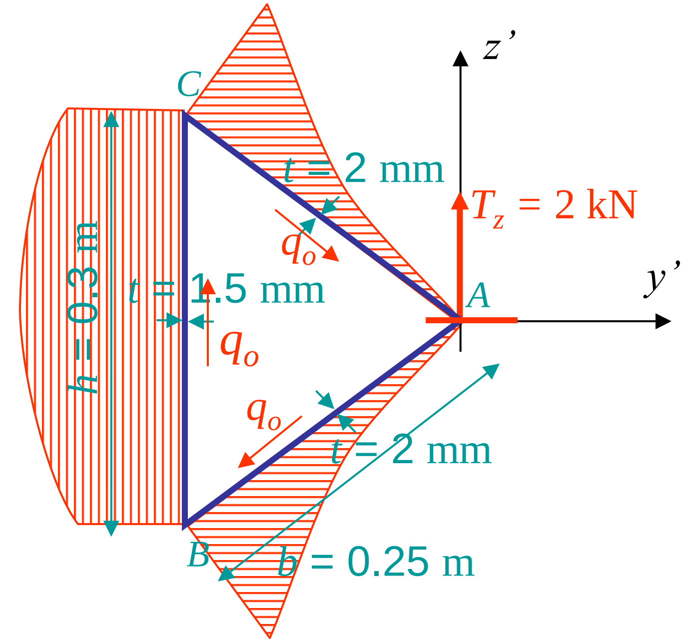
$$\Rightarrow \begin{cases} q_o^{BC}\left(z' = \frac{h}{2}\right) = 8.3\,10^3 \text{N} \cdot \text{m}^{-1} \\ q_o^{BC}\left(z' = -\frac{h}{2}\right) = 8.3\,10^3 \text{N} \cdot \text{m}^{-1} \\ q_o^{BC}(z' = 0) = 9.97\,10^3 \text{N} \cdot \text{m}^{-1} \end{cases}$$



- Closed shear flow
  - Starting from

$$q\left(s=0\right)=\frac{y_T T_z - z_T T_y - \oint p\left(s\right) q_o\left(s\right) ds}{2A_h}$$

- Give shear in anticlockwise direction around A




$$\Longrightarrow \quad q\left(A\right)=-\frac{\int_C^B q_o^{CB}\sqrt{b^2-\frac{h^2}{4}}\left(-dz'\right)}{2A_h}$$

$$\Longrightarrow \quad q\left(A\right)=-\frac{\int_{0.15}^{-0.15}\left(-9.967\,10^3+74.1\,10^3\;z'^2\right)\left(-dz'\right)\sqrt{0.25^2-0.15^2}}{2\frac{0.3\sqrt{0.25^2-0.15^2}}{2}}$$

$$\Longrightarrow \quad q\left(A\right)=-\frac{\left(-9.967\,10^3\;0.3+74.1\,10^3\;\frac{2\;0.15^3}{3}\right)}{0.3}=9.4\,10^3\;\mathrm{N}\cdot\mathrm{m}^{-1}$$

# Thin-walled composite beam

- Shear flow
  - Starting from

$$q_o^{AB}(z') = 370\,10^3\ z'^2\ \mathrm{N}\cdot\mathrm{m}^{-3}$$

$$q_o^{AC}(z') = -370\,10^3\ z'^2\ \mathrm{N}\cdot\mathrm{m}^{-3}$$

$$q_o^{BC}(z') = 9.967\ 10^3\mathrm{N}\cdot\mathrm{m}^{-1} - 74.1\,10^3\ z'^2\ \mathrm{N}\cdot\mathrm{m}^{-3}$$

  - With the anticlockwise constant flux

$$q(A) = 9.4\,10^3\ \mathrm{N}\cdot\mathrm{m}^{-1}$$

  - Total flux

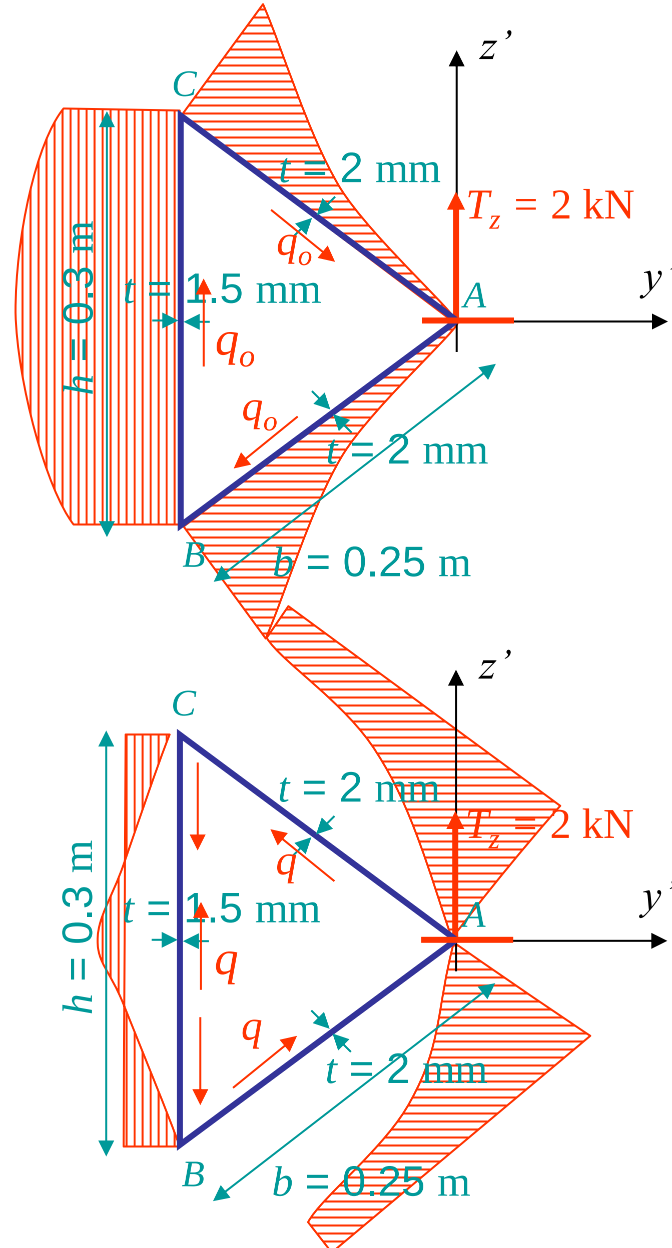
$$q^{AC}(z') = q_o^{AC}(z') + q(A) = -370\,10^3\ z'^2\ \mathrm{N}\cdot\mathrm{m}^{-3} + 9.4\,10^3\mathrm{N}\cdot\mathrm{m}^{-1}$$

$$\Rightarrow q^{AC}(0.15) = 1.1\,10^3\mathrm{N}\cdot\mathrm{m}^{-1}$$

$$q^{BA}(z') = -q_o^{AB}(z') + q(A) = -370\,10^3\ z'^2\ \mathrm{N}\cdot\mathrm{m}^{-3} + 9.4\,10^3\mathrm{N}\cdot\mathrm{m}^{-1}$$

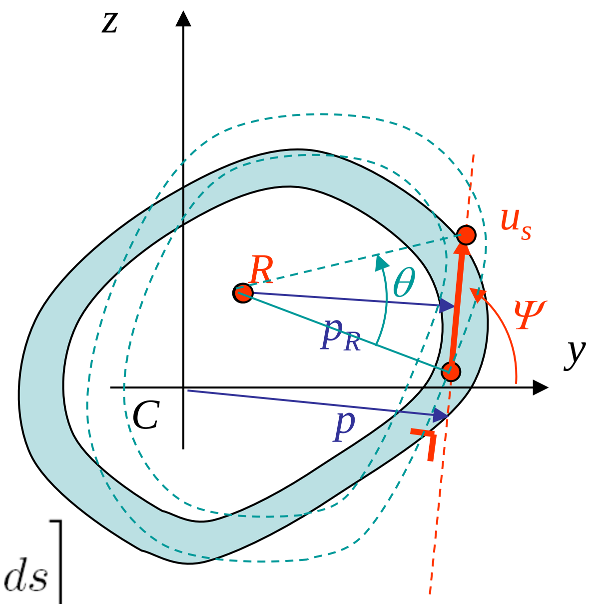
$$\Rightarrow q^{BA}(-0.15) = 1.1\,10^3\mathrm{N}\cdot\mathrm{m}^{-1}$$

$$q^{CB}(z') = -q_o^{BC}(z') + q(A) = -0.6\ 10^3\mathrm{N}\cdot\mathrm{m}^{-1} + 74.1\,10^3\ z'^2\ \mathrm{N}\cdot\mathrm{m}^{-3}$$



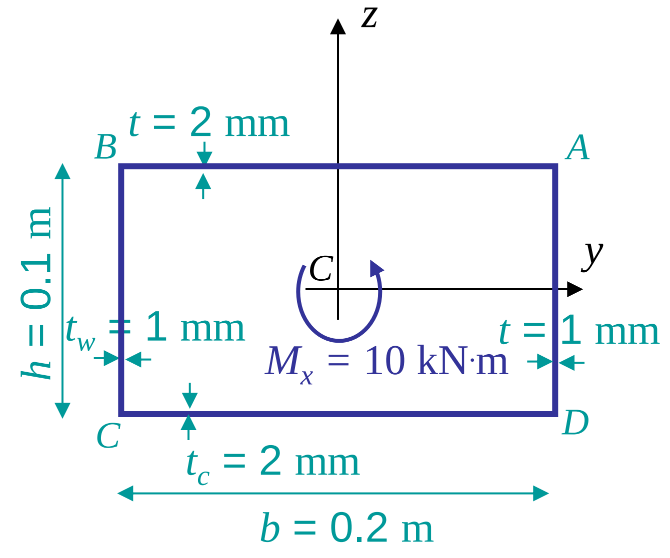
- Torsion
  - Closed-section beam
    - From previous analysis
      - Shear flow $M_x = 2A_h q$
      - Twist rate $\theta_{,x} = \frac{M_x}{4A_h^2}\oint \frac{1}{\mu t} ds$
      - Warping

$$\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \frac{M_x}{2A_h}\left[\int_0^s \frac{1}{\mu t} ds - \frac{A_{R_p}(s)}{A_h}\oint \frac{1}{\mu t} ds\right]$$

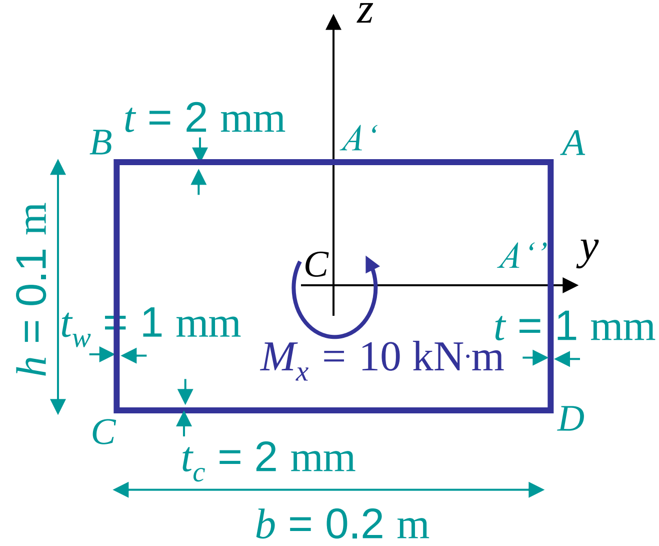



    - For composite beams
      - Twist rate $\theta_{,x} = \frac{M_x}{4A_h^2}\oint \frac{ds}{\mu_{XY}^i t_i}$
      - Rearranging terms:
        - » $M_x = \mu\bar{I}_T\theta_{,x}$ with the torsional stiffness $\mu\bar{I}_T = \frac{4A_h^2}{\oint \frac{ds}{\mu_{XY}^i t_i}}$
      - Warping $\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \frac{M_x}{2A_h}\left[\int_0^s \frac{ds}{\mu_{XY}^i t_i} - \frac{A_{Rp}(s)}{A_h}\oint \frac{ds}{\mu_{XY}^i t_i}\right]$

- Example
  - Doubly symmetrical closed-section
    - Covers AB & CD
      - $\mu_{XY}{}^{c}$ = 20 GPa
      - Thickness 2 mm
    - Webs BC & AD
      - $\mu_{XY}{}^{w}$ = 35 GPa
      - Thickness 1 mm
  - Shear flow in the section?
  - Warping distribution?



- Shear flow
  - $q = \frac{M_x}{2A_h} = \frac{10000}{2\;0.2\;0.1} = 250\;10^3\;\mathrm{N}\cdot\mathrm{m}^{-1}$
- Warping
  - $\oint \frac{ds}{\mu^i_{XY} t_i} = \int_A^B \frac{-dy}{\mu^c_{XY} t_c} + \int_B^C \frac{-dz}{\mu^w_{XY} t_w} + \int_C^D \frac{dy}{\mu^c_{XY} t_c} + \int_D^A \frac{dz}{\mu^w_{XY} t_w}$

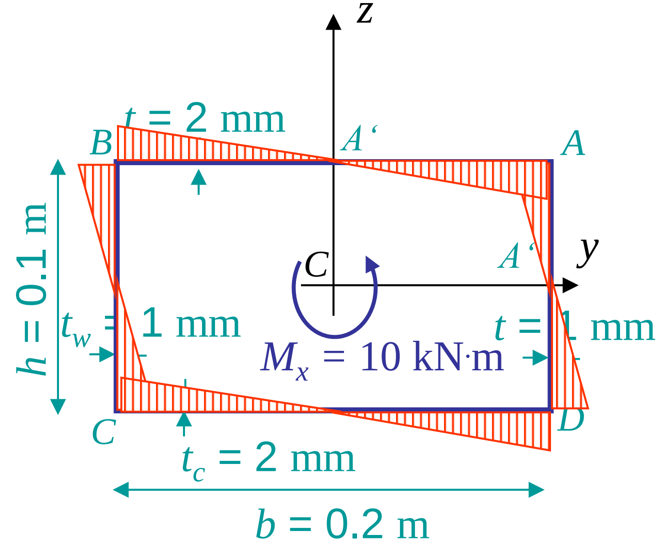



$$\Longrightarrow \oint \frac{ds}{\mu^i_{XY} t_i} = 2\int_{0.1}^{-0.1} \frac{-dy}{20\;10^9\;0.002} + 2\int_{0.05}^{-0.05} \frac{-dz}{35\;10^9\;0.001}$$

$$\Longrightarrow \oint \frac{ds}{\mu^i_{XY} t_i} = 15.71\;10^{-9}\;\mathrm{m}^2\cdot N^{-1}$$

  - By symmetry, warping is equal to zeros at mid sections (at $A$' & $A$'')

- Warping (2)
  - By symmetry: zero warping at $A$' & $A$''
  - Wall A''A
    - $$\int_{A''}^{s} \frac{ds}{\mu^i_{XY} t_i} = \int_0^z \frac{dz}{35\,10^9\ 0.001} = 28.6\,10^{-9}\ z\ \mathrm{m}\cdot N^{-1}$$
    - $$A_{R_p}(s) = \frac{1}{2}\int_0^z \frac{b\,dz}{2} = 0.05\ z\ \mathrm{m}$$




$$\Rightarrow \boldsymbol{u}_x(z)^{A''A} = \frac{M_x}{2A_h}\left[\int_{A''}^{s} \frac{ds}{\mu^i_{XY} t_i} - \frac{A_{Rp}(u)}{A_h}\oint \frac{ds}{\mu^i_{XY} t_i}\right]$$

$$\Rightarrow \boldsymbol{u}_x(z)^{A''A} = 250\,10^3\left[28.6\,10^{-9}\ z - \frac{0.05\ z}{0.2\ 0.1}\ 15.71\,10^{-9}\right] = -0.0027\ z$$

$$\Rightarrow \boldsymbol{u}_x(0.05)^{A''A} = -0.133\,10^{-3}\ \mathrm{m}$$

  - Other walls by symmetry

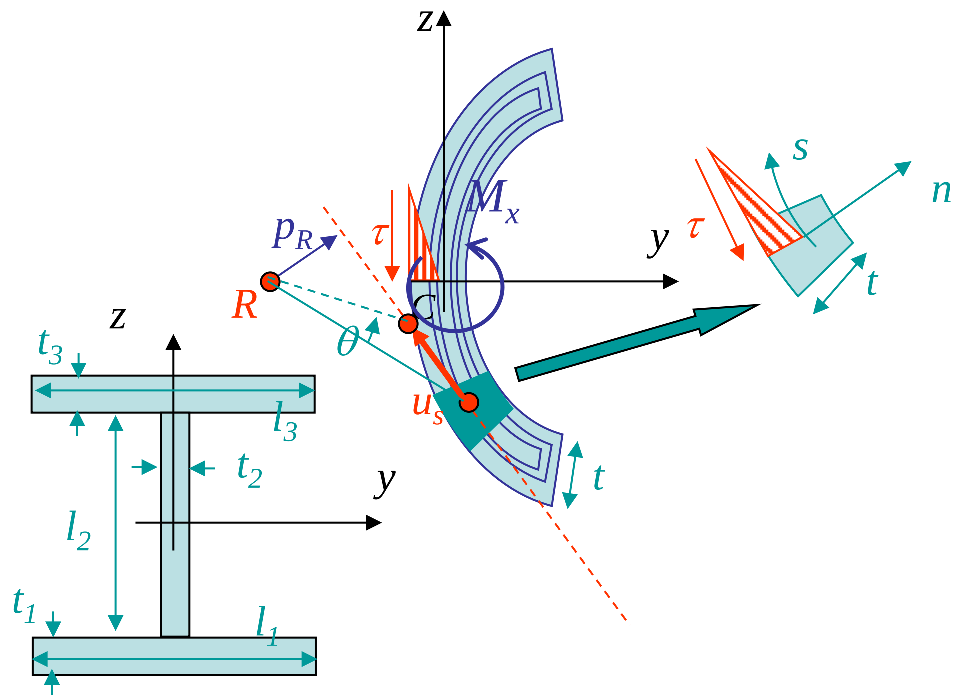
# Thin-walled composite beam

- Torsion (2)
  - Open-section beam
    - From previous analysis
      - Curved sections

$$C = \frac{M_x}{\theta_{,x}} = \frac{1}{3}\int \mu t^3 ds$$

      - Sum of rectangular sections

$$\frac{M_x}{\theta_{,x}} = \sum_i \frac{l_i t_i^3 \mu}{3}$$




    - For composite beams
      - Either

$$C = \mu \bar{I}_T = \frac{M_x}{\theta_{,x}} = \int_i \frac{\mu_{XY}^i t_i^3 ds}{3}$$

      - Or

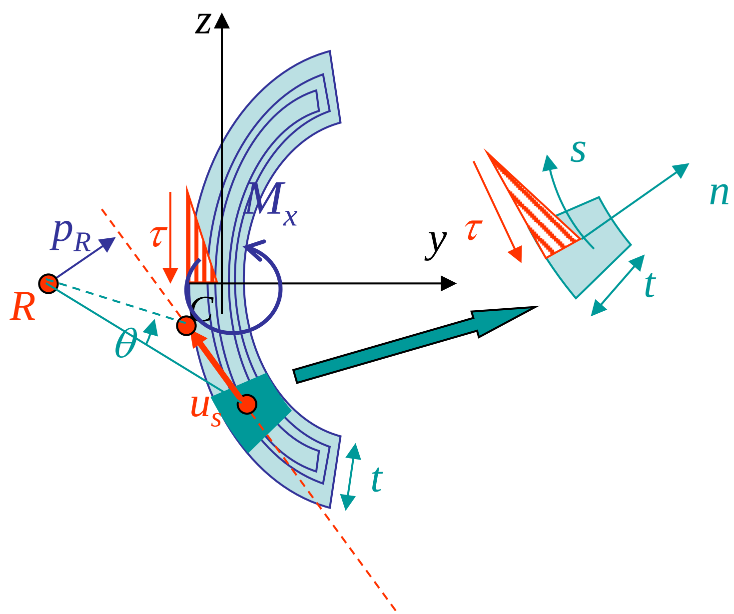
$$C = \mu \bar{I}_T = \frac{M_x}{\theta_{,x}} = \sum_i \frac{\mu_{XY}^i l_i t_i^3}{3}$$

    - Maximum shear keeps the same expression
      - With correct shear modulus: $\tau_{\max_i} = \mu_{XY}^i t_i \theta_{,x}$

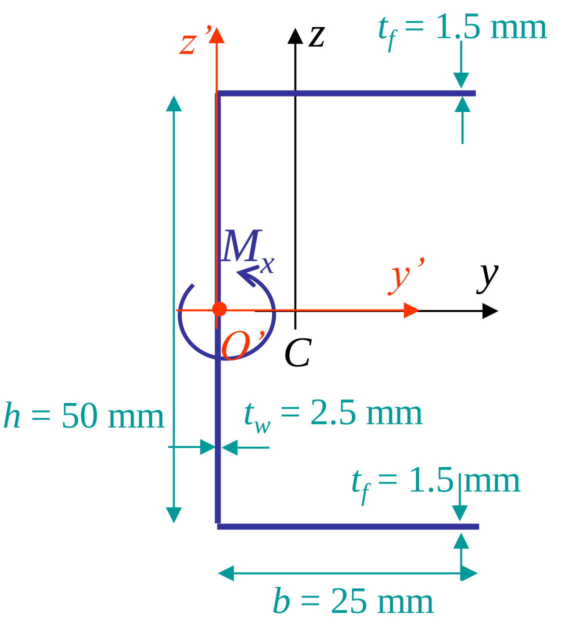
- Torsion (3)
  - Open-section beam (2)
    - Warping keeps the same expression

$$\boldsymbol{u}_x^s(s) = \boldsymbol{u}_x^s(0) - \theta_{,x} \int_0^s p_R ds' = \boldsymbol{u}_x^s(0) - 2A_{R_p}(s)\,\theta_{,x}$$

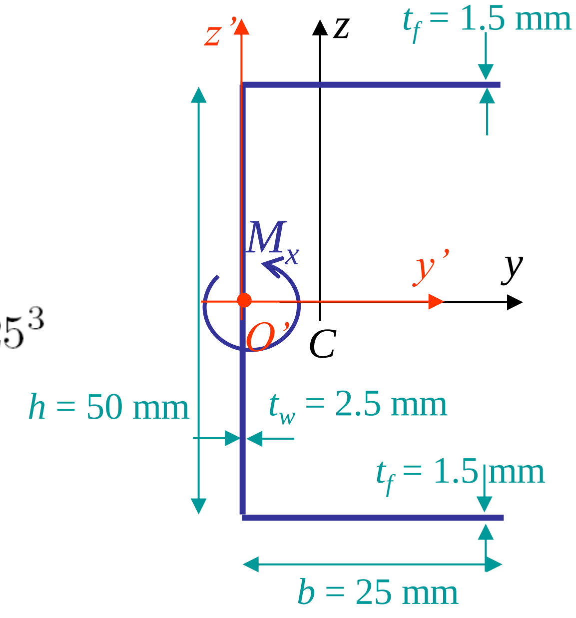
    - Requires position of the shear center $R$
      - Method of determination of the shear center has to be adapted using shear expressions for composite structures



- Example
  - U open section
  - Flanges
    - Shear modulus 20 GPa
    - Thickness 1.5 mm
  - Web
    - Shear modulus 15 GPa
    - Thickness 2.5 mm
  - Torque of 10 N·m
  - Maximum shear stress?



- Maximum shear stress
  - Torsional second moment of area




$$\mu \bar{I}_T = \sum_i \frac{\mu^i_{XY} l_i t_i{}^3}{3} = \frac{2}{3}\mu^f_{XY} b t_f^3 + \frac{1}{3}\mu^w_{XY} h t_w^3$$

$$\Rightarrow \mu \bar{I}_T = \frac{2}{3} 20\ 10^9\ 0.25\ 0.0015^3 + \frac{1}{3} 15\ 10^9\ 0.5\ 0.0025^3$$
$$= 50.3\ \mathrm{N}\cdot\mathrm{m}^2$$

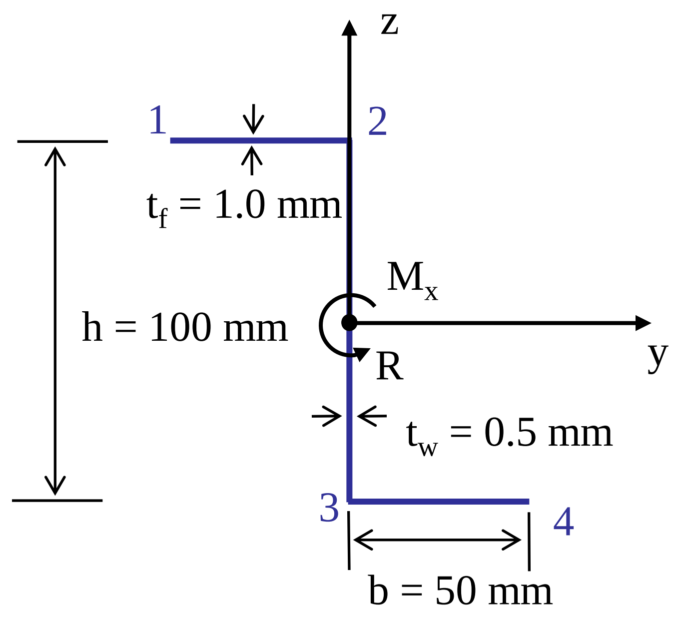
  - Twist rate

$$\Rightarrow \theta_{,x} = \frac{M_x}{\mu \bar{I}_T} = \frac{10}{50.3} = 0.199\ \mathrm{rad}\cdot\mathrm{m}^{-1}$$

  - Maximum shear stress reached

$$\Rightarrow \begin{cases} \tau^w_{\max} = \mu^w_{XY} t_w \theta_{,x} = 15\ 10^9\ 0.0025\ 0.199 = 74.6\ \mathrm{MPa} \\ \tau^f_{\max} = \mu^f_{XY} t_f \theta_{,x} = 20\ 10^9\ 0.0015\ 0.199 = 59.7\ \mathrm{MPa} \end{cases}$$

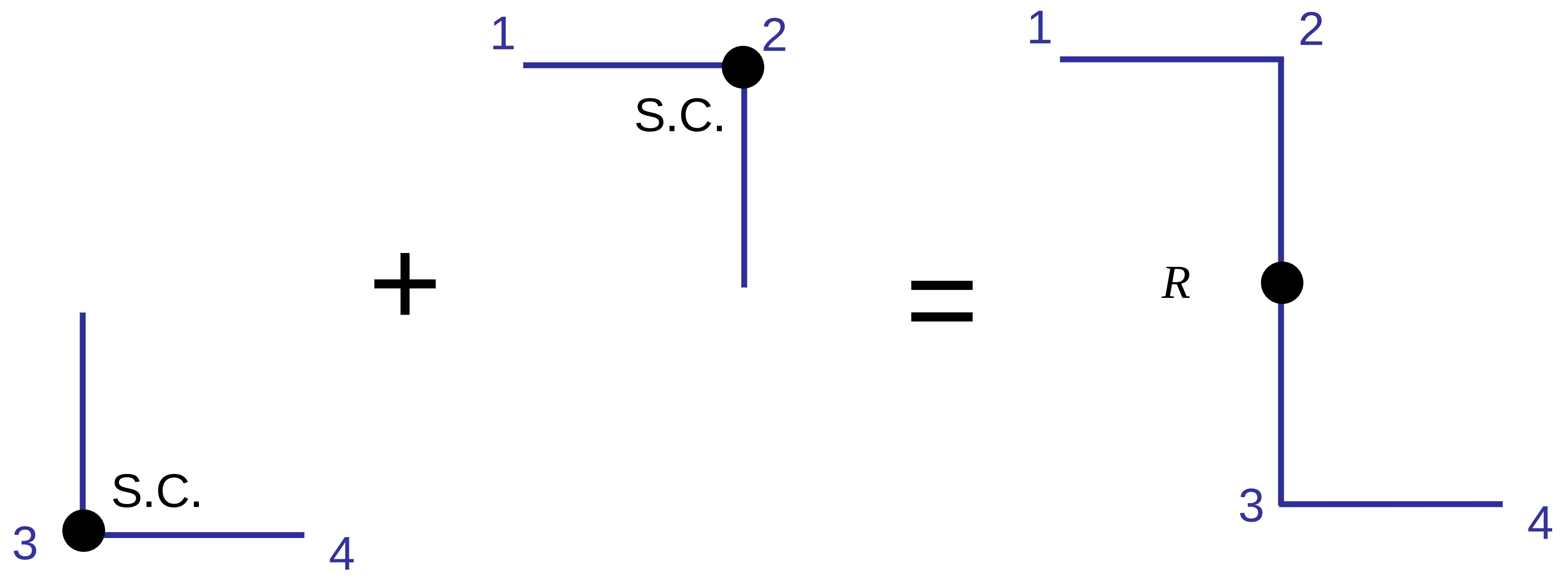
- Laminate shear moduli
  - 16 300 $N/mm^2$ for the flanges
  - 20 900 $N/mm^2$ for the web
- The beam is subjected to a torque of 0.5 $kN{\cdot}mm$
  - Rate of the twist?
  - Maximum shear stress?
  - Value of warping at point 1?



# References

- Lecture notes
  - Aircraft Structures for engineering students, T. H. G. Megson, Butterworth-Heinemann, An imprint of Elsevier Science, 2003, ISBN 0 340 70588 4
- Other references
  - Books
    - Mécanique des matériaux, C. Massonet & S. Cescotto, De boek Université, 1994, ISBN 2-8041-2021-X

- Shear center
  - For an angle section of the type shown below
    - Resultant internal shear loads pass through sides intersection
    - So shear centre (S. C.) is located at the intersection of the sides

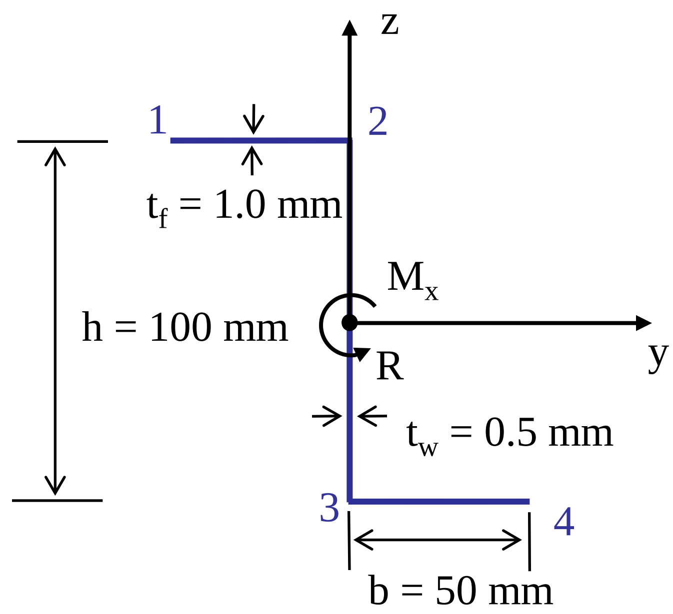



  - So that, by superposition, the shear centre of the actual beam is $R$

- Torsional rigidity
  - Open section

$$\mu \bar{I}_T = \sum_i \frac{\mu^i_{XY} l_i {t_i}^3}{3} = \frac{2}{3}\mu^f_{XY} b t_f^3 + \frac{1}{3}\mu^w_{XY} h t_w^3$$

$$\Rightarrow \mu \bar{I}_T = \frac{2}{3} * 16.3\, 10^9 * 0.05 * 0.001^3 + \frac{1}{3} * 20.9\, 10^9 * 0.1 * 0.005^3 = 0.63\ \mathrm{N \cdot m^2}$$




- Twist rate
  - $\theta_{,x} = \frac{M_x}{\mu \bar{I}_T} = \frac{0.5}{0.63} = 0.8\ \mathrm{rad \cdot m^{-1}}$

- Maximum shear stress (absolute value)

$$\begin{cases} \tau^w_{\max} = \mu^w_{XY} t_w \theta_{,x} = 20.9\, 10^9 * 0.005 * 0.8 = 8.4\ \mathrm{MPa} \\ \tau^f_{\max} = \mu^f_{XY} t_f \theta_{,x} = 16.3\, 10^9 * 0.001 * 0.8 = 13.0\ \mathrm{MPa} \end{cases}$$

# Exercise: Torsion of a thin-walled composite beam

- Warping
  - Shear center $R$
    - Warping = 0 (by definition)
    - Origin set to $R$

$$\Longrightarrow \boldsymbol{u}_x^s(s=R)=0$$

  - We can use

$$\boldsymbol{u}_x^s(s)=\boldsymbol{u}_x^s(0)-\theta_{,x}\int_0^s p_R ds'$$

    - Area swept at point 1:

$$\begin{aligned} A_{R_p}(s=1) &= \pm\text{hatched triangle area} \\ &= \pm 0.5 * 50\,10^{-3} * 50\,10^{-3} \\ &= \pm 1.25 * 10^{-3}\ \mathrm{m}^2 \end{aligned}$$

    - As $A_{R_p}(s=1)$ is positive because the scanning of the swept area is anticlockwise as well as the applied torque

$$\Longrightarrow \begin{aligned} \boldsymbol{u}_x^s(s=1) &= \boldsymbol{u}_x^s(s=R)-2*A_{R_p}(s=1)*\theta_{,x} \\ &= 0-2*1.25*10^{-3}*0.8 \\ &= -2\ \mathrm{mm} \end{aligned}$$