University of Liège
Aerospace & Mechanical Engineering

Aircraft Structures
Beams – Torsion & Section Idealization

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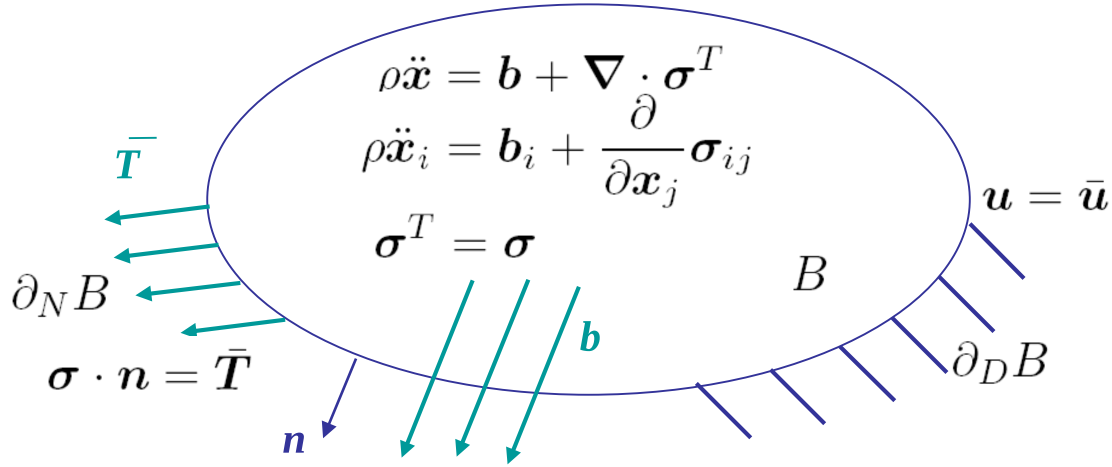
- Balance of body B
  - Momenta balance
    - Linear
    - Angular
  - Boundary conditions
    - Neumann
    - Dirichlet



$$\rho\ddot{\boldsymbol{x}} = \boldsymbol{b} + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^T$$

$$\rho\ddot{\boldsymbol{x}}_i = \boldsymbol{b}_i + \frac{\partial}{\partial \boldsymbol{x}_j}\boldsymbol{\sigma}_{ij}$$

$$\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$$

- Small deformations with linear elastic, homogeneous & isotropic material
  - (Small) Strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{\nabla}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{\nabla}\right)$, or $\begin{cases} \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{x}_i}\boldsymbol{u}_j + \frac{\partial}{\partial \boldsymbol{x}_j}\boldsymbol{u}_i\right) \\ \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\boldsymbol{u}_{j,i} + \boldsymbol{u}_{i,j}\right) \end{cases}$
  - Hooke's law $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon}$, or $\boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl}\boldsymbol{\varepsilon}_{kl}$

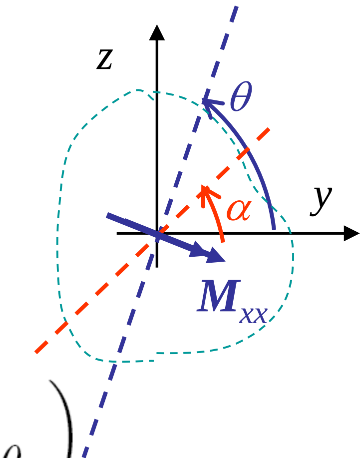
    with $\mathcal{H}_{ijkl} = \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda = K - 2\mu/3}\delta_{ij}\delta_{kl} + \underbrace{\frac{E}{1+\nu}}_{2\mu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$
  - Inverse law $\boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma}$

    with $\mathcal{G}_{ijkl} = \frac{1+\nu}{E}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right) - \frac{\nu}{E}\delta_{ij}\delta_{kl}$

Pure bending: linear elasticity summary

- General expression for unsymmetrical beams
  - Stress $\boldsymbol{\sigma}_{xx} = \kappa E z \cos\alpha - \kappa E y \sin\alpha$

    With $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{\kappa E} \begin{pmatrix} I_{yy} & -I_{yz} \\ -I_{yz} & I_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$

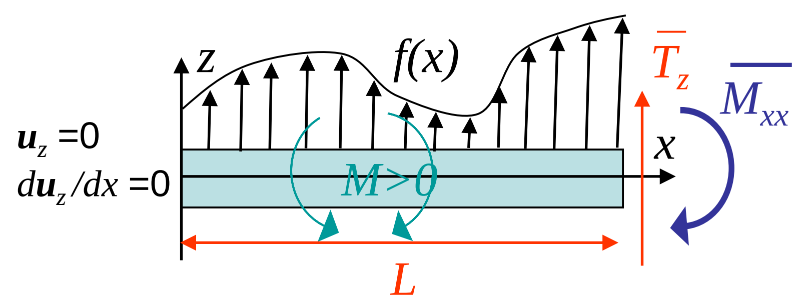


  - Curvature

$$\begin{pmatrix} -\boldsymbol{u}_{z,xx} \\ \boldsymbol{u}_{y,xx} \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{E\left(I_{yy}I_{zz} - I_{yz}I_{yz}\right)} \begin{pmatrix} I_{zz} & I_{yz} \\ I_{yz} & I_{yy} \end{pmatrix} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

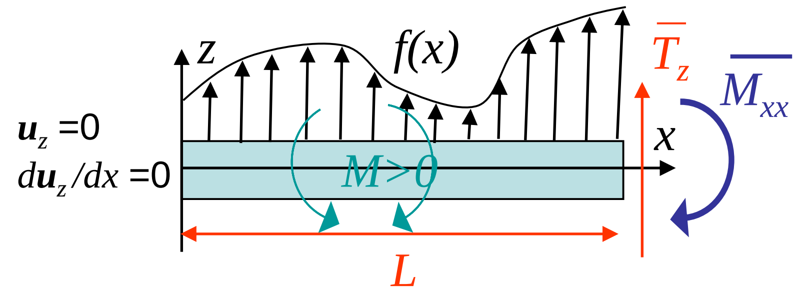
  - In the principal axes $I_{yz} = 0$

- Euler-Bernoulli equation in the principal axis
  - $\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f(x)$ for x in $[0\ L]$
  - BCs $\begin{cases} -\frac{\partial}{\partial x}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\Big|_{0,\,L} = \bar{T}_z\big|_{0,\,L} \\ -EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\Big|_{0,\,L} = \bar{M}_{xx}\big|_{0,\,L} \end{cases}$
  - Similar equations for $\boldsymbol{u}_y$

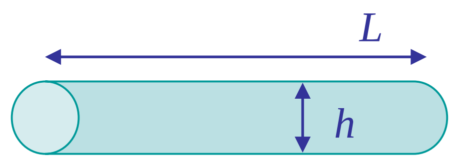


Beam shearing: linear elasticity summary

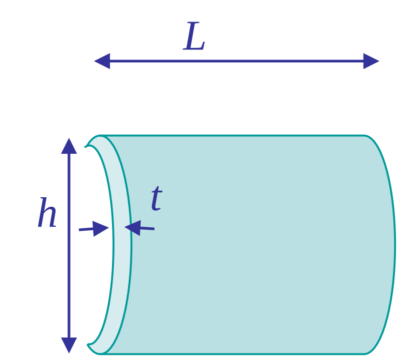
- General relationships
  - $$\begin{cases} f_z(x) = -\partial_x T_z = -\partial_{xx} M_y \\ f_y(x) = -\partial_x T_y = \partial_{xx} M_z \end{cases}$$



- Two problems considered
  - Thick symmetrical section



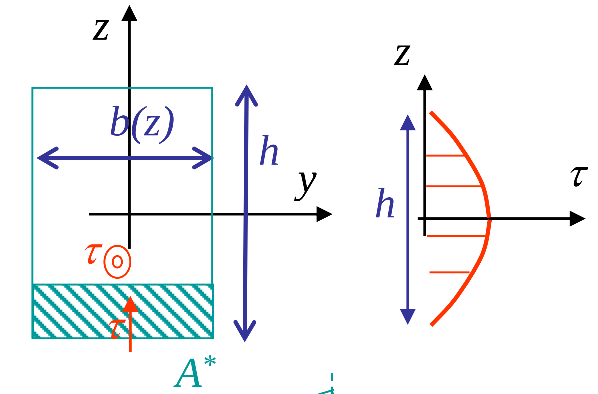
    - Shear stresses are small compared to bending stresses if $h/L << 1$
  - Thin-walled (unsymmetrical) sections

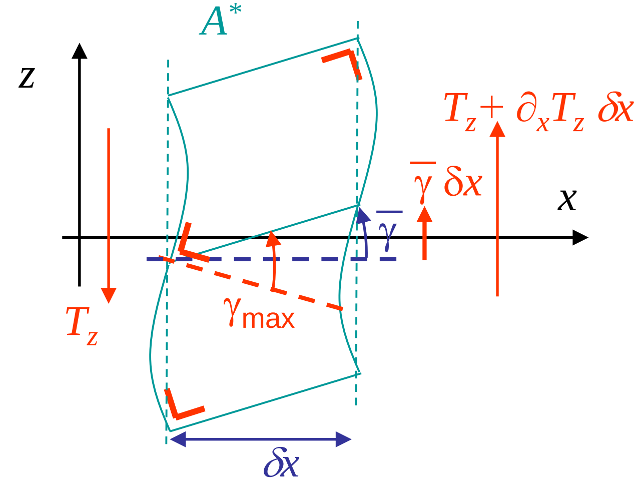


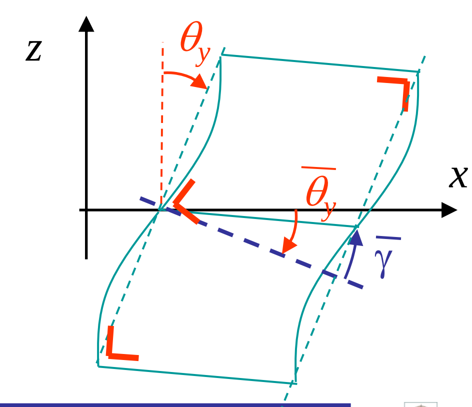
    - Shear stresses are not small compared to bending stresses
    - Deflection mainly results from bending stresses
    - 2 cases
      - Open thin-walled sections
        - » Shear = shearing through the shear center + torque
      - Closed thin-walled sections
        - » Twist due to shear has the same expression as torsion

Beam shearing: linear elasticity summary

- Shearing of symmetrical thick-section beams
  - Stress $\boldsymbol{\sigma}_{zx} = -\frac{T_z S_n(z)}{I_{yy} b(z)}$
    - With $S_n(z) = \int_{A^*} z dA$
    - Accurate only if $h > b$
  - Energetically consistent averaged shear strain
    - $\bar{\gamma} = \frac{T_z}{A'\mu}$ with $A' = \frac{1}{\int_A \frac{S_n^2}{I_{yy}^2 b^2} dA}$
    - Shear center on symmetry axes
  - Timoshenko equations
    - $\bar{\gamma} = 2\bar{\boldsymbol{\varepsilon}}_{xz} = \frac{\partial \boldsymbol{u}_x}{\partial z} + \frac{\partial \boldsymbol{u}_z}{\partial x} = \theta_y + \partial_x \boldsymbol{u}_z$ & $\kappa = \frac{\partial \theta_y}{\partial x}$
    - On [0 L]: $\begin{cases} \frac{\partial}{\partial_x}\left(EI\frac{\partial \theta_y}{\partial x}\right) - \mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right) = 0 \\ \frac{\partial}{\partial x}\left(\mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right)\right) = -f \end{cases}$



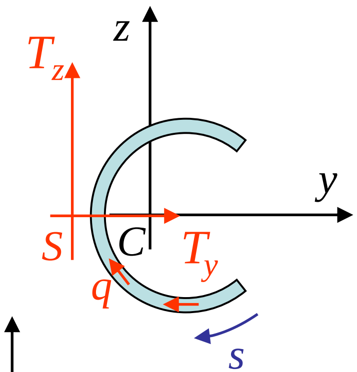


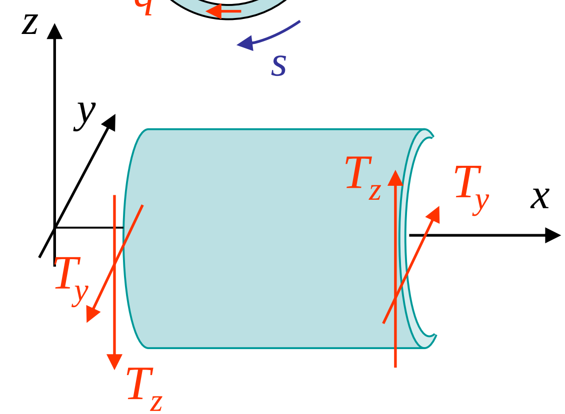


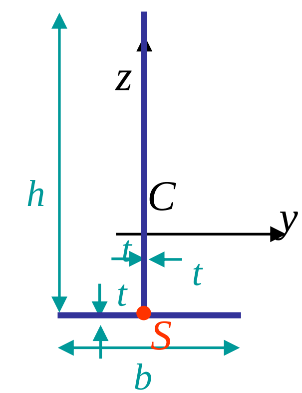
- Shearing of open thin-walled section beams
  - Shear flow $q = t\tau$
    - $$q\left(s\right) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s tzds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s tyds'$$
    - In the principal axes

$$q\left(s\right) = -\frac{T_z}{I_{yy}}\int_0^s tzds' - \frac{T_y}{I_{zz}}\int_0^s tyds'$$

  - Shear center S
    - On symmetry axes
    - At walls intersection
    - Determined by momentum balance
  - Shear loads correspond to
    - Shear loads passing through the shear center &
    - Torque







Beam shearing: linear elasticity summary

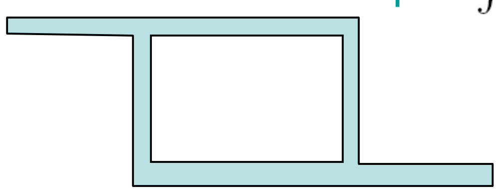
- Shearing of closed thin-walled section beams
  - Shear flow $q = t\tau$
    - $q(s) = q_o(s) + q(0)$
    - Open part (for anticlockwise of q, s)

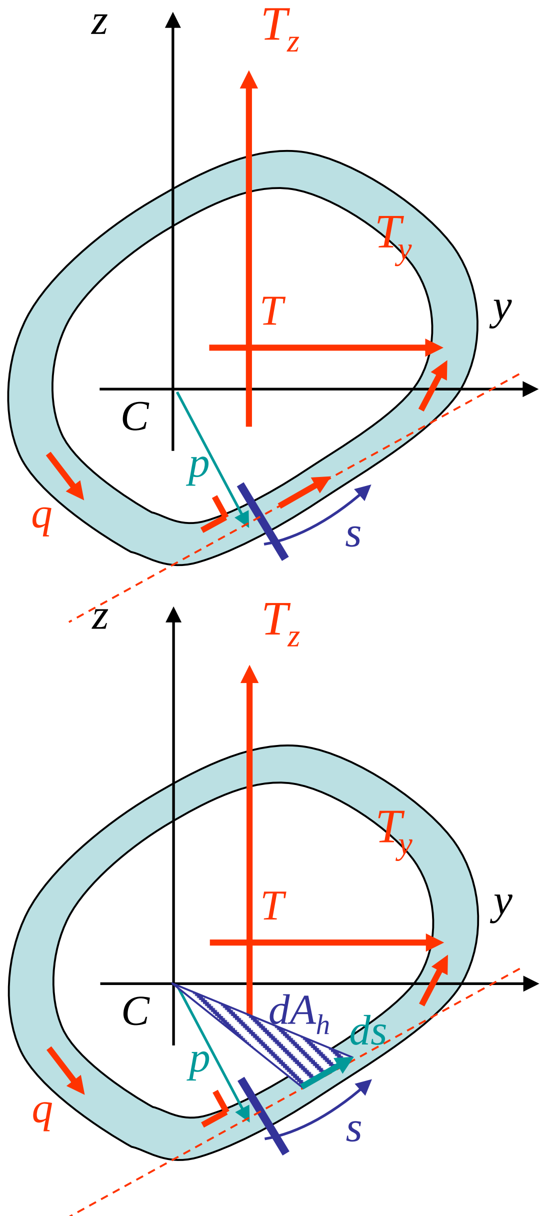
$$q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s t(s')\,z(s')\,ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s t(s')\,y(s')\,ds'$$

    - Constant twist part

$$q(s=0) = \frac{y_T T_z - z_T T_y - \oint p(s)\,q_o(s)\,ds}{2A_h}$$

    - The $q(0)$ is related to the closed part of the section, but there is a $q_o(s)$ in the open part which should be considered for the shear torque $\oint p(s)\,q_o(s)\,ds$

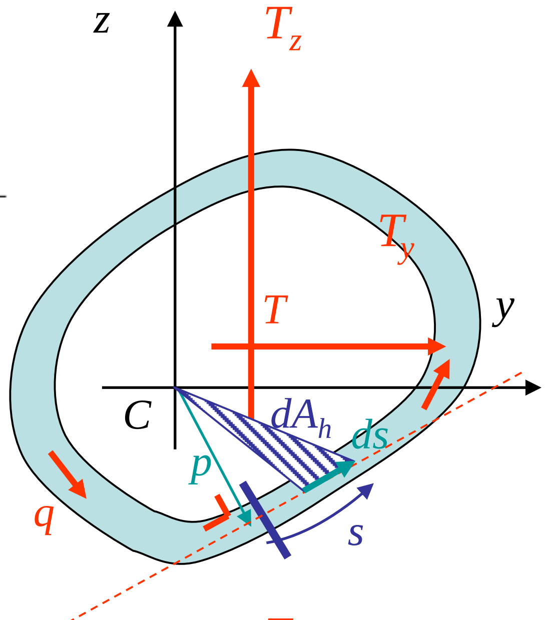


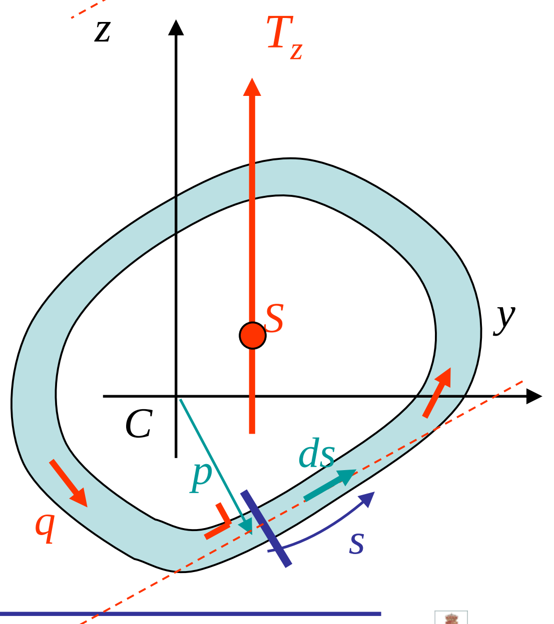


- Shearing of closed thin-walled section beams
  - Warping around twist center R
    - $$\boldsymbol{u}_x\left(s\right)=\boldsymbol{u}_x\left(0\right)+\int_0^s \frac{q}{\mu t}ds-\frac{1}{A_h}\oint \frac{q}{\mu t}ds\left\{A_{Cp}\left(s\right)+\frac{z_R\left[y\left(s\right)-y\left(0\right)\right]-y_R\left[z\left(s\right)-z\left(0\right)\right]}{2}\right\}$$
    - With $\boldsymbol{u}_x\left(0\right)=\frac{\oint t\boldsymbol{u}_x\left(s\right)ds}{\oint t\left(s\right)ds}$
      - $\boldsymbol{u}_x(0)=0$ for symmetrical section if origin on the symmetry axis
  - Shear center S
    - Compute q for shear passing thought S
    - Use

$$q\left(s=0\right)=\frac{y_T T_z-z_T T_y-\oint p\left(s\right)q_o\left(s\right)ds}{2A_h}$$

With point S=T

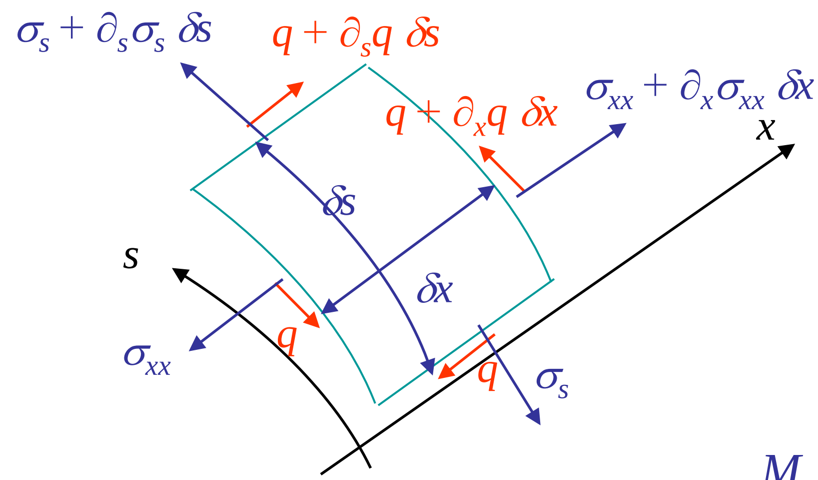




- General relationships
  - We have seen
    - $$(\boldsymbol{\sigma}_{xx} + \partial_x \boldsymbol{\sigma}_{xx} \delta x)\, t\delta s - \boldsymbol{\sigma}_{xx} t\delta s + (q + \partial_s q \delta s)\, \delta_x - q\delta x = 0$$

      $\Longrightarrow$ $t\partial_x \boldsymbol{\sigma}_{xx} + \partial_s q = 0$
    - $$(\boldsymbol{\sigma}_{s} + \partial_s \boldsymbol{\sigma}_{s} \delta s)\, t\delta x - \boldsymbol{\sigma}_{xx} t\delta x + (q + \partial_x q \delta x)\, \delta_s - q\delta s = 0$$

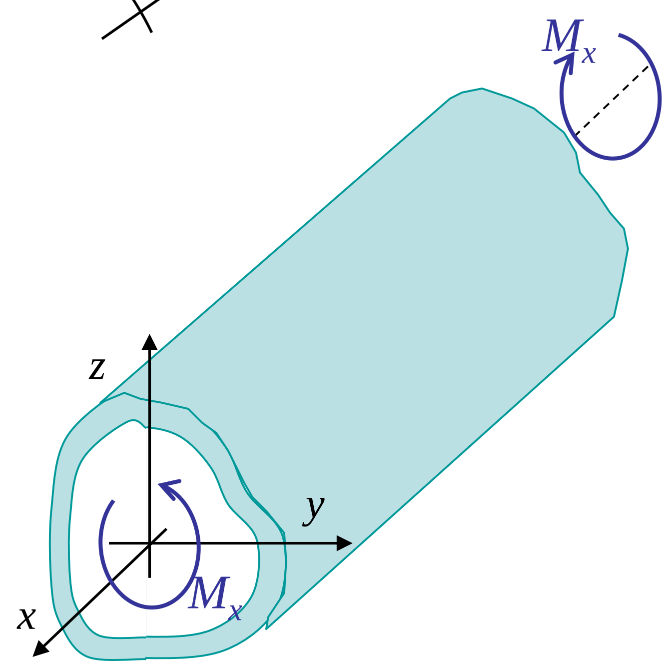
      $\Longrightarrow$ $t\partial_s \boldsymbol{\sigma}_{s} + \partial_x q = 0$



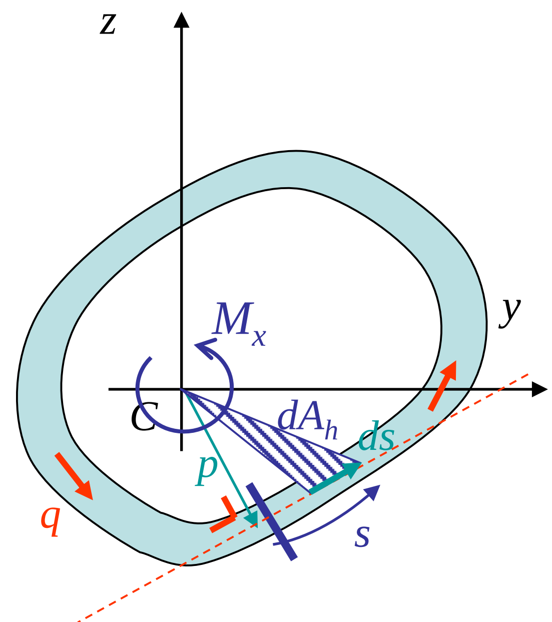
  - If the section is closed
    - Bredt assumption for closed sections:
      Stresses are constant on t, and if there is only a constant torque applied then $\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_{xx} = 0$

      $\Longrightarrow$ $\begin{cases} \partial_x q = 0 \\ \partial_s q = 0 \end{cases}$

      $\Longrightarrow$ Constant shear flow (not shear stress)



- Torque
  - As q due to torsion is constant
    - $M_x = \oint pq ds = q \oint p ds$ $\Longrightarrow$ $M_x = 2A_h q$



- Displacements
  - It has been established that
    - $\gamma = 2\boldsymbol{\varepsilon}_{xs} = \dfrac{\partial \boldsymbol{u}_s}{\partial x} + \dfrac{\partial \boldsymbol{u}_x}{\partial s}$
    - So in linear elasticity

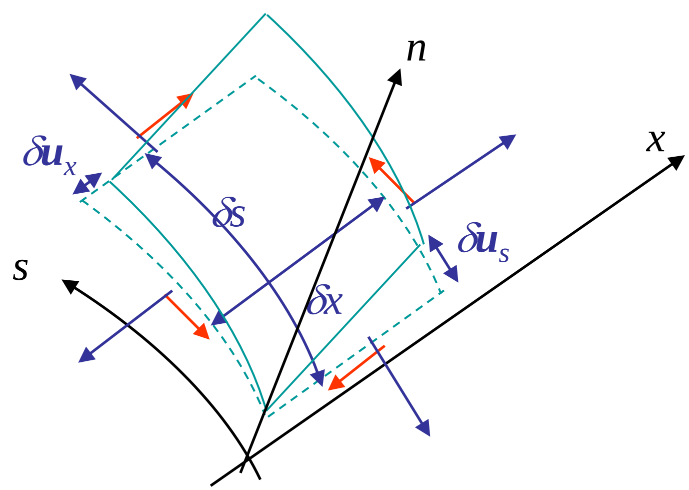
      $q = \mu t \left(\boldsymbol{u}_{s,x} + \boldsymbol{u}_{x,s}\right)$
  - But for pure torsion q is constant

$\Longrightarrow$ $0 = q_{,x} = \mu t \left(\boldsymbol{u}_{x,sx} + \boldsymbol{u}_{s,xx}\right)$

- Remark μt is not constant along s but it is assumed constant along x

$\Longrightarrow$ $\boldsymbol{\varepsilon}_{xx,s} + \boldsymbol{u}_{s,xx} = 0$

- As $\sigma_{xx}= \sigma_s$ =0 $\Longrightarrow$ $\boldsymbol{u}_{s,xx} = 0$



- Displacements (2)
  - It has been established that for a twist around the twist center R

$$\frac{\partial \boldsymbol{u}_s}{\partial x} = p\frac{\partial \theta}{\partial x} + \frac{\partial \boldsymbol{u}_y^C}{\partial x}\cos\Psi + \frac{\partial \boldsymbol{u}_z^C}{\partial x}\sin\Psi$$

  - As $\boldsymbol{u}_{s,xx} = 0$

$$0 = p\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \boldsymbol{u}_y^C}{\partial x^2}\cos\Psi + \frac{\partial^2 \boldsymbol{u}_z^C}{\partial x^2}\sin\Psi$$

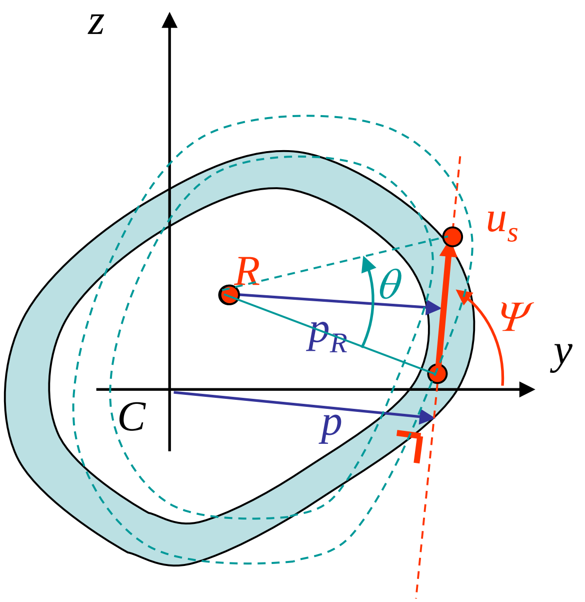
for all values of s (so all value of Ψ)

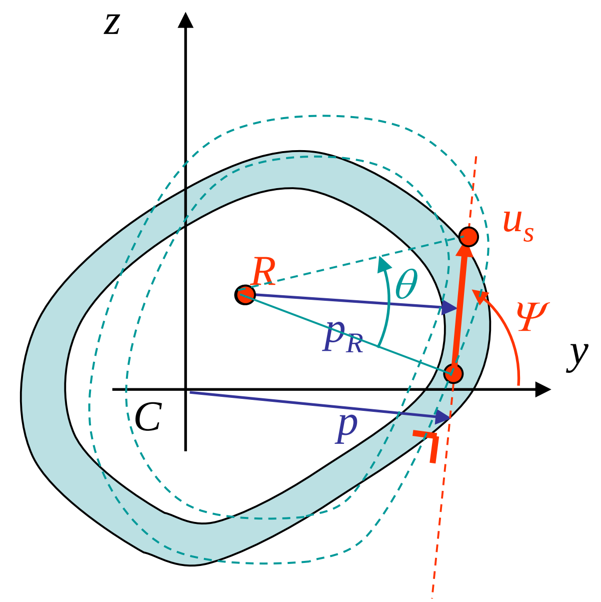
    - The only possible solution is

$$\frac{\partial^2 \theta}{\partial x^2} = 0 \ , \ \frac{\partial^2 \boldsymbol{u}_y^C}{\partial x^2} = 0 \ \& \ \frac{\partial^2 \boldsymbol{u}_z^C}{\partial x^2} = 0$$

  - So displacement fields related to torsion are linear with x

$$\Rightarrow \begin{cases} \theta = C_1 x + C_2 \\ \boldsymbol{u}_y^C = C_3 x + C_4 \\ \boldsymbol{u}_z^C = C_5 x + C_6 \end{cases}$$

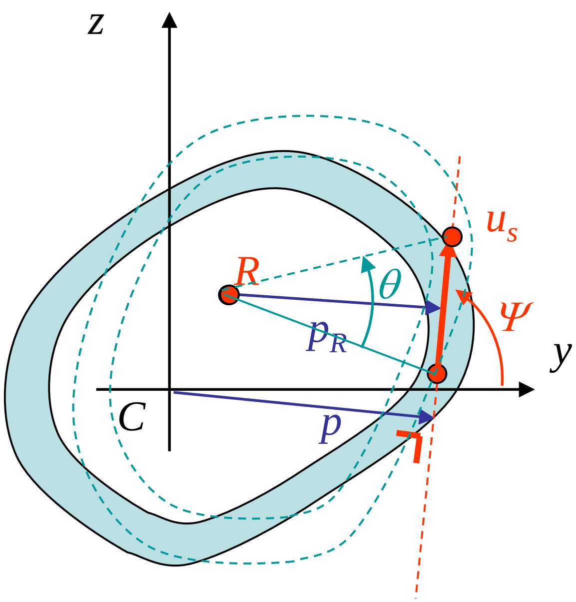




- Rate of twist
  - Use
    - Relation $\oint \frac{q}{\mu t} ds = 2A_h \frac{\partial \theta}{\partial x}$

      developed for shearing, but with q due to torsion constant on s
    - Torque expression $M_x = 2A_h q$
  - Twist
    - $\theta_{,x} = \frac{M_x}{4A_h^2} \oint \frac{1}{\mu t} ds$ constant with x

      $\Longrightarrow \theta = \frac{M_x}{4A_h^2} \oint \frac{1}{\mu t} ds x + C_2$
  - Torsion rigidity
    - $C = \frac{M_x}{\theta_{,x}} = \frac{4A_h^2}{\oint \frac{1}{\mu t} ds}$
    - Torsion second moment of area for constant μ : $I_T = \frac{4A_h^2}{\oint \frac{1}{t} ds} \leq I_p = \int_A r^2 dA$



- Warping
  - Use
    - Relation

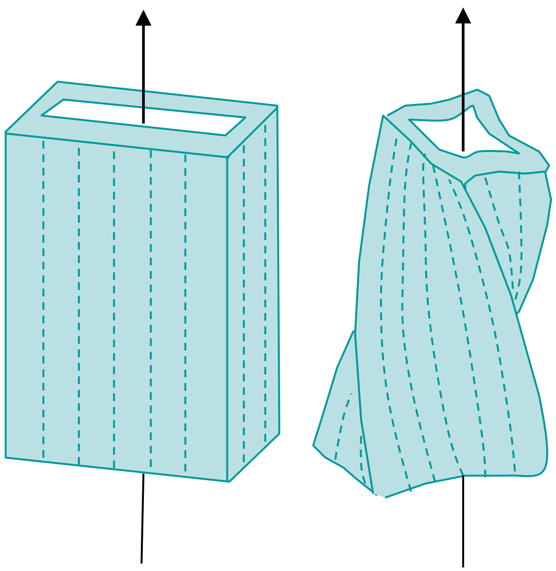
$$\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \int_0^s \frac{q}{\mu t} ds - \frac{A_{Rp}(s)}{A_h} \oint \frac{q}{\mu t} ds$$

developed for shearing, but with q due to torsion constant on s

    - Swept from twist center R $A_{Rp}(s) = \frac{1}{2}\int_0^s p_R ds$
    - Torque expression $M_x = 2A_h q$
  - Warp displacement
    - $\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \frac{M_x}{2A_h}\int_0^s \frac{1}{\mu t} ds - \frac{M_x A_{R_p}(s)}{2A_h^2} \oint \frac{1}{\mu t} ds$

$$\Rightarrow \boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \frac{M_x}{2A_h}\left[\int_0^s \frac{1}{\mu t} ds - \frac{A_{R_p}(s)}{A_h} \oint \frac{1}{\mu t} ds\right]$$

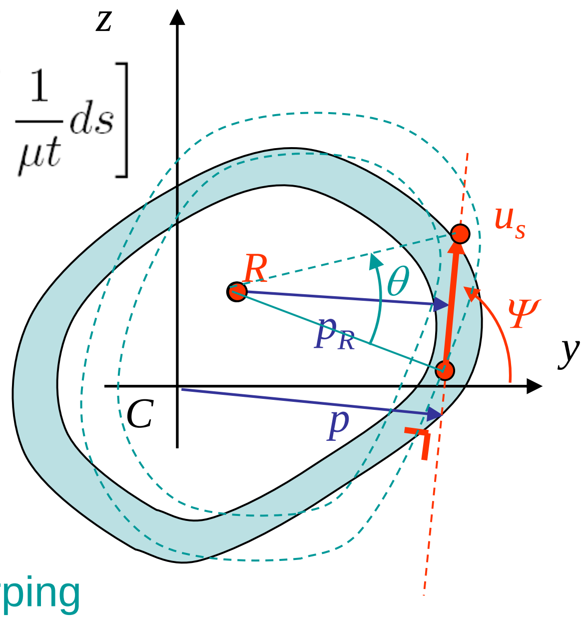
- Twist & Warping under pure torsion
  - Twist $\theta_{,x} = \frac{M_x}{4A_h^2} \oint \frac{1}{\mu t} ds$
  - Warp $\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \frac{M_x}{2A_h} \left[\int_0^s \frac{1}{\mu t} ds - \frac{A_{R_p}(s)}{A_h} \oint \frac{1}{\mu t} ds \right]$
  - Deformation
    - Plane surfaces are no longer plane
    - It has been assumed they keep the same projected shape + linear rotation
    - Longitudinal strains are equal to zero
      - All sections possess identical warping
      - Longitudinal generators keep the same length although subjected to axial displacement



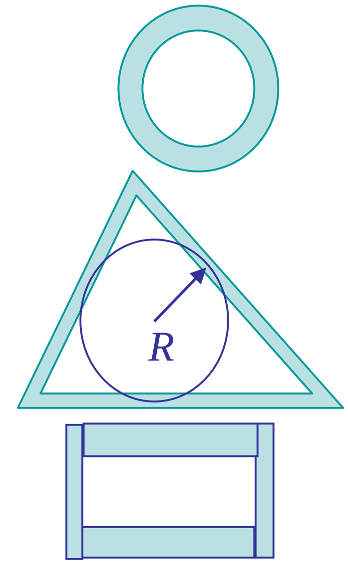
Torsion of closed thin-walled section beams

- Zero warping under pure torsion
  - Warp $\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \frac{M_x}{2A_h}\left[\int_0^s \frac{1}{\mu t} ds - \frac{A_{R_p}(s)}{A_h} \oint \frac{1}{\mu t} ds\right]$
  - Zero warping condition requires
    - $\frac{\int_0^s \frac{1}{\mu t} ds}{\oint \frac{1}{\mu t} ds} = \frac{\frac{1}{2}\int_0^s p_R ds}{A_h}$ for all values of s

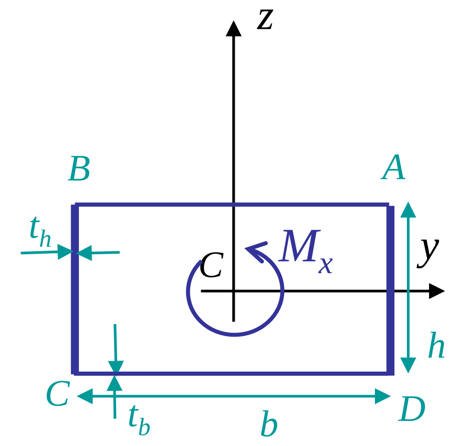
      $\Longrightarrow \frac{1}{\mu t \oint \frac{1}{\mu t} ds} = \frac{p_R}{2A_h} \Longrightarrow p_R \mu t = \frac{2A_h}{\oint \frac{1}{\mu t} ds}$



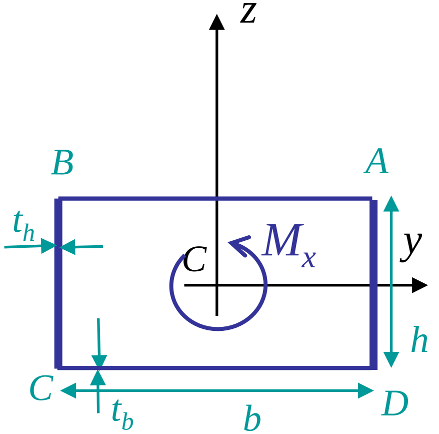
    - As right member is constant the condition of zero warping is $p_R \mu t$ constant with respect to s
    - Solutions at constant shear modulus
      - Circular pipe of constant thickness
      - Triangular section of constant t (p_R is the radius of the inscribed circle which origin coincides with the twist center)
      - Rectangular section with $t_h\, b = t_b\, h$



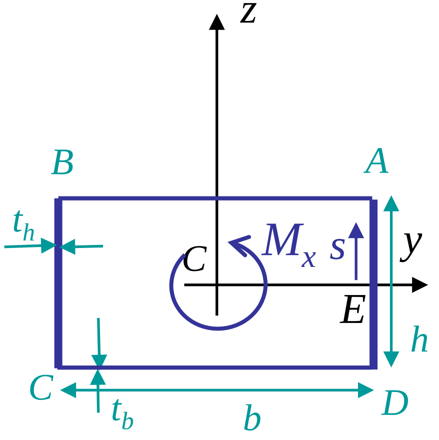
- Example
  - Doubly symmetrical rectangular closed section
  - Constant shear modulus
  - Twist rate?
  - Warping distribution?



- Twist rate
  - As the section is doubly symmetrical, the twist center is also the section centroid C
  - Twist rate $\theta_{,x} = \frac{M_x}{4A_h^2}\oint \frac{1}{\mu t} ds$
    - $A_h = hb$
    - $\oint \frac{1}{t} ds = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{t_h} dz + \int_{\frac{b}{2}}^{-\frac{b}{2}} \frac{1}{t_b}(-dy) + \int_{\frac{h}{2}}^{-\frac{h}{2}} \frac{1}{t_h}(-dz) + \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{1}{t_b} dy = \frac{2h}{t_h} + \frac{2b}{t_b}$
    - $\theta_{,x} = \frac{M_x}{2\mu h^2 b^2}\left(\frac{h}{t_h} + \frac{b}{t_b}\right)$
  - For a beam of length L and constant section $\frac{\theta}{LM_x} = \frac{\frac{h}{t_h} + \frac{b}{t_b}}{2\mu h^2 b^2}$
    - Torsion rigidity $C = \left(\frac{\frac{h}{t_h} + \frac{b}{t_b}}{2\mu h^2 b^2}\right)^{-1} = \mu I_T \leq \mu I_p$



- Warping
  - As the section is doubly symmetrical, the twist center is also the section centroid C
  - Warping
    - It can be set up to 0 at point E
      - By symmetry it will be equal to zero wherever a symmetry axis intercept the wall



- $$\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \frac{M_x}{2A_h}\left[\int_0^s \frac{1}{\mu t} ds - \frac{A_{R_p}(s)}{A_h}\oint \frac{1}{\mu t} ds\right]$$
- $A_h = hb$ & $\oint \frac{1}{t} ds = \frac{2h}{t_h} + \frac{2b}{t_b}$
- On part EA
  - $\int_0^s \frac{1}{t} ds = \int_0^z \frac{1}{t_h} dz = \frac{z}{t_h}$ & $A_{R_p} = \int_0^s \frac{p_R}{2} ds = \int_0^z \frac{b}{4} dz = \frac{bz}{4}$

$$\Longrightarrow \boldsymbol{u}_x(z)^{EA} = \frac{M_x}{2\mu hb}\left[\frac{z}{t_h} - \frac{bz}{4bh}\left(\frac{2h}{t_h} + \frac{2b}{t_b}\right)\right]$$

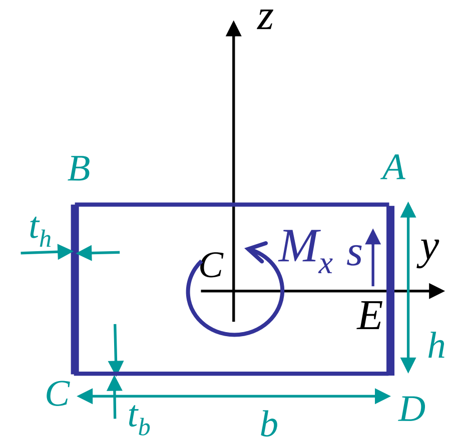
- Warping (2)
  - On part EA

$$\boldsymbol{u}_x(z)^{EA} = \frac{M_x}{2\mu hb}\left[\frac{z}{t_h} - \frac{bz}{4bh}\left(\frac{2h}{t_h} + \frac{2b}{t_b}\right)\right]$$

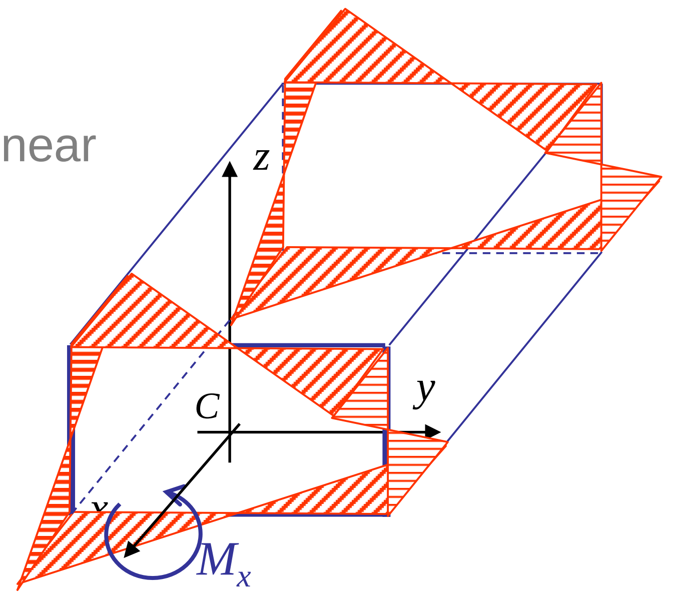
$$\Rightarrow \boldsymbol{u}_x(z)^{EA} = \frac{M_x z}{2\mu hb}\left[\frac{1}{t_h} - \frac{1}{2h}\frac{ht_b + bt_h}{t_h t_b}\right]$$

$$\Rightarrow \boldsymbol{u}_x(z)^{EA} = \frac{M_x z}{2\mu hb}\frac{ht_b - bt_h}{2ht_h t_b}$$

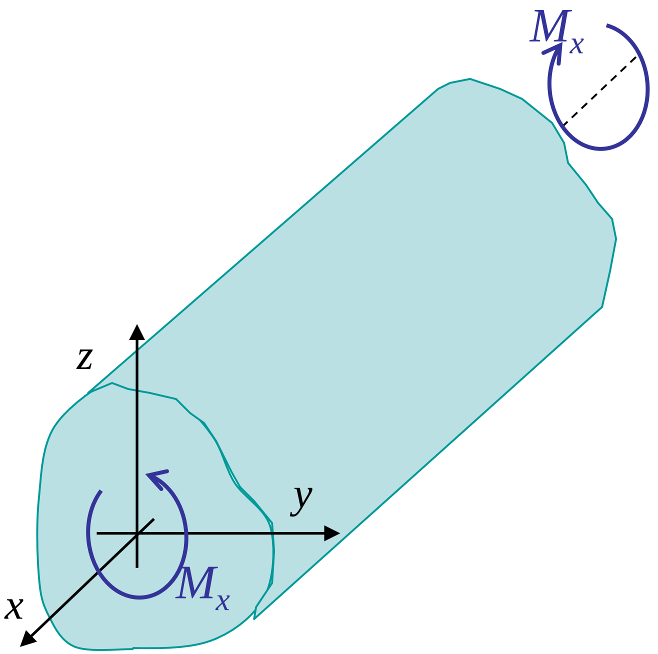
$$\Rightarrow \boldsymbol{u}_x(z)^{EA} = \frac{M_x z}{4\mu h^2 b}\left(\frac{h}{t_h} - \frac{b}{t_b}\right)$$

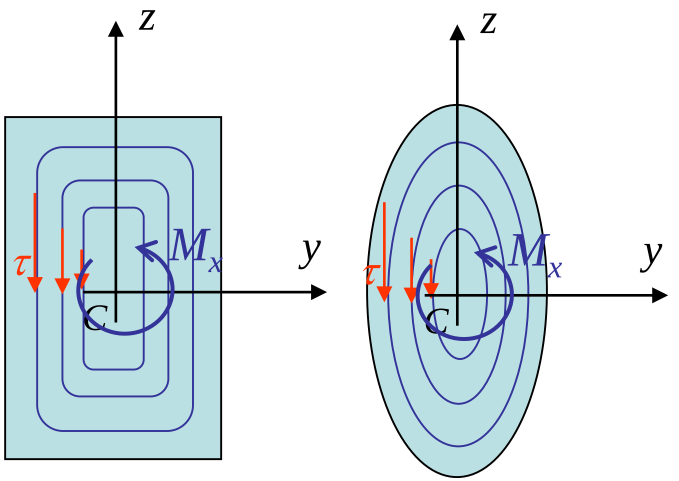


  - So using symmetry and as distribution is linear
    - $$\begin{cases} \boldsymbol{u}_x^A = \boldsymbol{u}_x^C = \dfrac{M_x}{8\mu hb}\left(\dfrac{h}{t_h} - \dfrac{b}{t_b}\right) \\ \boldsymbol{u}_x^B = \boldsymbol{u}_x^D = \dfrac{M_x}{8\mu hb}\left(\dfrac{b}{t_b} - \dfrac{h}{t_h}\right) \end{cases}$$
    - Zero warping if $b\, t_h = h\, t_b$

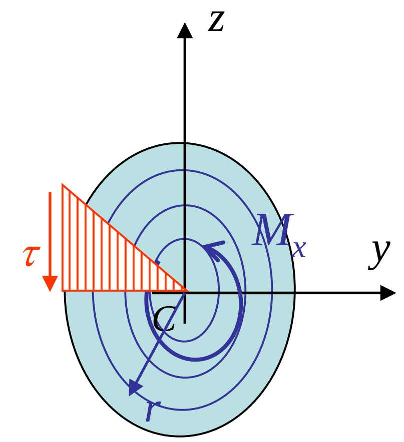


- Torsion of a thick section
  - The problem can be solved explicitly by recourse to a stress function
  - Hydrodynamic analogy
    - Shear stresses have the same expression than the velocity in a rotational flow in a box of same section





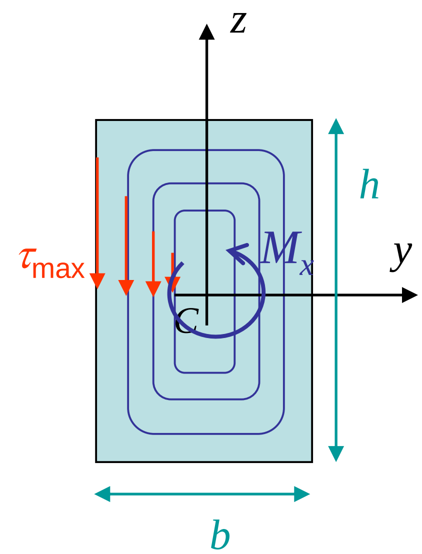
- Torsion of a thick circular section
  - Exact solution of the problem
    - By symmetry there is no warping
      - $\Longrightarrow$ sections remain plane
      - $\Longrightarrow$ $\gamma = r\theta_{,x}$



  - In linear elasticity
    - Shear stresses $\tau = \mu\gamma = r\mu\theta_{,x}$
    - Torque $M_x = \int_A r\tau dA = \int_A \mu r^2 dA\theta_{,x}$
    - Torsion rigidity $C = \frac{M_x}{\theta_{,x}} = \int_A \mu r^2 dA$
      - At constant shear modulus (required for symmetry): $C = \mu I_p$
      - For circular cross sections (only) $I_p = I_T$
    - Maximum shear stress $\tau_{\max} = \frac{M_x r_{\max}}{I_p}$

- Torsion of a rectangular section
  - Exact solution of the problem with stress function
    - Assumptions
      - Linear elasticity
      - Constant shear modulus
    - Maximum stress at mid position of larger edge
      - $\tau_{\max} = \dfrac{M_x}{\alpha h b^2}$
    - Torsion rigidity (constant μ)
      - $C = \dfrac{M_x}{\theta_{,x}} = \beta h b^3 \mu$

        $\Longrightarrow I_T = \beta h b^3$



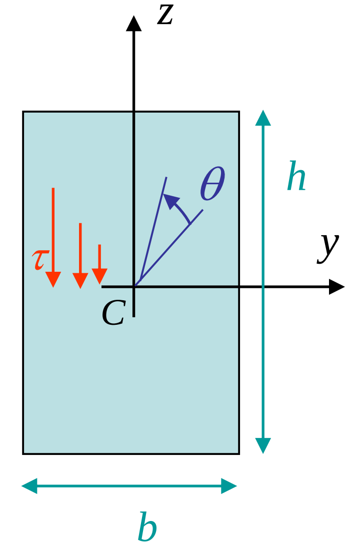
h/b	1	1.5	2	4	∞
α	0.208	0.231	0.246	0.282	1/3
β	0.141	0.196	0.229	0.281	1/3

- Approximation for $h>>b$
  - $C = \dfrac{M_x}{\theta_{,x}} = \dfrac{hb^3\mu}{3} \Longrightarrow I_T = \dfrac{hb^3}{3}$
  - $\tau_{xy} = 0$ & $\tau_{xz} = 2\mu y \theta_{,x}$
  - $\tau_{\max} = \dfrac{3M_x}{hb^2}$

- Torsion of a rectangular section (2)
  - Warping
    - As $\begin{cases} \gamma_{xz} = \boldsymbol{u}_{x,z} + \boldsymbol{u}_{z,x} = \dfrac{\tau_{xz}}{\mu} \\ \gamma_{xy} = \boldsymbol{u}_{y,x} + \boldsymbol{u}_{x,y} = \dfrac{\tau_{xy}}{\mu} \end{cases}$
    - For a rigid rotation (first order approximation)
      - $\boldsymbol{u}_{x,z} = \dfrac{\tau_{xz}}{\mu} - \boldsymbol{u}_{z,x} = \dfrac{\tau_{xz}}{\mu} - \dfrac{\partial}{\partial x}(\theta y)$

        $\Longrightarrow \boldsymbol{u}_{x,z} = \dfrac{\tau_{xz}}{\mu} - y\theta_{,x}$
      - $\boldsymbol{u}_{x,y} = \dfrac{\tau_{xy}}{\mu} - \boldsymbol{u}_{y,x} = \dfrac{\tau_{xy}}{\mu} - \dfrac{\partial}{\partial x}(-\theta z)$

        $\Longrightarrow \boldsymbol{u}_{x,y} = \dfrac{\tau_{xy}}{\mu} + z\theta_{,x}$
    - For a thin rectangular section
      - $\tau_{xy} = 0$ & $\tau_{xz} = 2\mu y\theta_{,x}$
      - $\boldsymbol{u}_{x,y} = \dfrac{\tau_{xy}}{\mu} + z\theta_{,x} \Longrightarrow \boldsymbol{u}_x = zy\theta_{,x} + C_1 z + C_2$
      - Doubly symmetrical section $\Longrightarrow \boldsymbol{u}_x = zy\theta_{,x}$

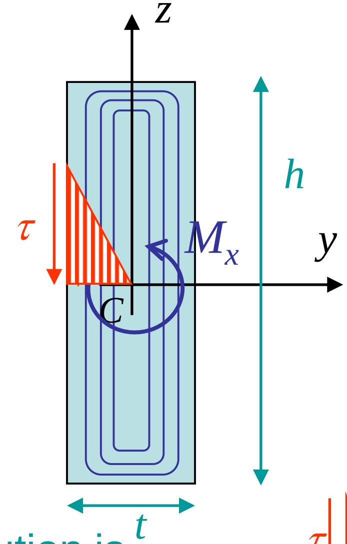


Torsion of open thin-walled section beams

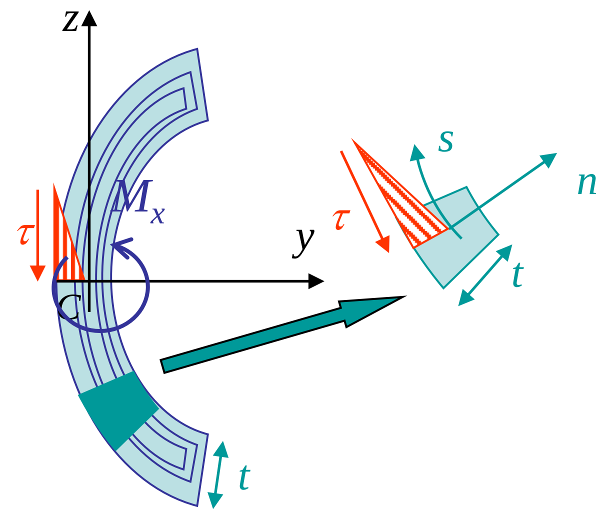
- Rectangle approximation of open thin-walled section beams
  - Thin rectangle
    - $\tau_{xy} = 0$ & $\tau_{xz} = 2\mu y \theta_{,x}$
    - For constant shear modulus

      $$C = \frac{M_x}{\theta_{,x}} = \frac{h t^3 \mu}{3} \Longrightarrow I_T = \frac{h t^3}{3}$$

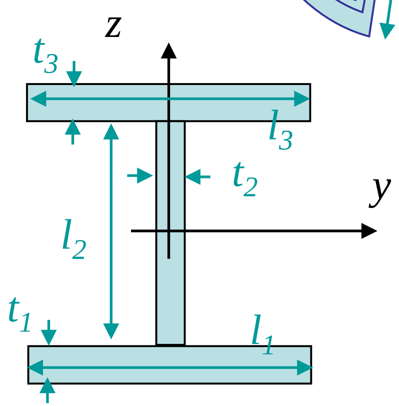
    - Warping $\boldsymbol{u}_x = z y \theta_{,x}$



  - Thin curved section
    - If t << curvature an **approximate** solution is
      - $\tau_{xs} = 2\mu n \theta_{,x}$
      - $C = \frac{M_x}{\theta_{,x}} = \frac{1}{3}\int \mu t^3 ds$



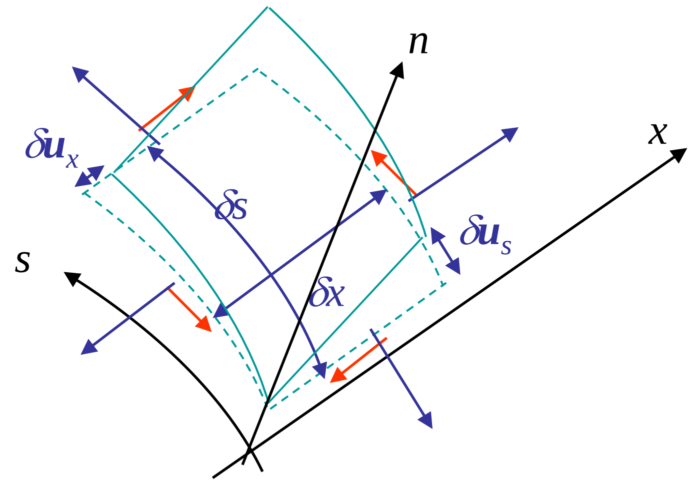
  - Open section composed of thin rectangles
    - Same approximation
      - $\tau_{\max_i} = \mu t_i \theta_{,x}$
      - $\frac{M_x}{\theta_{,x}} = \sum_i \frac{l_i t_i^3 \mu}{3}$

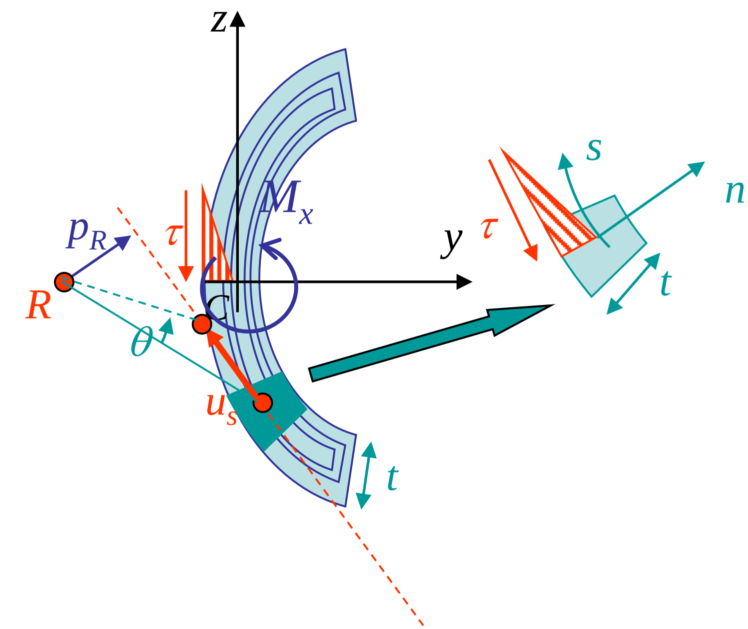


- Warping
  - Warping around *s*-axis
    - Thin rectangle $\boldsymbol{u}_x = zy\theta_{,x} + C_1 z + C_2$
    - Here C_i are not equal to 0
    - Part around s-axis $\boldsymbol{u}_x^t = ns\theta_{,x}$
  - Warping of the *s*-line (*n*=0)
    - We found $\gamma = 2\boldsymbol{\varepsilon}_{xs} = \dfrac{\partial \boldsymbol{u}_s}{\partial x} + \dfrac{\partial \boldsymbol{u}_x}{\partial s}$
    - If *R* is the twist center
      - $\dfrac{\partial \boldsymbol{u}_s}{\partial x} = p_R \theta_{,x}$

        $\Longrightarrow \tau_{xs} = \mu\gamma = \mu \dfrac{\partial \boldsymbol{u}_x}{\partial s} + \mu p_R \theta_{,x}$
      - As $\tau_{xs} = 2\mu n \theta_{,x}$ $\Longrightarrow$ $\tau_{xs}(n=0) = 0$

        $\Longrightarrow \dfrac{\partial \boldsymbol{u}_x}{\partial s} = -p_R \theta_{,x}$
    - Eventually *s*-axis warp (usually the larger)

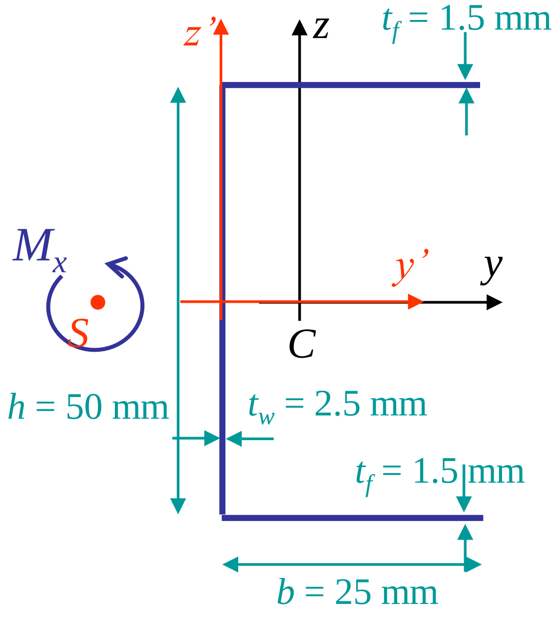




$$\boldsymbol{u}_x^s(s) = \boldsymbol{u}_x^s(0) - \theta_{,x} \int_0^s p_R ds' = \boldsymbol{u}_x^s(0) - 2A_{R_p}(s)\,\theta_{,x}$$

Torsion of open thin-walled section beams

- Example
  - U open section
  - Constant shear modulus (25 GPa)
  - Torque of 10 N·m
  - Maximum shear stress?
  - Warping distribution?



- Maximum shear stress
  - Torsion second moment of area

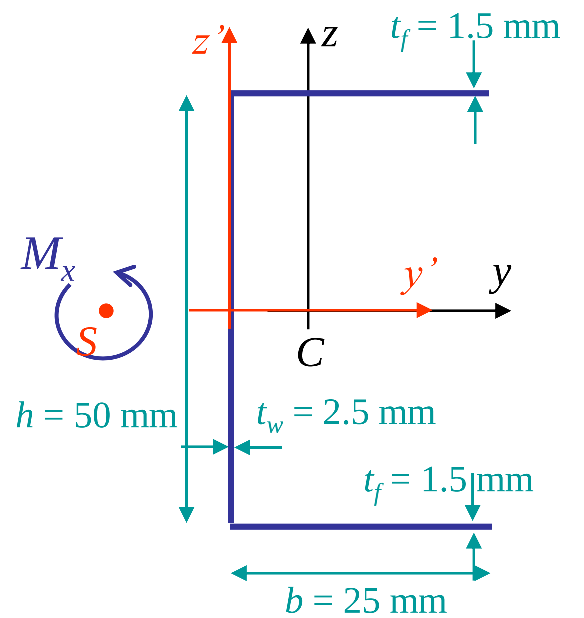
$$I_T = \sum \frac{l_i t_i^3}{3} = \frac{2}{3} b t_f^3 + \frac{h t_w^3}{3}$$
$$= \frac{2\ 0.025\ 0.0015^3 + 0.05\ 0.0025^3}{3} = 0.317\ 10^{-9} \mathrm{m}^4$$

  - Twist rate

$$\theta_{,x} = \frac{M_x}{\mu I_T} = \frac{10}{25\ 0.317} = 1.26\ \mathrm{rad} \cdot \mathrm{m}^{-1}$$

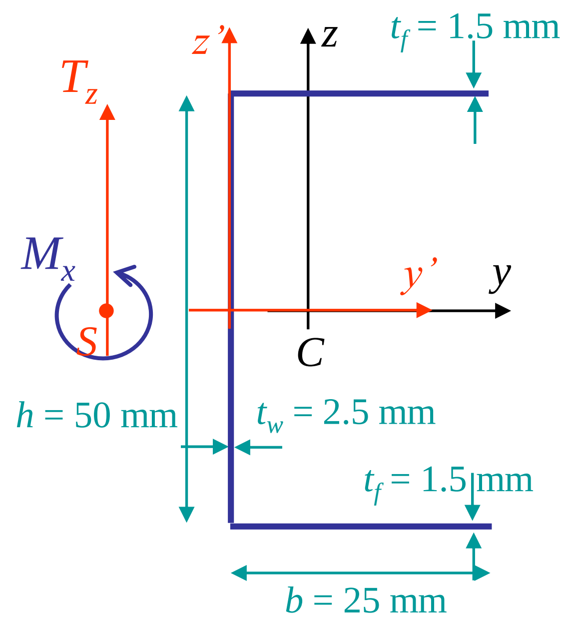
  - Maximum shear stress reached in web

$$\tau_{\max} = \pm 2\mu \frac{t_w}{2} \theta_{,x}$$
$$= \pm 25\ 10^9\ 0.00251.26 = \pm 78.9\ \mathrm{MPa}$$



- Twist center
  - Zero-warping point
  - Free ends so the shear center S corresponds to twist center R
    - See lecture on structural discontinuities
  - By symmetry, lies on Oy axis
  - Apply Shear T_z to obtained y'_S
  - Shear flow for symmetrical section
    - $q(s) = -\frac{T_z}{I_{yy}} \int_0^s t z ds'$
    - With $I_{yy} = \frac{t_w h^3}{12} + 2\frac{h^2}{4} t_f b$

$$= \frac{0.0025\ 0.05^3}{12} + \frac{0.05^2}{2}\ 0.0015\ 0.025 = 72.9\ 10^{-9}\ \mathrm{m}^4$$



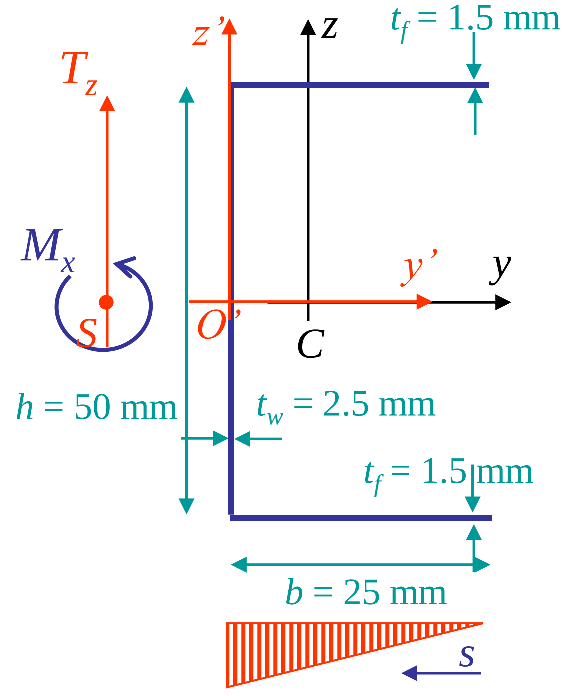
- Twist center (2)
  - Shear flow for symmetrical section (2)
    - $q\left(s\right) = -\frac{T_z}{I_{yy}}\int_0^s tzds'$
    - On lower flange
      - $$q_f\left(y'\right) = -\frac{T_z}{I_{yy}}\int_b^{y'} t_f\left(-\frac{h}{2}\right)\left(-dy'\right) = \frac{T_z t_f h}{2I_{yy}}\left(b-y'\right)$$
    - Momentum due to shear flow
      - Zero web contribution around *O*'
      - Top and lower flanges have the same contribution

$$M_{O'} = h\frac{-bq_f\left(y'=0\right)}{2} = -\frac{T_z t_f h^2 b^2}{4I_{yy}}$$
$$= -T_z\frac{0.0015\ 0.05^2\ 0.025^2}{4\ 72.9\ 10^{-9}} = -8.04 \text{ mm } T_z$$

    - Moment balance

$$M_{O'} = -8.04 \text{ mm } T_z = y'_S T_z \Longrightarrow y'_S = -8.04 \text{ mm}$$

    - Be carefull: clockwise orientation of q, s



- Warping of s-axis
  - $\boldsymbol{u}_x^s(s) = \boldsymbol{u}_x^s(0) - 2A_{R_p}(s)\,\theta_{,x}$
  - Origin in O' as by symmetry $\boldsymbol{u}_x(O')=0$
    - On $O'A$ branch
      - Area swept is positive

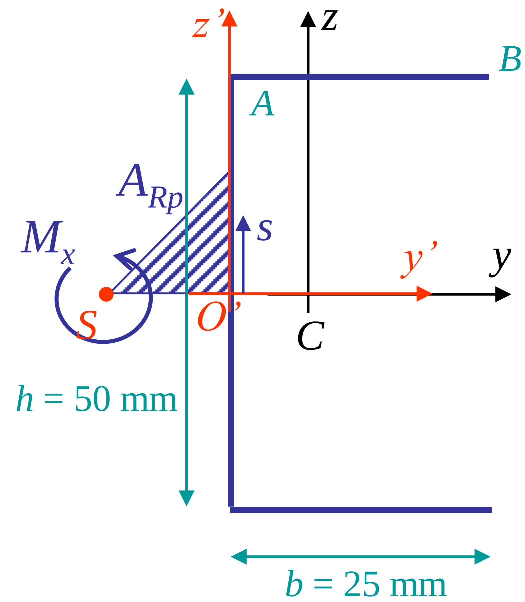
$$\boldsymbol{u}_x^{s,O'A}(z') = -\int_{O'}^{s} p_R ds \theta_{,x} = -|y'_S|\, z'\theta_{,x}$$
$$= -0.00804\ 1.26z' = -0.0101z'$$

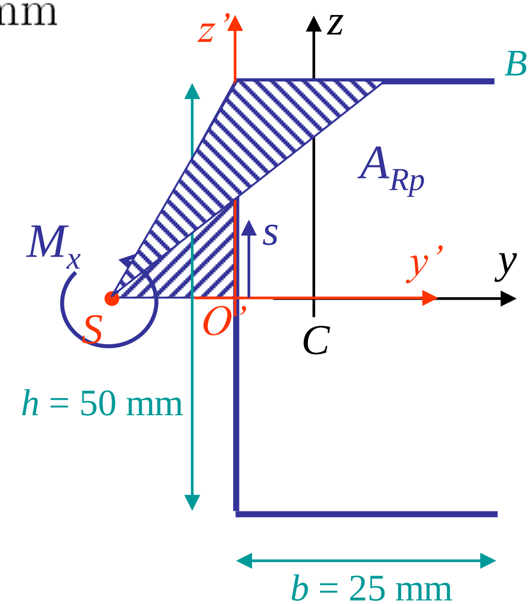
      - At point A

$$\boldsymbol{u}_x^{s,A} = -0.0101\frac{h}{2} = -0.0101\ 0.025 = -0.25\ \mathrm{mm}$$

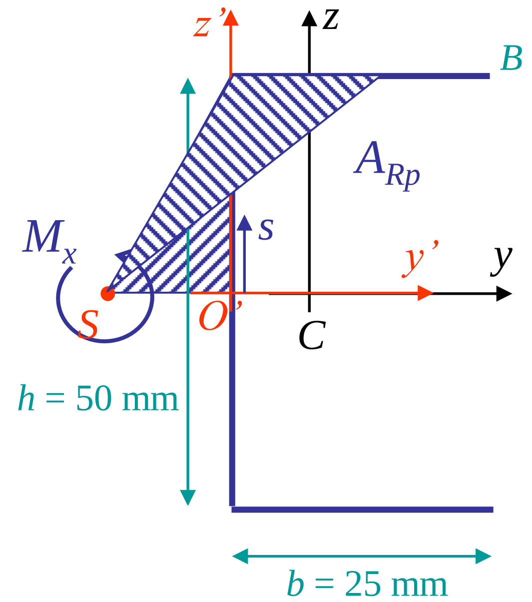
    - On AB branch
      - Area swept is negative

$$\boldsymbol{u}_x^{s,AB}(y') = \boldsymbol{u}_x^{s,A} - \int_A^s p_R ds\theta_{,x}$$
$$= -0.25\ \mathrm{mm} + \int_0^{y'} \frac{h}{2} dy''\theta_{,x}$$





- Warping of s-axis (2)
  - $\boldsymbol{u}_x^s(s) = \boldsymbol{u}_x^s(0) - 2A_{R_p}(s)\,\theta_{,x}$
  - Origin in O' as by symmetry $\boldsymbol{u}_x(O')=0$ (2)
    - On AB branch
      - Area swept is negative



$$\boldsymbol{u}_x^{s,AB}(y') = \boldsymbol{u}_x^{s,A} - \int_A^s p_R ds \theta_{,x}$$
$$= -0.25 \text{ mm} + \int_0^{y'} \frac{h}{2} dy'' \theta_{,x}$$
$$\boldsymbol{u}_x^{s,AB}(y') = -0.25 \text{ mm} + \frac{h\theta_{,x}}{2} y'$$
$$= -0.25 \text{ mm} + 0.025\ 1.26\ y' = -0.25 \text{ mm} + 0.0315\ y'$$

      - At point B

$$\boldsymbol{u}_x^{s,B} = -0.25 \text{ mm} + 0.0315\ 0.025 = 0.54 \text{ mm}$$

    - Branches for z'<0 obtained by symmetry

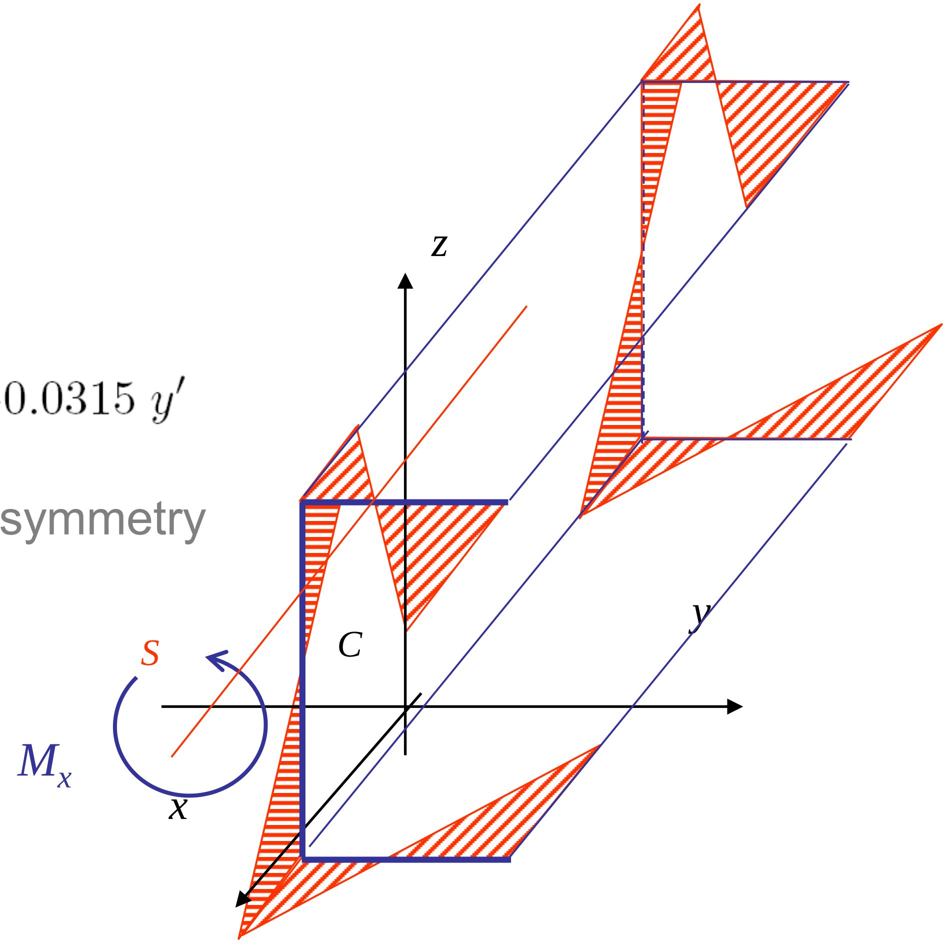
- Warping of *s*-axis (3)
  - On *O'A* branch

$$\boldsymbol{u}_x^{s,O'A}(z') = -0.0101 z'$$

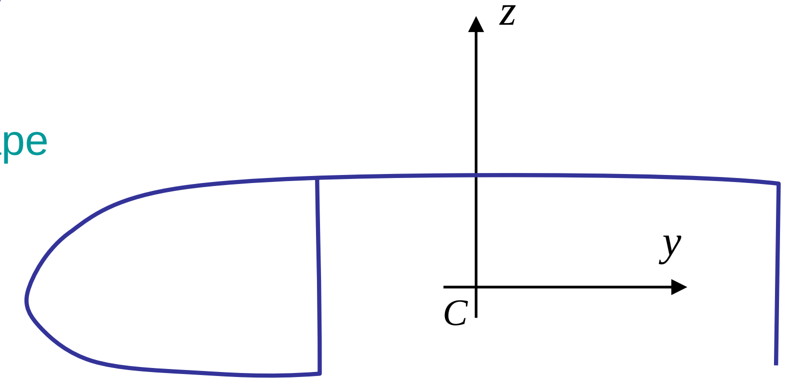
  - On *AB* branch

$$\boldsymbol{u}_x^{s,AB}(y') = -0.25\ \mathrm{mm} + 0.0315\ y'$$

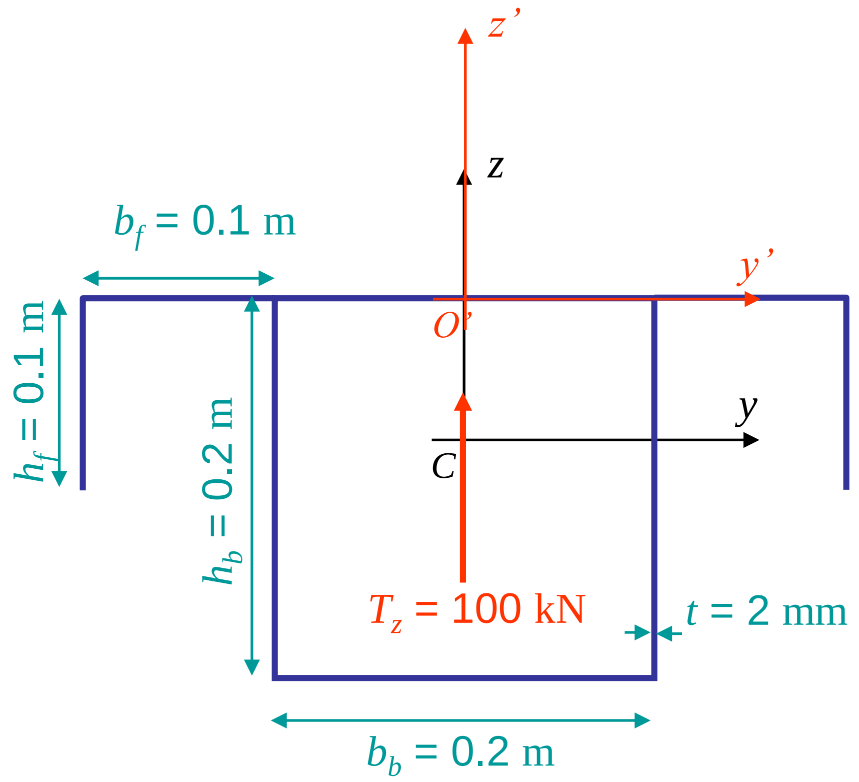
  - Branches for $z'<0$ obtained by symmetry



- Wing section near an undercarriage bay
  - Bending
    - There was no assumption on section shape
    - Use same formula
  - Shearing
    - Shear center has to be evaluated for the complete section
    - Shearing results into a shear load passing through this center & a torque
    - Shear flow has different expression in open & closed parts of the section
  - Torsion
    - Rigidity of open section can be neglected most of the time
    - But stress in open section can be high

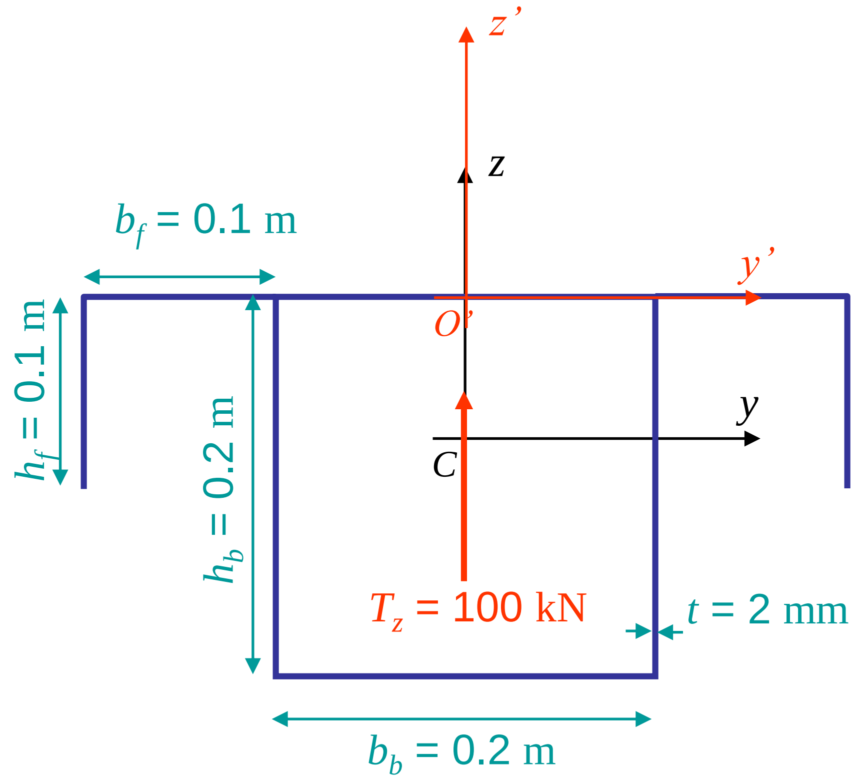


- Example
  - Simply symmetrical section
  - Constant thickness
  - Shear stress?



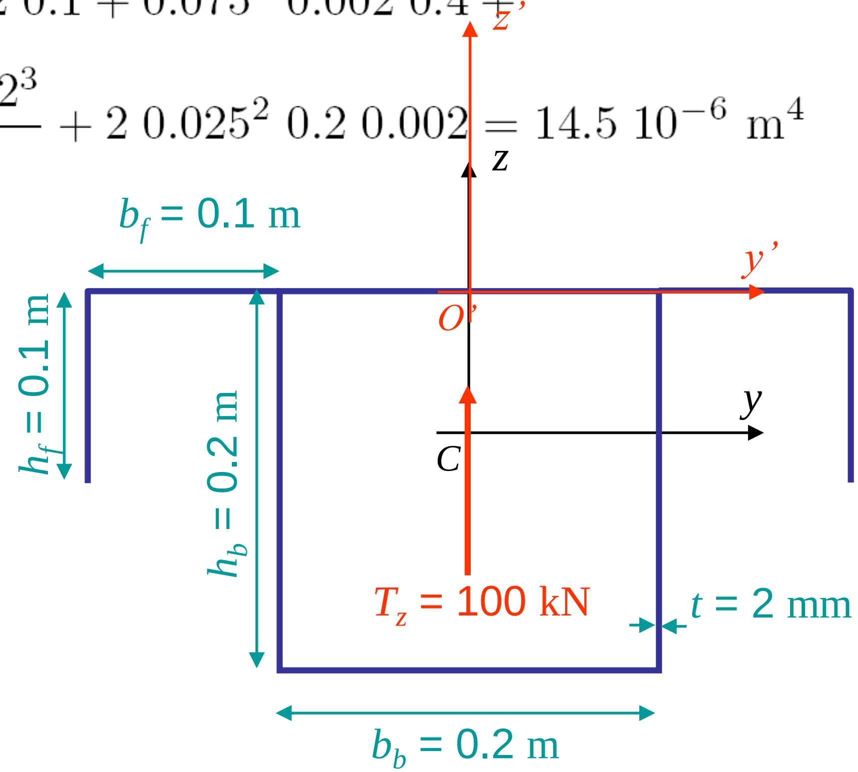
- Centroid
  - By symmetry, on $O'_{z'}$ axis
  - z'_C ?
    - $z'_C t\left(2h_f + 2b_f + 2b_b + 2h_b\right) = 2h_f t\left(-\frac{h_f}{2}\right) + b_b t\left(-h_b\right) + 2h_b t\left(-\frac{h_b}{2}\right)$

$$\Longrightarrow z'_C = -\frac{2h_f\frac{h_f}{2} + b_b h_b + 2h_b\frac{h_b}{2}}{2h_f + 2b_f + 2b_b + 2h_b} = -\frac{0.01 + 0.04 + 0.04}{0.2 + 0.2 + 0.4 + 0.4} = -0.075 \text{ m}$$



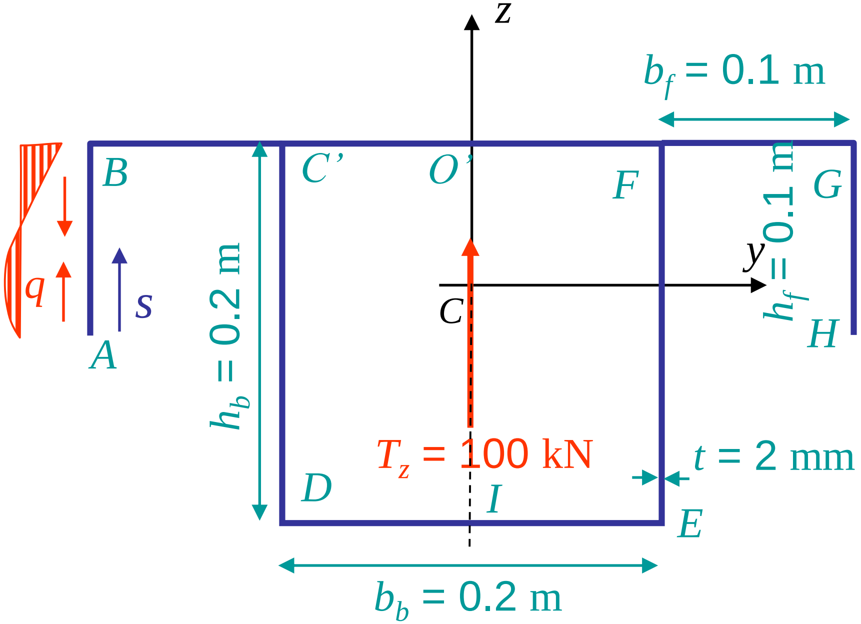
- Second moment of area
  - As z'_C = -0.075 m
  - $$I_{yy} = 2\frac{th_f^3}{12} + 2\left(-\frac{h_f}{2} - z'_C\right)^2 th_f + \left(-z'_C\right)^2 t\left(2b_f + b_b\right) + \left(-h_b - z'_C\right)^2 tb_b + 2\frac{th_b^3}{12} + 2\left(-\frac{h_b}{2} - z'_C\right)^2 h_b t$$

$$\Rightarrow I_{yy} = 2\frac{0.002\ 0.1^3}{12} + 2\ 0.025^2\ 0.002\ 0.1 + 0.075^2\ 0.002\ 0.4 + 0.125^2\ 0.002\ 0.2 + 2\frac{0.002\ 0.2^3}{12} + 2\ 0.025^2\ 0.2\ 0.002 = 14.5\ 10^{-6}\ \mathrm{m}^4$$



- Shear flow
  - As I_{xy} = 0 & as shear center on Cz
    - $q(s) = q_o(s) + q(0)$

      with $q_o(s) = -\frac{T_z}{I_{yy}} \int_0^s tz ds$
    - At A & H shear stress has to be zero
      - If origin on A, q(0) = 0
      - Corresponds to an open section



  - Branch AB

$$q^{AB}(s) = -\frac{T_z}{I_{yy}} \int_{-h_f - z'_C}^{z} t z'' dz''$$

$$= -\frac{T_z t}{2 I_{yy}} \left[z^2 - \left(-h_f - z'_C\right)^2 \right]$$

$$\Longrightarrow q^{AB}(z) = -\frac{100\ 10^3\ 0.002}{2\ 14.5\ 10^{-6}} \left[z^2 - 0.025^2\right] = 4310\ \mathrm{N \cdot m^{-1}} - 6.9\ 10^6\ \mathrm{N \cdot m^{-3}} z^2$$

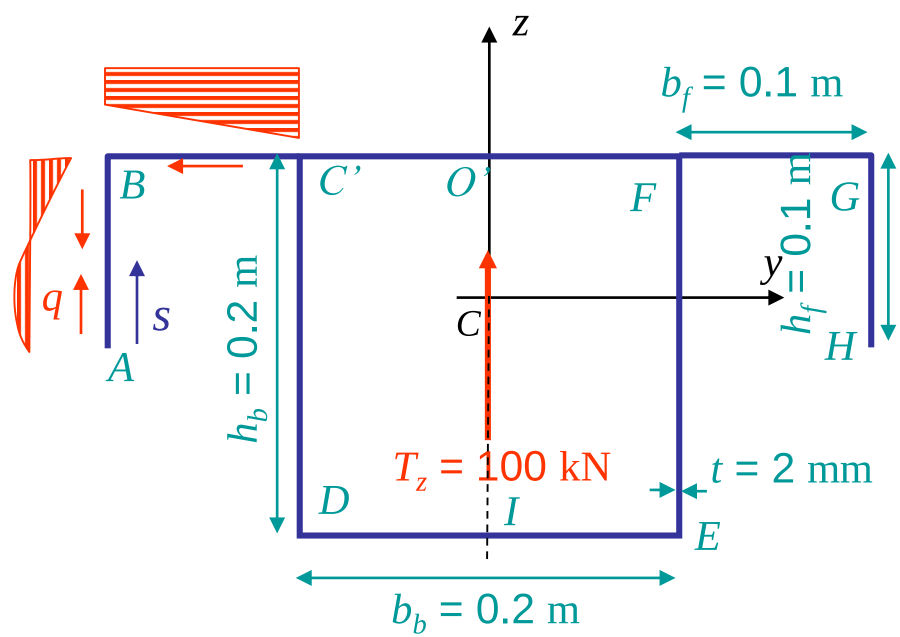
$$\Longrightarrow q^B = q^{AB}(0.075) = 4310 - 6.9\ 10^6\ 0.075^2 = -34.5\ 10^3\ \mathrm{N \cdot m^{-1}}$$

- Shear flow (2)
  - Branch *BC'*
    - $q^{BC'}(s) = q^B - \frac{T_z}{I_{yy}} \int_{-b_f - \frac{b_b}{2}}^{y} t\,(-z'_C)\, dy'' = q^B + \frac{T_z t z'_C}{I_{yy}} \left[y + b_f + \frac{b_b}{2} \right]$

$$\Longrightarrow q^{BC'}(y) = -34.5\ 10^3\ \mathrm{N{\cdot}m^{-1}} - \frac{100\ 10^3\ 0.002\ 0.075}{14.5\ 10^{-6}} [y + 0.2]$$
$$= -241.4\ 10^3\ \mathrm{N{\cdot}m^{-1}} - 1.034\ 10^6\ \mathrm{N{\cdot}m^{-2}}\, y$$

    - $q^{C';\,BC'} = q^{BC'}(-0.1) = -241.4\ 10^3 + 103.4\ 10^3 = -138\ 10^3\ \mathrm{N \cdot m^{-1}}$

  - Branches *FG* & *GH*
    - By symmetry



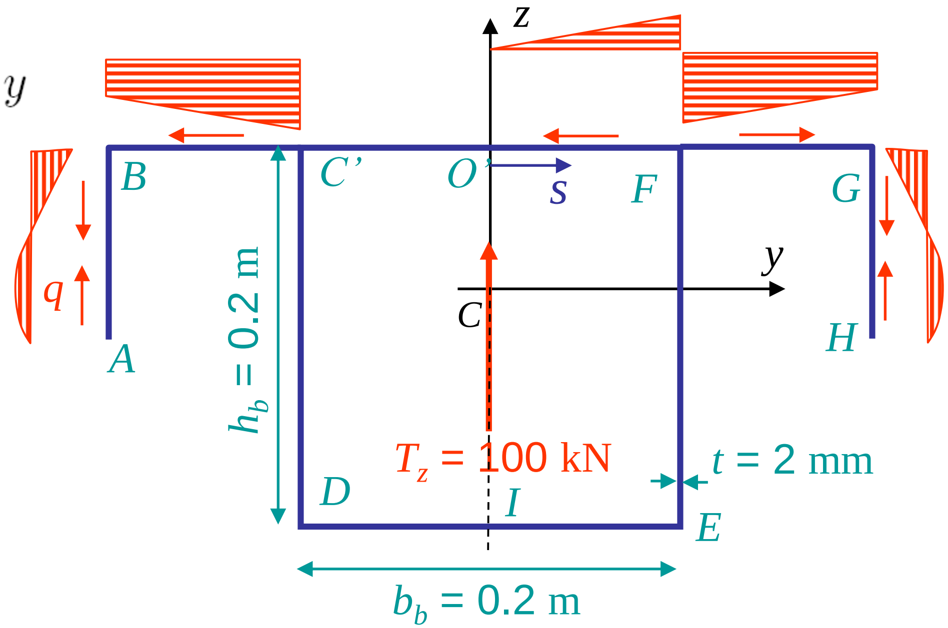
- Shear flow (3)
  - Closed part: $q(s) = q_o(s) + q(0)$
    - With $q_o(s) = -\frac{T_z}{I_{yy}} \int_0^s tz ds$ & $q(s=0) = \frac{y_T T_z - z_T T_y - \oint p(s) q_o(s) ds}{2A_h}$
  - Let us fix the origin at O'
    - By symmetry $q(0) = 0$ (if not the formula would have required anticlockwise s, q)

      $\Rightarrow$ $q = q_o(s)$
  - Branch $O'F$

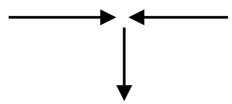
$$q^{O'F} = -\frac{T_z}{I_{yy}} \int_0^y t(-z'_C)\, dy = \frac{T_z t y z'_C}{I_{yy}} y$$

$$q^{O'F}(y) = -\frac{100\ 10^3\ 0.002\ 0.075}{14.5\ 10^{-6}} y = -1.03\ 10^6\ y\ \mathrm{N \cdot m^{-2}}$$

$$q^{F;\ O'F} = q^{O'F}(0.1) = -103\ 10^3\ \mathrm{N \cdot m^{-1}}$$



- Shear flow (4)
  - Branch *FE*
    - Shear flux should be conserved at point F



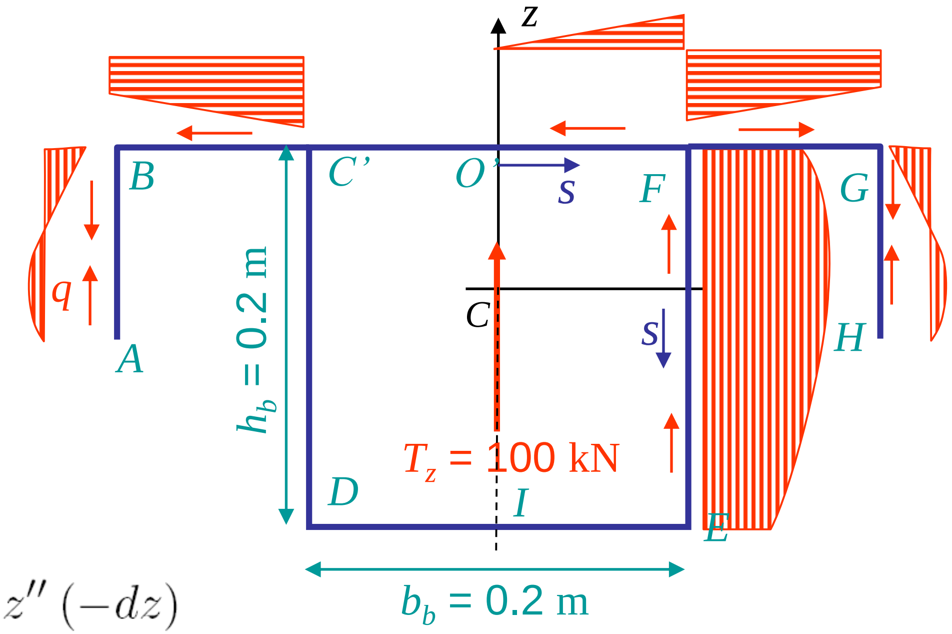
$$q^{F;\,FE} = q^{F;\,O'F} + q^{F;\,GF}$$
$$= -241\ 10^3\ \mathrm{N}\cdot\mathrm{m}^{-1}$$

    - Shear flux on branch

$$\Rightarrow q^{FE} = q^{F;\,FE} - \frac{T_z}{I_{yy}}\int_{-z'_C}^{z} t z''\,(-dz)$$
$$= q^{F;\,FE} + \frac{T_z t}{2 I_{yy}}\left(z^2 - z'^2_C\right)$$

$$\Rightarrow q^{FE}(z) = -241\ 10^3 + \frac{100\ 10^3\ 0.002}{2\ 14.5\ 10^{-6}}\left(z^2 - 0.075^2\right)$$
$$= 6.9\ 10^6\ z^2\ \mathrm{N{\cdot}m^{-3}} - 279.8\ 10^3\ \mathrm{N{\cdot}m^{-1}}$$

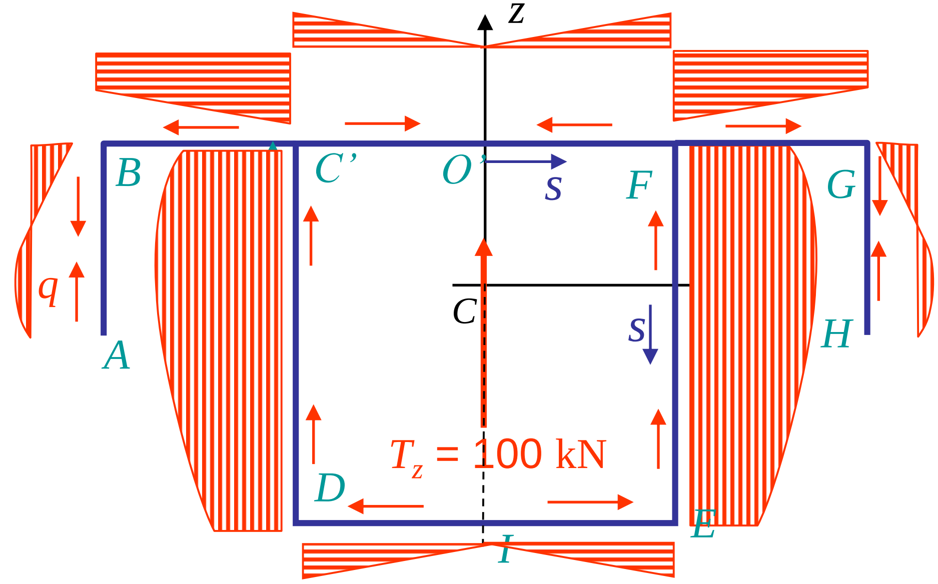
$$\Rightarrow \begin{cases} q^E = q^{FE}(-0.125) = -6.9\ 10^6\ 0.125^2 - 279.8\ 10^3 = -172\ 10^3\ \mathrm{N}\cdot\mathrm{m}^{-1} \\ \max\limits_z q^{FE}(z) = q^{FE}(0) = -279.8\ 10^3\ \mathrm{N}\cdot\mathrm{m}^{-1} \end{cases}$$



- Shear flow (5)
  - Branch *EI*
    - $q^{EI}(s) = q^E - \frac{T_z}{I_{yy}} \int_{\frac{b_b}{2}}^{y} t\left(-h_b - z'_C\right)(-dy) = q^E + \frac{T_z t\left(h_b + z'_C\right)}{I_{yy}} \left(\frac{b_b}{2} - y\right)$

$$\Longrightarrow q^{EI}(y) = -172\ 10^3 + \frac{100\ 10^3\ 0.002\ 0.125}{14.5\ 10^{-6}}(0.1 - y) = -1.72\ 10^6\ y\ \mathrm{N \cdot m^{-2}}$$

  - Other branches by symmetry



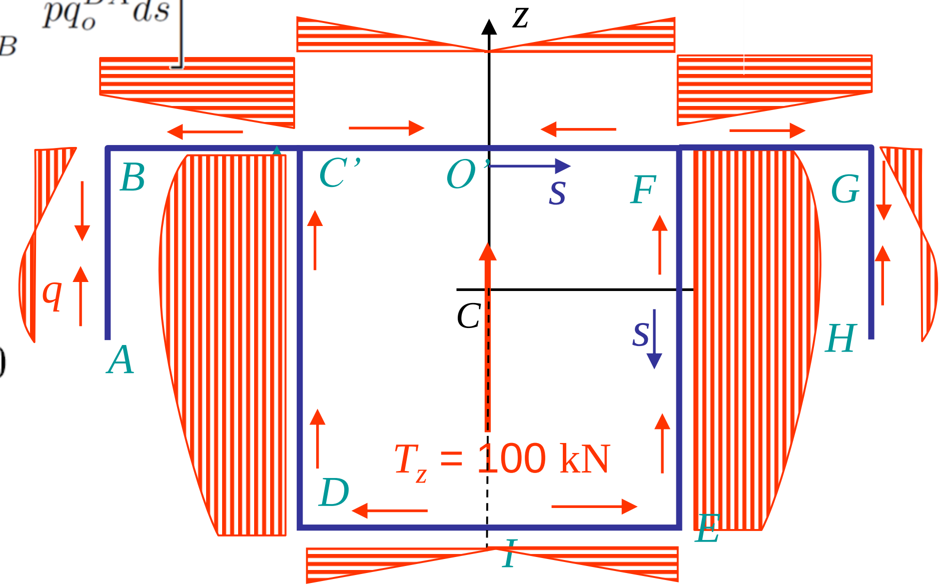
- Shear flow (6)
  - Remark, if symmetry had not been used, shear stress at O' should be computed (but require anticlockwise s and q for these signs of y_T & z_T)
    - $q(s=0) = \frac{y_T T_z - z_T T_y - \oint p(s)\, q_o(s)\, ds}{2A_h} \Longrightarrow q(O') = -\frac{1}{2A_h}\oint p q_o(s)\, ds$

$$\Longrightarrow q(O') = -\frac{1}{2b_b h_b}\left[\int_F^{O'} p q_o^{FO'} ds + \int_E^F p q_o^{EF} ds + \int_I^E p q_o^{IE} ds + \int_D^I p q_o^{DI} ds + \int_{C'}^D p q_o^{C'D} ds + \int_{O'}^{C'} p q_o^{O'C'} ds + \int_H^G p q_o^{HG} ds + \int_G^F p q_o^{GF} ds + \int_{C'}^B p q_o^{C'B} ds + \int_B^A p q_o^{BA} ds\right]$$

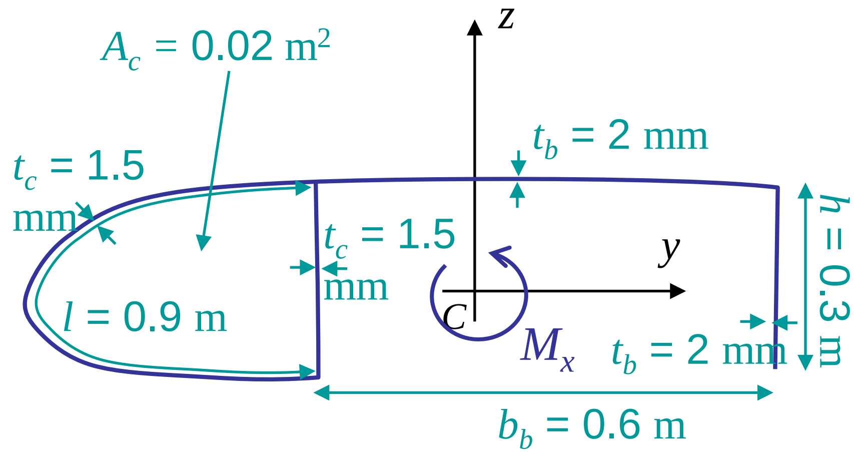
  - With
    - $p^{O'F} = p^{C'O'}$ & $q^{O'F} = -q^{C'O'}$ & $ds^{O'F} = ds^{C'O'} \Longrightarrow$

$$\int_F^{O'} p q_o^{FO'} ds + \int_{O'}^{C'} p q_o^{O'C'} ds = 0$$

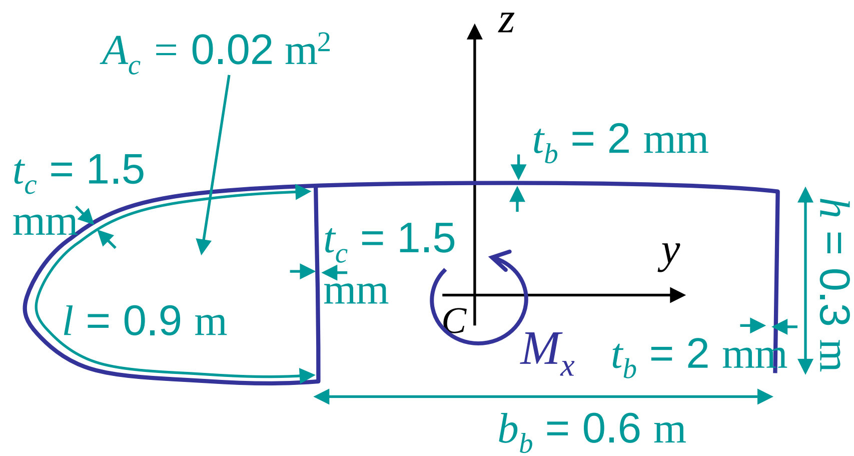
    - etc



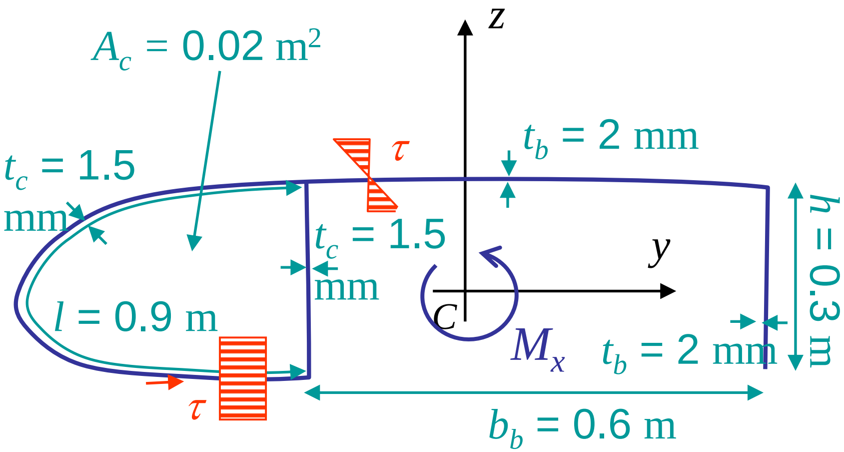
- Example
  - Closed nose cell
    - 0.02 m^2 – area
    - 0.9 m – outer length
  - Open bay
  - Constant shear modulus
    μ = 25 GPa
  - Torque 10 kN·m
  - Twist rate?
  - Shear stress?



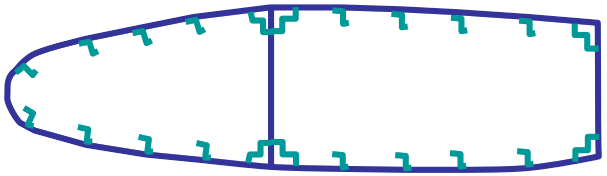
- Twist rate
  - As an approximation the 2 torsion rigidities are added
  - Cell
    - Closed section with constant μ
      - $I_{T,\text{ closed}} = \dfrac{4A_h^2}{\oint \frac{1}{t} ds}$
      - $\mu I_{Tc} = \dfrac{4\mu A_c^2 t_c}{l+h} = \dfrac{4\ 0.02^2\ 0.0015\ 25\ 10^9}{1.2} = 50\ 10^3 \quad \text{N}\cdot\text{m}^2$
  - Bay
    - Open section with constant μ
      - $I_{T,\text{ open}} = \sum_i \dfrac{l_i t_i^3}{3}$
      - $\mu I_{Tb} = \dfrac{\mu t_b^3}{3}(b_b + h) = \dfrac{25\ 10^9\ 0.002^3\ 0.9}{3} = 60 \quad \text{N}\cdot\text{m}^2$
  - Twist rate
    - $\mu I_T = 50060 \quad \text{N}\cdot\text{m}^2$
    - $\theta_{,x} = \dfrac{M_x}{\mu I_T} = \dfrac{10^4}{50060} = 0.1998 \quad \text{rad}\cdot\text{m}^{-1}$

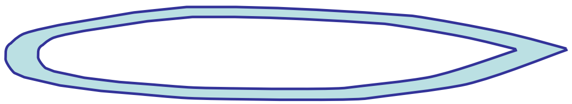


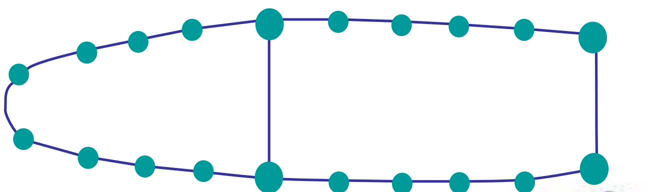
- Shear stress
  - Cell
    - Closed section ($M_x = 2A_h q$)
    - $q_c = \frac{M_x}{2A_c} = \frac{10^4}{2\ 0.02}$
      $= 250\ 10^3\ \ \mathrm{N \cdot m^{-1}}$
    - $\tau_c = \frac{q_c}{t_c} = \frac{250\ 10^3}{0.0015} = 166.7\ \ \mathrm{MPa}$
  - Bay
    - Open section ($\tau_{\max_i} = \mu t_i \theta_{,x}$)
    - $\tau_{b,\max} = \mu t_b \theta_{,x} = 25\ 10^9\ 0.002\ 0.1998 = 9.99\ \ \mathrm{MPa}$

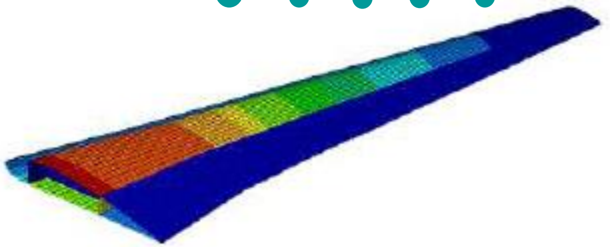


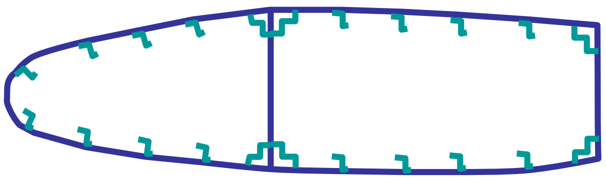
- Example 2-spar wing (one cell)
  - Stringers to stiffen thin skins
  - Angle section form spar flanges
- Design stages
  - Conceptual
    - Define the plane configuration
      - Span, airfoil profile, weights, …
    - Analyses should be fast and simple
      - Formula, statistics, …
  - Preliminary design
    - Starting point: conceptual design
    - Define more variables
      - Number of stringers, stringer area, …
    - Analyses should remain fast and simple
      - Use beam idealization
        - » See today
      - FE model of thin structures
        - » See next lectures
  - Detailed design
    - All details should be considered (rivets, …)
    - Most accurate analyses (3D, non-linear, FE)



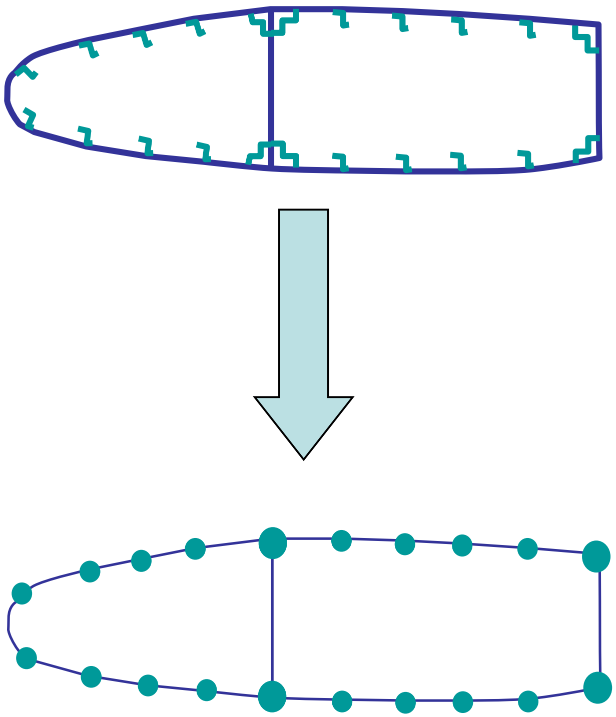








- Principle of idealization
  - Booms
    - Stringers, spar flanges, …
      - Have small sections compared to airfoil
      - Direct stress due to wing bending is almost constant in each of these
      - They are replaced by concentrated area called booms
    - Booms
      - Have their centroid on the skin
      - Are carrying most direct stress due to beam bending
  - Skin
    - Skin is essentially carrying shear stress
    - It can be assumed
      - That skin is carrying only shear stress
      - If direct stress carrying capacity of skin is reported to booms by appropriate modification of their area



- Panel idealization
  - Skin panel
    - Thickness t_D, width b
    - Carrying direct stress **linearly distributed**
  - Replaced by
    - Skin without thickness
    - 2 booms of area A_1 and A_2
  - Booms' area **depending on loading**
    - Moment around boom 2

$$\boldsymbol{\sigma}_{xx}^{2} t_D b \frac{b}{2} + \frac{\left(\boldsymbol{\sigma}_{xx}^{1} - \boldsymbol{\sigma}_{xx}^{2}\right)}{2} t_D b \frac{2b}{3} = \boldsymbol{\sigma}_{xx}^{1} A_1 b$$

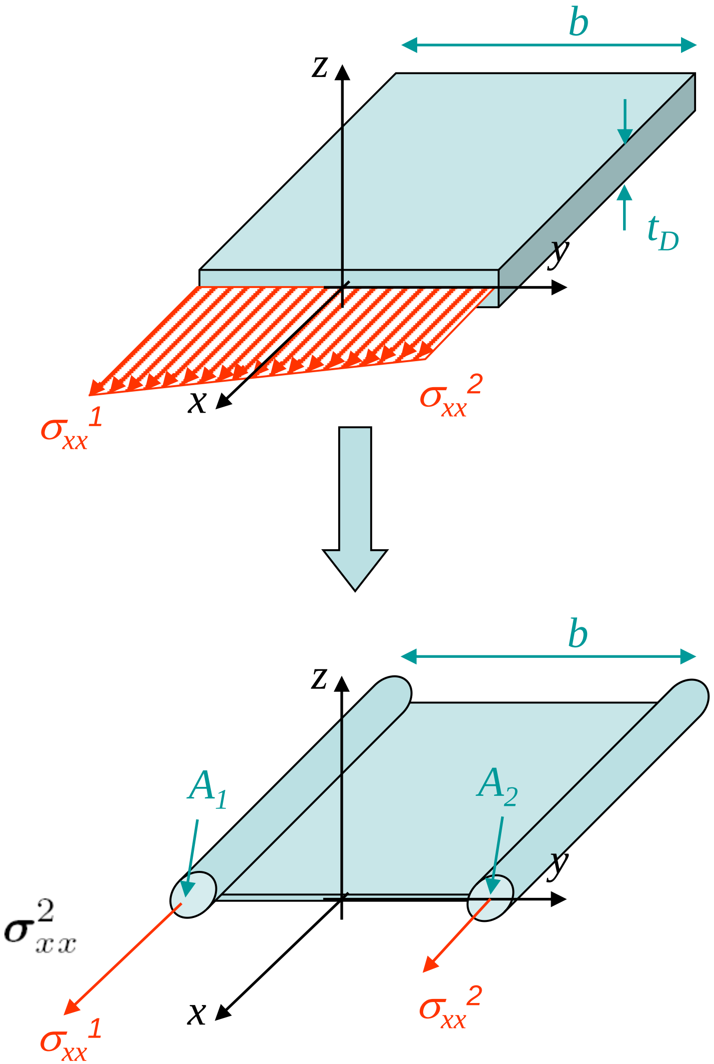
$$\Longrightarrow A_1 = \frac{t_D b}{6}\left(2 + \frac{\boldsymbol{\sigma}_{xx}^{2}}{\boldsymbol{\sigma}_{xx}^{1}}\right)$$

- Total axial loading

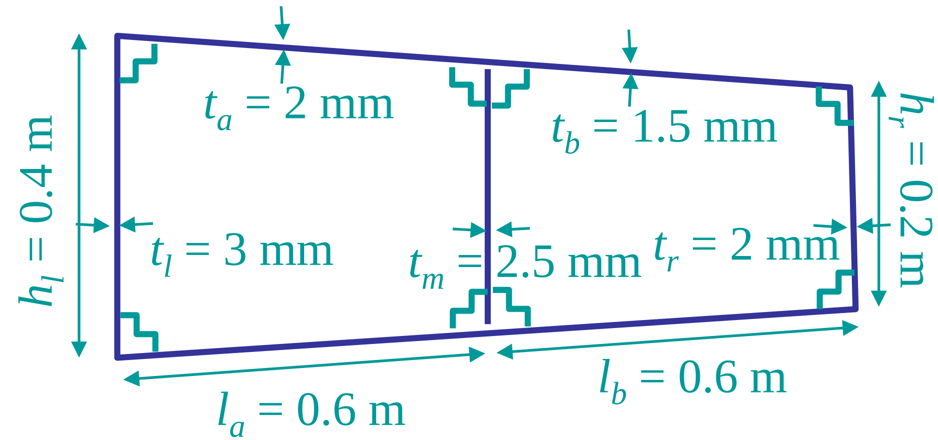
$$\boldsymbol{\sigma}_{xx}^{2} t_D b + \left(\boldsymbol{\sigma}_{xx}^{1} - \boldsymbol{\sigma}_{xx}^{2}\right)\frac{t_D b}{2} = A_1 \boldsymbol{\sigma}_{xx}^{1} + A_2 \boldsymbol{\sigma}_{xx}^{2}$$

$$\Longrightarrow A_2 = t_D b + \left(\frac{\boldsymbol{\sigma}_{xx}^{1}}{\boldsymbol{\sigma}_{xx}^{2}} - 1\right)\frac{t_D b}{2} - A_1 \frac{\boldsymbol{\sigma}_{xx}^{1}}{\boldsymbol{\sigma}_{xx}^{2}}$$

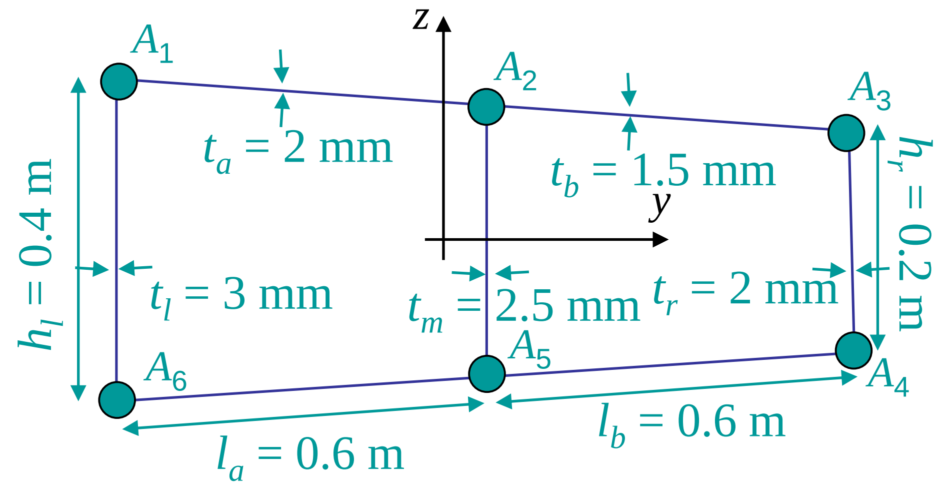
$$\Longrightarrow A_2 = \frac{t_D b}{6}\left(2 + \frac{\boldsymbol{\sigma}_{xx}^{1}}{\boldsymbol{\sigma}_{xx}^{2}}\right)$$



- Example
  - 2-cell box wing section
  - Simply symmetrical
  - Angle section of 300 mm^2

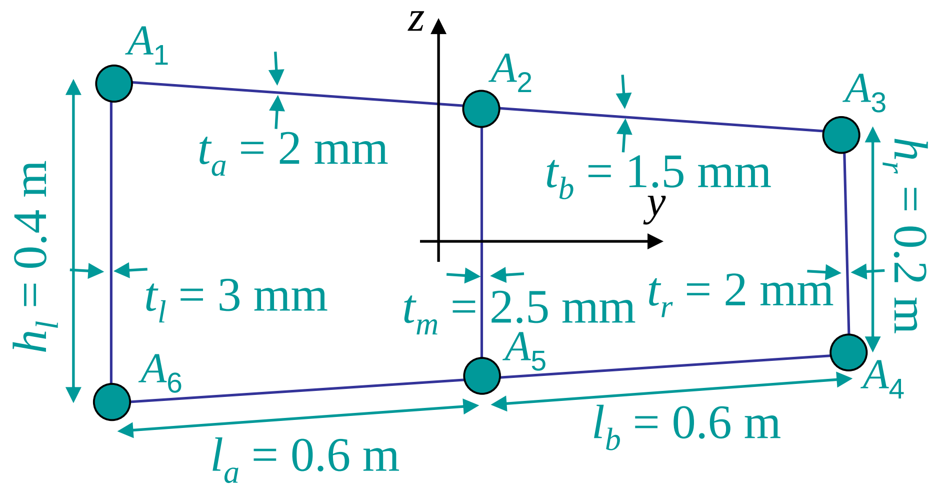


  - Idealization of this section to resist to bending moment?
    - Bending moment along y-axis
    - 6 direct-stress carrying booms
    - Shear-stress-only carrying skin panels



- Booms' area
  - Bending moment
    - Along y-axis
    - Stress proportional to z

      $\Rightarrow$ stress distribution is linear on each section edge
  - Contributions
    - Flange(s)' area
    - Reported skin parts
      - Use formula for linear distribution



- $$A_1 = 300\ 10^{-6} + \frac{0.003\ 0.4}{6}\left(2 + \frac{-0.2}{0.2}\right) + \frac{0.002\ 0.6}{6}\left(2 + \frac{0.15}{0.2}\right)$$

  $$\Rightarrow A_6 = A_1 = 1.05\ 10^{-3}\ \ \mathrm{m}^2$$

- $$A_2 = 2\ 300\ 10^{-6} + \frac{0.002\ 0.6}{6}\left(2 + \frac{0.2}{0.15}\right) + \frac{0.0015\ 0.6}{6}\left(2 + \frac{0.1}{0.15}\right) + \frac{0.0025\ 0.3}{6}\left(2 + \frac{-0.15}{0.15}\right) \Rightarrow A_2 = A_4 = 1.79\ 10^{-3}\ \ \mathrm{m}^2$$

- $$A_3 = 300\ 10^{-6} + \frac{0.0015\ 0.6}{6}\left(2 + \frac{0.15}{0.1}\right) + \frac{0.002\ 0.2}{6}\left(2 + \frac{-0.1}{0.1}\right)$$

  $$\Rightarrow A_4 = A_3 = 0.892\ 10^{-3}\ \ \mathrm{m}^2$$

Section idealization consequences

- Consequence on bending
  - Idealization depends on the loading case
    - Booms area are dependent on the loading case
  - Direct stress due to bending is carried by booms only
    - For bending the axial load is equal to zero

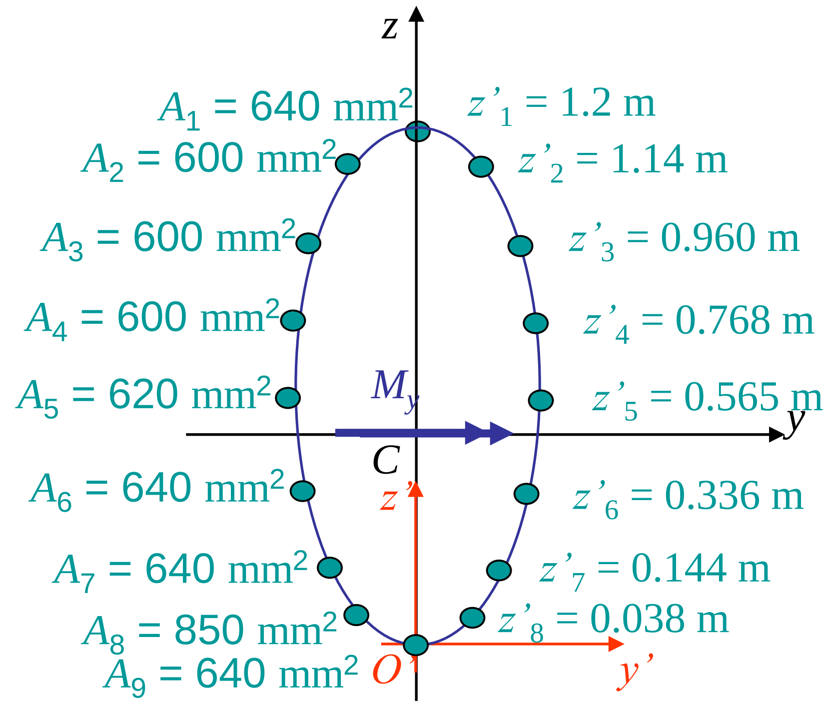
$$\Longrightarrow \boldsymbol{N}_x = \int_A \boldsymbol{\sigma}_{xx} dA = \sum_i \boldsymbol{\sigma}^i_{xx} A_i = 0$$

    - But direct stress depends on the distance z from neutral axis

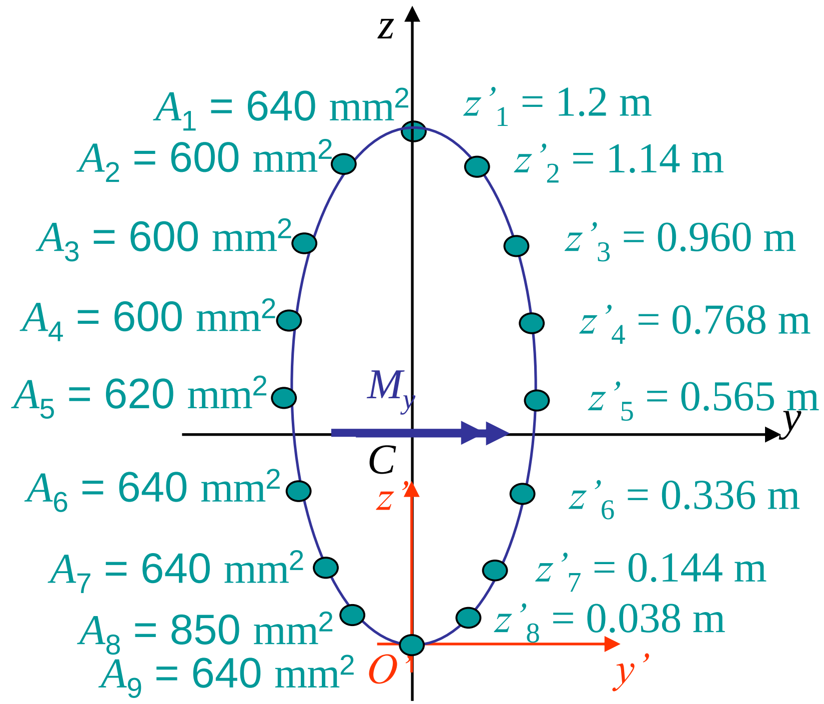
$$\boldsymbol{\sigma}^i_{xx} = \kappa E z_i \Longrightarrow \sum_i z_i A_i = 0$$

  - It can be concluded that for open or closed sections, the position of the neutral axis, and thus the second moments of area
    - Refer to the direct stress carrying area only
    - Depend on the loading case only

- Example
  - Idealized fuselage section
    - Simply symmetrical
    - Direct stress carrying booms
    - Shear stress carrying skin panels
  - Subjected to a bending moment
    - $M_y = 100$ kN·m
  - Stress in each boom?



- Centroid
  - Of idealized section



$$z'_c \left(A_1 + 2\sum_{i=2}^{8} A_i + A_9 \right) = A_1 z'_1 + 2\sum_{i=2}^{8} A_i z'_i$$

$$\Rightarrow z'_c = \frac{1}{6\ 0.00064 + 6\ 0.0006 + 2\ 0.00062 + 2\ 0.00085}$$

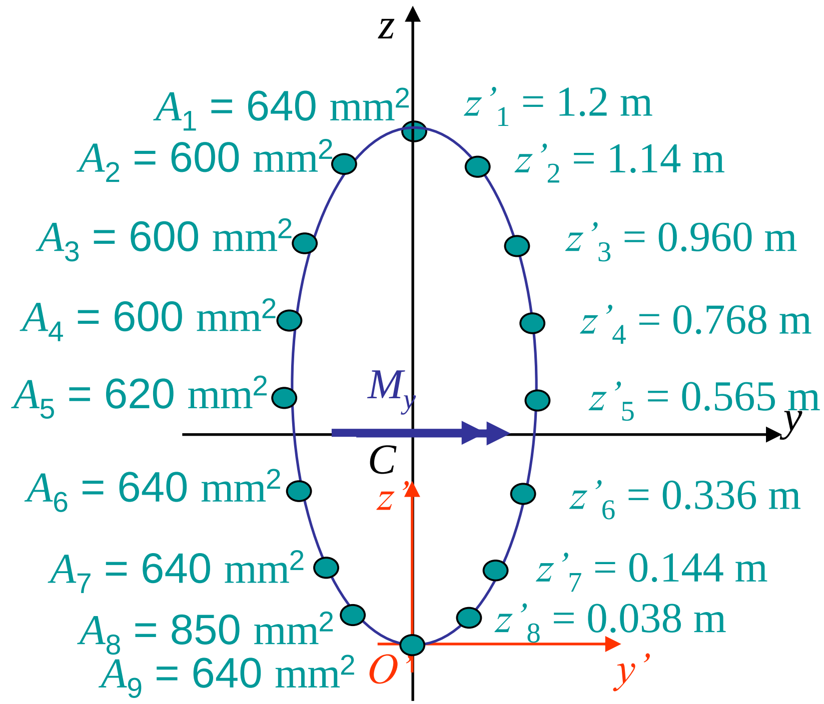
$$[1.2\ 0.00064 + 2\ (1.14 + 0.96 + 0.768)\ 0.0006 + 2\ 0.565\ 0.00062 +$$

$$2\ (0.336 + 0.144)\ 0.00064 + 2\ 0.038\ 0.00085]$$

$$\Rightarrow z'_C = \frac{0.0055892}{0.01038} = 0.54 \text{ m}$$

- Second moment of area
  - Of idealized section

$$I_{yy} = A_1 \left(z'_1 - z'_C\right)^2 + 2\sum_{i=2}^{8} A_i \left(z'_i - z'_C\right)^2 + A_9 \left(z'_9 - z'_C\right)$$

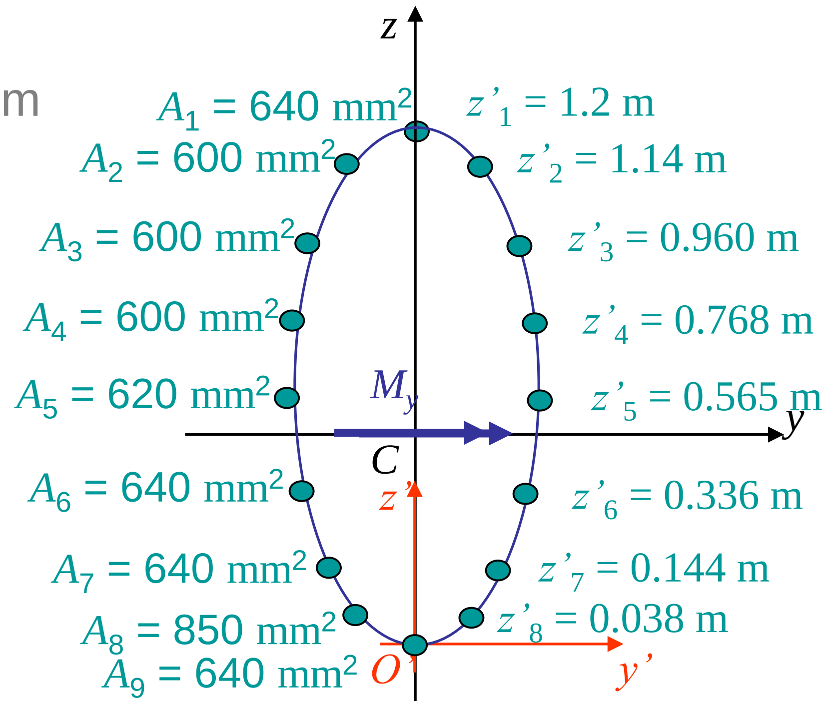


$$\Rightarrow I_{yy} = 0.00064\ 0.66^2 + 2\ 0.0006\ \left(0.6^2 + 0.42^2 + 0.228^2\right) + 2\ 0.00062\ 0.025^2 + 2\ 0.00064\ \left(\left(-0.204\right)^2 + \left(-0.396\right)^2\right) + 2\ 0.00085\ \left(-0.502\right)^2 + 0.00064\ \left(-0.54\right)^2$$

$$\Rightarrow I_{yy} = 1.855\ 10^{-3}\ \ \mathrm{m}^4$$

- Stress distribution
  - Stress assumed constant in each boom
  - As we are in the principal axes

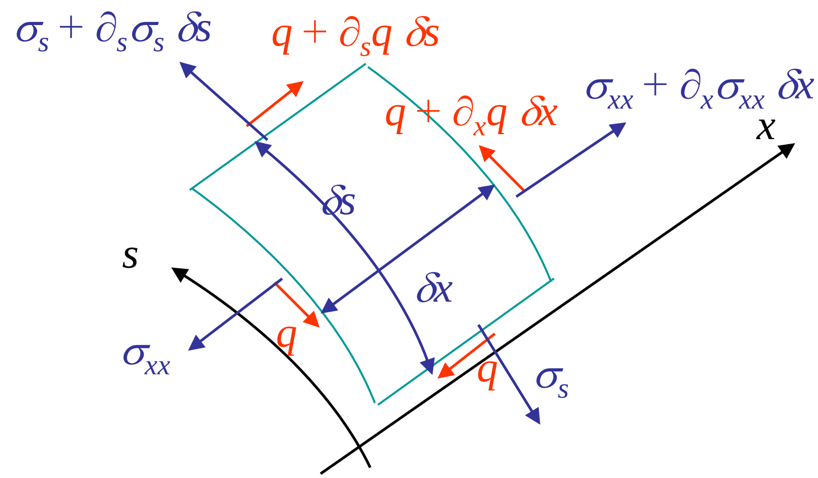
$$\boldsymbol{\sigma}_{xx}^{i} = \frac{M_y z_i}{I_{yy}} = \frac{M_y}{I_{yy}}\left(z'_i - z'_C\right)$$



$$\begin{cases} \boldsymbol{\sigma}_{xx}^{1} = \dfrac{100\ 10^3}{1.855\ 10^{-3}} 0.66 = 35.6\ \text{ MPa} \\ \boldsymbol{\sigma}_{xx}^{2} = \dfrac{100\ 10^3}{1.855\ 10^{-3}} 0.6 = 32.3\ \text{MPa} \\ \boldsymbol{\sigma}_{xx}^{3} = \dfrac{100\ 10^3}{1.855\ 10^{-3}} 0.42 = 22.6\ \text{MPa} \\ \boldsymbol{\sigma}_{xx}^{4} = \dfrac{100\ 10^3}{1.855\ 10^{-3}} 0.228 = 12.3\ \text{MPa} \\ \boldsymbol{\sigma}_{xx}^{5} = \dfrac{100\ 10^3}{1.855\ 10^{-3}} 0.025 = 1.35\ \text{MPa} \\ \boldsymbol{\sigma}_{xx}^{6} = -\dfrac{100\ 10^3}{1.855\ 10^{-3}} 0.204 = -11.0\ \text{MPa} \end{cases}$$

$$\begin{cases} \boldsymbol{\sigma}_{xx}^{7} = -\dfrac{100\ 10^3}{1.855\ 10^{-3}} 0.396 = -21.3\ \text{MPa} \\ \boldsymbol{\sigma}_{xx}^{8} = -\dfrac{100\ 10^3}{1.855\ 10^{-3}} 0.502 = -27.1\ \text{MPa} \\ \boldsymbol{\sigma}_{xx}^{9} = -\dfrac{100\ 10^3}{1.855\ 10^{-3}} 0.54 = -29.1\ \text{MPa} \end{cases}$$

- Consequence on open-thin-walled section shearing
  - Classical formula
    - $q(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s tz ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s ty ds'$
    - Results from integration of balance equation $t\partial_x \boldsymbol{\sigma}_{xx} + \partial_s q = 0$
      - With $\boldsymbol{\sigma}_{xx} = \frac{(I_{zz}M_y + I_{yz}M_z)z - (I_{yz}M_y + I_{yy}M_z)y}{I_{yy}I_{zz} - I_{yz}^2}$



  - So consequences are
    - Terms $\int_0^s t(s')z(s')ds'$ & $\int_0^s t(s')y(s')ds'$ should account for the direct stress-carrying parts only (which is not the case of shear-carrying-only skin panels)
    - Expression of the shear flux should be modified to account for discontinuities encountered between booms and shear-carrying-only skin panels

- Consequence on open-thin-walled section shearing (2)
  - Equilibrium of a boom of an idealized section

$$\left(\boldsymbol{\sigma}_{xx} + \partial_x \boldsymbol{\sigma}_{xx}\delta x\right) A_i - \boldsymbol{\sigma}_{xx} A_i + q_{i+1}\delta x - q_i \delta x = 0$$

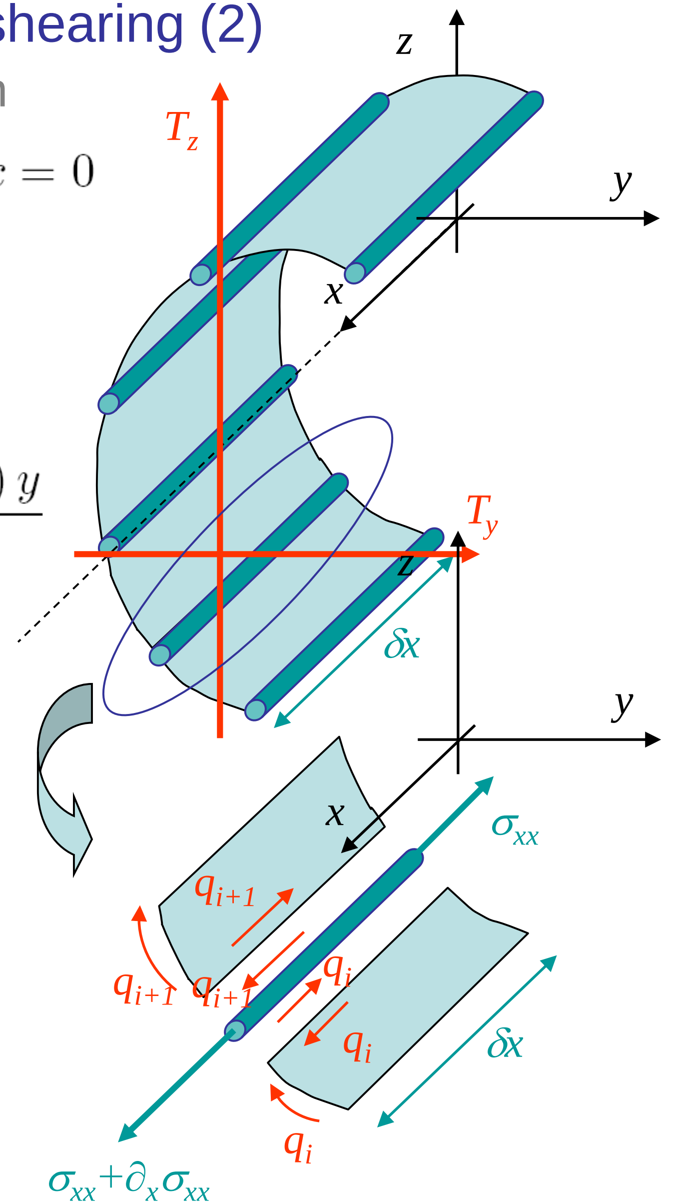
$$\Longrightarrow q_{i+1} - q_i = -\partial_x \boldsymbol{\sigma}_{xx} A_i$$

  - Lecture on beam shearing
    - Direct stress reads

$$\boldsymbol{\sigma}_{xx} = \frac{\left(I_{zz} M_y + I_{yz} M_z\right) z - \left(I_{yz} M_y + I_{yy} M_z\right) y}{I_{yy} I_{zz} - I_{yz}^2}$$

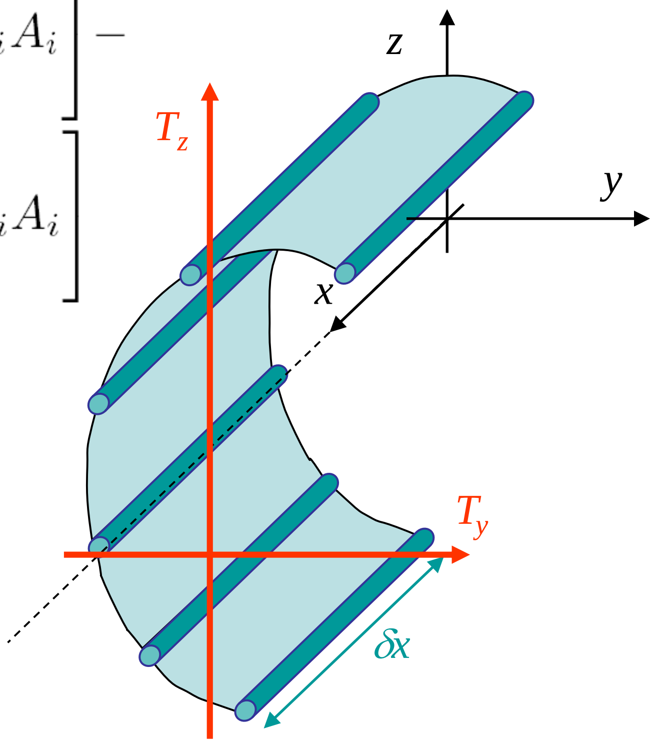
    - With $T_z = \partial_x M_y$ & $T_y = -\partial_x M_z$
  - Eventually
    - $q_{i+1} - q_i = -\frac{I_{zz} T_z - I_{yz} T_y}{I_{yy} I_{zz} - I_{yz}^2} z_i A_i - \frac{I_{yy} T_y - I_{yz} T_z}{I_{yy} I_{zz} - I_{yz}^2} y_i A_i$

      (no sum on i)

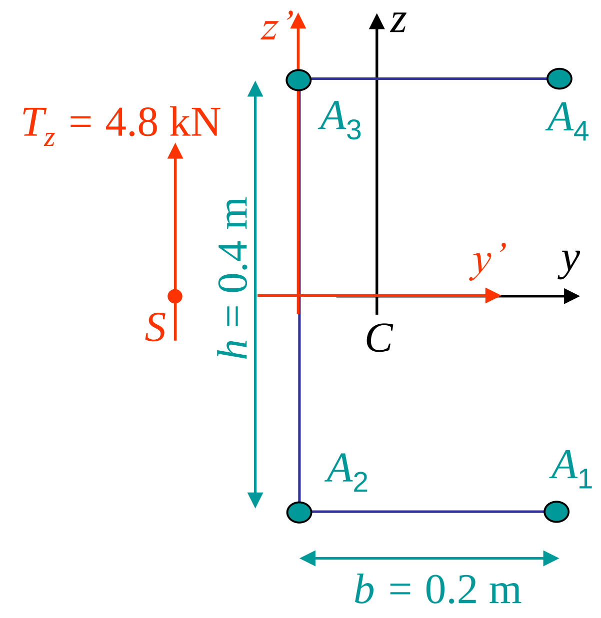


- Consequence on open-thin-walled section shearing (3)
  - Shear flow

$$q(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\boldsymbol{\sigma}} z ds + \sum_{i:\, s_i \leq s} z_i A_i\right] - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\boldsymbol{\sigma}} y ds + \sum_{i:\, s_i \leq s} y_i A_i\right]$$



- Example
  - Idealized U shape
    - Booms of 300 mm²- area each
    - Booms are carrying all the direct stress
    - Skin panels are carrying all the shear flow
  - Shear load passes through the shear center
  - Shear flow?



- Shear flow
  - Simple symmetry ⟹ principal axes

$$\Rightarrow q(s) = -\frac{T_z}{I_{yy}}\left[\int_0^s t_{\text{direct }\sigma} z\, ds + \sum_{i:\ s_i \leq s} z_i A_i\right]$$

  - Only booms are carrying direct stress

$$\Rightarrow q(s) = -\frac{T_z}{I_{yy}} \sum_{i:\ s_i \leq s} z_i A_i$$

  - Second moment of area

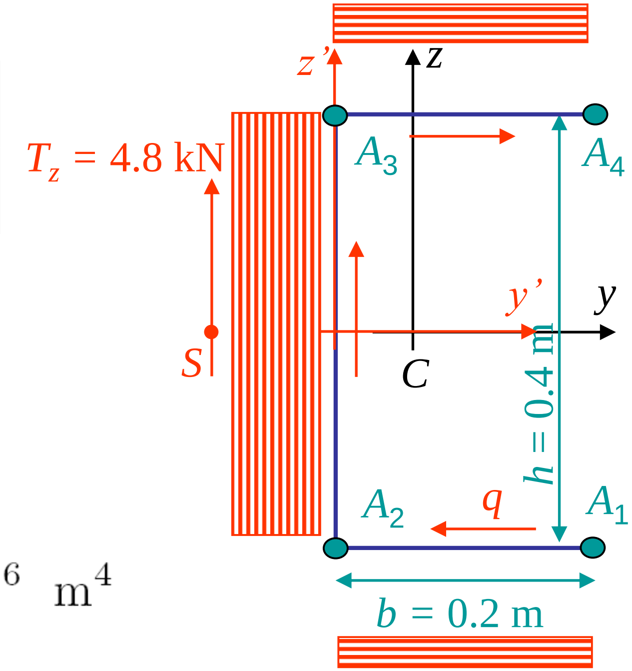
$$I_{yy} = \sum_i A_i z_i^2 = 4\ 300\ 10^{-6}\ 0.2^2 = 48\ 10^{-6}\ \ \mathrm{m}^4$$

  - Shear flow

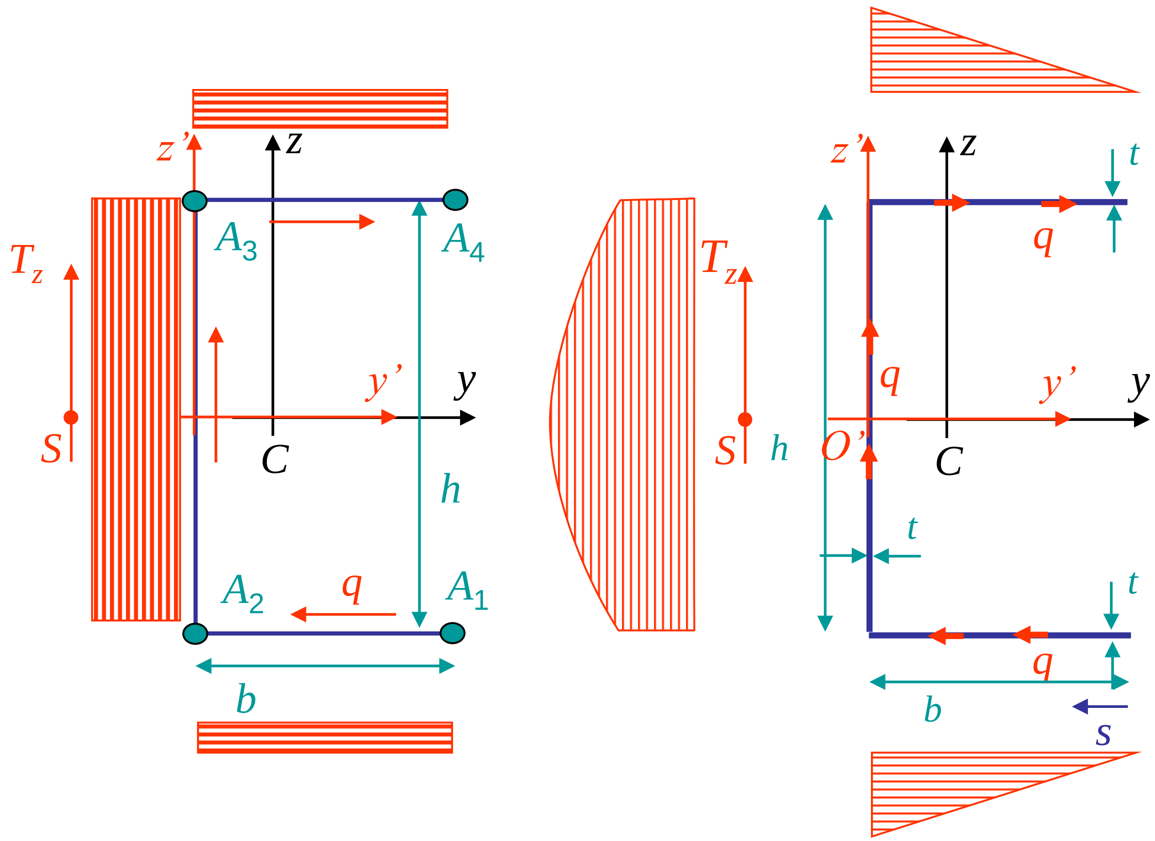
$$q^{12}(s) = -\frac{T_z}{I_{yy}} A_1 z_1 = -\frac{4.8\ 10^3}{48\ 10^{-6}} 300\ 10^{-6}(-0.2) = 6\ 10^3\ \ \mathrm{N \cdot m^{-1}}$$

$$q^{23}(s) = -\frac{T_z}{I_{yy}} (A_1 z_1 + A_2 z_2) = -\frac{4.8\ 10^3}{48\ 10^{-6}} 300\ 10^{-6}(-0.4) = 12\ 10^3\ \ \mathrm{N \cdot m^{-1}}$$

$$q^{34}(s) = -\frac{T_z}{I_{yy}} (A_1 z_1 + A_2 z_2 + A_3 z_3) = -\frac{4.8\ 10^3}{48\ 10^{-6}} 300\ 10^{-6}(-0.2) = 6\ 10^3\ \ \mathrm{N \cdot m^{-1}}$$



- Comparison with uniform U section
  - We are actually capturing the **average** value in each branch

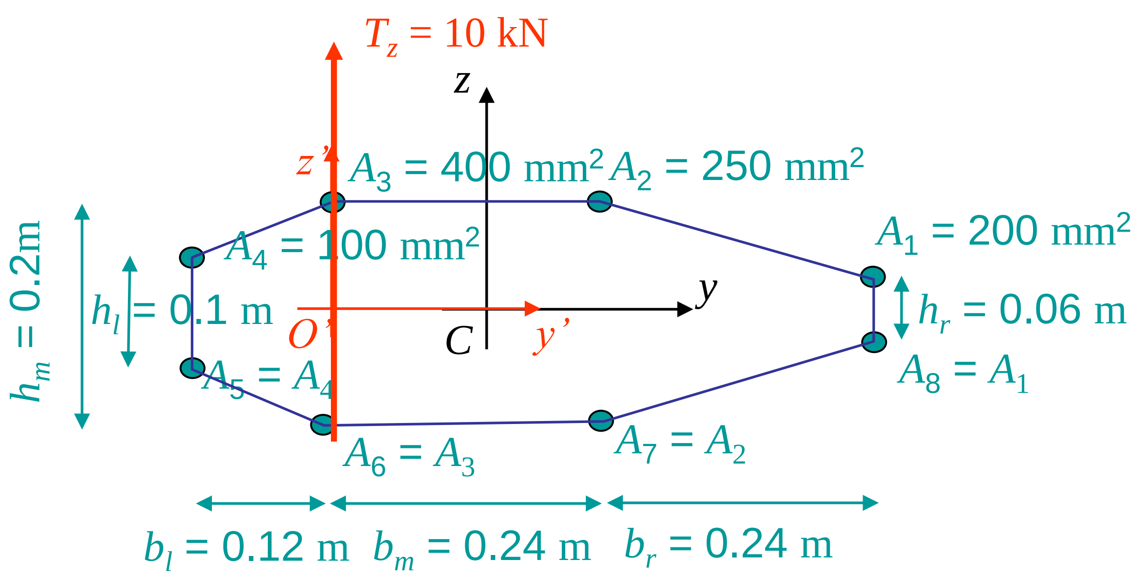


- Consequence on closed-thin-walled section shearing
  - Classical formula
    - $q(s) = q_o(s) + q(0)$
    - With $q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2} \int_0^s t(s')\, z(s')\, ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2} \int_0^s t(s')\, y(s')\, ds'$
    - And $q(s=0) = \frac{y_T T_z - z_T T_y - \oint p(s)\, q_o(s)\, ds}{2A_h}$

      for anticlockwise q and s
  - So consequences are the same as for open section
    - $q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2} \left[\int_0^s t_{\text{direct}\ \boldsymbol{\sigma}} z\, ds + \sum_{i:\ s_i \leq s} z_i A_i\right] - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2} \left[\int_0^s t_{\text{direct}\ \boldsymbol{\sigma}} y\, ds + \sum_{i:\ s_i \leq s} y_i A_i\right]$

Section idealization consequences

- Example
  - Idealized wing section
    - Simply symmetrical
    - Booms are carrying all the direct stress
    - Skin panels are carrying all the shear flow
  - Shear load passes through booms 3 & 6
  - Shear flow?

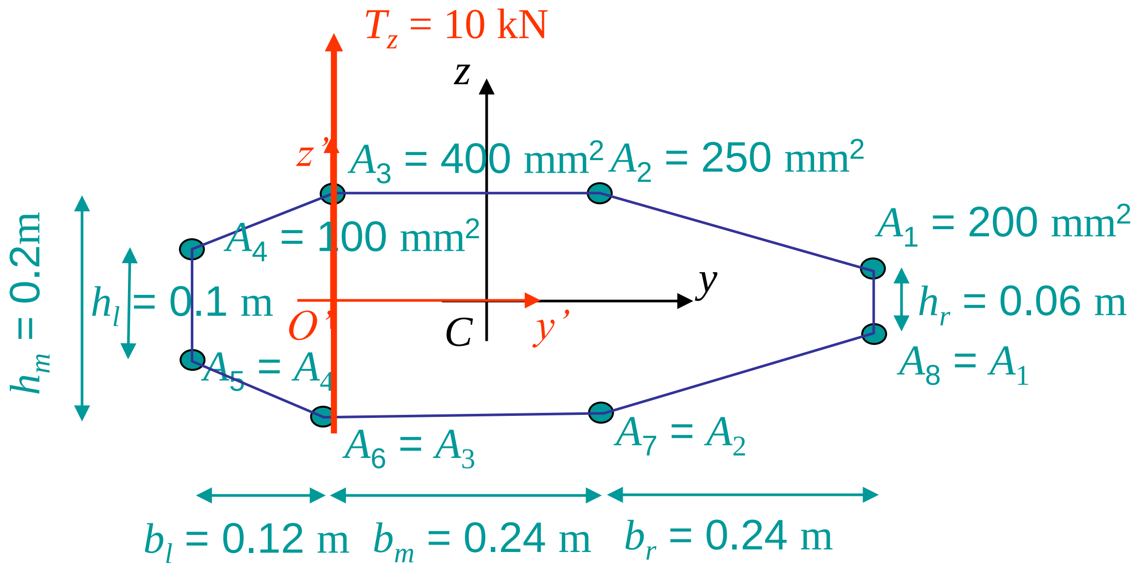


- Open part of shear flow
  - Symmetrical section
    - Shear center & centroid on Cy axis
    - I_{xy} = 0 (we are in the principal axes)
    - Only booms are carrying direct stress

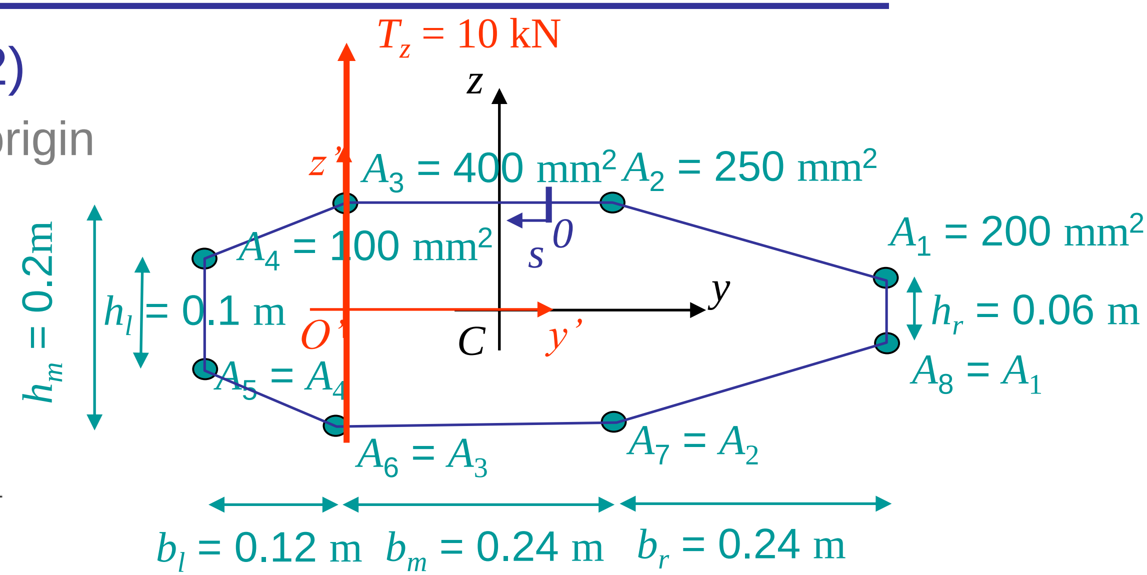
$$q_o(s) = -\frac{T_z}{I_{yy}} \sum_{i:\ s_i \leq s} z_i A_i$$

  - Second moment of area

$$I_{yy} = \sum_{i=1}^{8} A_i z_i^2 = 2\ 10^{-6}\ \left(200\ 0.03^2 + 250\ 0.1^2 + 400\ 0.1^2 + 100\ 0.05^2\right)$$
$$= 13.86\ 10^{-6}\ \ \mathrm{m}^4$$



- Open part of shear flow (2)
  - Choose (arbitrarily) the origin between boom 2 and 3



$$q_o^{03} = 0$$

$$q_o^{34} = -\frac{T_z}{I_{yy}} A_3 z_3 = -\frac{10^4}{13.86\ 10^{-6}} 0.0004\ 0.1 = -28.9\ 10^3 \quad \mathrm{N \cdot m^{-1}}$$

$$q_o^{45} = -\frac{10^4}{13.86\ 10^{-6}} (0.0004\ 0.1 + 0.0001\ 0.05) = -32.5\ 10^3 \quad \mathrm{N \cdot m^{-1}}$$

$$q_o^{56} = -\frac{10^4}{13.86\ 10^{-6}} [0.0004\ 0.1 + 0.0001\ (0.05 - 0.05)] = -28.9\ 10^3 \quad \mathrm{N \cdot m^{-1}}$$

$$q_o^{67} = -\frac{10^4}{13.86\ 10^{-6}} [0.0004\ \ (0.1 - 0.1) + 0.0001\ (0.05 - 0.05)] = 0$$

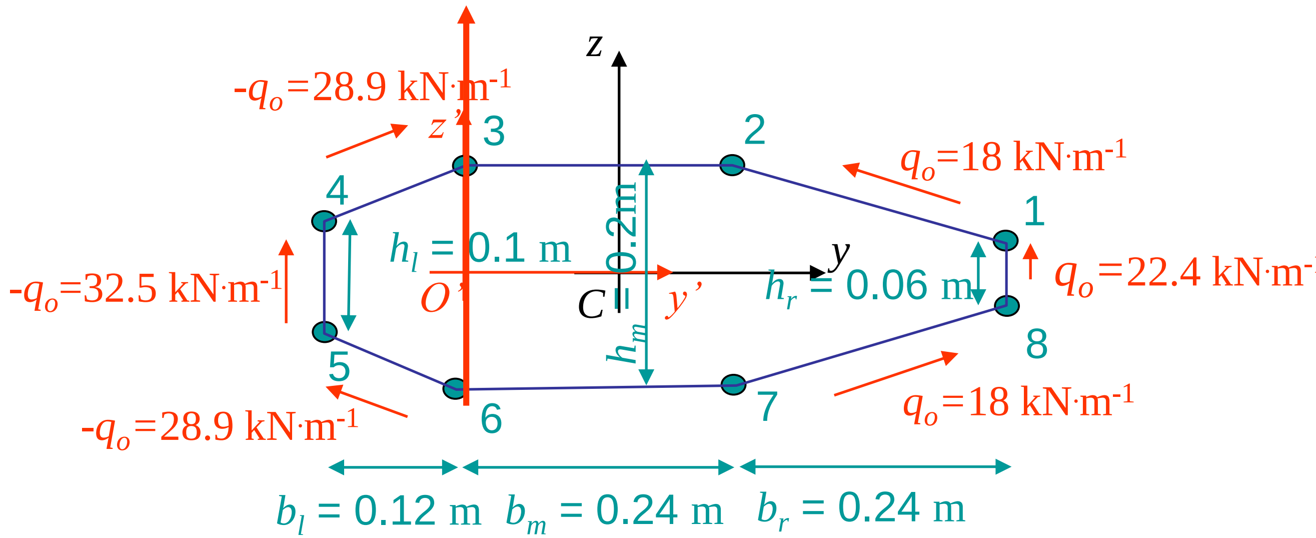
$$q_o^{78} = -\frac{10^4}{13.86\ 10^{-6}} [\ldots - 0.00025\ 0.1] = 18\ 10^3 \quad \mathrm{N \cdot m^{-1}}$$

$$q_o^{81} = -\frac{10^4}{13.86\ 10^{-6}} [\ldots - 0.00025\ 0.1 - 0.0002\ 0.03] = 22.4\ 10^3 \quad \mathrm{N \cdot m^{-1}}$$

- Open part of shear flow (3)
  - Choose (arbitrarily) the origin between boom 2 and 3 (2)

$$q_o^{12} = -\frac{10^4}{13.86\ 10^{-6}} [\ldots - 0.00025\ 0.1 + 0.0002\ (0.03 - 0.03)] = 18\ 10^3\ \ \mathrm{N} \cdot \mathrm{m}^{-1}$$

$$q_o^{20} = -\frac{10^4}{13.86\ 10^{-6}} [\ldots + 0.00025\ (0.1 - 0.1) + 0.0002\ (0.03 - 0.03)] = 0$$

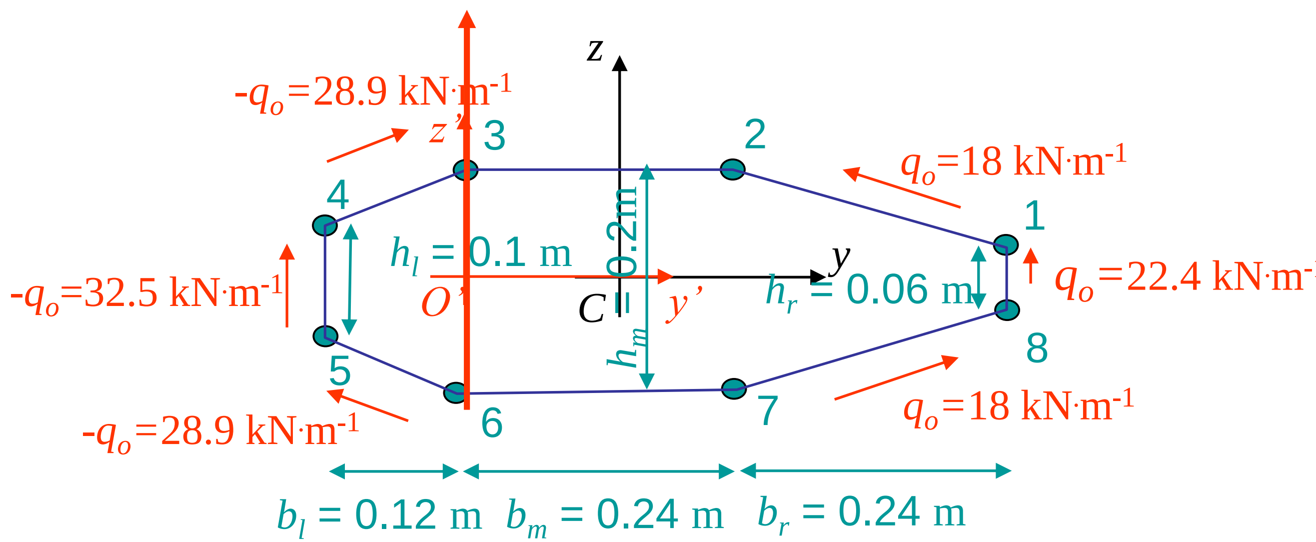


- Constant part of shear flow
  - $q(0) = \frac{y_T T_z - \oint p q_o ds}{2A_h}$ (anticlockwise s, q)
  - If origin is chosen at point O' $\Longrightarrow$ $q(0) = -\frac{\oint p_{O'} q_o ds}{2A_h}$
    - With

$$A_h = b_l \frac{h_m + h_l}{2} + b_m h_m + b_r \frac{h_m + h_r}{2} = 0.12\ 0.15 + 0.24\ 0.2 + 0.24\ 0.13 = 0.0972 \ \ \mathrm{m}^2$$

&

$$\oint p_{O'} q_o ds = q_o^{34} p_{O'}^{34} l^{34} + q_o^{45} p_{O'}^{45} l^{45} + q_o^{56} p_{O'}^{56} l^{56} + q_o^{78} p_{O'}^{78} l^{78} + q_o^{81} p_{O'}^{81} l^{81} + q_o^{12} p_{O'}^{12} l^{12}$$

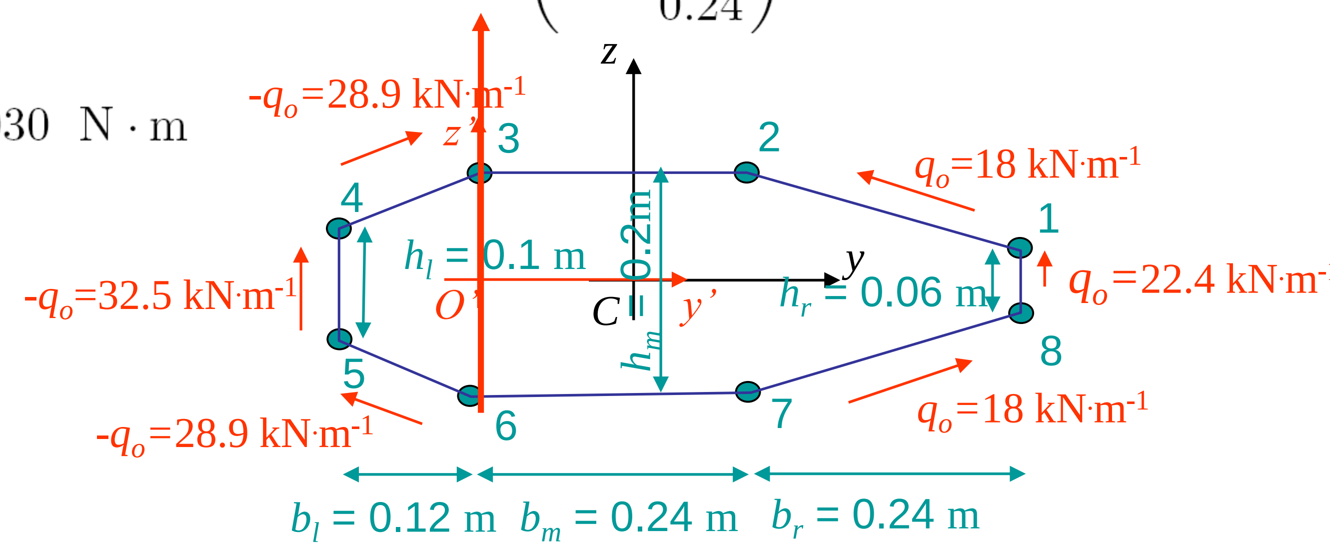


- Constant part of shear flow (2)

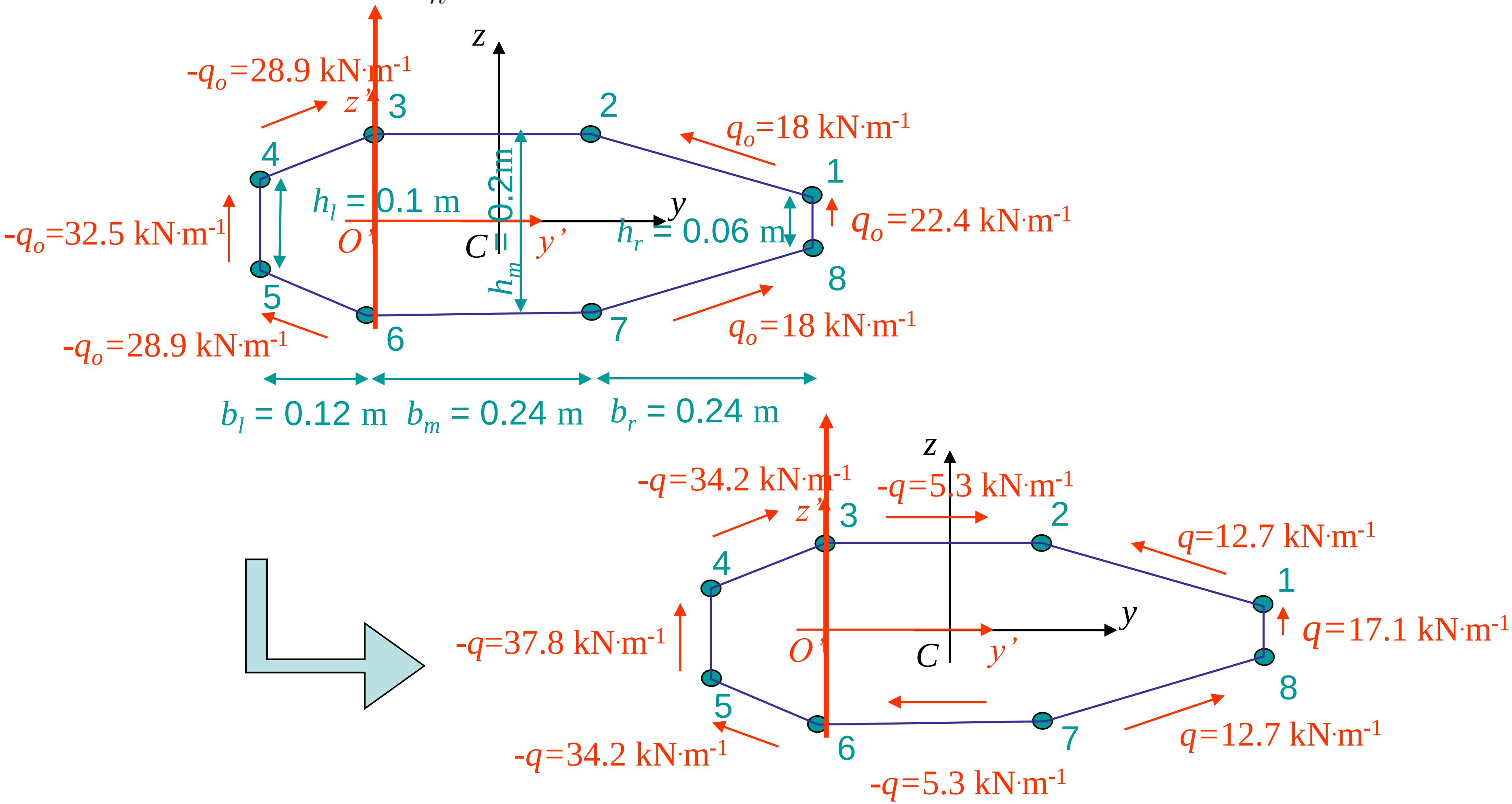
$$-\oint p_{O'} q_o ds = q_o^{34} p_{O'}^{34} l^{34} + q_o^{45} p_{O'}^{45} l^{45} + q_o^{56} p_{O'}^{56} l^{56} + q_o^{78} p_{O'}^{78} l^{78} + q_o^{81} p_{O'}^{81} l^{81} + q_o^{12} p_{O'}^{12} l^{12}$$

$$\begin{aligned}\oint p_{O'} q_o ds \quad = \quad & -28900 \ \cos\left(\operatorname{atan}\frac{0.05}{0.12}\right) 0.1 \sqrt{0.12^2 + 0.05^2} - 32500\ 0.12\ 0.1 \ - \\ & 28900 \ \cos\left(\operatorname{atan}\frac{0.05}{0.12}\right) 0.1 \sqrt{0.12^2 + 0.05^2} + \\ & 18000 \ \cos\left(\operatorname{atan}\frac{0.07}{0.24}\right) (0.1 + 0.07) \sqrt{0.24^2 + 0.07^2} + \\ & 22400\ 0.48\ 0.06 \ + 18000 \ \cos\left(\operatorname{atan}\frac{0.07}{0.24}\right) (0.1 + 0.07) \sqrt{0.24^2 + 0.07^2}\end{aligned}$$

$$\Rightarrow \oint p_{O'} q_o ds = 1030 \ \ \mathrm{N \cdot m}$$



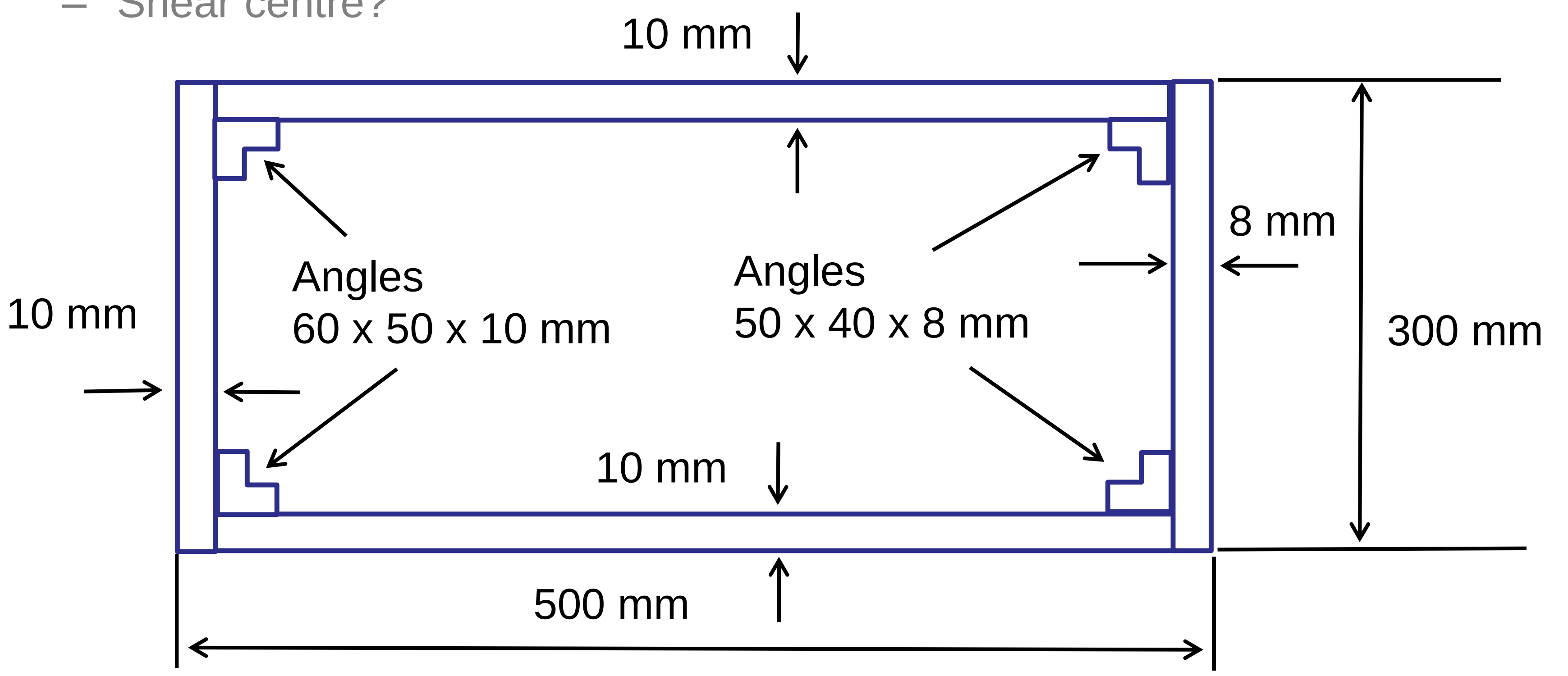
- Total shear flow
  - $q(0) = -\frac{\oint p_{O'} q_o ds}{2A_h} = -\frac{1030}{2\ 0.0972} = -5.3\ 10^3\ \ \mathrm{N \cdot m^{-1}}$



Section idealization consequences

- Consequence on torsion
  - If no axial constraint
    - Torsion analysis does not involve axial stress
    - So torsion is unaffected by the structural idealization

- Box section
  - Arrangement of
    - Direct stress carrying booms positioned at the four corners and
    - Panels which are assumed to carry only shear stresses
    - Constant shear modulus
  - Shear centre?



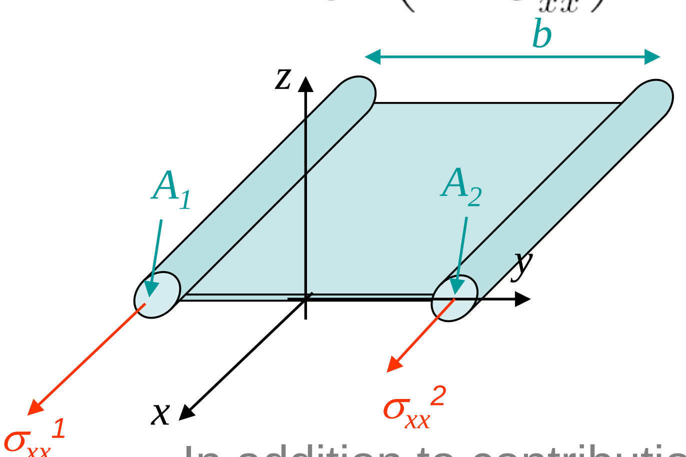
References

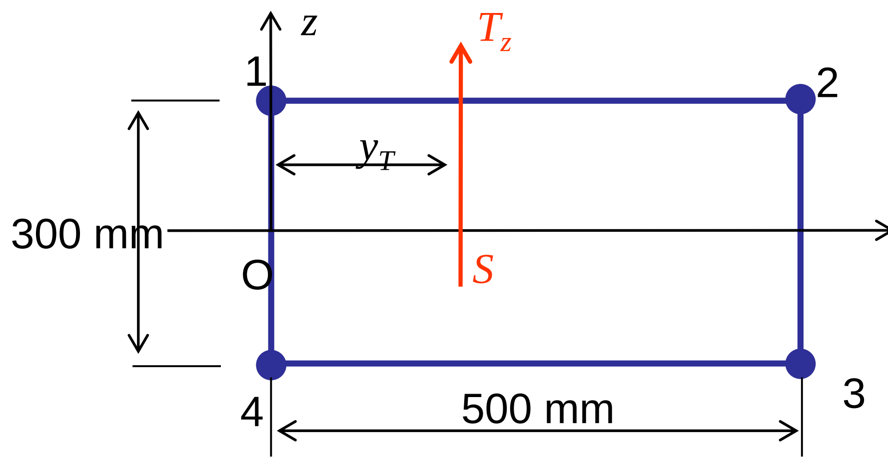
- Lecture notes
  - Aircraft Structures for engineering students, T. H. G. Megson, Butterworth-Heinemann, An imprint of Elsevier Science, 2003, ISBN 0 340 70588 4
- Other references
  - Books
    - Mécanique des matériaux, C. Massonet & S. Cescotto, De boek Université, 1994, ISBN 2-8041-2021-X

- As shear center lies on Oy by symmetry we consider T_Z
  - Section is required to resist bending moments in a vertical plane
  - Direct stress at any point is directly proportional to the distance from the horizontal axis of symmetry, i.e. axis y
  - The distribution of direct stress in all the panels will be linear so that we can use the relation below

$$A_1 = \frac{t_D b}{6}\left(2 + \frac{\boldsymbol{\sigma}_{xx}^2}{\boldsymbol{\sigma}_{xx}^1}\right)$$

$$A_2 = \frac{t_D b}{6}\left(2 + \frac{\boldsymbol{\sigma}_{xx}^1}{\boldsymbol{\sigma}_{xx}^2}\right)$$

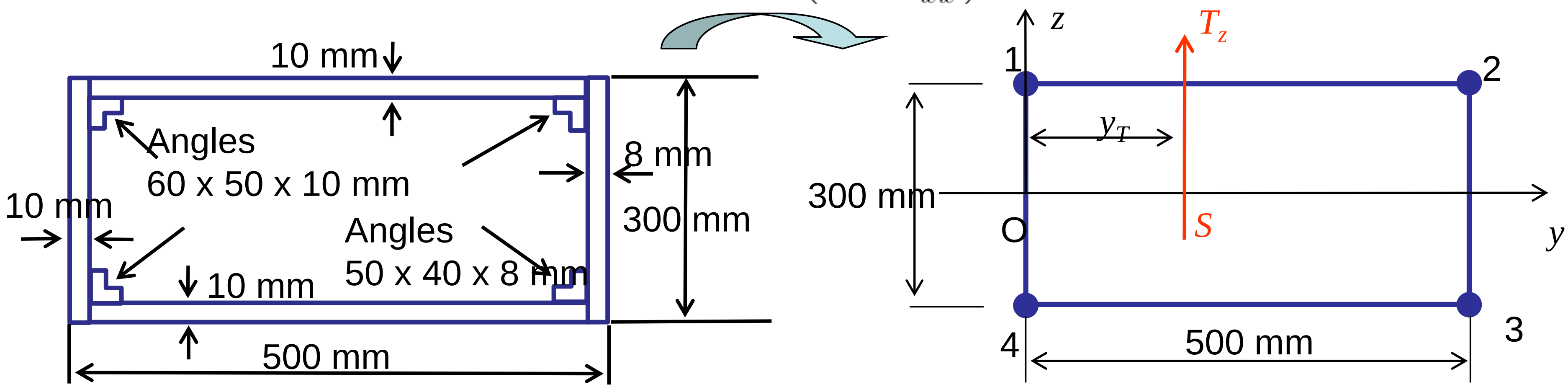




  - In addition to contributions from adjacent panels, booms areas include the existing spar flanges

- Booms area

$$A_1 = \frac{t_D b}{6}\left(2 + \frac{\boldsymbol{\sigma}_{xx}^2}{\boldsymbol{\sigma}_{xx}^1}\right)$$

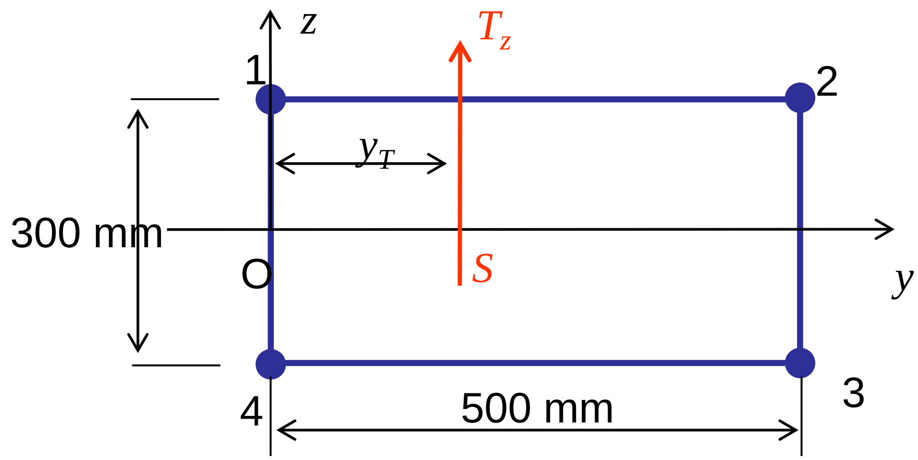


$$
\begin{aligned}
A_1 &= 60 \times 10 + 40 \times 10 + \frac{10\times300}{6}\left(2 + \frac{\sigma_{xx}^4}{\sigma_{xx}^1}\right) + \frac{10\times500}{6}\left(2 + \frac{\sigma_{xx}^2}{\sigma_{xx}^1}\right)\\
&= 60 \times 10 + 40 \times 10 + \frac{10\times300}{6}(2-1) + \frac{10\times500}{6}(2+1) &= 4000\ mm^4
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= 50 \times 8 + 32 \times 8 + \frac{8\times300}{6}\left(2 + \frac{\sigma_{xx}^3}{\sigma_{xx}^2}\right) + \frac{10\times500}{6}\left(2 + \frac{\sigma_{xx}^1}{\sigma_{xx}^2}\right)\\
&= 50 \times 8 + 32 \times 8 + \frac{10\times300}{6}(2-1) + \frac{10\times500}{6}(2+1) &= 3656\ mm^4
\end{aligned}
$$

- By symmetry
  - $A_3 = A_2 = 3656 \text{ mm}^2$
  - $A_4 = A_1 = 4000 \text{ mm}^2$

- Shear flow
  - Booms area
    - $A_3 = A_2 = 3656\ \mathrm{mm}^2$
    - $A_4 = A_1 = 4000\ \mathrm{mm}^2$
  - By symmetry $I_{yz} = 0$

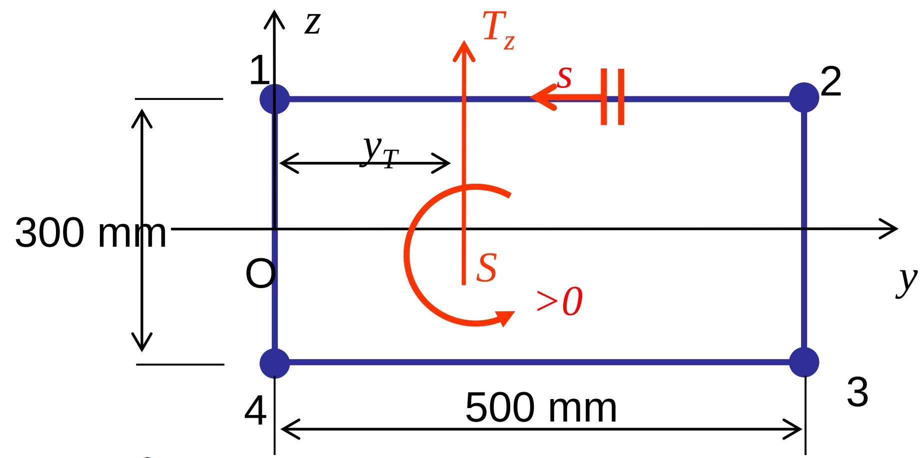


$$\Longrightarrow\ q(s) = -\frac{T_z}{I_{yy}} \sum_{i:\, s_i \leq s} z_i A_i + q(0)$$

As only booms resist direct stress

$$I_{yy} = \sum_{i=1}^{4} A_i z_i^2 = 2 \times 4000 \times 150^2 + 2 \times 3656 \times 150^2 = 344.5 \times 10^6\ mm^4$$

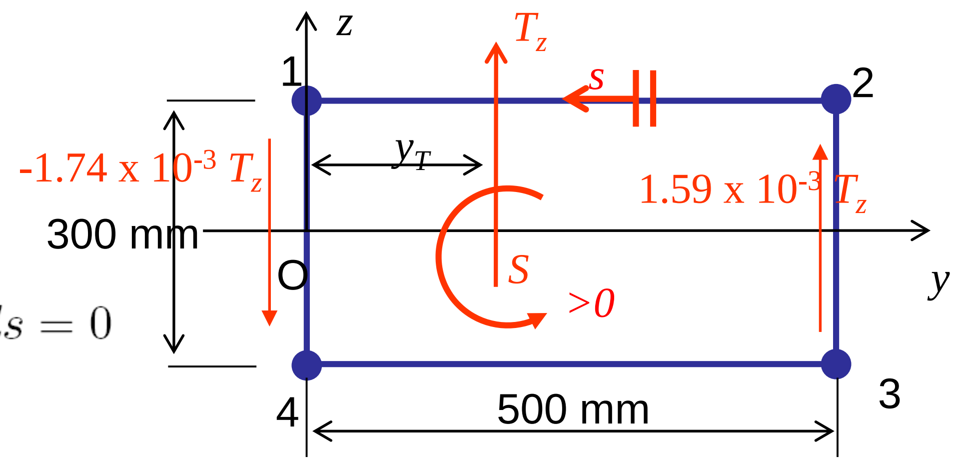
- Open shear flow
  - $q_o(s) = -\frac{T_z}{I_{yy}} \sum_{i:\, s_i \leq s} z_i A_i$



$$
\begin{cases}
q_0^{21} = 0 \\
q_0^{14} = -\frac{T_z}{I_{yy}} \times 4000 \times 150 = -1.74 \times 10^{-3}\, T_z \\
q_0^{43} = 0 \quad \text{(by symmetry)} \\
q_0^{32} = -\frac{T_z}{I_{yy}} \times 3656 \times -150 = 1.59 \times 10^{-3}\, T_z
\end{cases}
$$

- Constant shear flow
  - Load through the shear center

$\Rightarrow$ no torsion



$$\Rightarrow \oint \frac{q}{\mu t} ds = 2A_h \frac{\partial \theta}{\partial x}$$

$$\Rightarrow \oint \frac{q}{\mu t} ds = \oint \frac{q_o(s) + q(0)}{\mu t} ds = 0$$

$$\Rightarrow q(0) = -\frac{\oint \frac{q_0(s)}{t} ds}{\oint \frac{1}{t} ds}$$

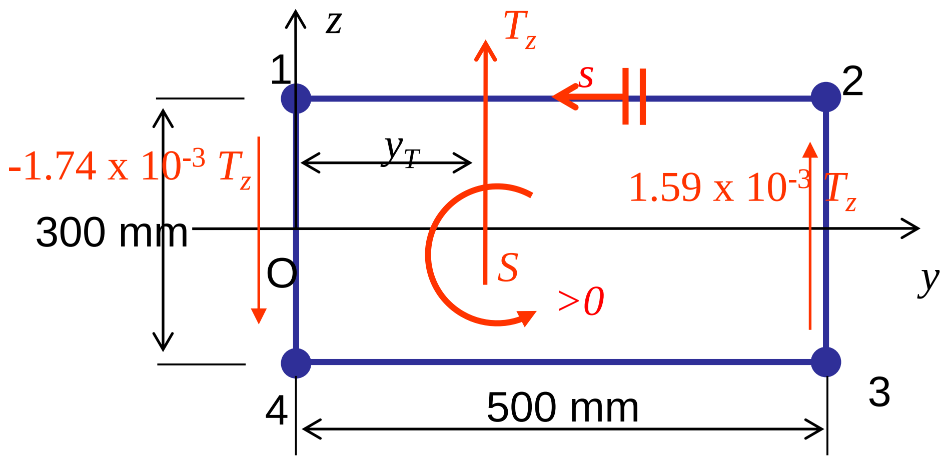
- With $\oint \frac{q_0(s)}{t} ds = q_0^{14} \times \frac{l^{14}}{t_{14}} + q_0^{32} \times \frac{l^{32}}{t_{32}}$

$$= -1.74 \times 10^{-3} T_z \times \frac{300}{10} + 1.59 \times 10^{-3} T_z \times \frac{300}{8} = 7.425 \times 10^{-3} T_z$$

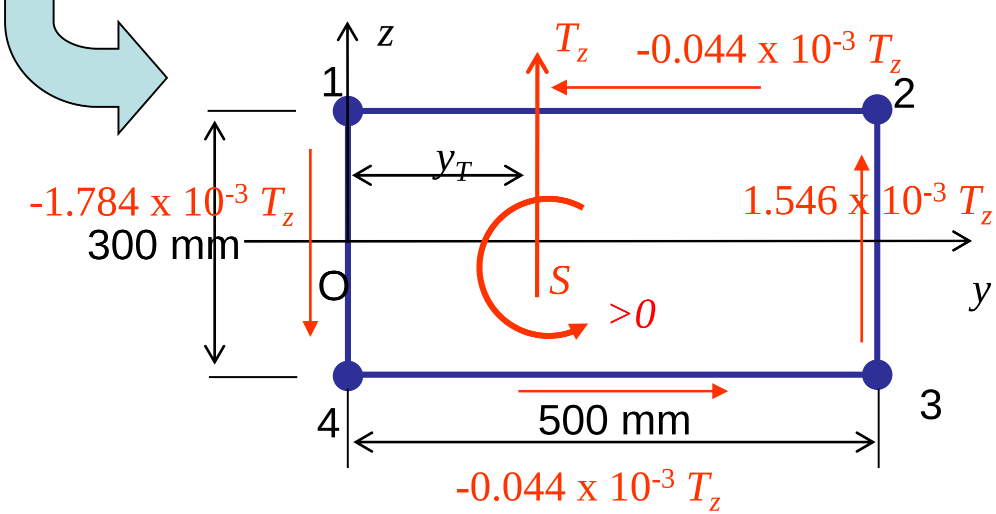
and $\oint \frac{1}{t} ds = 2 \times \frac{500}{10} + \frac{300}{10} + \frac{300}{8} = 167.5$

$$\Rightarrow q(0) = -0.044 \times 10^{-3} T_z$$

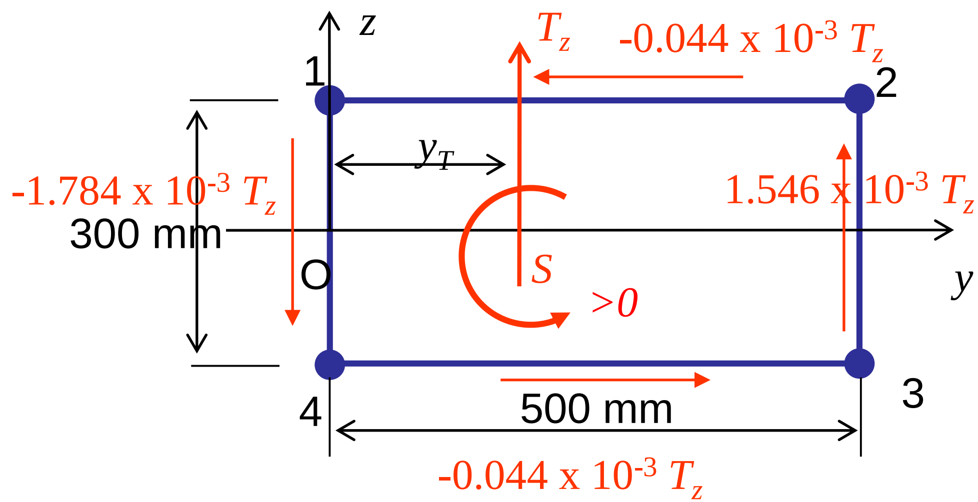
- Total shear flow



$$q\left(0\right) = -0.044 \times 10^{-3}\, T_z$$



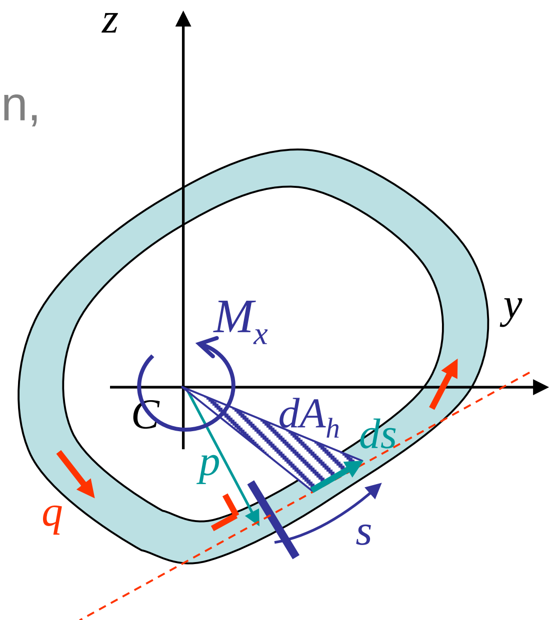
- Shear center
  - Moment around O
    - Due to shear flow
    - Should be balanced by the external loads



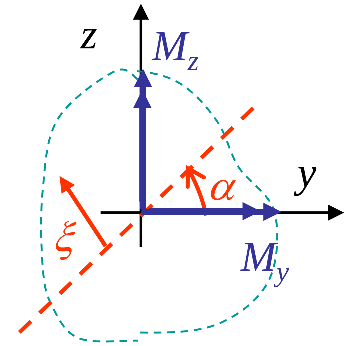
$$y_T T_z = 1.546 \times 10^{-3}\, T_z \times 300 \times 500 - 2 \times 0.044 \times 10^{-3}\, T_z \times 500 \times 150$$

$$\Longrightarrow y_T = 225 \text{ mm}$$

- Twist due to torsion
  - As torsion analysis remains valid for idealized section, one could use the twist rate
    - Closed section $\begin{cases} \theta_{,x} = \dfrac{M_x}{4A_h^2} \oint \dfrac{1}{\mu t} ds \\ M_x = 2A_h q \end{cases}$
    - Open section $\begin{cases} C = \dfrac{M_x}{\theta_{,x}} = \dfrac{1}{3} \int \mu t^3 ds \\ \tau_{xs} = 2\mu n \theta_{,x} \end{cases}$
  - In general
    - $\Delta\theta = \int_0^L \dfrac{M_x}{C} dx$
    - $\tau \propto M_x$
    - $\gamma = \dfrac{\tau}{\mu}$
  - How can we compute deflection for other loading cases?



- Symmetrical bending
  - For pure bending we found $\boldsymbol{\sigma}_{xx} = \kappa E \xi$
  - Therefore the virtual work reads



- $$\int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \boldsymbol{\varepsilon}_{xx} dA dx = \int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \left(\frac{\boldsymbol{\sigma}_{xx}}{E} \right) dA dx$$
$$= \int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \left(\frac{\kappa E \xi}{E} \right) dA dx$$

  - Let us assume Cz symmetrical axis, M_z= 0 & pure bending (M_y constant)

-

$$= M_y \delta \int_0^L \left(-\boldsymbol{u}_{z,xx} \right) dx = -M_y \delta \Delta \boldsymbol{u}_{z,x}$$

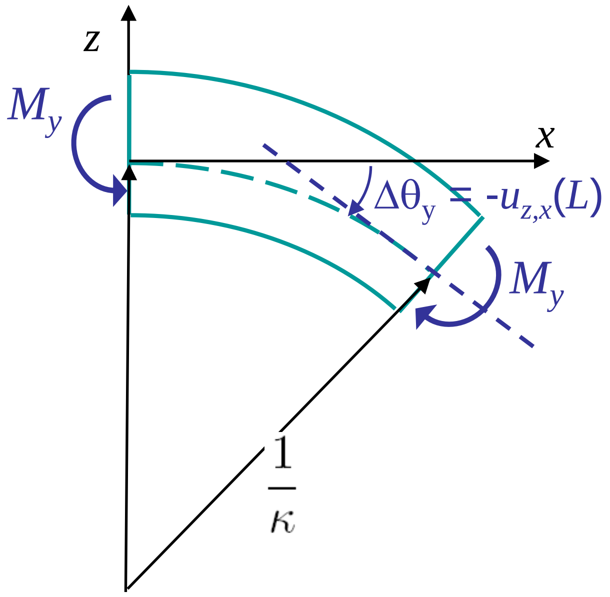
- Consider a unit applied moment, and $\sigma^{(1)}$ the corresponding stress distribution
  - $$\int_0^L \int_A \boldsymbol{\sigma}_{xx}^{(1)} \boldsymbol{\varepsilon}_{xx} dA dx = \int_0^L \int_A \boldsymbol{\sigma}_{xx}^{(1)} \frac{\boldsymbol{\sigma}_{xx}}{E} dA dx = -\Delta \boldsymbol{u}_{z,x}$$
- The energetically conjugated displacement (angle for bending) can be found by integrating the strain distribution multiplied by the unit-loading stress distribution

- Virtual displacement
  - Expression for pure bending

$$\int_0^L \int_A \boldsymbol{\sigma}_{xx}^{(1)} \frac{\boldsymbol{\sigma}_{xx}}{E} dA dx = -\Delta \boldsymbol{u}_{z,x}$$

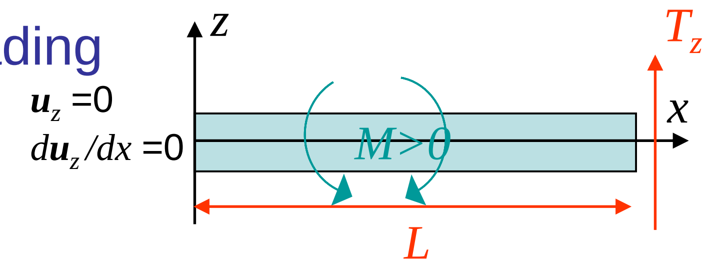
  - In linear elasticity the general formula of virtual displacements reads

$$\int_0^L \int_A \boldsymbol{\sigma}^{(1)} : \boldsymbol{\varepsilon} dA dx = P^{(1)} \Delta_P$$



- $\boldsymbol{\sigma}^{(1)}$ is the stress distribution corresponding to a (unit) load $P^{(1)}$
- Δ_P is
  - The energetically conjugated displacement to P
  - In the direction of $P^{(1)}$
  - Corresponds to the strain distribution $\boldsymbol{\varepsilon}$

- Symmetrical bending due to extremity loading
  - Example Cz symmetrical axis, M_z= 0 & bending due to extremity load



- $$\int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \boldsymbol{\varepsilon}_{xx} dA dx = \int_0^L \int_A \boldsymbol{\sigma}_{xx} z dA \delta \left(-\boldsymbol{u}_{z,xx}\right) dx = \int_0^L M_y \delta \left(-\boldsymbol{u}_{z,xx}\right) dx$$

- Case of a semi-cantilever beam

$$\int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \boldsymbol{\varepsilon}_{xx} dA dx = \int_0^L T_z \left(x - L\right) \delta \left(-\boldsymbol{u}_{z,xx}\right) dx$$

$$= T_z \cancel{\left[\left(L - x\right) \delta \boldsymbol{u}_{z,x}\right]_0^L} + T_z \int_0^L \delta \boldsymbol{u}_{z,x} dx = T_z \delta \Delta \boldsymbol{u}_z$$

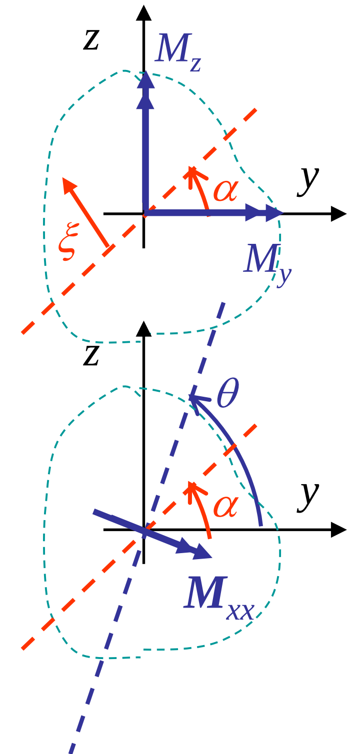
- Eventually

$$\Delta \boldsymbol{u}_z = \int_0^L \int_A \boldsymbol{\sigma}_{xx}^{(1)} \boldsymbol{\varepsilon}_{xx} dA dx$$

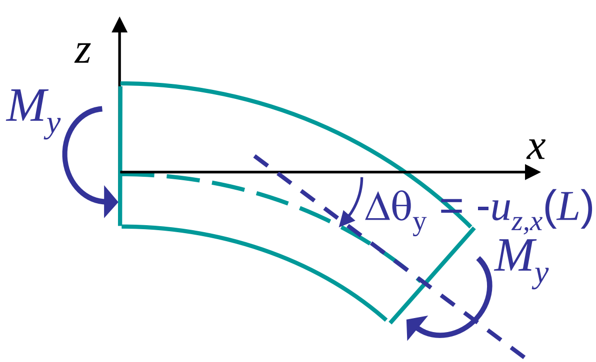
- $\boldsymbol{\sigma}^{(1)}$ is the stress distribution corresponding to a (unit) load $T_z^{(1)}$
- Δu_z is the energetically conjugated displacement to T_z in the direction of $T_z^{(1)}$ that corresponds to the strain distribution $\boldsymbol{\varepsilon}$

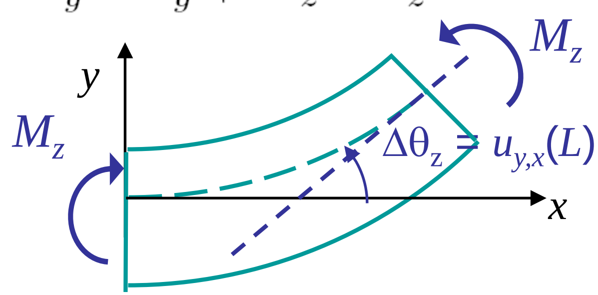
- General pure bending
  - If neutral axis is α-inclined
    - $$\int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \boldsymbol{\varepsilon}_{xx} dA dx = \int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \left(\frac{\kappa E \xi}{E} \right) dA dx$$
    - With $\xi = z \cos\alpha - y \sin\alpha$
    - It has been shown that $\begin{cases} \dfrac{\partial^2 \boldsymbol{u}_y}{\partial x^2} = \dfrac{\partial^2 \xi}{\partial x^2} \sin\alpha = \kappa \sin\alpha \\ \dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2} = -\dfrac{\partial^2 \xi}{\partial x^2} \cos\alpha = -\kappa \cos\alpha \end{cases}$

    $$\Longrightarrow \kappa \xi = \kappa z \cos\alpha - \kappa y \sin\alpha = -\boldsymbol{u}_{z,xx} z - \boldsymbol{u}_{y,xx} y$$



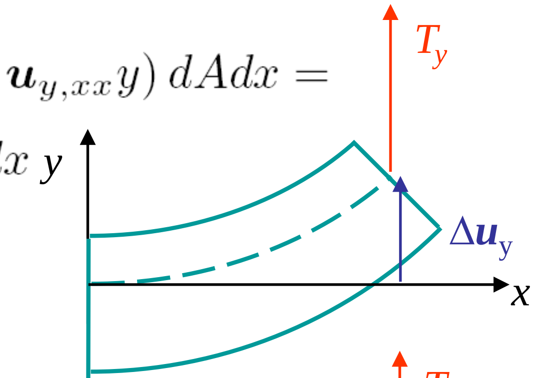
  - Eventually, as M is constant with x
    - $$\int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \left(\frac{\boldsymbol{\sigma}_{xx}}{E} \right) dA dx = \int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \left(-\boldsymbol{u}_{z,xx} z - \boldsymbol{u}_{y,xx} y \right) dA dx =$$
    $$-M_y \delta \Delta \boldsymbol{u}_{z,x} + M_z \delta \Delta \boldsymbol{u}_{y,x} = M_y \delta \Delta \theta_y + M_z \delta \Delta \theta_z$$

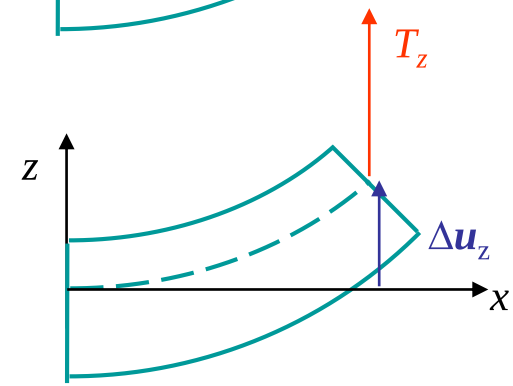




- General bending due to extremity loading
  - Bending moment depends on x
    - $\int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \left(\frac{\boldsymbol{\sigma}_{xx}}{E}\right) dA dx = \int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \left(-\boldsymbol{u}_{z,xx} z - \boldsymbol{u}_{y,xx} y\right) dA dx =$

      $\int_0^L \left(-M_y \delta \Delta \boldsymbol{u}_{z,xx} + M_z \delta \Delta \boldsymbol{u}_{y,xx}\right) dx$





    - Integration by parts

$$\int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \left(\frac{\boldsymbol{\sigma}_{xx}}{E}\right) dA dx =$$

$$\int_0^L (L-x) \left[T_z \delta \Delta \boldsymbol{u}_{z,xx} + T_y \delta \Delta \boldsymbol{u}_{y,xx}\right] dx =$$

$$\left[(L-x)\left(T_z \delta \Delta \boldsymbol{u}_{z,x} + T_y \delta \Delta \boldsymbol{u}_{y,x}\right)\right]_0^L +$$

$$\int_0^L \left[T_z \delta \Delta \boldsymbol{u}_{z,x} + T_y \delta \Delta \boldsymbol{u}_{y,x}\right] dx$$

    - Semi-cantilever beam

$$\int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \left(\frac{\boldsymbol{\sigma}_{xx}}{E}\right) dA dx = T_z \delta \Delta \boldsymbol{u}_z + T_y \delta \Delta \boldsymbol{u}_y = \boldsymbol{T} \cdot \delta \Delta \boldsymbol{u}$$

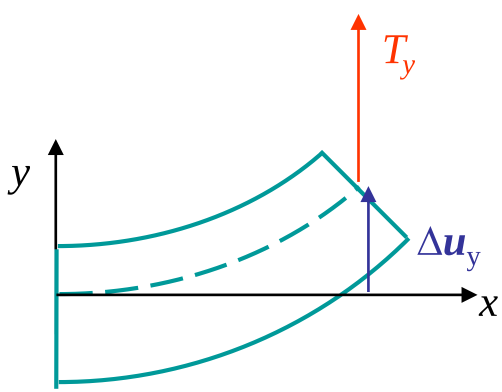
Annex 1: Deflection of open and closed section beams

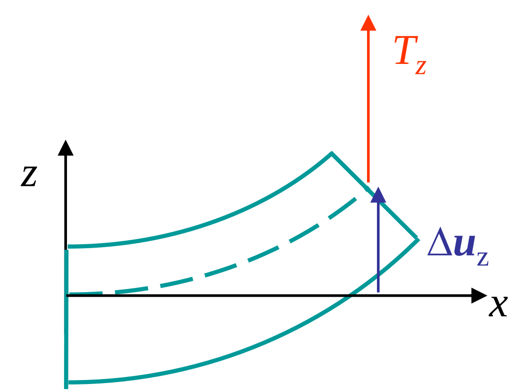
- General bending due to extremity loading (2)
  - Virtual displacement method
    - $\int_0^L \int_A \boldsymbol{\sigma}_{xx}^{(1)} \boldsymbol{\varepsilon}_{xx} dA dx = \Delta_P u$
    - With $\boldsymbol{\sigma}^{(1)}$ due to the (unit) moments $M^{(1)}$ resulting from the unit extremity loading

$$\boldsymbol{\sigma}_{xx}^{(1)} = \frac{\left(I_{zz} M_y^{(1)} + I_{yz} M_z^{(1)}\right) z - \left(I_{yz} M_y^{(1)} + I_{yy} M_z^{(1)}\right) y}{I_{yy} I_{zz} - I_{yz}^2}$$

    - With $\Delta_P u$ displacement in the direction of the unit extremity loading and corresponding to the strain distribution

$$\boldsymbol{\varepsilon}_{xx} = \frac{1}{E} \frac{\left(I_{zz} M_y + I_{yz} M_z\right) z - \left(I_{yz} M_y + I_{yy} M_z\right) y}{I_{yy} I_{zz} - I_{yz}^2}$$





- General bending due to extremity loading (3)
  - Virtual displacement method (2)
    - After developments, and if $\Delta_P u$ is the displacement in the direction of $T^{(1)} = 1$

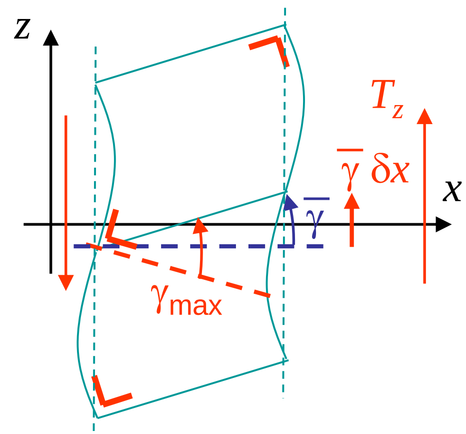
$$\begin{aligned}\Delta_P u &= \int_0^L \int_A \boldsymbol{\sigma}_{xx}^{(1)} \boldsymbol{\varepsilon}_{xx} dA dx \\ &= \frac{1}{E\left(I_{yy}I_{zz} - I_{yz}^2\right)^2} \int_0^L \int_A \left[\left(I_{zz}M_y^{(1)} + I_{yz}M_z^{(1)}\right) z - \left(I_{yz}M_y^{(1)} + I_{yy}M_z^{(1)}\right) y\right] \\ &\quad \left[\left(I_{zz}M_y + I_{yz}M_z\right) z - \left(I_{yz}M_y + I_{yy}M_z\right) y\right] dA dx\end{aligned}$$

$$\Longrightarrow \Delta_P u = \frac{1}{E\left(I_{yy}I_{zz} - I_{yz}^2\right)^2} \int_0^L \left\{\left(I_{zz}M_y^{(1)} + I_{yz}M_z^{(1)}\right)\left(I_{zz}M_y + I_{yz}M_z\right) I_{yy} + \right.$$
$$\left(I_{yz}M_y^{(1)} + I_{yy}M_z^{(1)}\right)\left(I_{yz}M_y + I_{yy}M_z\right) I_{zz} - \left(I_{zz}M_y^{(1)} + I_{yz}M_z^{(1)}\right)$$
$$\left.\left(I_{yz}M_y + I_{yy}M_z\right) I_{yz} - \left(I_{yz}M_y^{(1)} + I_{yy}M_z^{(1)}\right)\left(I_{zz}M_y + I_{yz}M_z\right) I_{yz}\right\} dx$$

- In the principal axes $I_{yz} = 0$

$$\Delta_P u = \frac{1}{E I_{yy} I_{zz}} \int_0^L \left\{I_{zz}M_y^{(1)} M_y + I_{yy}M_z^{(1)} M_z\right\} dx$$

- Shearing
  - Internal energy variation
    - $\int_0^L \int_A \tau\delta\gamma dA dx = \int_0^L \int_A \tau\delta\frac{\tau}{\mu} dA dx = \int_0^L \int_s q\delta\frac{q}{\mu t} ds dx$
  - Variation of the work of external forces
    - $\int_0^L \int_A \tau\delta\gamma dA dx = \int_0^L \int_s t\tau\delta\left(\partial_x \boldsymbol{u}_s + \partial_s \boldsymbol{u}_x\right) ds dx$
    - Defining the average deformation of a section
      - See use of A' for thick beams
      - Vectorial value
      - $\int_0^L \int_A \tau\delta\gamma dA dx = \int_0^L \int_s t\tau\delta\partial_x \bar{\boldsymbol{u}}_s \cdot d\boldsymbol{s} dx = \int_0^L \left(\int_s t\tau d\boldsymbol{s}\right) \cdot \delta\partial_x \bar{\boldsymbol{u}}_s dx$
      - Applied shear loading $\boldsymbol{T} = \int_s t\tau d\boldsymbol{s}$



$$\Rightarrow \int_0^L \int_A \tau\delta\gamma dA dx = \int_0^L \boldsymbol{T} \cdot \delta\partial_x \bar{\boldsymbol{u}} dx = \boldsymbol{T} \cdot \delta\Delta\bar{\boldsymbol{u}}$$

- Shearing (2)
  - Virtual work $$\int_0^L \int_s q^{(1)} \frac{q}{\mu t} ds dx = \boldsymbol{T}^{(1)} \bar{\Delta} \boldsymbol{u} = \Delta_T u$$
    - With $\Delta_T u$ the average deflection of the section in the direction of the applied unit shear load
    - With $q^{(1)}$ the shear flux distribution resulting from this applied unit shear load

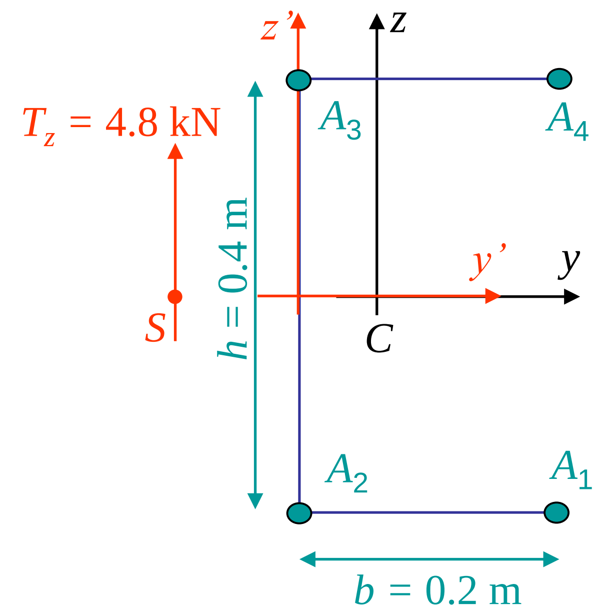
$$q^{(1)}(s) = -\frac{I_{zz}T_z^{(1)} - I_{yz}T_y^{(1)}}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct } \boldsymbol{\sigma}} z ds + \sum_{i:\ s_i \leq s} z_i A_i\right] - \frac{I_{yy}T_y^{(1)} - I_{yz}T_z^{(1)}}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct } \boldsymbol{\sigma}} y ds + \sum_{i:\ s_i \leq s} y_i A_i\right] + \left\{q^{(1)}(0)\right\}$$

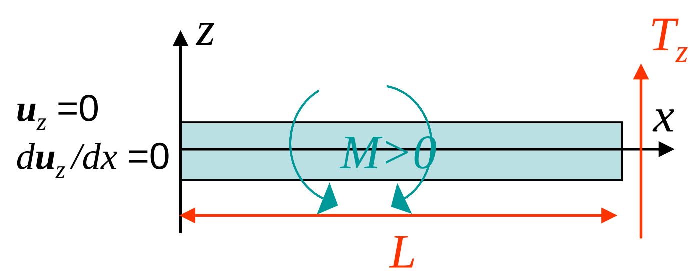
    - With q the shear flux distribution corresponding to the deflection $\Delta_T u$

$$q(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct } \boldsymbol{\sigma}} z ds + \sum_{i:\ s_i \leq s} z_i A_i\right] - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct } \boldsymbol{\sigma}} y ds + \sum_{i:\ s_i \leq s} y_i A_i\right] + \{q(0)\}$$

    - $\{q(0)\}$ meaning only for closed sections

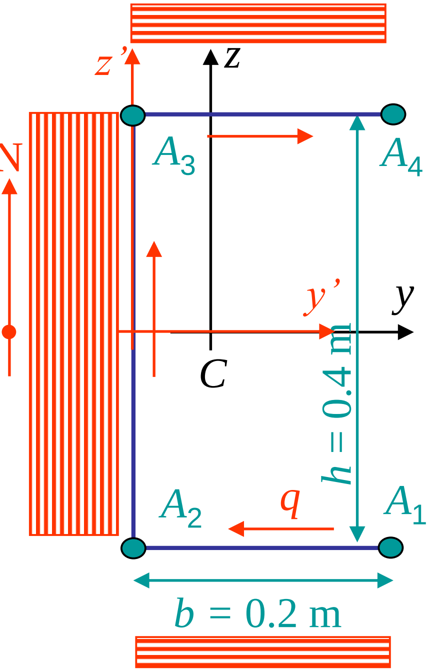
- Example
  - Idealized U shape
    - Booms of 300-mm²- area each
    - Booms are carrying all the direct stress
    - Skin panels are carrying all the shear flow
    - Actual skin thickness is 1 mm
  - Beam length of 2 m
    - Shear load passes through the shear center at one beam extremity
    - Other extremity is clamped
  - Material properties
    - E = 70 GPa
    - μ = 30 GPa
  - Deflection ?





- Shear flow (already solved)
  - Simple symmetry $\Rightarrow$ principal axes

$$\Rightarrow q(s) = -\frac{T_z}{I_{yy}}\left[\int_0^s t_{\text{direct }\boldsymbol{\sigma}} z ds + \sum_{i:\, s_i \leq s} z_i A_i\right]$$



  - Only booms are carrying direct stress

$$\Rightarrow q(s) = -\frac{T_z}{I_{yy}} \sum_{i:\, s_i \leq s} z_i A_i$$

  - Second moment of area

$$I_{yy} = \sum_i A_i z_i^2 = 4\ 300\ 10^-6\ 0.2^2 = 48\ 10^{-6} \quad \text{m}^4$$

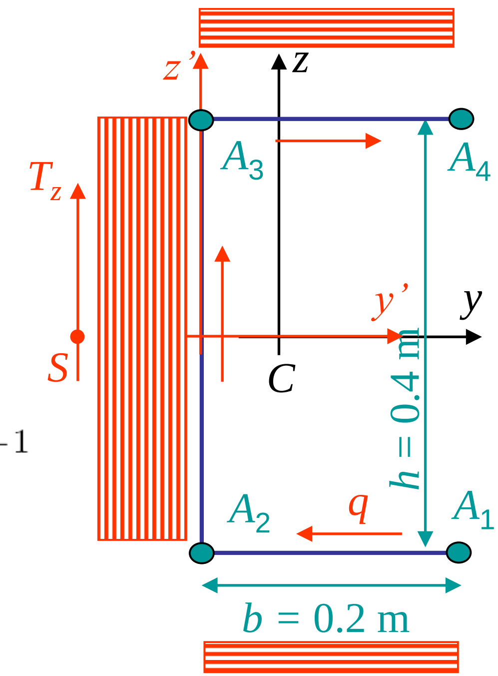
  - Shear flow

$$q^{12}(s) = -\frac{T_z}{I_{yy}} A_1 z_1 = -\frac{4.8\ 10^3}{48\ 10^{-6}} 300\ 10^{-6}(-0.2) = 6\ 10^3 \quad \text{N}\cdot\text{m}^{-1}$$

$$q^{23}(s) = -\frac{T_z}{I_{yy}}(A_1 z_1 + A_2 z_2) = -\frac{4.8\ 10^3}{48\ 10^{-6}} 300\ 10^{-6}(-0.4) = 12\ 10^3 \quad \text{N}\cdot\text{m}^{-1}$$

$$q^{34}(s) = -\frac{T_z}{I_{yy}}(A_1 z_1 + A_2 z_2 + A_3 z_3) = -\frac{4.8\ 10^3}{48\ 10^{-6}} 300\ 10^{-6}(-0.2) = 6\ 10^3 \quad \text{N}\cdot\text{m}^{-1}$$

- Unit shear flow
  - Same argumentation as before but with T_z = 1 N



$$q^{(1),\ 12}(s) = -\frac{1\ \mathrm{N}}{I_{yy}} A_1 z_1 = -\frac{1}{48\ 10^{-6}} 300\ 10^{-6}\,(-0.2) = 1.25\ \ \mathrm{N}\cdot\mathrm{m}^{-1}$$

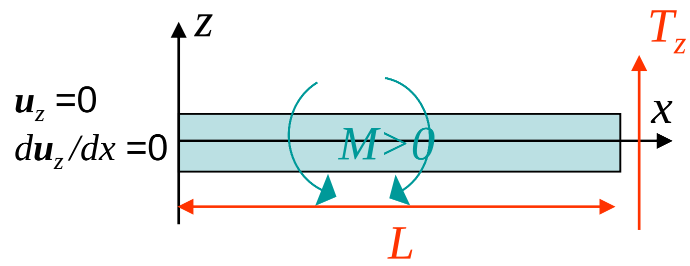
$$q^{(1),\ 23}(s) = -\frac{1\ \mathrm{N}}{I_{yy}}\left(A_1 z_1 + A_2 z_2\right) = -\frac{1}{48\ 10^{-6}} 300\ 10^{-6}\,(-0.4) = 2.5\ \ \mathrm{N}\cdot\mathrm{m}^{-1}$$

$$q^{(1),\ 34}(s) = -\frac{1\ \mathrm{N}}{I_{yy}}\left(A_1 z_1 + A_2 z_2 + A_3 z_3\right) = -\frac{1}{48\ 10^{-6}} 300\ 10^{-6}\,(-0.2) = 1.25\ \ \mathrm{N}{\cdot}\mathrm{m}^{-1}$$

- Displacement due to shearing
  - $\Delta_T u = \int_0^L \int_s q^{(1)} \frac{q}{\mu t} ds dx = 2 \int_s q^{(1)} \frac{q}{30\ 10^9 0.001} ds$

$$\Rightarrow \Delta_T u = \frac{2}{30\ 10^9 0.001}\left[6000\ 1.25\ 0.2 + 12000\ 2.5\ 0.4 + 6000\ 1.25\ 0.2\right] = 10^{-3}\ \ \mathrm{m}$$

- Bending
  - Moment due to extremity load
    - $\begin{cases} M_y = (x-L)\,T_z \\ M_y^{(1)} = (x-L) \end{cases}$



  - Deflection due to extremity load
    - In the principal axes

$$\Longrightarrow \Delta_P u = \frac{1}{E}\int_0^L \frac{M_y^{(1)} M_y}{I_{yy}} dx = \frac{T_z}{I_{yy}E}\int_0^L (x-L)^2\, dx = \frac{T_z L^3}{3 I_{yy} E}$$

$$\Longrightarrow \Delta_P u = \frac{4.8\ 10^3\ 2^3}{3\ 48\ 10^{-6}\ 70\ 10^9} = 0.00381 \text{ m}$$

- Total deflection
  - No torsion as shear load passes through the shear center
  - $\delta \boldsymbol{u}_z = \Delta_T u + \Delta_P u = 0.00481 \text{ m}$