Aircraft Structures
Beams – Torsion & Section Idealization

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Elasticity

- **Balance of body** $B$
  - Momenta balance
    - Linear
    - Angular
  - Boundary conditions
    - Neumann
    - Dirichlet

\[ \rho \ddot{\mathbf{x}} = \mathbf{b} + \nabla \cdot \mathbf{\sigma}^T \]
\[ \rho \ddot{x}_i = b_i + \frac{\partial}{\partial x_j} \sigma_{ij} \]
\[ \mathbf{\sigma}^T = \mathbf{\sigma} \]

- **Small deformations with linear elastic, homogeneous & isotropic material**

  - (Small) Strain tensor
    \[ \varepsilon = \frac{1}{2} \left( \nabla \otimes \mathbf{u} + \mathbf{u} \otimes \nabla \right), \]
    or
    \[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} u_j + \frac{\partial}{\partial x_j} u_i \right) \]
  
  - Hooke’s law
    \[ \mathbf{\sigma} = \mathcal{H} : \varepsilon \]
    , or
    \[ \sigma_{ij} = \mathcal{H}_{ijkl} \varepsilon_{kl} \]
  
  with
  \[ \mathcal{H}_{ijkl} = \frac{E \nu}{(1 + \nu) (1 - 2\nu)} \delta_{ij} \delta_{kl} + \frac{E}{1 + \nu} \left( \frac{1}{2} \delta_{ik} \delta_{jl} + \frac{1}{2} \delta_{il} \delta_{jk} \right) \]

  - Inverse law
    \[ \varepsilon = \mathcal{G} : \mathbf{\sigma} \]
    \[ \lambda = K - 2\mu/3 \]
    \[ 2\mu \]

  with
  \[ \mathcal{G}_{ijkl} = \frac{1 + \nu}{E} \left( \frac{1}{2} \delta_{ik} \delta_{jl} + \frac{1}{2} \delta_{il} \delta_{jk} \right) - \frac{\nu}{E} \delta_{ij} \delta_{kl} \]
Pure bending: linear elasticity summary

- **General expression for unsymmetrical beams**
  - Stress \( \sigma_{xx} = \kappa E \varepsilon \cos \alpha - \kappa E \gamma \sin \alpha \)
    
    With \( \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} = \frac{\| M_{xx} \|}{\kappa E} \begin{pmatrix} I_{yy} & -I_{yz} \\ -I_{yz} & I_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix} \)
  - Curvature
    \( \begin{pmatrix} -u_{zz,xx} \\ u_{yy,xx} \end{pmatrix} = \frac{\| M_{xx} \|}{E \left( I_{yy} I_{zz} - I_{yz} I_{yz} \right)} \begin{pmatrix} I_{zz} & I_{yz} \\ I_{yz} & I_{yy} \end{pmatrix} \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix} \)
  - In the principal axes \( I_{yz} = 0 \)

- **Euler-Bernoulli equation in the principal axis**
  
  \[- \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 u_z}{\partial x^2} \right) = f(x) \text{ for } x \text{ in } [0, L] \]
  
  \[- \frac{\partial}{\partial x} \left( EI \frac{\partial^2 u_z}{\partial x^2} \right) \bigg|_{0, L} = T_z \bigg|_{0, L} \]
  
  \[- EI \frac{\partial^2 u_z}{\partial x^2} \bigg|_{0, L} = M_{xx} \bigg|_{0, L} \]
  
  Similar equations for \( u_y \)
Beam shearing: linear elasticity summary

- **General relationships**
  \[
  f_z(x) = -\partial_x T_z = -\partial_{xx} M_y \\
  f_y(x) = -\partial_x T_y = \partial_{xx} M_z 
  \]

- **Two problems considered**
  - Thick symmetrical section
    - Shear stresses are small compared to bending stresses if \( h/L << 1 \)
  - Thin-walled (unsymmetrical) sections
    - Shear stresses are not small compared to bending stresses
    - Deflection mainly results from bending stresses
    - 2 cases
      - Open thin-walled sections
        » Shear = shearing through the shear center + torque
      - Closed thin-walled sections
        » Twist due to shear has the same expression as torsion
Beam shearing: linear elasticity summary

- **Shearing of symmetrical thick-section beams**
  - Stress \( \sigma_{xz} = -\frac{T_z S_n(z)}{I_{yy} b(z)} \)
  - With \( S_n(z) = \int_{A^*} z dA \)
  - Accurate only if \( h > b \)
  - Energetically consistent averaged shear strain \( z \)
    - \( \bar{\gamma} = \frac{T_z}{A' \mu} \) with \( A' = \frac{1}{\int A \frac{S_n^2}{T_{yy} b^2} dA} \)
    - Shear center on symmetry axes
  - Timoshenko equations
    - \( \bar{\gamma} = 2 \bar{\varepsilon}_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \theta_y + \partial_x u_z \) & \( \kappa = \frac{\partial \theta_y}{\partial x} \)
    - On \([0 L]:\)
      \[
      \left\{ \begin{array}{l}
      \frac{\partial}{\partial x} \left( EI \frac{\partial \theta_y}{\partial x} \right) - \mu A' \left( \theta_y + \partial_x u_z \right) = 0 \\
      \frac{\partial}{\partial x} \left( \mu A' \left( \theta_y + \partial_x u_z \right) \right) = -f
      \end{array} \right.
      \]
Beam shearing: linear elasticity summary

- **Shearing of open thin-walled section beams**
  - Shear flow \( q = \frac{T}{t} \)
    
    \[
    q(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2} \int_0^s tzs' ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2} \int_0^s tyds'
    \]
  - In the principal axes
    
    \[
    q(s) = -\frac{T_z}{I_{yy}} \int_0^s tzs' ds' - \frac{T_y}{I_{zz}} \int_0^s tyds'
    \]
  - Shear center \( S \)
    - On symmetry axes
    - At walls intersection
    - Determined by momentum balance
  - Shear loads correspond to
    - Shear loads passing through the shear center &
    - Torque
Beam shearing: linear elasticity summary

- Shearing of closed thin-walled section beams
  - Shear flow \( q = t\tau \)
  - \( q(s) = q_o(s) + q(0) \)
  - Open part (for anticlockwise of \( q, s \))
    \[
    q_o(s) = -\frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2} \int_0^s t(s') z(s') ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2} \int_0^s t(s') y(s') ds'
    \]
  - Constant twist part
    \[
    q(s = 0) = \frac{y_T T_z - z_T T_y - \int p(s) q_o(s) ds}{2A_h}
    \]
  - The \( q(0) \) is related to the closed part of the section, but there is a \( q_o(s) \) in the open part which should be considered for the shear torque \( \int p(s) q_o(s) ds \)
Beam shearing: linear elasticity summary

- Shearing of closed thin-walled section beams
  
  - Warping around twist center $R$
    
    $\mathbf{u}_x (s) = \mathbf{u}_x (0) + \int_0^s \frac{q}{\mu t} ds - \frac{1}{A_h} \oint \frac{q}{\mu t} ds \left\{ A_{CP} (s) + \frac{z_R [y(s) - y(0)] - y_R [z(s) - z(0)]}{2} \right\}$

  - With $\mathbf{u}_x (0) = 0$ for symmetrical section if origin on the symmetry axis

  - Shear center $S$
    
    - Compute $q$ for shear passing through $S$
    
    - Use
      
      $q(s = 0) = \frac{y_T T_z - z_T T_y - \oint p(s) q_o (s) ds}{2 A_h}$

    With point $S=T$
Torsion of closed thin-walled section beams

- **General relationships**
  - We have seen
    - \((\sigma_{xx} + \partial_x \sigma_{xx} \delta x) t \delta s - \sigma_{xx} t \delta s + (q + \partial_s q \delta s) \delta_x - q \delta x = 0\)
      \[\implies t \partial_x \sigma_{xx} + \partial_s q = 0\]
    - \((\sigma_s + \partial_s \sigma_s \delta s) t \delta x - \sigma_{xx} t \delta x + (q + \partial_x q \delta x) \delta_s - q \delta s = 0\)
      \[\implies t \partial_s \sigma_s + \partial_x q = 0\]
  - If the section is closed
    - **Bredt assumption for closed sections:** Stresses are constant on \(t\), and if there is only a constant torque applied then \(\sigma_s = \sigma_{xx} = 0\)
      \[\begin{cases} 
      \partial_x q = 0 \\
      \partial_s q = 0 
      \end{cases}\]
      \[\implies \text{Constant shear flow (not shear stress)}\]
Torsion of closed thin-walled section beams

• Torque
  – As $q$ due to torsion is constant
    • $M_x = \int p q d s = q \int p d s \quad \Rightarrow \quad M_x = 2 A_h q$

• Displacements
  – It has been established that
    • $\gamma = 2 \epsilon_{x s} = \frac{\partial u_s}{\partial x} + \frac{\partial u_x}{\partial s}$
    • So in linear elasticity
      
      $q = \mu t (u_{s,x} + u_{x,s})$
      
      – But for pure torsion $q$ is constant
        
        $0 = q_{,x} = \mu t (u_{x,sx} + u_{s,xx})$
        
        • Remark $\mu t$ is not constant along $s$ but it is assumed constant along $x$
        
        $\epsilon_{x x,s} + u_{s,xx} = 0$
        
        • As $\sigma_{xx} = \sigma_s = 0 \quad \Rightarrow \quad u_{s,xx} = 0$
Displacements (2)

- It has been established that for a twist around the twist center $R$

$$\frac{\partial u_s}{\partial x} = p \frac{\partial \theta}{\partial x} + \frac{\partial u_y^C}{\partial x} \cos \Psi + \frac{\partial u_z^C}{\partial x} \sin \Psi$$

- As $u_{s,xx} = 0$

$$0 = p \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 u_y^C}{\partial x^2} \cos \Psi + \frac{\partial^2 u_z^C}{\partial x^2} \sin \Psi$$

for all values of $s$ (so all value of $\Psi$)

- The only possible solution is

$$\frac{\partial^2 \theta}{\partial x^2} = 0 \ , \ \frac{\partial^2 u_y^C}{\partial x^2} = 0 \ \& \ \frac{\partial^2 u_z^C}{\partial x^2} = 0$$

- So displacement fields related to torsion are linear with $x$

$$\begin{align*}
\theta &= C_1 x + C_2 \\
u_y^C &= C_3 x + C_4 \\
u_z^C &= C_5 x + C_6
\end{align*}$$
• Rate of twist
  - Use
    • Relation \( \int \frac{q}{\mu t} ds = 2A_h \frac{\partial \theta}{\partial x} \)
devolved for shearing, but with \( q \) due
to torsion constant on \( s \)
    • Torque expression \( M_x = 2A_h q \)
  - Twist
    • \( \theta_s = \frac{M_x}{4A_h^2} \int \frac{1}{\mu t} ds \) constant with \( x \)
    \[ \theta = \frac{M_x}{4A_h^2} \int \frac{1}{\mu t} ds x + C_2 \]
  - Torsion rigidity
    • \( C = \frac{M_x}{\theta_s} = \frac{4A_h^2}{\int \frac{1}{\mu t} ds} \)
    • Torsion second moment of area for constant \( \mu \):
      \[ I_T = \frac{4A_h^2}{\int \frac{1}{t} ds} \leq I_p = \int_A r^2 dA \]
Warping

- Use

  • Relation

  \[ u_x(s) = u_x(0) + \int_0^s \frac{q}{\mu t} \, ds - \frac{A_{R_p}(s)}{A_h} \int_0^s \frac{q}{\mu t} \, ds \]

  developed for shearing, but with \( q \) due to torsion constant on \( s \)

  • Swept from twist center \( R \)

  \[ A_{R_p}(s) = \frac{1}{2} \int_0^s p_{R} \, ds \]

  • Torque expression

  \[ M_x = 2A_h q \]

- Warp displacement

  • \( u_x(s) = u_x(0) + \frac{M_x}{2A_h} \int_0^s \frac{1}{\mu t} \, ds - \frac{M_x A_{R_p}(s)}{2A_h^2} \int \frac{1}{\mu t} \, ds \]

  \[ \Longrightarrow u_x(s) = u_x(0) + \frac{M_x}{2A_h} \left[ \int_0^s \frac{1}{\mu t} \, ds - \frac{A_{R_p}(s)}{A_h} \int \frac{1}{\mu t} \, ds \right] \]
Torsion of closed thin-walled section beams

- **Twist & Warping under pure torsion**
  - **Twist** \( \theta_x = \frac{M_x}{4A_h^2} \int \frac{1}{\mu t} ds \)
  - **Warp** \( u_x(s) = u_x(0) + \frac{M_x}{2A_h} \left[ \int_0^s \frac{1}{\mu t} ds - \frac{A_{R_p}(s)}{A_h} \int_0^s \frac{1}{\mu t} ds \right] \)
  - **Deformation**
    - Plane surfaces are no longer plane
    - It has been assumed they keep the same projected shape + linear rotation
    - Longitudinal strains are equal to zero
      - All sections possess identical warping
      - Longitudinal generators keep the same length although subjected to axial displacement
• Zero warping under pure torsion

  - Warp $u_x(s) = u_x(0) + \frac{M_x}{2A_h} \left[ \int_0^s \frac{1}{\mu t} ds - \frac{A_{R_\psi}}{A_h} \int_0^s \frac{1}{\mu t} ds \right]

  - Zero warping condition requires

    \[ \int_0^s \frac{1}{\mu t} ds = \frac{1}{2} \int_0^s p_R ds \]

    for all values of $s$

    \[ \frac{1}{\mu t} \int_0^s \frac{1}{\mu t} ds = \frac{p_R}{2A_h} \quad \Rightarrow \quad p_R \mu t = \frac{2A_h}{\int_0^s \frac{1}{\mu t} ds} \]

  - As right member is constant the condition of zero warping is $p_R \mu t$ constant with respect to $s$

  - Solutions at constant shear modulus

    - Circular pipe of constant thickness
    - Triangular section of constant $t$

      ($p_R$ is the radius of the inscribed circle which origin coincides with the twist center)

    - Rectangular section with $t_h b = t_b h$
Example

- Doubly symmetrical rectangular closed section
- Constant shear modulus
- Twist rate?
- Warping distribution?
Torsion of closed thin-walled section beams

- **Twist rate**
  - As the section is doubly symmetrical, the twist center is also the section centroid $C$
  - Twist rate $\theta_x = \frac{M_x}{4A_h^2} \int \frac{1}{\mu t} \, ds$
    - $A_h = h b$
    - $\int \frac{1}{t} \, ds = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{t_h} \, dz + \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{1}{t_b} \, (-dy) + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{t_h} \, (-dz) + \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{1}{t_b} \, dy = \frac{2h}{t_h} + \frac{2b}{t_b}$
    - $\theta_x = \frac{M_x}{2\mu h^2 b^2} \left( \frac{h}{t_h} + \frac{b}{t_b} \right)$
  - For a beam of length $L$ and constant section $\frac{\theta}{LM_x} = \frac{\frac{h}{t_h} + \frac{b}{t_b}}{2\mu h^2 b^2}$
    - Torsion rigidity $C = \left( \frac{\frac{h}{t_h} + \frac{b}{t_b}}{2\mu h^2 b^2} \right)^{-1} = \mu I_T \leq \mu I_P$
Torsion of closed thin-walled section beams

- **Warping**
  - As the section is doubly symmetrical, the twist center is also the section centroid \( C \)
  - **Warping**
    - It can be set up to 0 at point \( E \)
      - By symmetry it will be equal to zero wherever a symmetry axis intercept the wall

- \( \mathbf{u}_x (s) = \mathbf{u}_x (0) + \frac{M_x}{2A_h} \left[ \int_0^s \frac{1}{\mu t} \, ds - \frac{A_{R_p} (s)}{A_h} \int_0^s \frac{1}{\mu t} \, ds \right] \)

- \( A_h = h b \) & \( \int \frac{1}{t} \, ds = \frac{2h}{t_h} + \frac{2b}{t_b} \)

- On part \( EA \)

\[-\int_0^s \frac{1}{t} \, ds = \int_0^z \frac{1}{t_h} \, dz = \frac{z}{t_h} \quad \& \quad A_{R_p} = \int_0^s \frac{p R}{2} \, ds = \int_0^z \frac{b}{4} \, dz = \frac{bz}{4} \]

\[ \mathbf{u}_x (z)_{EA} = \frac{M_x}{2 \mu h b} \left[ \frac{z}{t_h} - \frac{bz}{4bh} \left( \frac{2h}{t_h} + \frac{2b}{t_b} \right) \right] \]
Torsion of closed thin-walled section beams

• Warping (2)
  
  – On part $EA$

  \[
  \mathbf{u}_x(z)^{EA} = \frac{M_x}{2\mu hb} \left[ \frac{z}{t_h} - \frac{bz}{4bh} \left( \frac{2h}{t_h} + \frac{2b}{t_b} \right) \right]
  \]

  \[
  \mathbf{u}_x(z)^{EA} = \frac{M_x z}{2\mu hb} \left[ \frac{1}{t_h} - \frac{1}{2h} \frac{ht_b + bt_h}{t_h t_b} \right]
  \]

  \[
  \mathbf{u}_x(z)^{EA} = \frac{M_x z}{2\mu hb} \frac{ht_b - bt_h}{2ht_h t_b}
  \]

  \[
  \mathbf{u}_x(z)^{EA} = \frac{M_x z}{4\mu h^2 b} \left( \frac{h}{t_h} - \frac{b}{t_b} \right)
  \]

  – So using symmetry and as distribution is linear

\[
\begin{align*}
\mathbf{u}_x^A &= \mathbf{u}_x^C = \frac{M_x}{8\mu hb} \left( \frac{h}{t_h} - \frac{b}{t_b} \right) \\
\mathbf{u}_x^B &= \mathbf{u}_x^D = \frac{M_x}{8\mu hb} \left( \frac{b}{t_b} - \frac{h}{t_h} \right)
\end{align*}
\]

• Zero warping if $b t_h = h t_b$
• Torsion of a thick section
  – The problem can be solved explicitly by recourse to a stress function
  – Hydrodynamic analogy
    • Shear stresses have the same expression than the velocity in a rotational flow in a box of same section
Torsion of thick section

- Torsion of a thick circular section
  - Exact solution of the problem
    - By symmetry there is no warping
      - Sections remain plane
    - $\gamma = r\theta, x$
  - In linear elasticity
    - Shear stresses $\tau = \mu \gamma = r\mu \theta, x$
    - Torque $M_x = \int_A r\tau dA = \int_A \mu r^2 dA \theta, x$
    - Torsion rigidity $C = \frac{M_x}{\theta, x} = \int_A \mu r^2 dA$
      - At constant shear modulus (required for symmetry): $C = \mu I_p$
      - For circular cross sections (only) $I_p = I_T$
    - Maximum shear stress $\tau_{\text{max}} = \frac{M_x r_{\text{max}}}{I_p}$
**Torsion of thick section**

- **Torsion of a rectangular section**
  - Exact solution of the problem with stress function
    - **Assumptions**
      - Linear elasticity
      - Constant shear modulus
    - Maximum stress at mid position of larger edge
      - \( \tau_{\text{max}} = \frac{M_x}{\alpha h b^2} \)
    - Torsion rigidity (constant \( \mu \))
      - \( C = \frac{M_x}{\theta_{,x}} = \beta h b^3 \mu \)
        - \( IT = \beta h b^3 \)
    - Approximation for \( h \gg b \)
      - \( C = \frac{M_x}{\theta_{,x}} = \frac{h b^3 \mu}{3} \)
        - \( IT = \frac{h b^3}{3} \)
      - \( \tau_{xy} = 0 \) & \( \tau_{xz} = 2 \mu y \theta_{,x} \)
      - \( \tau_{\text{max}} = \frac{3M_x}{hb^2} \)

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
h/b & 1 & 1.5 & 2 & 4 & \infty \\
\hline
\alpha & 0.208 & 0.231 & 0.246 & 0.282 & 1/3 \\
\beta & 0.141 & 0.196 & 0.229 & 0.281 & 1/3 \\
\hline
\end{array}
\]
Torsion of thick section

- Torsion of a rectangular section (2)
  - Warping
    \[
    \begin{align*}
    \gamma_{xz} &= u_{x,z} + u_{z,x} = \frac{\tau_{xz}}{\mu} \\
    \gamma_{xy} &= u_{y,x} + u_{x,y} = \frac{\tau_{xy}}{\mu}
    \end{align*}
    \]
  - As
    \[
    u_{x,z} = \frac{\tau_{xz}}{\mu} - u_{z,x} = \frac{\tau_{xz}}{\mu} - \frac{\partial}{\partial x} (\theta y)
    \]
    \[
    u_{x,z} = \frac{\tau_{xz}}{\mu} - y \theta_x
    \]
  - For a rigid rotation (first order approximation)
    \[
    u_{x,y} = \frac{\tau_{xy}}{\mu} - u_{y,x} = \frac{\tau_{xy}}{\mu} - \frac{\partial}{\partial x} (-\theta z)
    \]
    \[
    u_{x,y} = \frac{\tau_{xy}}{\mu} + z \theta_x
    \]
  - For a thin rectangular section
    \[
    \tau_{xy} = 0 \quad \text{&} \quad \tau_{xz} = 2\mu y \theta_x
    \]
    \[
    u_{x,y} = \frac{\tau_{xy}}{\mu} + z \theta_x \quad \implies \quad u_x = z y \theta_x + C_1 z + C_2
    \]
    \[
    u_x = z y \theta_x
    \]
    \[
    \text{Doubly symmetrical section} \quad \implies \quad u_x = z y \theta_x
    \]
Torsion of open thin-walled section beams

- **Rectangle approximation of open thin-walled section beams**
  - Thin rectangle
    - \( \tau_{xy} = 0 \quad \& \quad \tau_{xz} = 2\mu y\theta_{,x} \)
  - For constant shear modulus
    - \( C = \frac{M_x}{\theta_{,x}} = \frac{ht^3\mu}{3} \quad \Rightarrow \quad I_T = \frac{ht^3}{3} \)
  - Warping \( u_x = zy\theta_{,x} \)
  - Thin curved section
    - If \( t << \) curvature an approximate solution is
      - \( \tau_{xs} = 2\mu n\theta_{,x} \)
      - \( C = \frac{M_x}{\theta_{,x}} = \frac{1}{3} \int \mu t^3 ds \)
  - Open section composed of thin rectangles
    - Same approximation
      - \( \tau_{\text{max}_i} = \mu t_i \theta_{,x} \)
      - \( M_x = \theta_{,x} = \sum_i \frac{l_i t_i^3 \mu}{3} \)
Warping

- Warping around $s$-axis
  - Thin rectangle $u_x = z y \theta_{,x} + C_1 z + C_2$
  - Here $C_i$ are not equal to 0
  - Part around $s$-axis $u^t_x = n s \theta_{,x}$
- Warping of the $s$-line ($n=0$)
  - We found $\gamma = 2 \varepsilon_{xs} = \frac{\partial u_s}{\partial x} + \frac{\partial u_x}{\partial s}$
  - If $R$ is the twist center
    - $\frac{\partial u_s}{\partial x} = p_R \theta_{,x}$
    - $\tau_{xs} = \mu \gamma = \mu \frac{\partial u_x}{\partial s} + \mu p_R \theta_{,x}$
    - As $\tau_{xs} = 2 \mu n \theta_{,x}$ $\tau_{xs}(n=0) = 0$
    - $\frac{\partial u_x}{\partial s} = -p_R \theta_{,x}$
  - Eventually $s$-axis warp (usually the larger)

$$u^s_x(s) = u^s_x(0) - \theta_{,x} \int_0^s p_R ds' = u^s_x(0) - 2 A_{R_p}(s) \theta_{,x}$$
• Example
  - U open section
  - Constant shear modulus (25 GPa)
  - Torque of 10 N·m
  - Maximum shear stress?
  - Warping distribution?
• Maximum shear stress
  - Torsion second moment of area
  
  \[ I_T = \sum \frac{l_i t_i^3}{3} = \frac{2}{3} bt_f^3 + \frac{ht_w^3}{3} = \frac{2 \times 0.025 \times 0.0015^3 + 0.05 \times 0.0025^3}{3} = 0.317 \times 10^{-9} \text{m}^4 \]
  - Twist rate
  
  \[ \theta_{,x} = \frac{M_x}{\mu I_T} = \frac{10}{25 \times 0.317} = 1.26 \text{ rad} \cdot \text{m}^{-1} \]
  - Maximum shear stress reached in web
  
  \[ \tau_{\text{max}} = \pm 2\mu \frac{t_w}{2} \theta_{,x} = \pm 25 \times 10^9 \times 0.00251.26 = \pm 78.9 \text{ MPa} \]
Torsion of open thin-walled section beams

- **Twist center**
  - Zero-warping point
  - Free ends so the shear center $S$ corresponds to twist center $R$
    - See lecture on structural discontinuities
  - By symmetry, lies on $Oy$ axis
  - Apply Shear $T_z$ to obtained $y'$
  - Shear flow for symmetrical section

- \[ q(s) = -\frac{T_z}{I_{yy}} \int_0^s tzs'ds' \]

- With \[ I_{yy} = \frac{twh^3}{12} + \frac{2h^2}{4}tfb \]
  \[ = \frac{0.0025}{12} \frac{0.05^3}{2} + \frac{0.05^2}{2} 0.0015 0.025 = 72.9 \times 10^{-9} m^4 \]
Torsion of open thin-walled section beams

- **Twist center (2)**
  - Shear flow for symmetrical section (2)
    
    \[ q(s) = -\frac{T_z}{I_{yy}} \int_0^s t \ z \ ds' \]

    - **On lower flange**
      
      \[ q_f(y') = -\frac{T_z}{I_{yy}} \int_b^{y'} t_f \left( -\frac{h}{2} \right) (-dy') \]
      
      \[ = \frac{T_z t_f h}{2I_{yy}} (b - y') \]

    - **Momentum due to shear flow**
      
      - Zero web contribution around \( O' \)
      
      - Top and lower flanges have the same contribution

      \[ M_{O'} = h \frac{-b q_f(y' = 0)}{2} = -\frac{T_z t_f h^2 b^2}{4I_{yy}} \]

      \[ = -T_z \frac{0.0015 \ 0.05^2 \ 0.025^2}{4 \ 72.9 \ 10^{-9}} = -8.04 \text{ mm } T_z \]

    - **Moment balance**

      \[ M_{O'} = -8.04 \text{ mm } T_z = y'ST_z \quad y'_S = -8.04 \text{ mm} \]

    - **Be careful:** clockwise orientation of \( q, s \)
Warping of $s$-axis

- $u^s_x(s) = u^s_x(0) - 2A_{R_p}(s)\theta_x$
- Origin in $O'$ as by symmetry $u_x(O')=0$
  - On $O'A$ branch
    - Area swept is positive
      \[
      u^s_{x,O'A}(z') = -\int_{O'}^s p_Rds\theta_x = -|y'_S|z'\theta_x
      \]
      \[
      = -0.00804 \times 1.26z' = -0.0101z'
      \]
    - At point $A$
      \[
      u^s_{x,A} = -0.0101 \frac{h}{2} = -0.0101 \times 0.025 = -0.25 \text{ mm}
      \]
  - On $AB$ branch
    - Area swept is negative
      \[
      u^s_{x,AB}(y') = u^s_{x,A} - \int_{A}^{s} p_Rds\theta_x
      \]
      \[
      = -0.25 \text{ mm} + \int_{0}^{y'} \frac{h}{2}dy''\theta_x
      \]
• Warping of $s$-axis (2)
  
  - $u^s_x(s) = u^s_x(0) - 2AR_p(s)\theta_{,x}$
  - Origin in O’ as by symmetry $u_x(O’)=0$ (2)

  • On $AB$ branch
    
    - Area swept is negative

    $$u^{s,AB}_x(y') = u^{s,A}_x - \int_A^s p_R ds\theta_{,x}$$

    $$= -0.25 \text{ mm} + \int_0^{y'} \frac{h}{2} dy''\theta_{,x}$$

    $$u^{s,AB}_x(y') = -0.25 \text{ mm} + \frac{h\theta_{,x}}{2} y'$$

    $$= -0.25 \text{ mm} + 0.25 \times 1.26 \times y' = -0.25 \text{ mm} + 0.315 \times y'$$

    - At point $B$

    $$u^{s,B}_x = -0.25 \text{ mm} + 0.0315 \times 0.25 = 0.54 \text{ mm}$$

  • Branches for $z’<0$ obtained by symmetry
Torsion of open thin-walled section beams

- Warping of $s$-axis (3)
  - On $O'A$ branch
    \[ u_{x}^{s,O'A}(z') = -0.0101z' \]
  - On $AB$ branch
    \[ u_{x}^{s,AB}(y') = -0.25 \text{ mm} + 0.0315y' \]
  - Branches for $z'<0$ obtained by symmetry
Combined open and closed section beams

- **Wing section near an undercarriage bay**
  - **Bending**
    - There was no assumption on section shape
    - Use same formula
  - **Shearing**
    - Shear center has to be evaluated for the complete section
    - Shearing results into a shear load passing through this center & a torque
    - Shear flow has different expression in open & closed parts of the section
  - **Torsion**
    - Rigidity of open section can be neglected most of the time
    - But stress in open section can be high
Combined open and closed section beams

• Example
  – Simply symmetrical section
  – Constant thickness
  – Shear stress?

![Diagram showing a section with dimensions and forces](image)

- $b_f = 0.1 \text{ m}$
- $h_f = 0.1 \text{ m}$
- $b_b = 0.2 \text{ m}$
- $h_b = 0.2 \text{ m}$
- $T_z = 100 \text{ kN}$
- $t = 2 \text{ mm}$
Combined open and closed section beams

- **Centroid**
  - By symmetry, on $O'_{z'}$ axis
  - $z'_C$?
  
  \[ z'_C t (2h_f + 2b_f + 2b_b + 2h_b) = 2h_f t \left( -\frac{h_f}{2} \right) + b_b t (-h_b) + 2h_b t \left( -\frac{h_b}{2} \right) \]

  \[ z'_C = - \frac{2h_f \frac{h_f}{2} + b_b h_b + 2h_b \frac{h_b}{2}}{2h_f + 2b_f + 2b_b + 2h_b} = - \frac{0.01 + 0.04 + 0.04}{0.2 + 0.2 + 0.4 + 0.4} = -0.075 \text{ m} \]
Combined open and closed section beams

- **Second moment of area**
  
  - As \( z'_C = -0.075 \text{ m} \)
  
  \[
  I_{yy} = 2 \frac{th_f^3}{12} + 2 \left( -\frac{h_f}{2} - z'_C \right)^2 th_f + \left( -z'_C \right)^2 t (2b_f + b_b) + (-h_b - z'_C)^2 tb_b + 2 \frac{th_b^3}{12} + 2 \left( -\frac{h_b}{2} - z'_C \right)^2 h_b t
  \]

  \[
  I_{yy} = 2 \frac{0.002 \times 0.1^3}{12} + 2 \times 0.025^2 \times 0.002 \times 0.1 + 0.075^2 \times 0.002 \times 0.4 + (-0.125^2 \times 0.002 \times 0.2 + 2 \times 0.002 \times 0.2^3) + 2 \times 0.025^2 \times 0.2 \times 0.002 = 14.5 \times 10^{-6} \text{ m}^4
  \]

\[
\begin{align*}
  h_f &= 0.1 \text{ m} \\
  h_b &= 0.2 \text{ m} \\
  b_f &= 0.1 \text{ m} \\
  b_b &= 0.2 \text{ m} \\
  t &= 2 \text{ mm} \\
  T_z &= 100 \text{ kN}
\end{align*}
\]
• **Shear flow**
  
  – As $I_{xy} = 0$ & as shear center on $C_z$

  $q(s) = q_o(s) + q(0)$

  with $q_o(s) = -\frac{T_z}{I_{yy}} \int_0^s tz ds$

• At $A$ & $H$ shear stress has to be zero
  
  – If origin on $A$, $q(0) = 0$

  – Corresponds to an open section

• Branch $AB$

\[
q^{AB}(s) = -\frac{T_z}{I_{yy}} \int^{z'}_{-h_f-z'_C} tz'' dz'' = -\frac{T_z t}{2I_{yy}} \left[z^2 - (-h_f - z'_C)^2\right]
\]

\[
q^{AB}(z) = -\frac{100 \times 10^3}{2 \times 14.5 \times 10^{-6}} \left[z^2 - 0.025^2\right] = 4310 \text{ N} \cdot \text{m}^{-1} - 6.9 \times 10^6 \text{ N} \cdot \text{m}^{-3} z^2
\]

\[
q^B = q^{AB}(0.075) = 4310 - 6.9 \times 10^6 \times 0.075^2 = -34.5 \times 10^3 \text{ N} \cdot \text{m}^{-1}
\]
**Shear flow (2)**

- **Branch \(BC'\)**
  
  \[
  q^{BC'}(s) = q^B - \frac{T_z}{I_{yy}} \int_{-b_f - \frac{b_b}{2}}^{y} t (-z_C') \, dy'' = q^B + \frac{T_z t z_C'}{I_{yy}} \left[ y + b_f + \frac{b_b}{2} \right]
  \]

  \[
  q^{BC'}(y) = -34.5 \times 10^3 \text{ N} \cdot \text{m}^{-1} - \frac{100 \times 10^3 \times 0.002 \times 0.075}{14.5 \times 10^{-6}} [y + 0.2]
  \]

  \[
  = -241.4 \times 10^3 \text{ N} \cdot \text{m}^{-1} - 1.034 \times 10^6 \text{ N} \cdot \text{m}^{-2} y
  \]

- **\(q^{C'}: BC'\)**
  
  \[
  q^{C'} = q^{BC'}(-0.1) = -241.4 \times 10^3 + 103.4 \times 10^3 = -138 \times 10^3 \text{ N} \cdot \text{m}^{-1}
  \]

- **Branches \(FG \& GH\)**

  - By symmetry

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**Aircraft Structures - Beam - Torsion & Section Idealization**

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\(2013-2104\)
Shear flow (3)

- Closed part: \( q(s) = q_o(s) + q(0) \)
  - With \( q_o(s) = -\frac{T_z}{I_{yy}} \int_0^s tzds \) & \( q(s = 0) = \frac{y_T T_z - z_T T_y - \int p(s) q_o(s) ds}{2A_h} \)

- Let us fix the origin at \( O' \)
  - By symmetry \( q(0) = 0 \) (if not the formula would have required anticlockwise \( s, q \))

\[ q = q_o(s) \]

- Branch \( O'F \)

\[ q^{O'F} = -\frac{T_z}{I_{yy}} \int_0^y t (-z'_{C}) dy = \frac{T_z t y z'_{C}}{I_{yy}} y \]

\[ q^{O'F}(y) = -\frac{100 \times 10^3 \times 0.002 \times 0.075}{14.5 \times 10^{-6}} y = -1.03 \times 10^6 \text{ } y \text{ } \text{N} \cdot \text{m}^{-2} \]

\[ q^{F; \ O'F} = q^{O'F}(0.1) = -103 \times 10^3 \text{ } \text{N} \cdot \text{m}^{-1} \]
Combined open and closed section beams

- **Shear flow (4)**
  
  - **Branch FE**
    
    - Shear flux should be conserved at point $F$
      
      $$q^{F; FE} = q^{F; O'F} + q^{F; GF}$$
      
      $$= -241 \times 10^3 \text{ N} \cdot \text{m}^{-1}$$
    
    - Shear flux on branch
      
      $$q^{FE} = q^{F; FE} - \frac{T_z}{I_{yy}} \int_{-z_C'}^{z} tZ'' (-dz)$$
      
      $$= q^{F; FE} + \frac{T_z}{2I_{yy}} (z^2 - z_C'^2)$$
      
      $$q^{FE} (z) = -241 \times 10^3 + \frac{100 \times 10^3 \times 0.002}{2 \times 145 \times 10^{-6}} (z^2 - 0.075^2)$$
      
      $$= 6.9 \times 10^6 z^2 \text{ N} \cdot \text{m}^{-3} - 279.8 \times 10^3 \text{ N} \cdot \text{m}^{-1}$$
      
      $$q^E = q^{FE} (-0.125) = -6.9 \times 10^6 \times 0.125^2 - 279.8 \times 10^3 = -172 \times 10^3 \text{ N} \cdot \text{m}^{-1}$$
      
      $$\max_{z} q^{FE} (z) = q^{FE} (0) = -279.8 \times 10^3 \text{ N} \cdot \text{m}^{-1}$$
Combined open and closed section beams

- Shear flow (5)
  - Branch $EI$
    - $q^{EI}(s) = q^E - \frac{T_z}{I_{yy}} \int_{\frac{b_b}{2}}^{y} t \left(-h_b - z'_C\right) (-dy) = q^E + \frac{T_z t (h_b + z'_C)}{I_{yy}} \left(\frac{b_b}{2} - y\right)$
    - $q^{EI}(y) = -172 \times 10^3 + \frac{100 \times 10^3 \times 0.002 \times 0.125}{14.5 \times 10^{-6}} (0.1 - y) = -1.72 \times 10^6 \ y \ N \cdot m^{-2}$
  - Other branches by symmetry
• Shear flow (6)

- Remark, if symmetry had not been used, shear stress at $O'$ should be computed (but require anticlockwise $s$ and $q$ for these signs of $y_T$ & $z_T$)

\[
q(s = 0) = \frac{y_T T_z - z_T T_y}{2A_h} \cdot p(s) q_o(s) ds
\]

\[
q(O') = -\frac{1}{2A_h} \int p q_o(s) ds
\]

- With

\[
p^{OF} = p^{CO'} \quad \text{&} \quad q^{OF} = -q^{CO'} \quad \&
\]

\[
ds^{OF} = ds^{CO'}
\]

\[
\int_F^{O'} p q_o^{FO'} ds + \int_{O'}^{C'} p q_o^{O'C'} ds = 0
\]

- etc
Example

- Closed nose cell
  - 0.02 m² – area
  - 0.9 m – outer length
- Open bay
- Constant shear modulus
  \( \mu = 25 \text{ GPa} \)
- Torque 10 kN·m
- Twist rate?
- Shear stress?
Combined open and closed section beams

- **Twist rate**
  - As an approximation the 2 torsion rigidities are added
  - **Cell**
    - Closed section with constant $\mu$
      - $I_{T,\text{closed}} = \frac{4A_c h^2}{\int \frac{1}{t} ds}$
      - $\mu I_{Tc} = \frac{4\mu A_c^2 t_c}{l + h}$
    - $\mu I_{Tc} = \frac{4 \mu A_c^2 t_c}{l + h} = \frac{4 \cdot 0.02^2 \cdot 0.0015 \cdot 25 \cdot 10^9}{1.2} = 50 \cdot 10^3 \text{ N} \cdot \text{m}^2$
  - **Bay**
    - Open section with constant $\mu$
      - $I_{T,\text{open}} = \sum \frac{l_i t_i^3}{3}$
      - $\mu I_{Tb} = \frac{\mu t_b^3}{3} (b_b + h) = \frac{25 \cdot 10^9 \cdot 0.002^3 \cdot 0.9}{3} = 60 \text{ N} \cdot \text{m}^2$
  - **Twist rate**
    - $\mu I_T = 50060 \text{ N} \cdot \text{m}^2$
    - $\theta_{x} = \frac{M_x}{\mu I_T} = \frac{10^4}{50060} = 0.1998 \text{ rad} \cdot \text{m}^{-1}$
Combined open and closed section beams

- **Shear stress**
  - **Cell**
    - Closed section \(M_x = 2 A_l q\)
    - \[q_c = \frac{M_x}{2A_c} = \frac{10^4}{2 \times 0.02} = 250 \times 10^3 \text{ N} \cdot \text{m}^{-1}\]
    - \[\tau_c = \frac{q_c}{t_c} = \frac{250 \times 10^3}{0.0015} = 166.7 \text{ MPa}\]
  - **Bay**
    - Open section \(\tau_{\text{max}} = \mu t_i \theta_{ix}\)
    - \[\tau_{b,\text{max}} = \mu t_b \theta_{ix} = 25 \times 10^9 \times 0.002 \times 0.1998 = 9.99 \text{ MPa}\]
• Example 2-spar wing (one cell)
  – Stringers to stiffen thin skins
  – Angle section form spar flanges

• Design stages
  – Conceptual
    • Define the plane configuration
      – Span, airfoil profile, weights, …
    • Analyses should be fast and simple
      – Formula, statistics, …
  – Preliminary design
    • Starting point: conceptual design
    • Define more variables
      – Number of stringers, stringer area, …
    • Analyses should remain fast and simple
      – Use beam idealization
        » See today
      – FE model of thin structures
        » See next lectures
  – Detailed design
    • All details should be considered (rivets, …)
    • Most accurate analyses (3D, non-linear, FE)
Wing section idealization

- Principle of idealization
  - Booms
    - Stringers, spar flanges, …
      - Have small sections compared to airfoil
      - Direct stress due to wing bending is almost constant in each of these
      - They are replaced by concentrated area called booms
  - Booms
    - Have their centroid on the skin
    - Are carrying most direct stress due to beam bending
  - Skin
    - Skin is essentially carrying shear stress
    - It can be assumed
      - That skin is carrying only shear stress
      - If direct stress carrying capacity of skin is reported to booms by appropriate modification of their area
Wing section idealization

**Panel idealization**

- **Skin panel**
  - Thickness $t_D$, width $b$
  - Carrying direct stress *linearly distributed*
- **Replaced by**
  - Skin without thickness
  - 2 booms of area $A_1$ and $A_2$
- **Booms’ area depending on loading**
  - Moment around boom 2
    \[
    \sigma_{xx}^2 t_D b \frac{b}{2} + \left( \frac{\sigma_{xx}^1 - \sigma_{xx}^2}{2} \right) t_D b \frac{2b}{3} = \sigma_{xx}^1 A_1 b
    \]
    \[\Rightarrow A_1 = \frac{t_D b}{6} \left( 2 + \frac{\sigma_{xx}^2}{\sigma_{xx}^1} \right)\]
  - Total axial loading
    \[
    \sigma_{xx}^2 t_D b + \left( \frac{\sigma_{xx}^1 - \sigma_{xx}^2}{2} \right) t_D b = A_1 \sigma_{xx}^1 + A_2 \sigma_{xx}^2
    \]
    \[\Rightarrow A_2 = t_D b + \left( \frac{\sigma_{xx}^1}{\sigma_{xx}^2} - 1 \right) \frac{t_D b}{2} - A_1 \frac{\sigma_{xx}^1}{\sigma_{xx}^2}
    \]
    \[\Rightarrow A_2 = \frac{t_D b}{6} \left( 2 + \frac{\sigma_{xx}^1}{\sigma_{xx}^2} \right)\]
Wing section idealization

- Example
  - 2-cell box wing section
  - Simply symmetrical
  - Angle section of 300 mm$^2$

- Idealization of this section to resist to bending moment?
  - Bending moment along $y$-axis
  - 6 direct-stress carrying booms
  - Shear-stress-only carrying skin panels
Wing section idealization

- **Booms’ area**
  - **Bending moment**
    - Along $y$-axis
    - Stress proportional to $z$
      - stress distribution is linear on each section edge
  - **Contributions**
    - Flange(s)’ area
    - Reported skin parts
      - Use formula for linear distribution
        - $A_1 = 300 \times 10^{-6} + \frac{0.003 \times 0.4}{6} \left( 2 + \frac{-0.2}{0.2} \right) + \frac{0.002 \times 0.6}{6} \left( 2 + \frac{0.15}{0.2} \right)$
        - $A_6 = A_1 = 1.05 \times 10^{-3}$ m$^2$
        - $A_2 = 2 \times 300 \times 10^{-6} + \frac{0.002 \times 0.6}{6} \left( 2 + \frac{0.2}{0.15} \right) + \frac{0.0015 \times 0.6}{6} \left( 2 + \frac{0.1}{0.15} \right) + \frac{0.0025 \times 0.3}{6} \left( 2 + \frac{-0.15}{0.15} \right)$
        - $A_2 = A_4 = 1.79 \times 10^{-3}$ m$^2$
        - $A_3 = 300 \times 10^{-6} + \frac{0.0015 \times 0.6}{6} \left( 2 + \frac{0.15}{0.1} \right) + \frac{0.002 \times 0.2}{6} \left( 2 + \frac{-0.1}{0.1} \right)$
        - $A_4 = A_3 = 0.892 \times 10^{-3}$ m$^2$
Section idealization consequences

• Consequence on bending
  – Idealization depends on the loading case
    • Booms area are dependent on the loading case
  – Direct stress due to bending is carried by booms only
    • For bending the axial load is equal to zero
      \[ N_x = \int_A \sigma_{xx} \, dA = \sum_i \sigma_{xx}^i A_i = 0 \]
    • But direct stress depends on the distance \( z \) from neutral axis
      \[ \sigma_{xx}^i = \kappa E z_i \sum_i z_i A_i = 0 \]
  – It can be concluded that for open or closed sections, the position of the neutral axis, and thus the second moments of area
    • Refer to the direct stress carrying area only
    • Depend on the loading case only
Section idealization consequences

• Example
  – Idealized fuselage section
    • Simply symmetrical
    • Direct stress carrying booms
    • Shear stress carrying skin panels
  – Subjected to a bending moment
    • $M_y = 100 \text{ kN}\cdot\text{m}$
  – Stress in each boom?
Section idealization consequences

- **Centroid**
  - Of idealized section

\[
 z'_c \left( A_1 + 2 \sum_{i=2}^{8} A_i + A_9 \right) = A_1 z'_1 + 2 \sum_{i=2}^{8} A_i z'_i
\]

\[
 z'_c = \frac{1}{6 \ 0.00064 + 6 \ 0.0006 + 2 \ 0.00062 + 2 \ 0.00085
[1.2 \ 0.00064 + 2 \ (1.14 + 0.96 + 0.768) \ 0.0006 + 2 \ 0.565 \ 0.00062 + 2 \ (0.336 + 0.144) \ 0.00064 + 2 \ 0.038 \ 0.00085]}
\]

\[
 z'_c = \frac{0.0055892}{0.01038} = 0.54 \text{ m}
\]
Section idealization consequences

- Second moment of area
  - Of idealized section

\[
I_{yy} = A_1 (z'_1 - z'_C)^2 + 2 \sum_{i=2}^{8} A_i (z'_i - z'_C)^2 + A_9 (z'_9 - z'_C)\]

\[
I_{yy} = 0.00064 \times 0.66^2 + 2 \times 0.0006 \times (0.6^2 + 0.42^2 + 0.228^2) + \\
2 \times 0.00062 \times 0.025^2 + 2 \times 0.00064 \times ((-0.204)^2 + (-0.396)^2) + \\
2 \times 0.00085 \times (-0.502)^2 + 0.00064 \times (-0.54)^2\]

\[
I_{yy} = 1.855 \times 10^{-3} \text{ m}^4
\]
Section idealization consequences

• Stress distribution
  – Stress assumed constant in each boom
  – As we are in the principal axes

\[ \sigma_{xx}^i = \frac{M_y z_i}{I_{yy}} = \frac{M_y}{I_{yy}} (z'_i - z'_C) \]

\[
\begin{align*}
\sigma_{xx}^1 &= \frac{100 \times 10^3}{1.855 \times 10^{-3}} 0.66 = 35.6 \text{ MPa} \\
\sigma_{xx}^2 &= \frac{100 \times 10^3}{1.855 \times 10^{-3}} 0.6 = 32.3 \text{ MPa} \\
\sigma_{xx}^3 &= \frac{100 \times 10^3}{1.855 \times 10^{-3}} 0.42 = 22.6 \text{ MPa} \\
\sigma_{xx}^4 &= \frac{100 \times 10^3}{1.855 \times 10^{-3}} 0.228 = 12.3 \text{ MPa} \\
\sigma_{xx}^5 &= \frac{100 \times 10^3}{1.855 \times 10^{-3}} 0.025 = 1.35 \text{ MPa} \\
\sigma_{xx}^6 &= -\frac{100 \times 10^3}{1.855 \times 10^{-3}} 0.204 = -11.0 \text{ MPa} \\
\sigma_{xx}^7 &= -\frac{100 \times 10^3}{1.855 \times 10^{-3}} 0.396 = -21.3 \text{ MPa} \\
\sigma_{xx}^8 &= -\frac{100 \times 10^3}{1.855 \times 10^{-3}} 0.502 = -27.1 \text{ MPa} \\
\sigma_{xx}^9 &= -\frac{100 \times 10^3}{1.855 \times 10^{-3}} 0.54 = -29.1 \text{ MPa}
\end{align*}
\]

\[ A_1 = 640 \text{ mm}^2 \quad z'_1 = 1.2 \text{ m} \]
\[ A_2 = 600 \text{ mm}^2 \quad z'_2 = 1.14 \text{ m} \]
\[ A_3 = 600 \text{ mm}^2 \quad z'_3 = 0.960 \text{ m} \]
\[ A_4 = 600 \text{ mm}^2 \quad z'_4 = 0.768 \text{ m} \]
\[ A_5 = 620 \text{ mm}^2 \quad z'_5 = 0.565 \text{ m} \]
\[ A_6 = 640 \text{ mm}^2 \quad z'_6 = 0.336 \text{ m} \]
\[ A_7 = 640 \text{ mm}^2 \quad z'_7 = 0.144 \text{ m} \]
\[ A_8 = 850 \text{ mm}^2 \quad z'_8 = 0.038 \text{ m} \]
\[ A_9 = 640 \text{ mm}^2 \]
Section idealization consequences

• Consequence on open-thin-walled section shearing
  – Classical formula
    \[ q(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2} \int_0^s t_z ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2} \int_0^s t_y ds' \]
  – Results from integration of balance equation
    \[ t\partial_x \sigma_{xx} + \partial_s q = 0 \]
    – With \( \sigma_{xx} = \frac{(I_{zz}M_y + I_{yz}M_z)\,z - (I_{yz}M_y + I_{yy}M_z)\,y}{I_{yy}I_{zz} - I_{yz}^2} \)
  – So consequences are
    • Terms \( \int_0^s t\,(s')\,z\,(s')\,ds' \) & \( \int_0^s t\,(s')\,y\,(s')\,ds' \) should account for the direct stress-carrying parts only (which is not the case of shear-carrying-only skin panels)
    • Expression of the shear flux should be modified to account for discontinuities encountered between booms and shear-carrying-only skin panels
Section idealization consequences

- Consequence on open-thin-walled section shearing (2)
  - Equilibrium of a boom of an idealized section
    \[
    (\sigma_{xx} + \partial_x \sigma_{xx} \delta x) A_i - \sigma_{xx} A_i + q_{i+1} \delta x - q_i \delta x = 0
    \]
    \[q_{i+1} - q_i = -\partial_x \sigma_{xx} A_i\]
  - Lecture on beam shearing
    - Direct stress reads
      \[
      \sigma_{xx} = \frac{(I_{zz} M_y + I_{yz} M_z) z - (I_{yz} M_y + I_{yy} M_z) y}{I_{yy} I_{zz} - I_{yz}^2}
      \]
      - With \( T_z = \partial_x M_y \) & \( T_y = -\partial_x M_z \)
    - Eventually
      \[
      q_{i+1} - q_i = -\frac{I_{zz} T_z - I_{yz} T_y}{I_{yy} I_{zz} - I_{yz}^2} z_i A_i - \frac{I_{yy} T_y - I_{yz} T_z}{I_{yy} I_{zz} - I_{yz}^2} y_i A_i
      \]
      (no sum on \(i\))
Section idealization consequences

- Consequence on open-thin-walled section shearing (3)

\[ q(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2} \left[ \int_0^s t_{\text{direct}} \sigma z ds + \sum_{i: s_i \leq s} z_i A_i \right] - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2} \left[ \int_0^s t_{\text{direct}} \sigma y ds + \sum_{i: s_i \leq s} y_i A_i \right] \]
Section idealization consequences

- **Example**
  - Idealized U shape
    - Booms of 300 mm$^2$ area each
    - Booms are carrying all the direct stress
    - Skin panels are carrying all the shear flow
  - Shear load passes through the shear center
  - Shear flow?

\[
T_z = 4.8 \text{ kN}
\]

\[
h = 0.4 \text{ m}
\]

\[
b = 0.2 \text{ m}
\]
Section idealization consequences

- **Shear flow**
  - Simple symmetry $\rightarrow$ principal axes
    
    $$ q(s) = -\frac{T_z}{I_{yy}} \left[ \int_{0}^{s} t_{\text{direct}} \sigma z ds + \sum_{i: s_i \leq s} z_i A_i \right] $$
  - Only booms are carrying direct stress
    
    $$ q(s) = -\frac{T_z}{I_{yy}} \sum_{i: s_i \leq s} z_i A_i $$
  - Second moment of area
    
    $$ I_{yy} = \sum_{i} A_i z_i^2 = 4 \times 300 \times 10^{-6} \times 0.2^2 = 48 \times 10^{-6} \, \text{m}^4 $$
  - Shear flow
    
    $$ q^{12}(s) = -\frac{T_z}{I_{yy}} A_1 z_1 = -\frac{4.8 \times 10^3}{48 \times 10^{-6}} 300 \times 10^{-6} (-0.2) = 6 \times 10^3 \, \text{N} \cdot \text{m}^{-1} $$
    $$ q^{23}(s) = -\frac{T_z}{I_{yy}} (A_1 z_1 + A_2 z_2) = -\frac{4.8 \times 10^3}{48 \times 10^{-6}} 300 \times 10^{-6} (-0.4) = 12 \times 10^3 \, \text{N} \cdot \text{m}^{-1} $$
    $$ q^{34}(s) = -\frac{T_z}{I_{yy}} (A_1 z_1 + A_2 z_2 + A_3 z_3) = -\frac{4.8 \times 10^3}{48 \times 10^{-6}} 300 \times 10^{-6} (-0.2) = 6 \times 10^3 \, \text{N} \cdot \text{m}^{-1} $$
Section idealization consequences

- Comparison with uniform U section
  - We are actually capturing the **average** value in each branch
Section idealization consequences

• Consequence on closed-thin-walled section shearing
  
  – Classical formula

    • \( q(s) = q_o(s) + q(0) \)

    • With \( q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2} \int_0^s t(s')z(s') \, ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2} \int_0^s t(s')y(s') \, ds' \)

    • And \( q(s = 0) = \frac{y_T T_z - z_T T_y - \int p(s) q_o(s) \, ds}{2A_h} \)

    for anticlockwise \( q \) and \( s \)

  – So consequences are the same as for open section

    • \( q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2} \left[ \int_0^s t_{\text{direct}} \sigma z \, ds + \sum_{i: s_i \leq s} z_i A_i \right] - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2} \left[ \int_0^s t_{\text{direct}} \sigma y \, ds + \sum_{i: s_i \leq s} y_i A_i \right] \)
Section idealization consequences

- Example
  - Idealized wing section
    - Simply symmetrical
    - Booms are carrying all the direct stress
    - Skin panels are carrying all the shear flow
  - Shear load passes through booms 3 & 6
  - Shear flow?

\[
\begin{align*}
A_1 &= 200 \text{ mm}^2 \\
A_2 &= 250 \text{ mm}^2 \\
A_3 &= 400 \text{ mm}^2 \\
A_4 &= 100 \text{ mm}^2 \\
A_5 &= A_4 \\
A_6 &= A_3 \\
A_7 &= A_2 \\
A_8 &= A_1 \\
T_z &= 10 \text{ kN} \\
h_i &= 0.1 \text{ m} \\
h_m &= 0.2 \text{ m} \\
h_r &= 0.06 \text{ m} \\
b_l &= 0.12 \text{ m} \\
b_m &= 0.24 \text{ m} \\
b_r &= 0.24 \text{ m}
\end{align*}
\]
Section idealization consequences

- **Open part of shear flow**
  - Symmetrical section
    - Shear center & centroid on $C_y$ axis
    - $I_{xy} = 0$ (we are in the principal axes)
    - Only booms are carrying direct stress

\[
q_o(s) = -\frac{T_z}{I_{yy}} \sum_{i: s_i \leq s} z_i A_i
\]

- Second moment of area

\[
I_{yy} = \sum_{i=1}^{8} A_i z_i^2 = 2 \times 10^{-6} \left( 200 \times 0.03^2 + 250 \times 0.1^2 + 400 \times 0.1^2 + 100 \times 0.05^2 \right)
= 13.86 \times 10^{-6} \text{ m}^4
\]
Section idealization consequences

- Open part of shear flow (2)
  - Choose (arbitrarily) the origin between boom 2 and 3

\[ q_{o3} = 0 \]
\[ q_{o34} = -\frac{T_z}{I_{yy}} A_3 z_3 \]
\[ = -\frac{10^4}{13.86 \times 10^{-6}} 0.0004 \times 0.1 \]
\[ = -28.9 \times 10^3 \text{ N} \cdot \text{m}^{-1} \]
\[ q_{o45} = -\frac{10^4}{13.86 \times 10^{-6}} (0.0004 \times 0.1 + 0.0001 \times 0.05) = -32.5 \times 10^3 \text{ N} \cdot \text{m}^{-1} \]
\[ q_{o56} = -\frac{10^4}{13.86 \times 10^{-6}} [0.0004 \times 0.1 + 0.0001 (0.05 - 0.05)] = -28.9 \times 10^3 \text{ N} \cdot \text{m}^{-1} \]
\[ q_{o67} = -\frac{10^4}{13.86 \times 10^{-6}} [0.0004 (0.1 - 0.1) + 0.0001 (0.05 - 0.05)] = 0 \]
\[ q_{o78} = -\frac{10^4}{13.86 \times 10^{-6}} [... - 0.00025 \times 0.1] = 18 \times 10^3 \text{ N} \cdot \text{m}^{-1} \]
\[ q_{o81} = -\frac{10^4}{13.86 \times 10^{-6}} [... - 0.00025 \times 0.1 - 0.0002 \times 0.03] = 22.4 \times 10^3 \text{ N} \cdot \text{m}^{-1} \]
Section idealization consequences

- Open part of shear flow (3)
  - Choose (arbitrarily) the origin between boom 2 and 3 (2)

\[
q_{o}^{12} = -\frac{10^4}{13.86 \times 10^{-6}} \left[ ... - 0.00025 \times 0.1 + 0.0002 \times (0.03 - 0.03) \right] = 18 \times 10^3 \text{ N} \cdot \text{m}^{-1}
\]

\[
q_{o}^{20} = -\frac{10^4}{13.86 \times 10^{-6}} \left[ ... + 0.00025 \times (0.1 - 0.1) + 0.0002 \times (0.03 - 0.03) \right] = 0
\]
Section idealization consequences

- **Constant part of shear flow**
  \[ q(0) = \frac{yTz - \int p\, q_o\, ds}{2A_h} \]  
  (anticlockwise \( s, q \))

- If origin is chosen at point \( O' \)
  \[ q(0) = -\frac{\int pO' q_o\, ds}{2A_h} \]

- With
  \[
  A_h = b_l \frac{h_m + h_l}{2} + b_m h_m + b_r \frac{h_m + h_r}{2} = 0.12 \ 0.15 + 0.24 \ 0.2 + 0.24 \ 0.13 = 0.0972 \ \text{m}^2
  \]

- \[
  \int pO' q_o\, ds = q_o^{34} pO' l^{34} + q_o^{45} pO' l^{45} + q_o^{56} pO' l^{56} + q_o^{78} pO' l^{78} + q_o^{81} pO' l^{81} + q_o^{12} pO' l^{12}
  \]

\( \text{b}_l = 0.12 \ \text{m} \quad \text{b}_m = 0.24 \ \text{m} \quad \text{b}_r = 0.24 \ \text{m} \)
Section idealization consequences

- Constant part of shear flow (2)

\[- \int p_O q_o ds = q_0^{34} p_O^{34} l^{34} + q_0^{45} p_O^{45} l^{45} + q_0^{56} p_O^{56} l^{56} + q_0^{78} p_O^{78} l^{78} + q_0^{81} p_O^{81} l^{81} + q_0^{12} p_O^{12} l^{12}\]

\[\int p_O q_o ds = -28900 \cos \left( \tan \frac{0.05}{0.12} \right) 0.1 \sqrt{0.12^2 + 0.05^2} - 32500 \times 0.12 \times 0.1 - \]

\[28900 \cos \left( \tan \frac{0.05}{0.12} \right) 0.1 \sqrt{0.12^2 + 0.05^2} + \]

\[18000 \cos \left( \tan \frac{0.07}{0.24} \right) (0.1 + 0.07) \sqrt{0.24^2 + 0.07^2} + \]

\[22400 \times 0.48 \times 0.06 + 18000 \cos \left( \tan \frac{0.07}{0.24} \right) (0.1 + 0.07) \sqrt{0.24^2 + 0.07^2}\]

\[\int p_O q_o ds = 1030 \text{ N} \cdot \text{m}\]

- \(q_o = 28.9 \text{ kN} \cdot \text{m}^{-1}\)
- \(q_o = 32.5 \text{ kN} \cdot \text{m}^{-1}\)
- \(q_o = 18 \text{ kN} \cdot \text{m}^{-1}\)
- \(q_o = 22.4 \text{ kN} \cdot \text{m}^{-1}\)

\(b_l = 0.12 \text{ m} \quad b_m = 0.24 \text{ m} \quad b_r = 0.24 \text{ m}\)
Section idealization consequences

- Total shear flow

\[ q(0) = - \frac{\int P_O q_o ds}{2 A_h} = - \frac{1030}{2 \times 0.0972} = -5.3 \times 10^3 \text{ N} \cdot \text{m}^{-1} \]

- \( q_o = 28.9 \text{ kN} \cdot \text{m}^{-1} \)
- \( h_l = 0.1 \text{ m} \)
- \( b_l = 0.12 \text{ m} \)
- \( b_m = 0.24 \text{ m} \)
- \( b_r = 0.24 \text{ m} \)
- \( h_m = 0.2 \text{ m} \)
- \( h_r = 0.06 \text{ m} \)

- \( q_o = 32.5 \text{ kN} \cdot \text{m}^{-1} \)
- \( q_o = 28.9 \text{ kN} \cdot \text{m}^{-1} \)
- \( q_o = 22.4 \text{ kN} \cdot \text{m}^{-1} \)
- \( q_o = 18 \text{ kN} \cdot \text{m}^{-1} \)

- \( q = 37.8 \text{ kN} \cdot \text{m}^{-1} \)
- \( q = 34.2 \text{ kN} \cdot \text{m}^{-1} \)
- \( q = 5.3 \text{ kN} \cdot \text{m}^{-1} \)
- \( q = 12.7 \text{ kN} \cdot \text{m}^{-1} \)
- \( q = 17.1 \text{ kN} \cdot \text{m}^{-1} \)
- \( q = 12.7 \text{ kN} \cdot \text{m}^{-1} \)
Section idealization consequences

• Consequence on torsion
  – If no axial constraint
    • Torsion analysis does not involve axial stress
    • So torsion is unaffected by the structural idealization
Exercise: Structural idealization

- Box section
  - Arrangement of
    - Direct stress carrying booms positioned at the four corners and
    - Panels which are assumed to carry only shear stresses
    - Constant shear modulus
  - Shear centre?

![Diagram of a box section with dimensions and angles](image-url)
References

• Lecture notes

• Other references
  – Books
Exercise: Structural idealization

- As shear center lies on $O_y$ by symmetry we consider $T_Z$
  - Section is required to resist bending moments in a vertical plane
  - Direct stress at any point is directly proportional to the distance from the horizontal axis of symmetry, i.e. axis $y$
  - The distribution of direct stress in all the panels will be linear so that we can use the relation below

$$A_1 = \frac{t_Db}{6} \left( 2 + \frac{\sigma_{xx}^2}{\sigma_{xx}^1} \right)$$

$$A_2 = \frac{t_Db}{6} \left( 2 + \frac{\sigma_{xx}^1}{\sigma_{xx}^2} \right)$$

- In addition to contributions from adjacent panels, booms areas include the existing spar flanges
Exercise: Structural idealization

- Booms area

\[ A_1 = \frac{t_D b}{6} \left( 2 + \frac{\sigma_{22}}{\sigma_{11}} \right) \]

\[ A_1 = 60 \times 10 + 40 \times 10 + \frac{10 \times 300}{6} \left( 2 + \frac{\sigma_{22}^4}{\sigma_{11}^2} \right) + \frac{10 \times 500}{6} \left( 2 + \frac{\sigma_{22}^2}{\sigma_{11}^2} \right) \]

\[ = 60 \times 10 + 40 \times 10 + \frac{10 \times 300}{6} (2 - 1) + \frac{10 \times 500}{6} (2 + 1) \]

\[ = 4000 \text{ mm}^4 \]

\[ A_2 = 50 \times 8 + 30 \times 8 + \frac{8 \times 300}{6} \left( 2 + \frac{\sigma_{22}^3}{\sigma_{11}^2} \right) + \frac{10 \times 500}{6} \left( 2 + \frac{\sigma_{22}^2}{\sigma_{11}^2} \right) \]

\[ = 60 \times 10 + 40 \times 10 + \frac{10 \times 300}{6} (2 - 1) + \frac{10 \times 500}{6} (2 + 1) \]

\[ = 3540 \text{ mm}^4 \]

- By symmetry
  - \( A_3 = A_2 = 3540 \text{ mm}^2 \)
  - \( A_4 = A_1 = 4000 \text{ mm}^2 \)
Exercise: Structural idealization

- **Shear flow**
  - Booms area
    - \( A_3 = A_2 = 3540 \text{ mm}^2 \)
    - \( A_4 = A_1 = 4000 \text{ mm}^2 \)
  - By symmetry \( I_{yz} = 0 \)

\[
q(s) = -\frac{T_z}{I_{yy}} \sum_{i: s_i \leq s} z_i A_i + q(0)
\]

As only booms resist direct stress

\[
I_{yy} = \sum_{i=1}^{4} A_i z_i^2 = 2 \times 4000 \times 150^2 + 2 \times 3540 \times 150^2 = 339 \times 10^6 \text{ mm}^4
\]
Exercise: Structural idealization

- Open shear flow

\[ q_o(s) = -\frac{T_z}{I_{yy}} \sum_{i: s_i \leq s} z_i A_i \]

\[ \begin{align*}
q_{01}^{21} &= 0 \\
q_{0}^{14} &= -\frac{T_z}{I_{yy}} \times 4000 \times 150 = -1.77 \times 10^{-3} T_z \\
q_{0}^{43} &= 0 \quad \text{(by symmetry)} \\
q_{0}^{32} &= -\frac{T_z}{I_{yy}} \times 3540 \times -150 = 1.57 \times 10^{-3} T_z
\end{align*} \]
Exercise: Structural idealization

- **Constant shear flow**
  - Load through the shear center
    - no torsion

  \[ \int \frac{q}{\mu t} \, ds = 2A_h \frac{\partial \theta}{\partial x} \]

  \[ \int \frac{q}{\mu t} \, ds = \int \frac{q_o(s) + q(0)}{\mu t} \, ds = 0 \]

  \[ q(0) = -\frac{\int \frac{q_o(s)}{t} \, ds}{\int \frac{1}{t} \, ds} \]

- **With**

  \[ \int \frac{q_o(s)}{t} \, ds = q_0^{14} \frac{l_1}{l_1} + q_0^{32} \frac{l_3}{l_3} \]

  \[ = -1.77 \times 10^{-3} T_z \times \frac{300}{10} + 1.57 \times 10^{-3} T_z \times \frac{300}{8} = 5.775 \times 10^{-3} T_z \]

  and

  \[ \int \frac{1}{t} \, ds = 2 \times \frac{500}{10} + \frac{300}{10} + \frac{300}{8} = 167.5 \]

  \[ q(0) = -0.034 \times 10^{-3} T_z \]
Exercise: Structural idealization

- Total shear flow

\[ q(0) = -0.034 \times 10^{-3} T_z \]
Exercise: Structural idealization

- **Shear center**
  - Moment around O
    - Due to shear flow
    - Should be balanced by the external loads

\[ y_T T_z = 1.536 \times 10^{-3} T_z \times 300 \times 500 - 2 \times 0.034 \times 10^{-3} T_z \times 500 \times 150 \]

\[ y_T = 225 \text{ mm} \]
Annex 1: Deflection of open and closed section beams

- **Twist due to torsion**
  - As torsion analysis remains valid for idealized section, one could use the twist rate
    
    \[
    \theta_{,x} = \frac{M_x}{4A_h^2} \int \frac{1}{\mu t} \, ds
    \]

    \[
    M_x = 2A_h q
    \]

  - **Closed section**
    \[
    C = \frac{M_x}{\theta_{,x}} = \frac{1}{3} \int \mu t^3 \, ds
    \]
    \[
    \tau_{xs} = 2\mu n \theta_{,x}
    \]

  - **Open section**

  - **In general**
    \[
    \Delta \theta = \int_0^L \frac{M_x}{C} \, dx
    \]
    \[
    \tau \propto M_x
    \]
    \[
    \gamma = \frac{\tau}{\mu}
    \]

  - How can we compute deflection for other loading cases?
Symmetrical bending

- For pure bending we found $\sigma_{xx} = \kappa E \xi$
- Therefore the virtual work reads

$$\int_0^L \int_A \sigma_{xx} \delta \varepsilon_{xx} dA dx = \int_0^L \int_A \sigma_{xx} \delta \left( \frac{\sigma_{xx}}{E} \right) dA dx = \int_0^L \int_A \sigma_{xx} \delta \left( \frac{\kappa E \xi}{E} \right) dA dx$$

- Let us assume $C_z$ symmetrical axis, $M_z = 0$ & pure bending ($M_y$ constant)

$$\int_0^L \int_A \sigma_{xx} \delta \varepsilon_{xx} dA dx = \int_0^L \int_A \sigma_{xx} z dA \delta (-u_{z,xx}) dx = M_y \delta \int_0^L (-u_{z,xx}) dx = -M_y \delta \Delta u_{z,x}$$

- Consider a unit applied moment, and $\sigma^{(1)}$ the corresponding stress distribution

$$\int_0^L \int_A \sigma^{(1)}_{xx} \varepsilon_{xx} dA dx = \int_0^L \int_A \sigma^{(1)}_{xx} \frac{\sigma_{xx}}{E} dA dx = -\Delta u_{z,x}$$

- The energetically conjugated displacement (angle for bending) can be found by integrating the strain distribution multiplied by the unit-loading stress distribution
Annex 1: Deflection of open and closed section beams

**Virtual displacement**

- Expression for pure bending

\[ \int_0^L \int_A \sigma_{xx}^{(1)} \frac{\sigma_{xx}}{E} dA dx = -\Delta u_{z,x} \]

- In linear elasticity the general formula of virtual displacements reads

\[ \int_0^L \int_A \sigma^{(1)} : \varepsilon dA dx = P^{(1)} \Delta P \]

- \( \sigma^{(1)} \) is the stress distribution corresponding to a (unit) load \( P^{(1)} \)
- \( \Delta P \) is
  - The energetically conjugated displacement to \( P \)
  - In the direction of \( P^{(1)} \)
  - Corresponds to the strain distribution \( \varepsilon \)
Symmetrical bending due to extremity loading

- Example $C_z$ symmetrical axis, $M_z = 0$ & bending due to extremity load

\[ \int_0^L \int_A \sigma_{xx} \delta \varepsilon_{xx} dA dx = \int_0^L \int_A \sigma_{xx} z dA \delta (-u_{z,xx}) dx = \int_0^L M_y \delta (-u_{z,xx}) dx \]

Case of a semi-cantilever beam

\[ \int_0^L \int_A \sigma_{xx} \delta \varepsilon_{xx} dA dx = \int_0^L T_z (x - L) \delta (-u_{z,xx}) dx \]

\[ = T_z [(L - x) \delta u_{z,x}]_0^L + T_z \int_0^L \delta u_{z,x} dx = T_z \delta \Delta u_z \]

Eventually

\[ \Delta u_z = \int_0^L \int_A \sigma^{(1)}_{xx} \varepsilon_{xx} dA dx \]

- $\sigma^{(1)}$ is the stress distribution corresponding to a (unit) load $T_z^{(1)}$
- $\Delta u_z$ is the energetically conjugated displacement to $T_z$ in the direction of $T_z^{(1)}$ that corresponds to the strain distribution $\varepsilon$
Annex 1: Deflection of open and closed section beams

- **General pure bending**
  - If neutral axis is $\alpha$-inclined
    \[
    \int_0^L \int_A \sigma_{xx} \delta \varepsilon_{xx} \, dA \, dx = \int_0^L \int_A \sigma_{xx} \delta \left( \frac{kE \xi}{E} \right) \, dA \, dx
    \]
    - With $\xi = z \cos \alpha - y \sin \alpha$
    - It has been shown that
      \[
      \begin{align*}
      \frac{\partial^2 u_y}{\partial x^2} &= \frac{\partial^2 \xi}{\partial x^2} \sin \alpha = \kappa \sin \alpha \\
      \frac{\partial^2 u_z}{\partial x^2} &= -\frac{\partial^2 \xi}{\partial x^2} \cos \alpha = -\kappa \cos \alpha
      \end{align*}
      \]
      \[
      \kappa \xi = \kappa z \cos \alpha - \kappa y \sin \alpha = -u_{z,xx}z - u_{y,xx}y
      \]
  - Eventually, as $M$ is constant with $x$
    \[
    \int_0^L \int_A \sigma_{xx} \delta \left( \frac{\sigma_y}{E} \right) \, dA \, dx = \int_0^L \int_A \sigma_{xx} \delta \left(-u_{z,xx}z - u_{y,xx}y\right) \, dA \, dx =
    \]
    \[
    -M_y \delta \Delta u_{z,x} + M_z \delta \Delta u_{y,x} = M_y \delta \Delta \theta_y + M_z \delta \Delta \theta_z
    \]
Annex 1: Deflection of open and closed section beams

- General bending due to extremity loading
  
  - Bending moment depends on \( x \)
    
    \[
    \int_0^L \int_A \sigma_{xx} \delta \left( \frac{\sigma_{xx}}{E} \right) \, dA \, dx = \int_0^L \int_A \sigma_{xx} \delta \left( -u_{z,xx} z - u_{y,xx} y \right) \, dA \, dx = \\
    \int_0^L (- M_y \delta \Delta u_{z,xx} + M_z \delta \Delta u_{y,xx}) \, dx \ \ y
    \]
  
  - Integration by parts
    
    \[
    \int_0^L \int_A \sigma_{xx} \delta \left( \frac{\sigma_{xx}}{E} \right) \, dA \, dx = \\
    \int_0^L (L - x) \left[ T_z \delta \Delta u_{z,xx} + T_y \delta \Delta u_{y,xx} \right] \, dx = \\
    [(L - x) (T_z \delta \Delta u_{z,x} + T_y \delta \Delta u_{y,x})]_0^L + \\
    \int_0^L [T_z \delta \Delta u_{z,x} + T_y \delta \Delta u_{y,x}] \, dx
    \]
  
  - Semi-cantilever beam
    
    \[
    \int_0^L \int_A \sigma_{xx} \delta \left( \frac{\sigma_{xx}}{E} \right) \, dA \, dx = T_z \delta \Delta u_z + T_y \delta \Delta u_y = T \cdot \delta \Delta u
    \]
Annex 1: Deflection of open and closed section beams

- General bending due to extremity loading (2)
  - Virtual displacement method
    \[ \int_0^L \int_A \sigma_{xx}^{(1)} \varepsilon_{xx} dA dx = \Delta_p u \]
    - With \( \sigma^{(1)} \) due to the (unit) moments \( M^{(1)} \) resulting from the unit extremity loading
      \[ \sigma_{xx}^{(1)} = \frac{\left( I_{zz} M_y^{(1)} + I_{yz} M_z^{(1)} \right) z - \left( I_{yz} M_y^{(1)} + I_{yy} M_z^{(1)} \right) y}{I_{yy} I_{zz} - I_{yz}^2} \]
    - With \( \Delta_{plu} \) displacement in the direction of the unit extremity loading and corresponding to the strain distribution
      \[ \varepsilon_{xx} = \frac{1}{E} \left( I_{zz} M_y + I_{yz} M_z \right) z - \left( I_{yz} M_y + I_{yy} M_z \right) y \]
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- General bending due to extremity loading (3)
  - Virtual displacement method (2)
    - After developments, and if $\Delta_{pu}$ is the displacement in the direction of $T^{(1)} = 1$

$$\Delta_{pu} = \int_0^L \int_A \sigma_{xx}^{(1)} \varepsilon_{xx} \ dA \ dx$$

$$= \frac{1}{E \left( I_{yy} I_{zz} - I_{yz}^2 \right)} \int_0^L \int_A \left[ \left( I_{zz} M_y^{(1)} + I_{yz} M_z^{(1)} \right) z - \left( I_{yz} M_y^{(1)} + I_{yy} M_z^{(1)} \right) y \right]$$

$$\left[ (I_{zz} M_y + I_{yz} M_z) z - (I_{yz} M_y + I_{yy} M_z) y \right] \ dA \ dx$$

$$\Delta_{pu} = \frac{1}{E \left( I_{yy} I_{zz} - I_{yz}^2 \right)} \int_0^L \left\{ \left( I_{zz} M_y^{(1)} + I_{yz} M_z^{(1)} \right) \left( I_{zz} M_y + I_{yz} M_z \right) I_{yy} + \left( I_{yz} M_y^{(1)} + I_{yy} M_z^{(1)} \right) \left( I_{yz} M_y + I_{yy} M_z \right) I_{zz} - \left( I_{zz} M_y^{(1)} + I_{yz} M_z^{(1)} \right) \left( I_{zz} M_y + I_{yz} M_z \right) I_{yz} \right\} \ dx$$

- In the principal axes $I_{yz} = 0$

$$\Delta_{pu} = \frac{1}{EI_{yy} I_{zz}} \int_0^L \left\{ I_{zz} M_y^{(1)} M_y + I_{yy} M_z^{(1)} M_z \right\} \ dx$$
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• Shearing
  – Internal energy variation
    \[ \int_0^L \int_A \tau \delta \gamma dA dx = \int_0^L \int_A \tau \delta \frac{T}{\mu} dA dx = \int_0^L \int_s q \delta \frac{q}{\mu t} ds dx \]
  – Variation of the work of external forces
    \[ \int_0^L \int_A \tau \delta \gamma dA dx = \int_0^L t \tau \delta (\partial_x u_s + \partial_x u_x) ds dx \]
  – Defining the average deformation of a section
    – See use of \( A' \) for thick beams
    – Vectorial value
      \[ \int_0^L \int_A \tau \delta \gamma dA dx = \int_0^L \int_s t \tau \delta \partial_x \bar{u}_s \cdot ds dx = \int_0^L \left( \int_s t \tau ds \right) \cdot \delta \partial_x \bar{u}_s dx \]
    – Applied shear loading \( T = \int_s t \tau ds \)

  \[ \int_0^L \int_A \tau \delta \gamma dA dx = \int_0^L T \cdot \delta \partial_x u dx = T \cdot \delta \Delta u \]
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- **Shearing (2)**
  - Virtual work
  \[
  \int_0^L \int_s q^{(1)} \frac{q}{\mu t} ds dx = T^{(1)} \Delta u = \Delta_T u
  \]
  - With \( \Delta_T u \) the average deflection of the section in the direction of the applied unit shear load
  - With \( q^{(1)} \) the shear flux distribution resulting from this applied unit shear load
  \[
  q^{(1)}(s) = -\left( \frac{I_{zz}T_z^{(1)} - I_{yz}T_y^{(1)}}{I_{yy}I_{zz} - I_{yz}^2} \right) \left[ \int_0^s t_{\text{direct}} \sigma z ds + \sum_{i: s_i \leq s} z_i A_i \right] - \left( \frac{I_{yy}T_y^{(1)} - I_{yz}T_z^{(1)}}{I_{yy}I_{zz} - I_{yz}^2} \right) \left[ \int_0^s t_{\text{direct}} \sigma y ds + \sum_{i: s_i \leq s} y_i A_i \right] + \{q^{(1)}(0)\}
  \]
  - With \( q \) the shear flux distribution corresponding to the deflection \( \Delta_T u \)
  \[
  q(s) = -\left( \frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2} \right) \left[ \int_0^s t_{\text{direct}} \sigma z ds + \sum_{i: s_i \leq s} z_i A_i \right] - \left( \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2} \right) \left[ \int_0^s t_{\text{direct}} \sigma y ds + \sum_{i: s_i \leq s} y_i A_i \right] + \{q(0)\}
  \]
  - \( \{q(0)\} \) meaning only for closed sections
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- Example
  - Idealized U shape
    - Booms of 300-mm$^2$ area each
    - Booms are carrying all the direct stress
    - Skin panels are carrying all the shear flow
    - Actual skin thickness is 1 mm
  - Beam length of 2 m
    - Shear load passes through the shear center at one beam extremity
    - Other extremity is clamped
  - Material properties
    - $E = 70$ GPa
    - $\mu = 30$ GPa
  - Deflection?

\[ T_z = 4.8 \text{ kN} \]

\[ h = 0.4 \text{ m} \]

\[ b = 0.2 \text{ m} \]
Shear flow (already solved)

- Simple symmetry ↔ principal axes

\[ q(s) = -\frac{T_z}{I_{yy}} \left[ \int_0^s t_{\text{direct}} \sigma z \, ds + \sum_{i: \ s_i \leq s} z_i A_i \right] \]

- Only booms are carrying direct stress

\[ q(s) = -\frac{T_z}{I_{yy}} \sum_{i: \ s_i \leq s} z_i A_i \]

- Second moment of area

\[ I_{yy} = \sum_i A_i z_i^2 = 4 \times 300 \times 10^{-6} \times 0.2^2 = 48 \times 10^{-6} \text{ m}^4 \]

- Shear flow

\[ q^{12}(s) = -\frac{T_z}{I_{yy}} A_1 z_1 = -\frac{4.8 \times 10^3}{48 \times 10^{-6}} \times 300 \times 10^{-6} (-0.2) = 6 \times 10^3 \text{ N} \cdot \text{m}^{-1} \]

\[ q^{23}(s) = -\frac{T_z}{I_{yy}} (A_1 z_1 + A_2 z_2) = -\frac{4.8 \times 10^3}{48 \times 10^{-6}} \times 300 \times 10^{-6} (-0.4) = 12 \times 10^3 \text{ N} \cdot \text{m}^{-1} \]

\[ q^{34}(s) = -\frac{T_z}{I_{yy}} (A_1 z_1 + A_2 z_2 + A_3 z_3) = -\frac{4.8 \times 10^3}{48 \times 10^{-6}} \times 300 \times 10^{-6} (-0.2) = 6 \times 10^3 \text{ N} \cdot \text{m}^{-1} \]
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- **Unit shear flow**
  - Same argumentation as before but with $T_z = 1 \text{ N}$

  \[
  q^{(1), 12} (s) = -\frac{1 \text{ N}}{I_{yy}} A_1 z_1 = -\frac{1}{48 \times 10^{-6}} 300 10^{-6} (-0.2) = 1.25 \text{ N} \cdot \text{m}^{-1}
  \]

  \[
  q^{(1), 23} (s) = -\frac{1 \text{ N}}{I_{yy}} (A_1 z_1 + A_2 z_2)
  = -\frac{1}{48 \times 10^{-6}} 300 10^{-6} (-0.4) = 2.5 \text{ N} \cdot \text{m}^{-1}
  \]

  \[
  q^{(1), 34} (s) = -\frac{1 \text{ N}}{I_{yy}} (A_1 z_1 + A_2 z_2 + A_3 z_3)
  = -\frac{1}{48 \times 10^{-6}} 300 10^{-6} (-0.2) = 1.25 \text{ N} \cdot \text{m}^{-1}
  \]

- **Displacement due to shearing**

  \[
  -\Delta_T u = \int_0^L \int_s q^{(1)} \frac{q}{\mu t} ds dx = 2 \int_s q^{(1)} \frac{q}{30 \times 10^9 0.001} ds
  \]

  \[
  \Delta_T u = \frac{2}{30 \times 10^9 0.001} \left[ 6000 \times 1.25 \times 0.2 + 12000 \times 2.5 \times 0.4 + 6000 \times 1.25 \times 0.2 \right] = 10^{-3} \text{ m}
  \]
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- **Bending**
  - Moment due to extremity load
    \[
    \begin{align*}
    M_y &= (x - L) T_z \\
    M_y^{(1)} &= (x - L)
    \end{align*}
    \]
  - Deflection due to extremity load
    - In the principal axes
      \[
      \Delta P u = \frac{1}{E} \int_0^L \frac{M_y^{(1)} M_y}{I_{yy}} \, dx = \frac{T_z}{I_{yy} E} \int_0^L (x - L)^2 \, dx = \frac{T_z L^3}{3 I_{yy} E}
      \]
      \[
      \Delta P u = \frac{4.8 \times 10^3 \times 2^3}{3 \times 48 \times 10^{-6} \times 70 \times 10^9} = 0.00381 \text{ m}
      \]

- **Total deflection**
  - No torsion as shear load passes through the shear center
  - \[\delta u_z = \Delta_T u + \Delta_P u = 0.00481 \text{ m}\]