University of Liège
Aerospace & Mechanical Engineering

Aircraft Structures
Beams - Bending & Shearing (Open Section)

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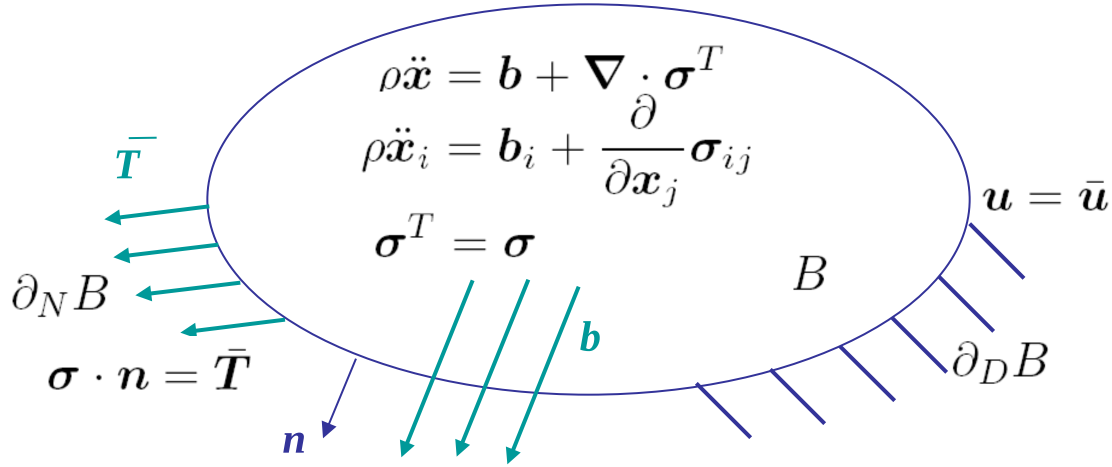
Elasticity

- Balance of body B
  - Momenta balance
    - Linear
    - Angular
  - Boundary conditions
    - Neumann
    - Dirichlet

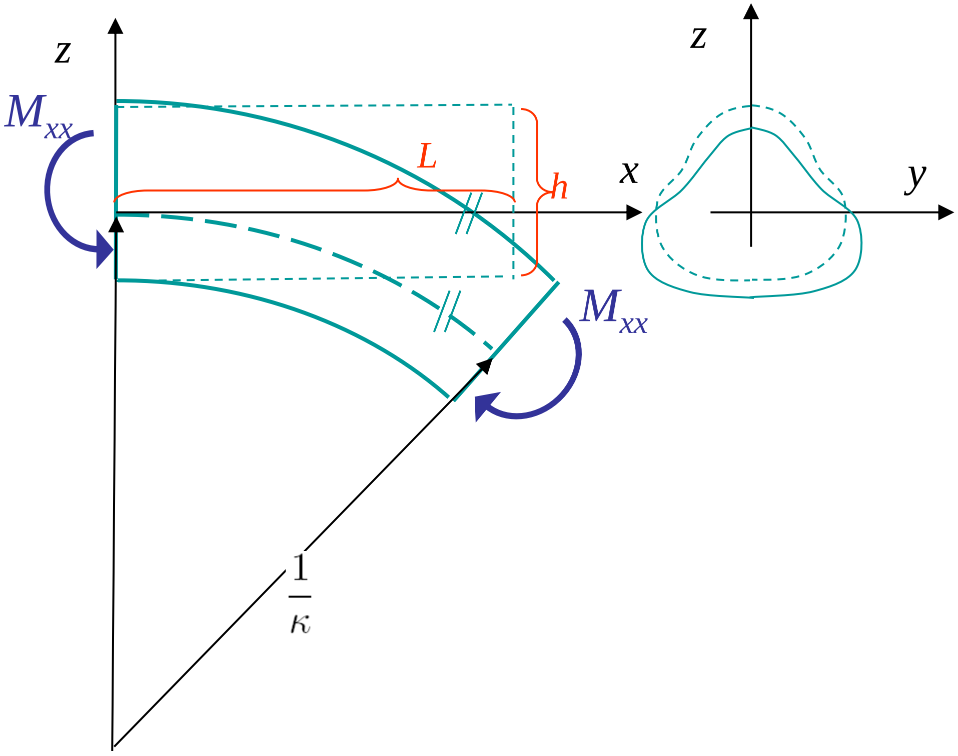


- Small deformations with linear elastic, homogeneous & isotropic material
  - (Small) Strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{\nabla}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{\nabla}\right)$, or $\begin{cases} \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{x}_i}\boldsymbol{u}_j + \frac{\partial}{\partial \boldsymbol{x}_j}\boldsymbol{u}_i\right) \\ \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\boldsymbol{u}_{j,i} + \boldsymbol{u}_{i,j}\right) \end{cases}$
  - Hooke's law $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon}$, or $\boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl}\boldsymbol{\varepsilon}_{kl}$

    with $\mathcal{H}_{ijkl} = \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda = K - 2\mu/3}\delta_{ij}\delta_{kl} + \underbrace{\frac{E}{1+\nu}}_{2\mu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$
  - Inverse law $\boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma}$

    with $\mathcal{G}_{ijkl} = \frac{1+\nu}{E}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right) - \frac{\nu}{E}\delta_{ij}\delta_{kl}$

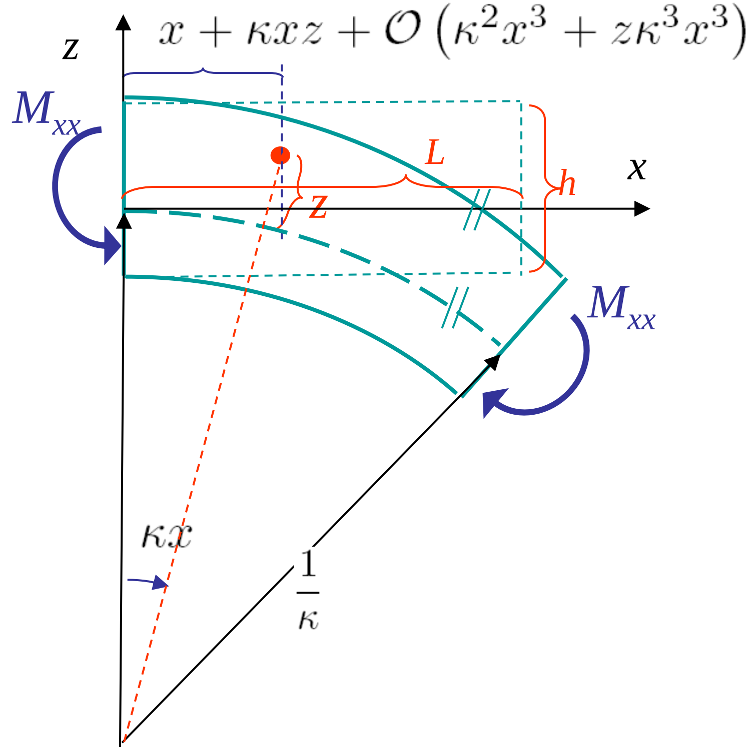
- Assumptions
  - Symmetrical beams
  - Cross-section remains plane and perpendicular to fibers
    - Bernoulli or Kirchhoff-Love theory
    - Only for thin structures (h/L << 1)
    - Limited bending: κL << 1
  - Linear elasticity
    - Small deformations
    - Homogeneous material
    - Hooke 's law
  - For pure bending
    - Vertical axis of symmetry
    - Constant curvature of the neutral plane $\kappa = -\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}$



Pure bending of symmetrical beams

- Kinematics
  - $\boldsymbol{u}_x = \kappa x z \Longrightarrow \varepsilon_{xx} = \kappa z$
- Linear elasticity
  - Hooke's law & stress-free edges
    - $\begin{cases} \boldsymbol{\sigma}_{xx} = \kappa E z \\ \boldsymbol{\sigma}_{yy} = \boldsymbol{\sigma}_{zz} = 0 \end{cases}$

$$\left(\frac{1}{\kappa} + z\right) \sin \kappa x = x + \kappa x z + \mathcal{O}\left(\kappa^2 x^3 + z \kappa^3 x^3\right)$$

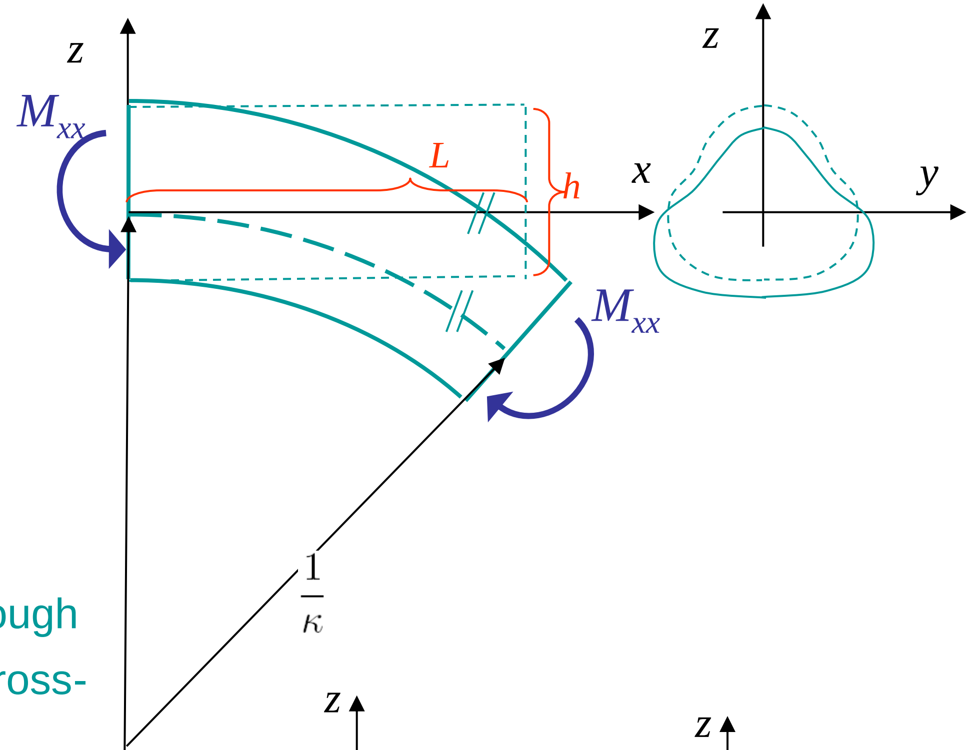


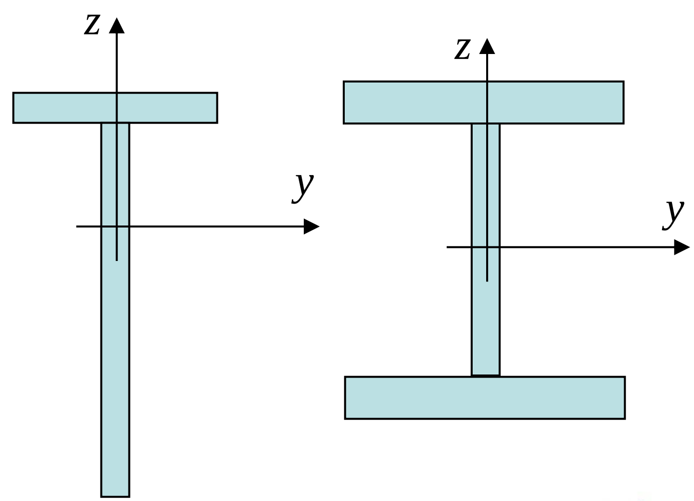
Pure bending of symmetrical beams

- Resultant forces
  - Tension

$$N_x = \int_A \boldsymbol{\sigma}_{xx} dydz = \kappa E \int_A z dydz$$

    - Should be equal to zero (pure bending)
    - So the neutral axis is defined such that the first moment of area $\int_A z dydz = 0$
      - The neutral axis passes through the centroid of area of the cross-section
      - As, here, Oz is a symmetry axis the neutral axis passes through the centroid of the cross-section

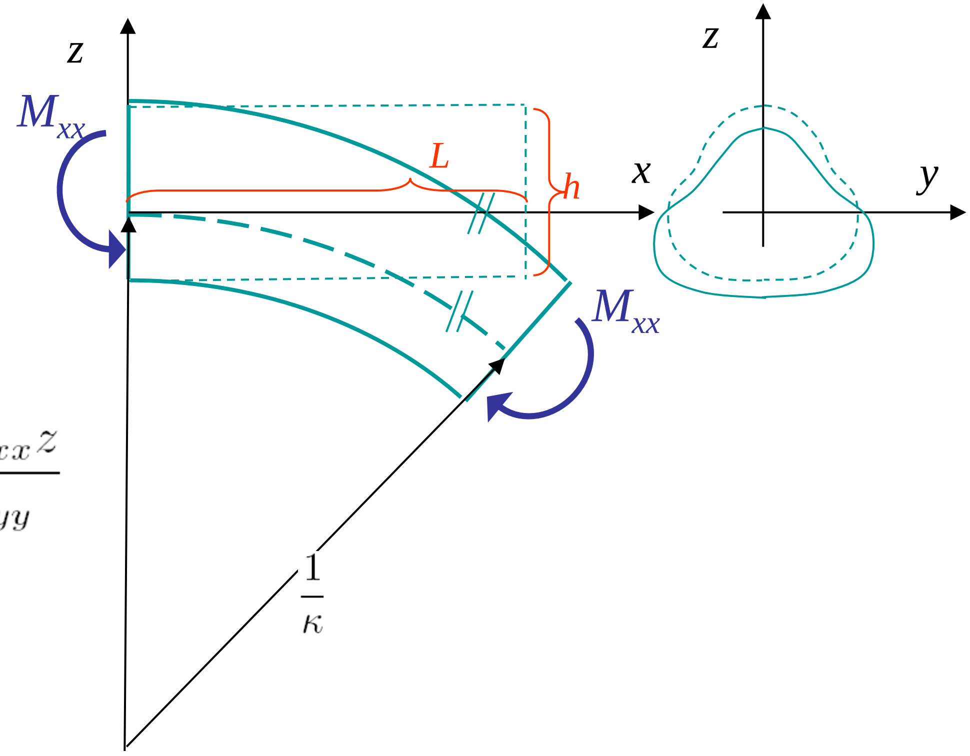




- Resultant forces (2)
  - Bending moment

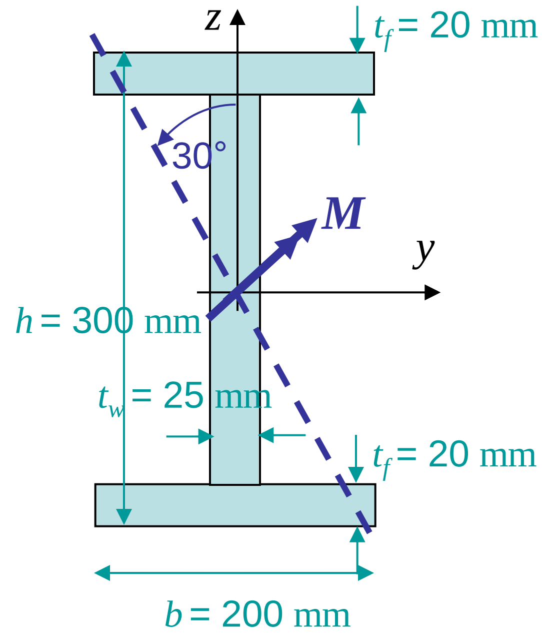
$$M_{xx} = \int_A \kappa E z^2 dy dz = \kappa E I_{yy}$$

- With $I_{yy} = \int_A z^2 dy dz$ the second moment of area
- As $\boldsymbol{\sigma}_{xx} = \kappa E z \Longrightarrow \boldsymbol{\sigma}_{xx} = \frac{M_{xx} z}{I_{yy}}$



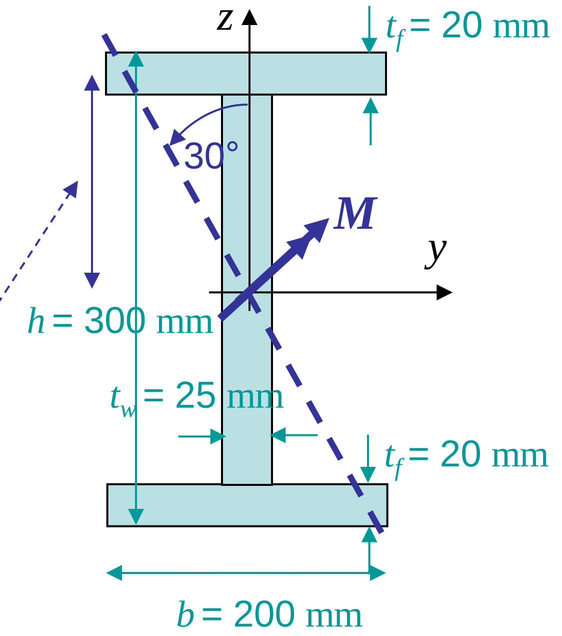
Pure bending of symmetrical beams

- Example
  - I-beam subjected to a bending moment
    - 100 kN·m
    - Applied in a 30°- inclined plane
  - Distribution of stress?
  - Neutral axis?



- Second moments of area
  - Doubly symmetrical cross-section

    ⟹ origin of the axes is the centroid



  - $$I_{yy} = 4\int_0^{\frac{t_w}{2}}\int_0^{\frac{h}{2}-t_f} z^2 dz dy + 4\int_0^{\frac{b}{2}}\int_{\frac{h}{2}-t_f}^{\frac{h}{2}} z^2 dz dy$$

    $$\Longrightarrow I_{yy} = \frac{(h-2t_f)^3 t_w}{12} + \frac{bh^3}{12} - \frac{2b\left(\frac{h}{2}-t_f\right)^3}{3}$$

    $$\Longrightarrow I_{yy} = \frac{(h-2t_f)^3 t_w}{12} + 2\left[\frac{bt_f^3}{12} + bt_f\left(\frac{h}{2}-\frac{t_f}{2}\right)^2\right]$$

    Transport theorem

    $$\Longrightarrow I_{yy} = \frac{0.26^3 0.025}{12} + 2\left[\frac{0.2\ 0.02^3}{12} + 0.2\ 0.02\ 0.140^2\right] = 1.94\,10^{-4}\ \mathrm{m}^4$$

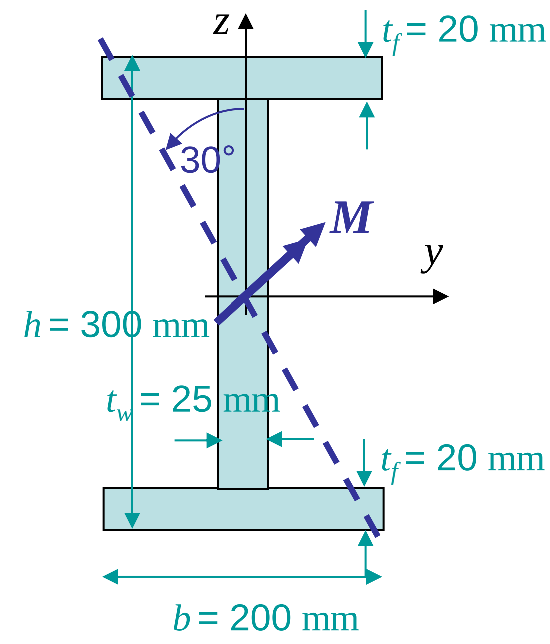
  - $$I_{zz} = \int_A y^2 dy dz = 2\frac{t_f b^3}{12} + \frac{(h-2t_f)\,t_w^3}{12}$$

    $$\Longrightarrow I_{zz} = 2\frac{0.02\ 0.2^3}{12} + \frac{0.26\ 0.025^3}{12} = 2.7\ 10^{-5}\ \mathrm{m}^4$$

- Bending moment components
  - $\boldsymbol{M}_{xx} \cdot \boldsymbol{E}_y = 10^5 \ \cos 30 = 50000\sqrt{3}$ Nm
  - $\boldsymbol{M}_{xx} \cdot \boldsymbol{E}_z = 10^5 \ \sin 30 = 50000$ Nm
- Stress distribution
  - Linear elasticity $\Rightarrow$ superposition
  - $\boldsymbol{\sigma}_{xx} = \frac{\boldsymbol{M}_{xx} \cdot \boldsymbol{E}_y z}{I_{yy}} - \frac{\boldsymbol{M}_{xx} \cdot \boldsymbol{E}_z y}{I_{zz}}$

$$\Rightarrow \boldsymbol{\sigma}_{xx} = \frac{50000\sqrt{3}z}{1.94\ 10^{-4}} - \frac{50000y}{2.7\ 10^{-5}}$$

$$= 447\ 10^6\ \mathrm{Pa} \cdot \mathrm{m}^{-1} z - 1851\ 10^6\ \mathrm{Pa} \cdot \mathrm{m}^{-1} y$$



- Stress distribution (2)
  - $\boldsymbol{\sigma}_{xx} = 447\ 10^6\ \mathrm{Pa \cdot m^{-1}} z - 1851\ 10^6\ \mathrm{Pa \cdot m^{-1}} y$
  - For $z = \pm 0.15\ \mathrm{m}$

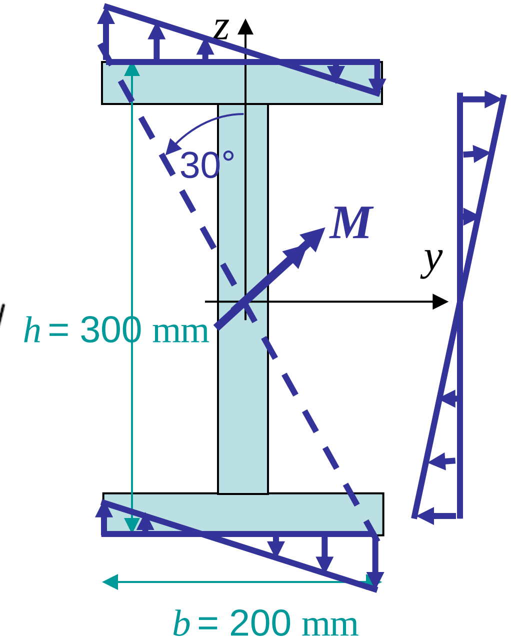
$$\boldsymbol{\sigma}_{xx}\left(z = \pm 0.15\right) = \pm 67\ 10^6\ \mathrm{Pa} - 1851\ 10^6\ \mathrm{Pa \cdot m^{-1}} y$$

$$\Longrightarrow \in \pm\left[-252;\ 118\right]\ \mathrm{MPa}$$

  - For $y = 0$

$$\boldsymbol{\sigma}_{xx}\left(y = \pm 0.\right) = 447\ 10^6\ \mathrm{Pa \cdot m^{-1}} z$$

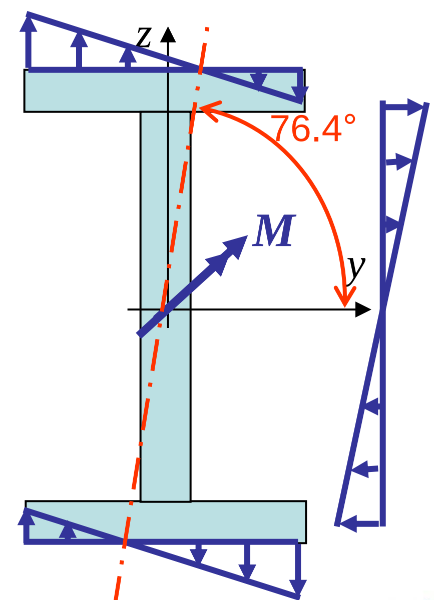
$$\Longrightarrow \in \pm\left[-67;\ 67\right]\ \mathrm{MPa}$$



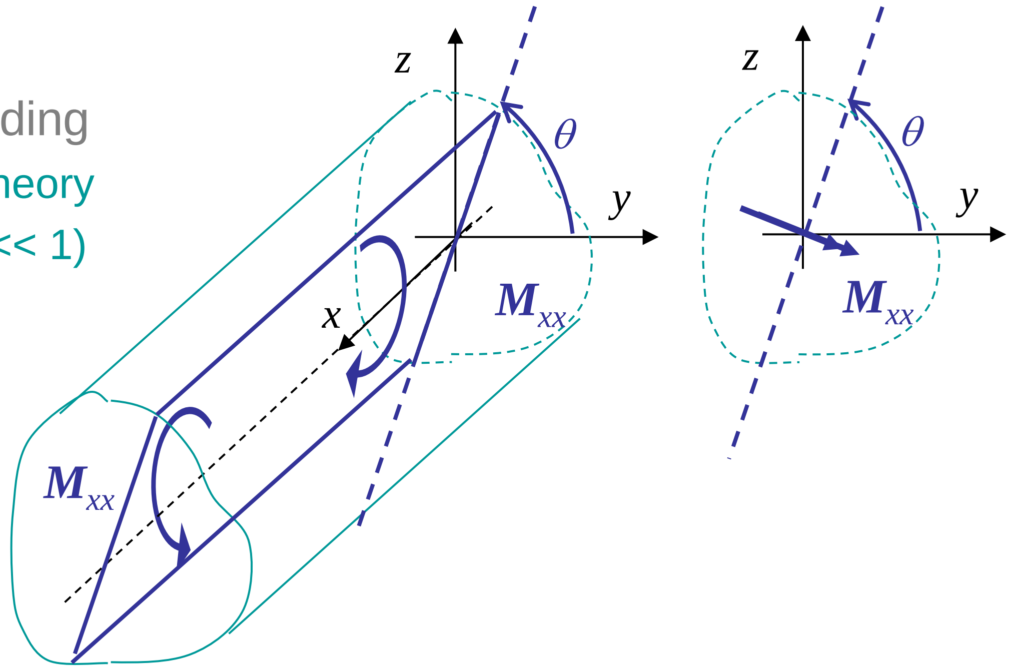
- Neutral axis
  - Such that $\sigma_{xx} = 0$

$$\Longrightarrow 0 = \boldsymbol{\sigma}_{xx} = 447\ 10^6\ \mathrm{Pa \cdot m^{-1}} z - 1851\ 10^6\ \mathrm{Pa \cdot m^{-1}} y$$

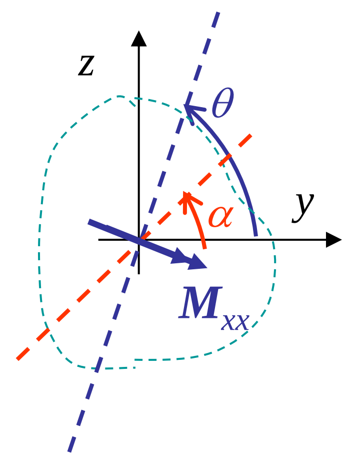
$$\Longrightarrow \frac{z}{y} = \frac{1851}{447} = \tan 76.4^o$$



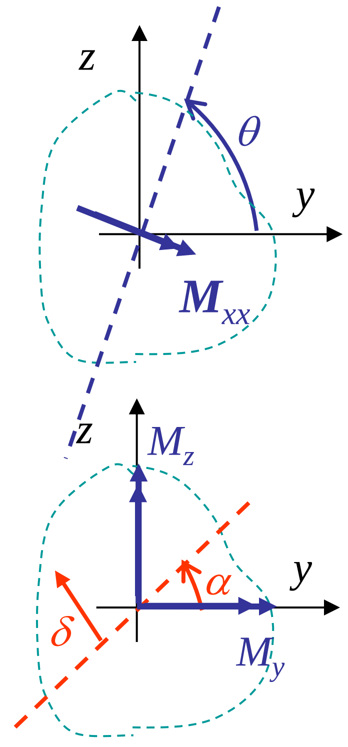
- Assumptions
  - Same as for symmetrical bending
    - Bernoulli or Kirchhoff-Love theory
    - Only for thin structures (h/L << 1)
    - Limited bending: κL << 1
    - Linear elasticity
    - Bending moment in a plane being θ-inclined



  - Let us assume
    - The existence of a neutral axis
      - Being α-inclined
      - Including the axes origin



- Bending moment components
  - $M_y = \boldsymbol{M}_{xx} \cdot \boldsymbol{E}_y = \|\boldsymbol{M}_{xx}\| \sin\theta$
  - $M_z = \boldsymbol{M}_{xx} \cdot \boldsymbol{E}_z = -\|\boldsymbol{M}_{xx}\| \cos\theta$



- Pure bending
  - Signed distance from neutral axis

    $\delta = z\cos\alpha - y\sin\alpha$
  - Plane section remains plane

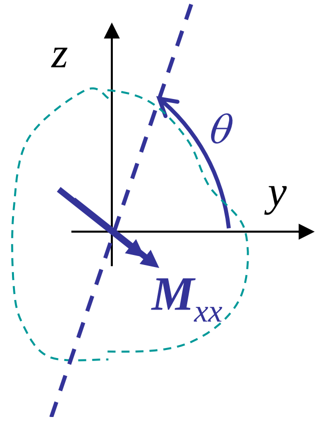
    $\boldsymbol{\sigma}_{xx} = \kappa E \delta$

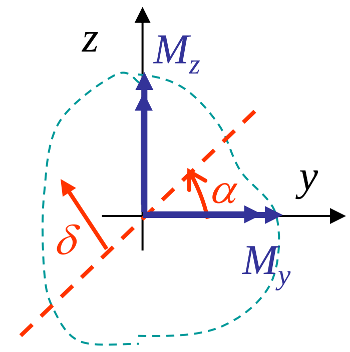
    $\Longrightarrow \quad \boldsymbol{\sigma}_{xx} = \kappa E z \cos\alpha - \kappa E y \sin\alpha$
  - Pure bending

$$N_x = \int_A \boldsymbol{\sigma}_{xx} dydz = \kappa E \int_A \left[z\cos\alpha - y\sin\alpha\right] dydz = 0$$

  - As $\int_A \delta dydz = 0$ the neutral axis still passes through the cross-section centroid

- Neutral axis
  - As $\begin{cases} M_y = \boldsymbol{M}_{xx} \cdot \boldsymbol{E}_y = \|\boldsymbol{M}_{xx}\| \sin\theta \\ M_z = \boldsymbol{M}_{xx} \cdot \boldsymbol{E}_z = -\|\boldsymbol{M}_{xx}\| \cos\theta \\ \boldsymbol{\sigma}_{xx} = \kappa E z \cos\alpha - \kappa E y \sin\alpha \end{cases}$





  - It yields
    - $M_y = \int_A z \boldsymbol{\sigma}_{xx} dy dz = \kappa E \int_A \left[z^2 \cos\alpha - yz \sin\alpha\right] dy dz$
      $= \kappa E I_{yy} \cos\alpha - \kappa E I_{yz} \sin\alpha = \|\boldsymbol{M}_{xx}\| \sin\theta$
    - $M_z = -\int_A y \boldsymbol{\sigma}_{xx} dy dz = \kappa E \int_A \left[-zy \cos\alpha + y^2 \sin\alpha\right] dy dz$
      $= -\kappa E I_{yz} \cos\alpha + \kappa E I_{zz} \sin\alpha = -\|\boldsymbol{M}_{xx}\| \cos\theta$
    - In tensorial form $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{\kappa E} \begin{pmatrix} I_{yy} & -I_{yz} \\ -I_{yz} & I_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$
  - So position of the neutral axis depends on
    - The geometry
    - The loading

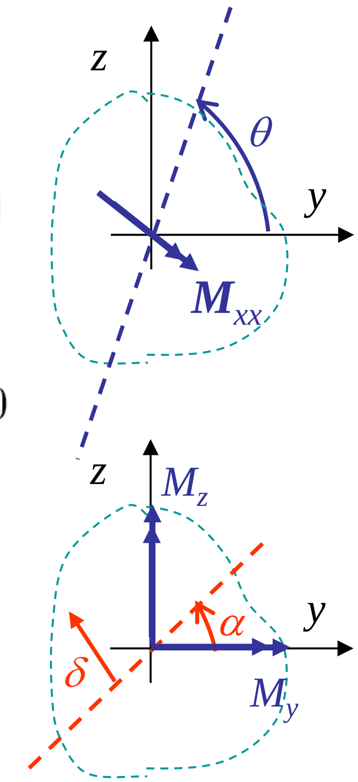
- Principal axes
  - As $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{\kappa E} \begin{pmatrix} I_{yy} & -I_{yz} \\ -I_{yz} & I_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$
  - The referential can be changed such that $\int_A yz dy dz = 0$
    - We are in the principal axes of the section
      - This referential is load-independent
    - The neutral axis is then obtained by

$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{\kappa E} \begin{pmatrix} \frac{\sin\theta}{I_{yy}} \\ -\frac{\cos\theta}{I_{zz}} \end{pmatrix}$$

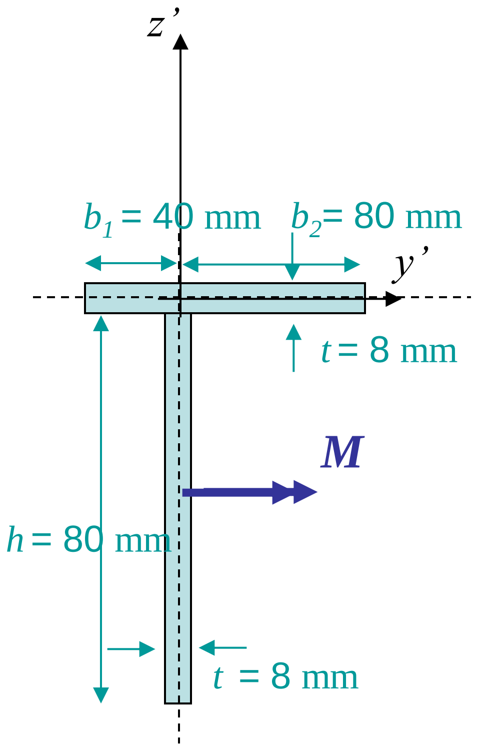
    - And the stresses $\boldsymbol{\sigma}_{xx} = \kappa E z \cos\alpha - \kappa E y \sin\alpha$ are rewritten as

$$\boldsymbol{\sigma}_{xx} = \|\boldsymbol{M}_{xx}\| \left[\frac{z}{I_{yy}} \sin\theta + \frac{y}{I_{zz}} \cos\theta \right]$$

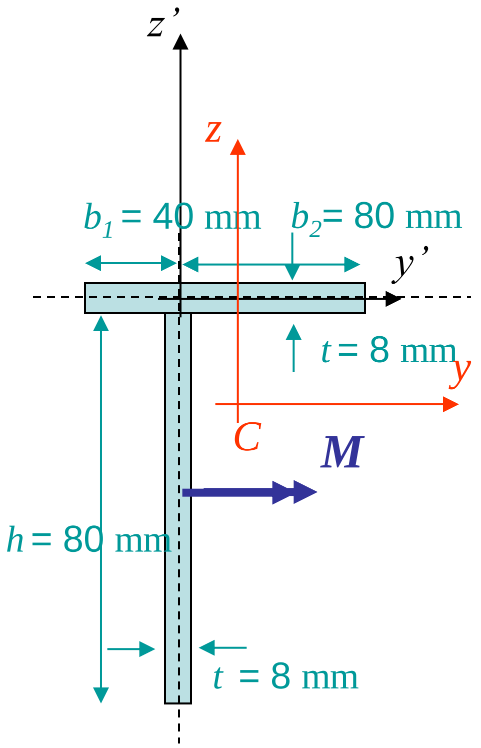
  - So in the principal axes, everything happens as for symmetrical loading



- Example
  - Beam subjected to a bending moment
    - 1.5 kN·m
    - In the vertical plane
  - Neutral axis?
  - Maximum stress due to bending ?



- Position of centroid
  - New frame *Cyz* linked to centroid *C*
    - $z'_C = \frac{\int_A z' dy' dz'}{\int_A dy' dz'} = \frac{-ht\frac{h+t}{2}}{ht + (b_1 + b_2)\, t}$
      $= \frac{-0.08\ 0.008\ 0.044}{0.08\ 0.008 + 0.12\ 0.008} = -0.0176 \text{ m}$
    - $y'_C = \frac{\int_A y' dy' dz'}{\int_A dy' dz'} = \frac{(b_1 + b_2)\, t\frac{b_2 - b_1}{2}}{ht + (b_1 + b_2)\, t}$
      $= \frac{0.12\ 0.008\ 0.02}{0.08\ 0.008 + 0.12\ 0.008} = 0.012 \text{ m}$



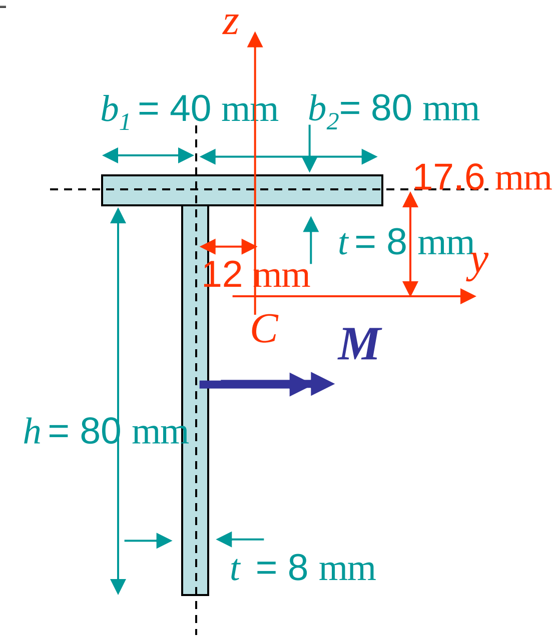
Pure bending of unsymmetrical beams

- Second moments of area
  - $I_{yy} = \int_A z^2 dydz = \frac{0.12\ 0.008^3}{12} + 0.0176^2\ 0.12\ 0.008 + \frac{0.08^3\ 0.008}{12} + (0.044 - 0.0176)^2\ 0.08\ 0.008$

$$\Rightarrow I_{yy} = 1.09\ 10^{-6}\ \mathrm{m}^4$$

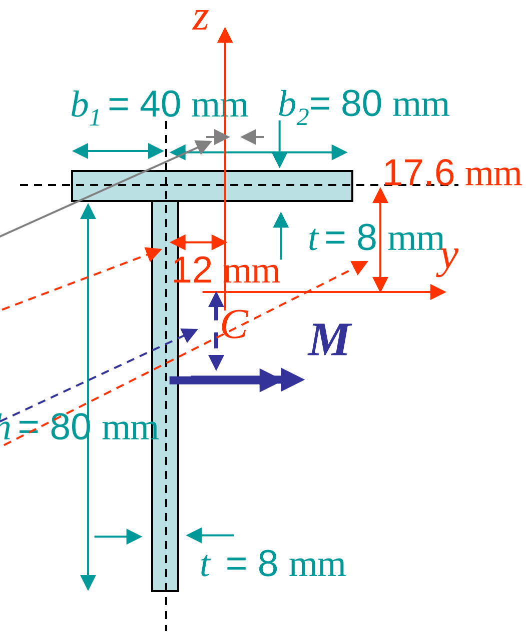
  - $I_{zz} = \int_A y^2 dydz = \frac{0.12^3\ 0.008}{12} + (0.02 - 0.012)^2\ 0.12\ 0.008 + \frac{0.08\ 0.008^3}{12} + 0.012^2\ 0.08\ 0.008$

$$\Rightarrow I_{zz} = 1.31\ 10^{-6}\ \mathrm{m}^4$$



Pure bending of unsymmetrical beams

- Second moments of area (2)
  - $I_{yz} = \int_A yz\,dy\,dz = \int_{-b_1-0.012}^{b_2-0.012} y\,dy \int_{0.0176-\frac{t}{2}}^{0.0176+\frac{t}{2}} z\,dz + \int_{-\frac{t}{2}-0.012}^{\frac{t}{2}-0.012} y\,dy \int_{0.0136-h}^{0.0136} z\,dz$



$$\Rightarrow I_{yz} = \frac{(b_2-0.012)^2-(b_1+0.012)^2}{2}\,\frac{\left(0.0176+\frac{t}{2}\right)^2-\left(0.0176-\frac{t}{2}\right)^2}{2} + \frac{\left(\frac{t}{2}-0.012\right)^2-\left(\frac{t}{2}+0.012\right)^2}{2}\,\frac{0.0136^2-(0.0136-h)^2}{2}$$

$$\Rightarrow I_{yz} = (b_2+b_1)\,t\left(\frac{b_2-b_1}{2}-0.012\right)0.0176 - th\,0.012\left(0.0136-\frac{h}{2}\right)$$

  - As each part is symmetrical, I_{yz} with respect to C is equal to the sum of
    - Area part times
    - y-distance from C of section part to global section C times
    - z-distance from C of section part to global section C

$$\Rightarrow I_{yz} = 0.120\ 0.008\ 0.008\ 0.0176 + 0.08\ 0.008\ 0.012\ 0.0264 = 3.38\ 10^{-7}\ \mathrm{m}^4$$

- Neutral axis
  - As

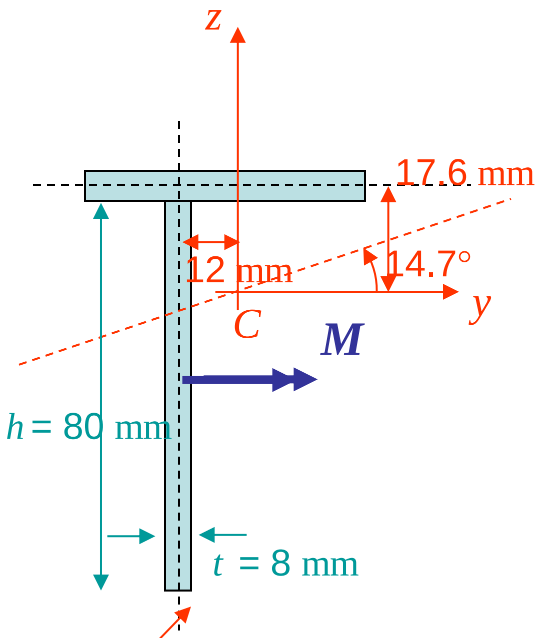
$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{\kappa E} \begin{pmatrix} I_{yy} & -I_{yz} \\ -I_{yz} & I_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

with $\theta = \pi/2$, so

$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{1.5\ 10^9}{\kappa E} \begin{pmatrix} 0.9972 \\ 0.2573 \end{pmatrix}$$

  - Neutral axis in inclined by 14.7° as

$$\Longrightarrow \tan\alpha = \frac{0.2573}{0.9972} = 0.258 = \tan 14.7^o$$



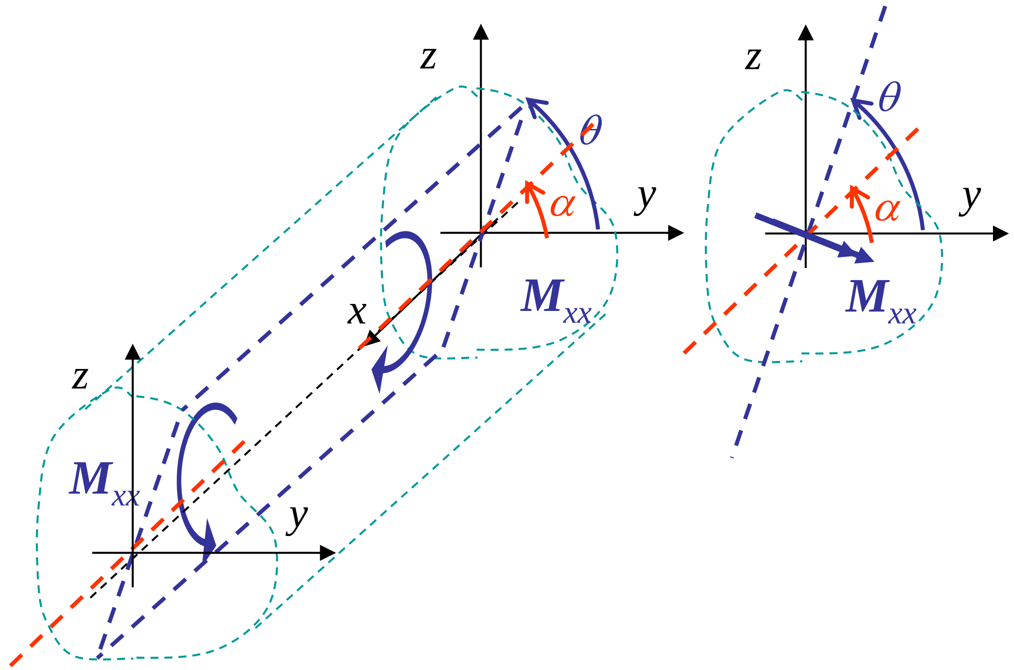
- Stresses
  - $\boldsymbol{\sigma}_{xx} = \kappa E z \cos\alpha - \kappa E y \sin\alpha =$

$$1.5\ 10^9\ 0.9972\ z - 1.5\ 10^9\ 0.2573 = 1500\ \text{MPa}\cdot\text{m}^{-1} z - 386\ \text{MPa}\cdot\text{m}^{-1} y$$

  - By inspection the maximum stress is reached

$$\Longrightarrow \max|\boldsymbol{\sigma}_{xx}| = \boldsymbol{\sigma}_{xx}(-0.008;\ -0.0664) = |-96.5|\ \text{MPa}$$

- Bending deflection
  - Let us consider
    - Unsymmetrical beam
    - Frame origin at the cross-section centroid
    - A bending moment in a planed being θ-inclined
    - A resulting neutral-axis being α-inclined

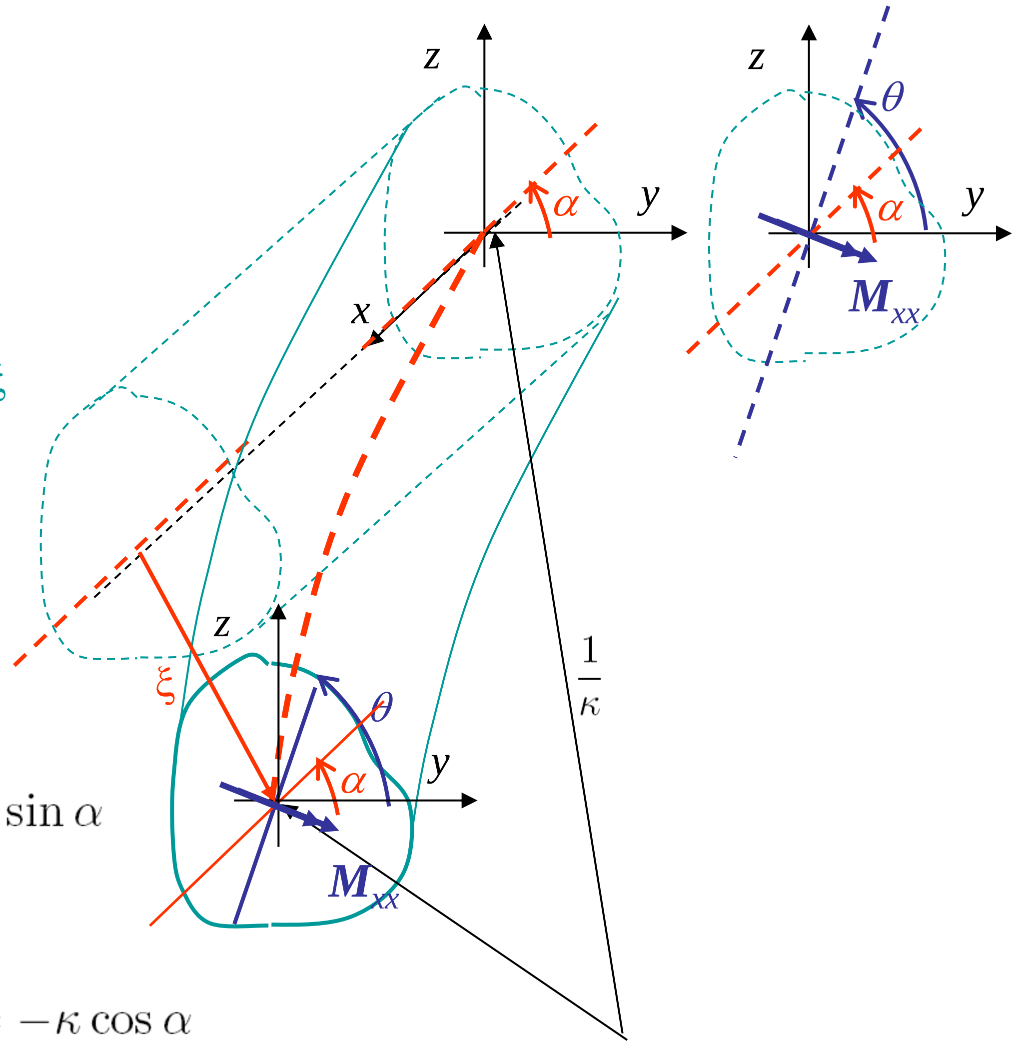


- Bending deflection (2)
  - Due to the bending moments
    - There is a deflection normal to the neutral-axis
    - The centroid is deflected by ξ
    - The neutral surface has a curvature κ with $\kappa = \frac{\partial^2 \xi}{\partial x^2}$
  - Deflection components
    - $\boldsymbol{u}_y = \xi \sin\alpha$

$$\Rightarrow \frac{\partial^2 \boldsymbol{u}_y}{\partial x^2} = \frac{\partial^2 \xi}{\partial x^2} \sin\alpha = \kappa \sin\alpha$$

    - $\boldsymbol{u}_z = -\xi \cos\alpha$

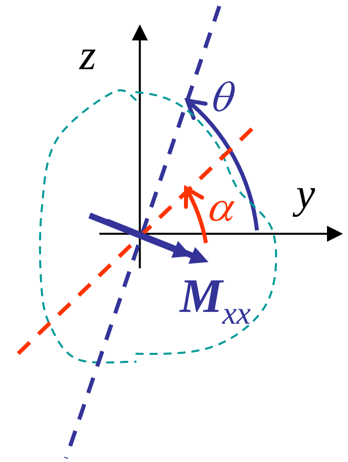
$$\Rightarrow \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} = -\frac{\partial^2 \xi}{\partial x^2} \cos\alpha = -\kappa \cos\alpha$$



- Bending deflection (3)
  - As
    - $\frac{\partial^2 \boldsymbol{u}_y}{\partial x^2} = \frac{\partial^2 \xi}{\partial x^2} \sin\alpha = \kappa \sin\alpha$
    - $\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} = -\frac{\partial^2 \xi}{\partial x^2} \cos\alpha = -\kappa \cos\alpha$
  - The neutral-axis is obtained by
    - $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{\kappa E} \begin{pmatrix} I_{yy} & -I_{yz} \\ -I_{yz} & I_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$

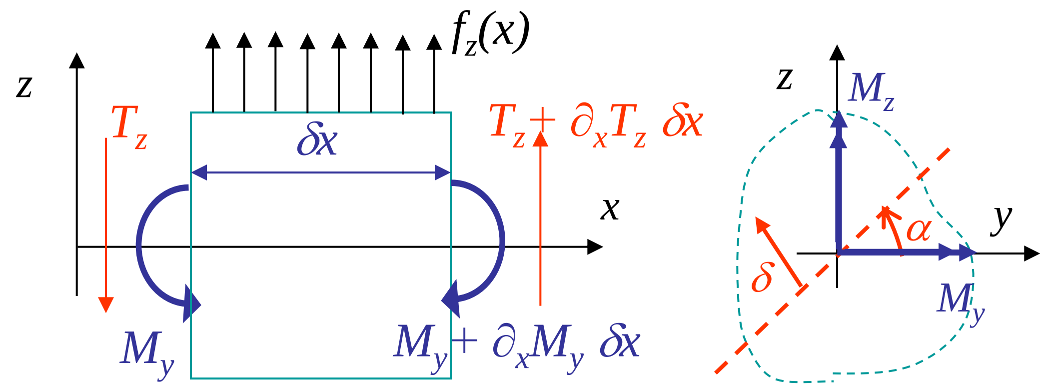
$$\Longrightarrow \begin{pmatrix} -\boldsymbol{u}_{z,xx} \\ \boldsymbol{u}_{y,xx} \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{E\left(I_{yy}I_{zz} - I_{yz}I_{yz}\right)} \begin{pmatrix} I_{zz} & I_{yz} \\ I_{yz} & I_{yy} \end{pmatrix} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

$$\& \begin{cases} M_y = \|\boldsymbol{M}_{xx}\| \sin\theta = -E\left(I_{yy}\boldsymbol{u}_{z,xx} + I_{yz}\boldsymbol{u}_{y,xx}\right) \\ M_z = -\|\boldsymbol{M}_{xx}\| \cos\theta = E\left(I_{yz}\boldsymbol{u}_{z,xx} + I_{zz}\boldsymbol{u}_{y,xx}\right) \end{cases}$$



- An unsymmetrical beam
  - Deflects both vertically & horizontally even for a loading in the vertical plane
  - Excepted in the principal axes (I_{yz}=0)

- Shearing-bending relationship



  – Linear balance equation

$$T_z = T_z + \partial_x T_z \delta x + f_z \delta x$$

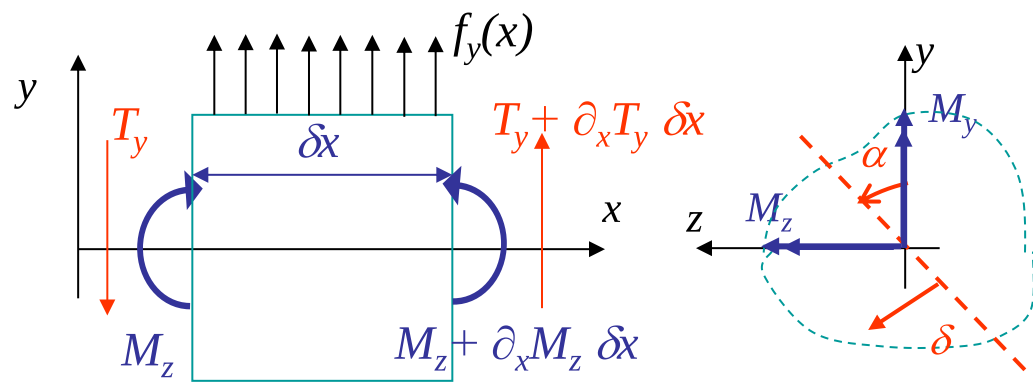
$$\Rightarrow \quad f_z\left(x\right) = -\partial_x T_z$$

  – Momentum balance equation

$$M_y + \partial_x M_y \delta x = M_y + f_z \delta x \frac{\delta_x}{2} + \left(T_z + \partial_x T_z \delta_x\right) \delta x$$

$$\Rightarrow \quad T_z = \partial_x M_y$$ as second order terms vanish

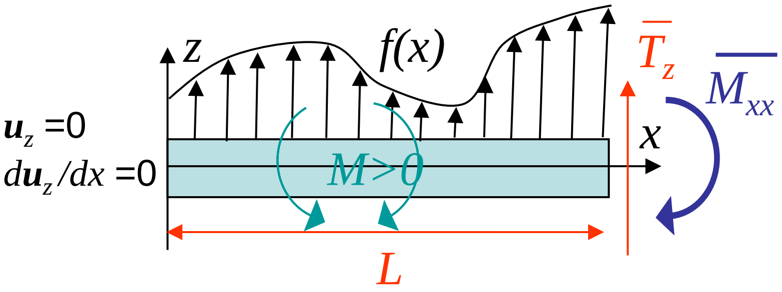
  – Eventually



    - $f_z\left(x\right) = -\partial_x T_z = -\partial_{xx} M_y$
    - $f_y\left(x\right) = -\partial_x T_y = \partial_{xx} M_z$

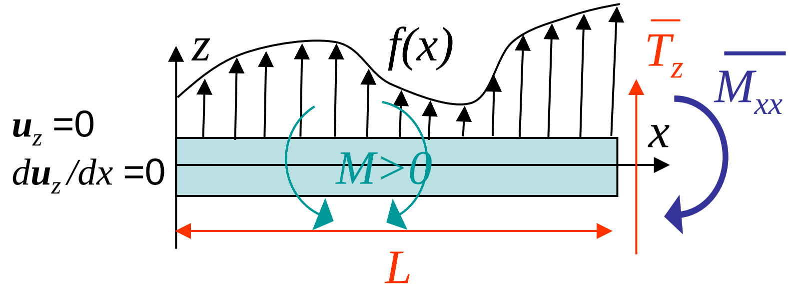
- Bending approximation: deflection equations
  - As $\begin{cases} f_z(x) = -\partial_x T_z = -\partial_{xx} M_y \\ f_y(x) = -\partial_x T_y = \partial_{xx} M_z \end{cases}$ & $\begin{cases} M_y = \|\boldsymbol{M}_{xx}\| \sin\theta = -E\left(I_{yy}\boldsymbol{u}_{z,xx} + I_{yz}\boldsymbol{u}_{y,xx}\right) \\ M_z = -\|\boldsymbol{M}_{xx}\| \cos\theta = E\left(I_{yz}\boldsymbol{u}_{z,xx} + I_{zz}\boldsymbol{u}_{y,xx}\right) \end{cases}$
  - Deflection equations read
    - Euler-Bernoulli equations for x in $[0; L]$

$$\begin{cases} f_z(x) = \partial_{xx}\left(EI_{yy}\boldsymbol{u}_{z,xx} + EI_{yz}\boldsymbol{u}_{y,xx}\right) \\ f_y(x) = \partial_{xx}\left(EI_{yz}\boldsymbol{u}_{z,xx} + EI_{zz}\boldsymbol{u}_{y,xx}\right) \end{cases}$$



    - Low order boundary conditions
      - Either on displacements $\bar{\boldsymbol{u}}_z|_{0,\ L} = \boldsymbol{u}_z|_{0,\ L}$ & $\bar{\boldsymbol{u}}_y|_{0,\ L} = \boldsymbol{u}_y|_{0,\ L}$
      - Or on shearing $\begin{cases} \bar{T}_y|_{0,\ L} = -\partial_x\left(EI_{yz}\boldsymbol{u}_{z,xx} + EI_{zz}\boldsymbol{u}_{y,xx}\right)|_{0,\ L} \\ \bar{T}_z|_{0,\ L} = -\partial_x\left(EI_{yy}\boldsymbol{u}_{z,xx} + EI_{yz}\boldsymbol{u}_{y,xx}\right)|_{0,\ L} \end{cases}$
    - High order boundary conditions
      - Either on rotations $\bar{\boldsymbol{u}}_{z,x}|_{0,\ L} = \boldsymbol{u}_{z,x}|_{0,\ L}$ & $\bar{\boldsymbol{u}}_{y,x}|_{0,\ L} = \boldsymbol{u}_{y,x}|_{0,\ L}$
      - Or on couple $\begin{cases} \bar{M}_y|_{0,\ L} = -\left(EI_{yy}\boldsymbol{u}_{z,xx} + EI_{yz}\boldsymbol{u}_{y,xx}\right)|_{0,\ L} \\ \bar{M}_z|_{0,\ L} = \left(EI_{yz}\boldsymbol{u}_{z,xx} + EI_{zz}\boldsymbol{u}_{y,xx}\right)|_{0,\ L} \end{cases}$

- Bending approximation: deflection equations with energy method
  - Same results can be obtained with an energy method
  - Let us assume we are in the principal axes and $M_z = 0$
    - As $\begin{cases} M_{xx} = \int_A \kappa E z^2 dy dz = \kappa E I \\ \kappa = -\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \end{cases}$



    - Internal energy variation

$$\delta E_{\text{int}} = \int_0^L \int_A \boldsymbol{\sigma}_{xx} \delta \boldsymbol{\varepsilon}_{xx} dA dx = \int_0^L \int_A E \kappa \delta \kappa z^2 dA dx =$$

$$\int_0^L \int_A E \delta \frac{\kappa^2}{2} z^2 dA dx = \delta \int_0^L \frac{M_{xx} \kappa}{2} dx$$

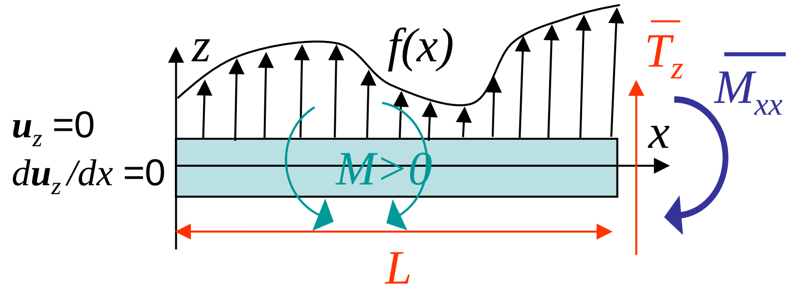
    - Work variation of external forces

$$\delta W_{\text{ext}} = \int_0^L f(x) \delta \boldsymbol{u}_z dx + \bar{T}_z \delta \boldsymbol{u}_z \Big]_0^L - \bar{M}_{xx} \frac{\partial \delta \boldsymbol{u}_z}{\partial x} \Big|_0^L$$

$$\Longrightarrow \int_0^L \frac{1}{2} E I \left(\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \right)^2 dx = \int_0^L f(x) \boldsymbol{u}_z dx + \bar{T}_z \boldsymbol{u}_z \Big]_0^L - \bar{M}_{xx} \frac{\partial \boldsymbol{u}_z}{\partial x} \Big|_0^L$$

- Bending approximation: deflection equations with energy method (2)
  - Energy conservation

$$\int_0^L \frac{1}{2} EI \left(\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)^2 dx = \int_0^L f(x)\, \boldsymbol{u}_z dx + \left.\bar{T}_z \boldsymbol{u}_z\right]_0^L - \left.\bar{M}_{xx} \frac{\partial \boldsymbol{u}_z}{\partial x}\right|_0^L$$



- Integration by parts of the internal energy variation

$$\delta E_{\text{int}} = \int_0^L EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \frac{\partial^2 \delta \boldsymbol{u}_z}{\partial x^2} dx = \left[EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \frac{\partial \delta \boldsymbol{u}_z}{\partial x}\right]_0^L - \int_0^L \frac{\partial}{\partial x}\left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) \frac{\partial \delta \boldsymbol{u}_z}{\partial x} dx$$

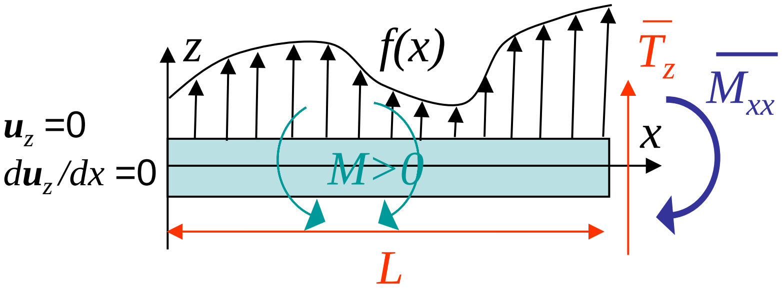
$$\delta E_{\text{int}} = \left[EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2} \frac{\partial \delta \boldsymbol{u}_z}{\partial x}\right]_0^L - \left[\frac{\partial}{\partial x}\left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) \delta \boldsymbol{u}_z\right]_0^L + \int_0^L \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) \delta \boldsymbol{u}_z dx$$

- Work variation of external forces

$$\delta W_{\text{ext}} = \int_0^L f(x)\, \delta \boldsymbol{u}_z dx + \left.\bar{T}_z \delta \boldsymbol{u}_z\right]_0^L - \left.\bar{M}_{xx} \frac{\partial \delta \boldsymbol{u}_z}{\partial x}\right]_0^L$$

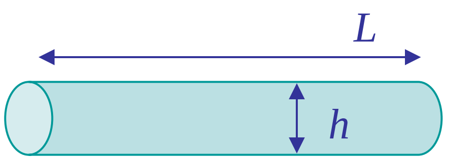
- Bending approximation: deflection equations with energy method (3)
  - Energy conservation (2)
    - As $\delta \boldsymbol{u}_z$ is arbitrary:

      ⟹ Euler-Bernoulli equations

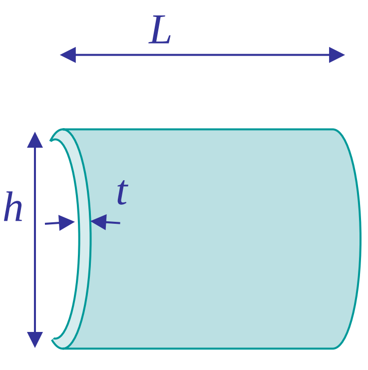


    - $$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f(x)$$ on [0, L] &
    - $$\begin{cases} -\dfrac{\partial}{\partial x}\left(EI\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\Bigg|_{0,\,L} = \bar{T}_z\big|_{0,\,L} \\ -EI\dfrac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\Bigg|_{0,\,L} = \bar{M}_{xx}\big|_{0,\,L} \end{cases}$$
    - Remark: if displacement or rotation constrained at x=0 or $x=L$
      - $\delta \boldsymbol{u}_z$ =0 is no longer arbitrary at this point
      - The boundary condition
        - » Becomes $\boldsymbol{u}_z$ = value or/and $\boldsymbol{u}_{z,x}$ = value
        - » Instead of being on shear or/and couple

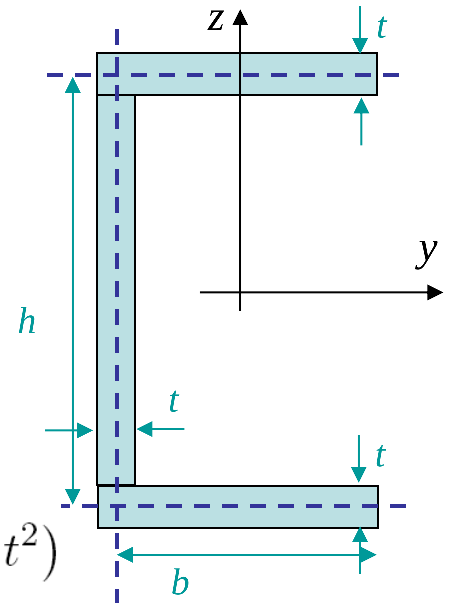
- Bending approximation: shearing
  - As
    - $f_z(x) = -\partial_x T_z = -\partial_{xx} M_y$
    - $f_y(x) = -\partial_x T_y = \partial_{xx} M_z$
  - There are no shearing loads only if the bending moment is constant
    - Cross section remains plane only if constant bending moments
    - Beam with cross-section dimensions << L
      - If bending variation is reduced (Saint-Venant)
      - Shear stress $O(T/bh)$ ~ bending stress x h/L
      - Shear stress can be neglected



    - For thin-walled cross sections (as plates/shells)
      - Shear stress $O(T/th)$ cannot be neglected
      - If bending variation is reduced (Saint-Venant)
      - ⇒ Deflection is primarily due to bending strain



- Second moments of area for thin-walled sections
  - Thin walled section
    - Thickness t << cross-section dimensions
    - Example I_{yy} of a U-thin walled cross-section

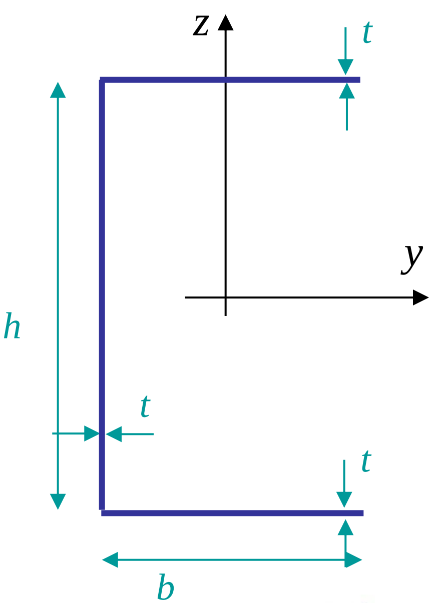


$$I_{yy} = 2\left(\frac{\left(b+\frac{t}{2}\right)t^3}{12} + \left(b+\frac{t}{2}\right)t\frac{h^2}{4}\right) + \frac{t\left(h-t\right)^3}{12}$$

$$\Longrightarrow I_{yy} = \frac{bth^2}{2} + \frac{th^3}{12} + \mathcal{O}\left(bt^3\right) + \mathcal{O}\left(ht^3\right) + \mathcal{O}\left(t^4\right) + \mathcal{O}\left(h^2t^2\right)$$

$$\Longrightarrow I_{yy} \simeq \frac{bth^2}{2} + \frac{th^3}{12}$$

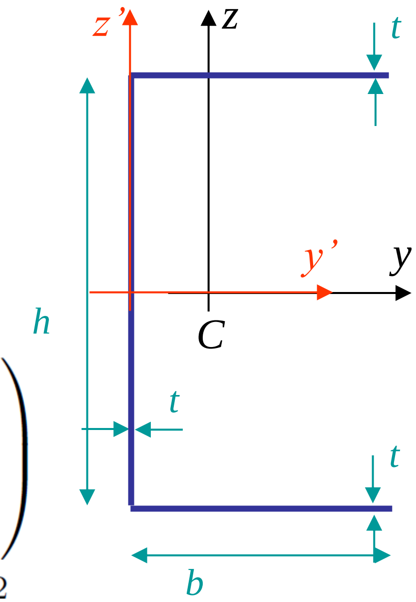
    - This result is obtained directly if the section is represented as a single line



$$I_{yy} = t\int_l z^2 dl = t\left(b\frac{h^2}{4} + \frac{h^3}{12} + b\frac{h^2}{4}\right) = \frac{bth^2}{2} + \frac{th^3}{12}$$

$$\Longrightarrow I_{yy} = \frac{th^3}{12}\left[1 + \frac{6b}{h}\right]$$

- Second moments of area for thin-walled sections (2)
  - Centroid $y'_C = \frac{2t\frac{b^2}{2}}{2bt + th} = \frac{b}{2 + \frac{h}{b}}$ & $z'_C = 0$
  - Second moments of area
    - $I_{yy} = \frac{th^3}{12}\left[1 + \frac{6b}{h}\right]$ & $I_{zz} = t\int_l y^2 dl$



$$\Rightarrow I_{zz} = t\left(2\frac{\left(b - \frac{b}{2+\frac{h}{b}}\right)^3 + \left(\frac{b}{2+\frac{h}{b}}\right)^3}{3} + h\left(\frac{b}{2+\frac{h}{b}}\right)^2\right)$$

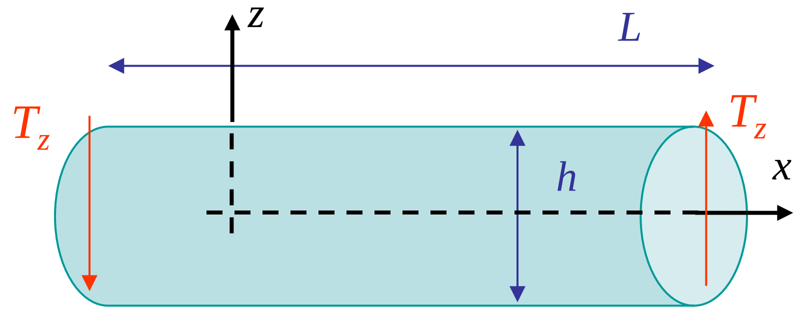
$$\Rightarrow I_{zz} = \frac{2tb^3}{3} - \frac{2tb^3}{2+\frac{h}{b}} + \frac{2tb^3}{\left(2+\frac{h}{b}\right)^2} + ht\left(\frac{b}{2+\frac{h}{b}}\right)^2$$

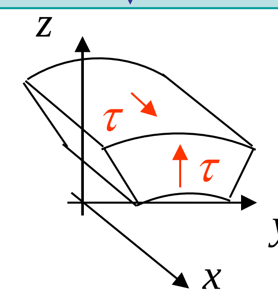
$$\Rightarrow I_{zz} = \frac{2tb^3}{12} + 2tb\left(\frac{b^2}{4} - \frac{b^2}{2+\frac{h}{b}} + \frac{b^2}{\left(2+\frac{h}{b}\right)^2}\right) + ht\left(\frac{b}{2+\frac{h}{b}}\right)^2$$

$$\Rightarrow I_{zz} = \frac{2tb^3}{12} + 2tb\left(\frac{b}{2} - \frac{b}{2+\frac{h}{b}}\right)^2 + ht\left(\frac{b}{2+\frac{h}{b}}\right)^2$$

- $I_{yz} = t\int_l yz\,dl$ null as Cy is a symmetry axis

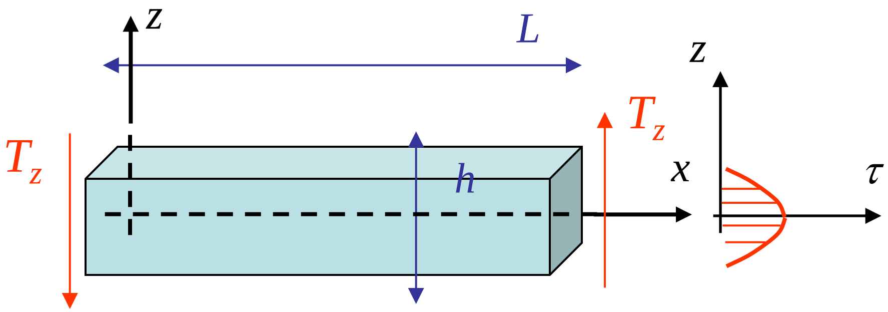
- Shear stress
  - A naïve solution is a uniform stress
    - $\tau = \boldsymbol{\sigma}_{xz} = \frac{T_z}{A}$



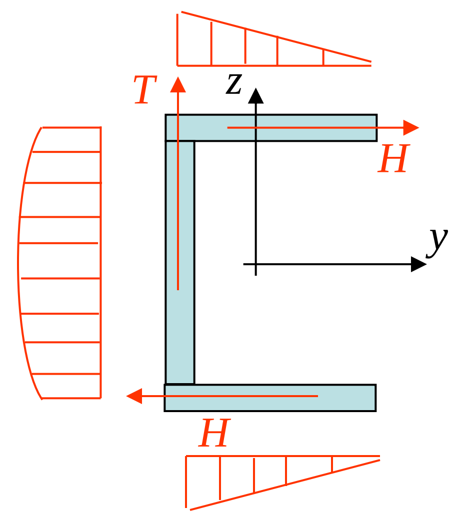


    - By reciprocity this would lead to $\boldsymbol{\sigma}_{zx} = \tau$
    - Impossible as the surface is stress-free
  - So the shear stress has to be tangent to the contour

For thick cross section

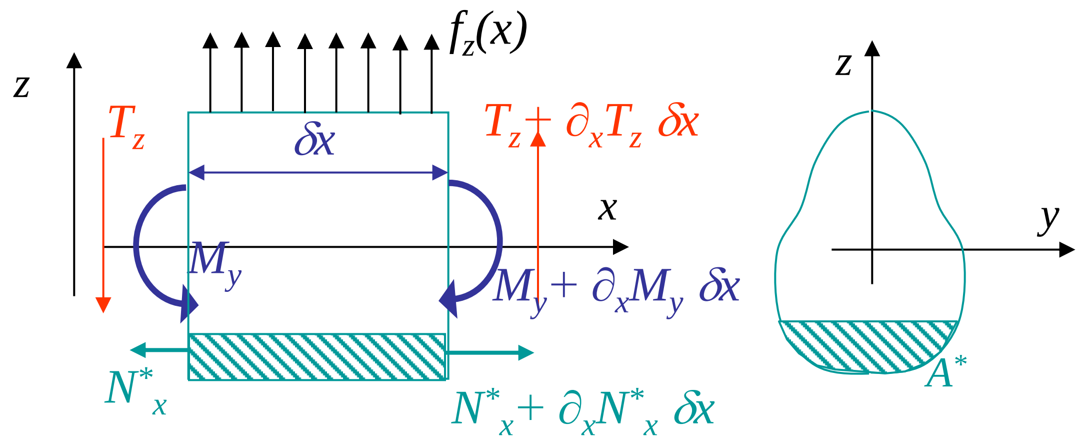


For thin-walled section



  - How to compute these profiles?

- Assumptions
  - Thick section
  - Symmetrical beam
    - $M_y = ||\boldsymbol{M}_{xx}||$
    - $M_z = 0$



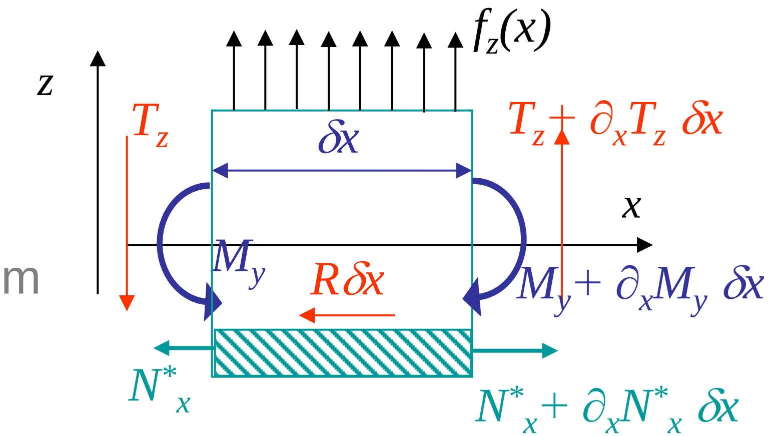
- Shear stress
  - From equilibrium (previous slide): $T_z = \partial_x M_y$
  - Let us consider the lower part of the beam (cross section A^*)
    - Normal force acting on this lower part extremities

$$\begin{cases} N_x^* = \int_{A^*} \boldsymbol{\sigma} dA = \dfrac{M_y}{I_{yy}} \int_{A^*} z dA \\ N_x^* + \partial_x N_x^* \delta x = \int_{A^*} (\boldsymbol{\sigma} + \partial_x \boldsymbol{\sigma} \delta x)\, dA = \dfrac{M_y + \partial_x M_y \delta x}{I_{yy}} \int_{A^*} z dA \end{cases}$$

$$\Longrightarrow \partial_x N_x^* = \frac{\partial_x M_y}{I_{yy}} \int_{A^*} z dA = \frac{T_z}{I_{yy}} \int_{A^*} z dA = \frac{T_z S_n(z)}{I_{yy}}$$

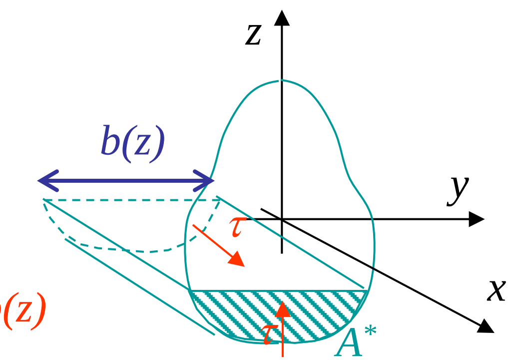
    - With the reduced moment of area $S_n(z) = \int_{A^*} z dA$

- Shear stress (2)
  - We found $\partial_x N_x^* = \dfrac{T_z S_n(z)}{I_{yy}}$



  - But lower part of the beam is at equilibrium
  - So the upper part should apply
    - On the lower part
    - A force $R\delta x$
    - With $R\delta x = \partial_x N_x^* \delta_x = \dfrac{T_z S_n(z)}{I_{yy}} \delta x$

$$\Longrightarrow R = \frac{T_z S_n(z)}{I_{yy}}$$



  - Shear stress
    - On the cut surface, assuming uniform shear stress on $b(z)$

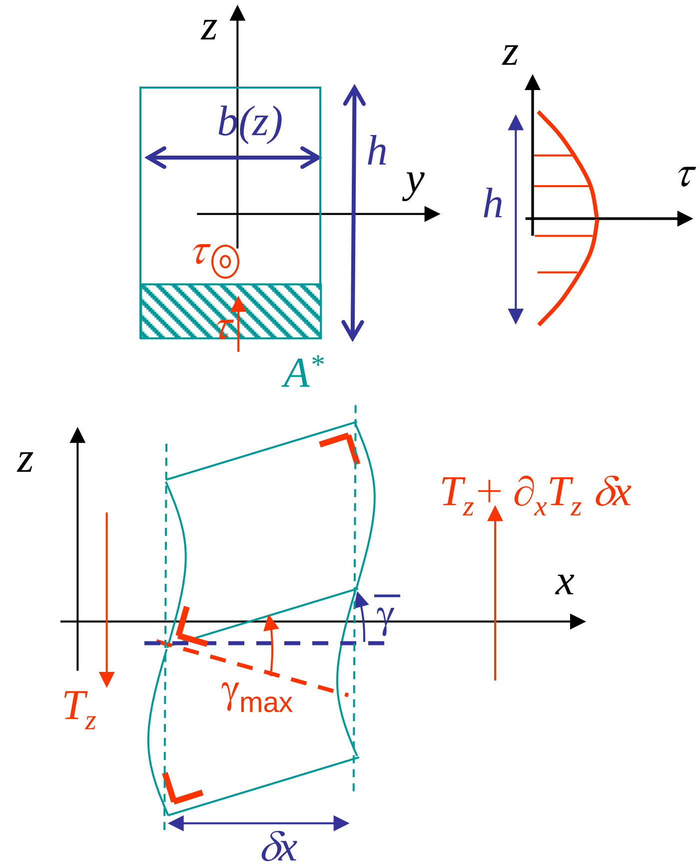
$$\boldsymbol{\sigma}_{xz} = \boldsymbol{\sigma}_{zx} = \tau = -\frac{R\delta x}{b(z)\,\delta x} = -\frac{T_z S_n(z)}{I_{yy} b(z)}$$

$$\Longrightarrow \boldsymbol{\sigma}_{zx} = -\frac{T_z S_n(z)}{I_{yy} b(z)}$$

- Deformation
  - Section shear strain in linear elasticity
    - $\gamma = 2\boldsymbol{\varepsilon}_{xz} = \frac{\boldsymbol{\sigma}_{xz}}{\mu} = \frac{\tau}{\mu}$
    - But shear stress is not uniform

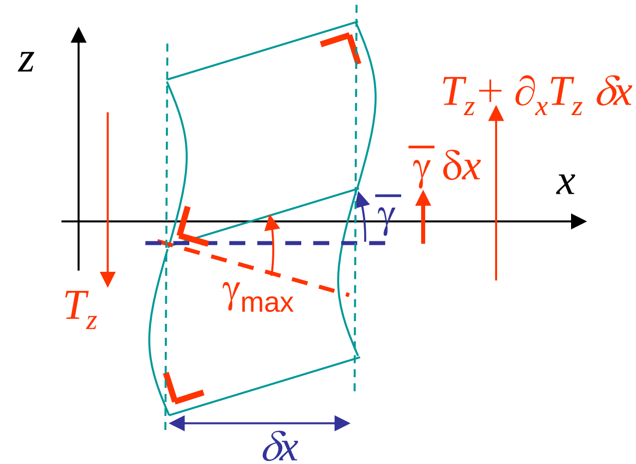
$$\boldsymbol{\sigma}_{zx} = -\frac{T_z S_n(z)}{I_{yy} b(z)}$$

      - $\gamma = 0$ on top and bottom surfaces
      - $\gamma_{\max} = \frac{\tau_{\max}}{\mu}$ at neutral axis
    - The average shear strain $\bar{\gamma}$ is defined as
      - The angle of the deformed neutral axis with its original direction



- Deformation (2)
  - Average $\bar{\gamma}$ obtained by energy method
    - Work of external forces
      - $\delta^2 W_{\text{ext}} = T_z \delta\bar{\gamma}\delta x$
      - If the shear stress was uniform one would have $\bar{\gamma} = \dfrac{T_z}{A\mu}$
      - To account for the non-uniformity, a corrected area A' is used $\bar{\gamma} = \dfrac{T_z}{A'\mu}$

$$\Longrightarrow \delta^2 W_{\text{ext}} = T_z \delta \frac{T_z}{A'\mu}\delta x = \delta \frac{T_z^2}{2A'\mu}\delta x$$



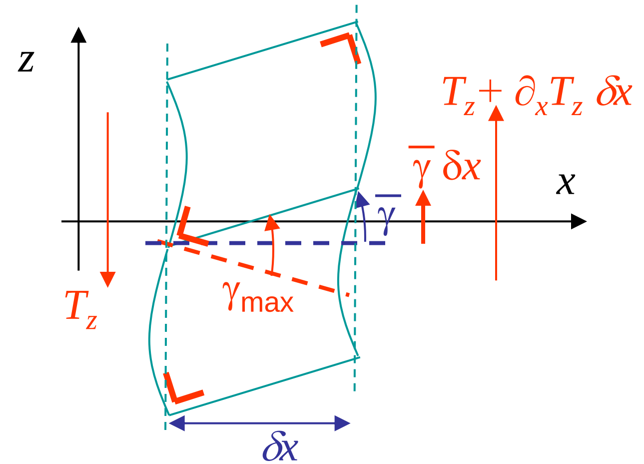
- Internal energy variation
  - $\delta^2 E_{\text{int}} = \displaystyle\int_A \left(\boldsymbol{\sigma}_{xz}\delta\boldsymbol{\varepsilon}_{xz} + \boldsymbol{\sigma}_{xz}\delta\boldsymbol{\varepsilon}_{xz}\right) dA\delta x = \int_A \frac{1}{\mu}\boldsymbol{\sigma}_{xz}\delta\boldsymbol{\sigma}_{xz} dA\delta x$
  - As $\boldsymbol{\sigma}_{zx} = -\dfrac{T_z S_n(z)}{I_{yy} b(z)}$

$$\Longrightarrow \delta^2 E_{\text{int}} = \int_A \frac{1}{2\mu}\delta\boldsymbol{\sigma}_{xz}^2 dA\delta x = \frac{1}{2}\delta\int_A \frac{T_z^2 S_n^2}{\mu I_{yy}^2 b^2} dA\delta x$$

- Deformation (3)
  - Average $\bar{\gamma}$ obtained by energy method (2)
    - Work of external forces = Internal energy variation

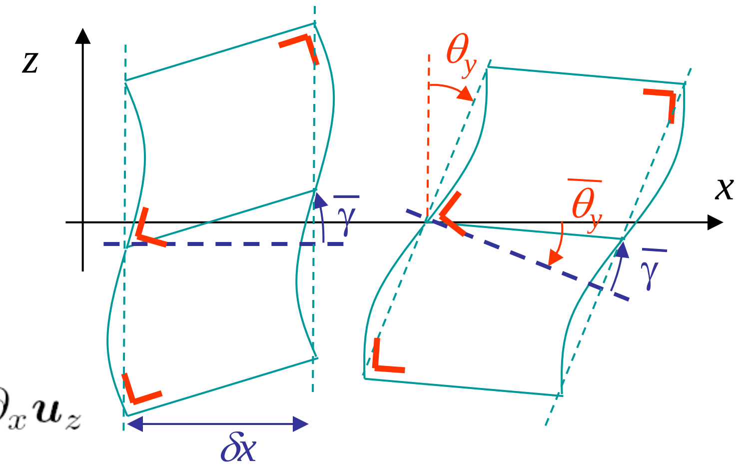
$$\delta^2 E_{\text{int}} = \frac{1}{2}\delta \int_A \frac{T_z^2 S_n^2}{\mu I_{yy}^2 b^2} dA \delta x = \delta \frac{T_z^2}{2A'\mu}\delta x = \delta^2 W_{\text{ext}}$$

$$\Longrightarrow \int_A \frac{S_n^2}{2\mu I_{yy}^2 b^2} dA = \frac{1}{2A'\mu}$$



  - Eventually
    - Average shear strain $\bar{\gamma} = \dfrac{T_z}{A'\mu}$
    - With $A' = \dfrac{1}{\int_A \frac{S_n^2}{I_{yy}^2 b^2} dA}$
    - Under the assumption of a constant shear stress on $b(z)$

- Deformation (4)
  - Shearing equations
    - Shearing is the difference between
      - Neutral fiber deflection
      - Orientation of the cross section
        - Angle θ_y

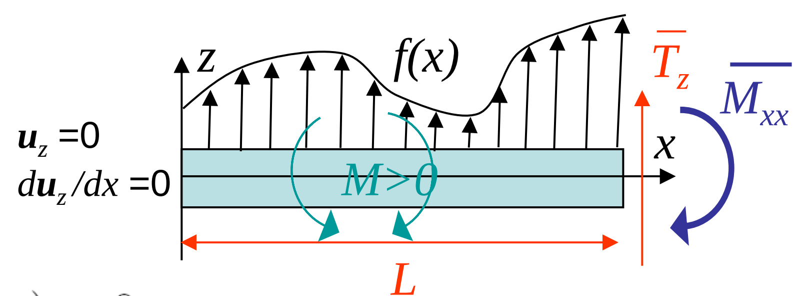


$$\Longrightarrow \bar{\gamma} = 2\bar{\boldsymbol{\varepsilon}}_{xz} = \frac{\partial \boldsymbol{u}_x}{\partial z} + \frac{\partial \boldsymbol{u}_z}{\partial x} = \theta_y + \partial_x \boldsymbol{u}_z$$

    - Curvature is then computed from the variation in cross-section direction

$$\Longrightarrow \kappa = \frac{\partial \theta_y}{\partial x}$$

    - Timoshenko equations on [0, L]



      - As $\partial_x M_{xx} - T_z = 0$

$$\Longrightarrow \frac{\partial}{\partial_x}\left(EI\frac{\partial \theta_y}{\partial x}\right) - \mu A' \left(\theta_y + \partial_x \boldsymbol{u}_z\right) = 0$$

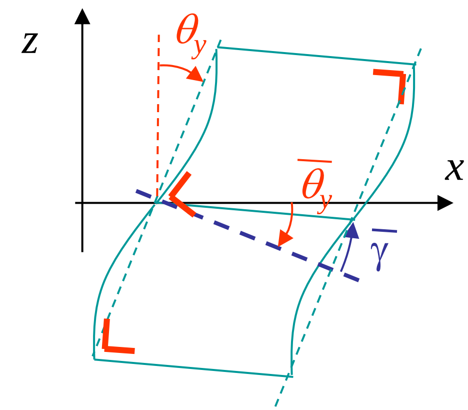
      - As $\partial_x T_z = -f$

$$\Longrightarrow \frac{\partial}{\partial x}\left(\mu A' \left(\theta_y + \partial_x \boldsymbol{u}_z\right)\right) = -f$$

- Bernoulli assumption
  - Timoshenko beam equations

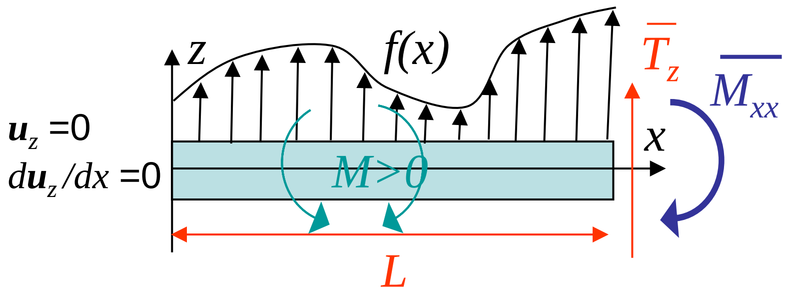
$$\frac{\partial}{\partial_x}\left(EI\frac{\partial\theta_y}{\partial x}\right) - \mu A' (\theta_y + \partial_x \boldsymbol{u}_z) = 0$$

$$\frac{\partial}{\partial x}\left(\mu A' \left(\theta_y + \partial_x \boldsymbol{u}_z\right)\right) = -f$$



  - Euler-Bernoulli equations

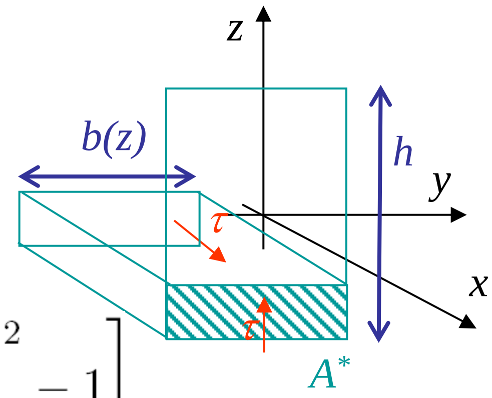
$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f\,(x)$$



    - It has been assumed that

      the cross-section remains planar
      and perpendicular to the neutral axis
    - $\boldsymbol{u}_{z,x} = -\theta_y$
    - Validity: $(EI)/(L^2 A'\mu) << 1$

- Example: shear stress for a rectangular section

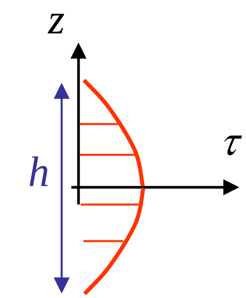


  – Using $\boldsymbol{\sigma}_{zx} = -\frac{T_z S_n(z)}{I_{yy} b(z)}$

    - Reduced moment of area

$$S_n(z) = \int_{A^*} z dA = b \int_{-\frac{h}{2}}^{z} z' dz' = b \frac{z^2 - \frac{h^2}{4}}{2} = \frac{bh^2}{8} \left[\left(\frac{2z}{h} \right)^2 - 1 \right]$$

    - Shear stress



$$\boldsymbol{\sigma}_{zx} = \frac{bh^2}{8} \left[1 - \left(\frac{2z}{h} \right)^2 \right] \frac{12 T_z}{b^2 h^3} = \frac{3 T_z}{2bh} \left[1 - \left(\frac{2z}{h} \right)^2 \right]$$

    - Maximum shear stress reached at z=0: $\tau_{\max} = \frac{3T_z}{2bh}$

  – Exact solution (elasticity theory):

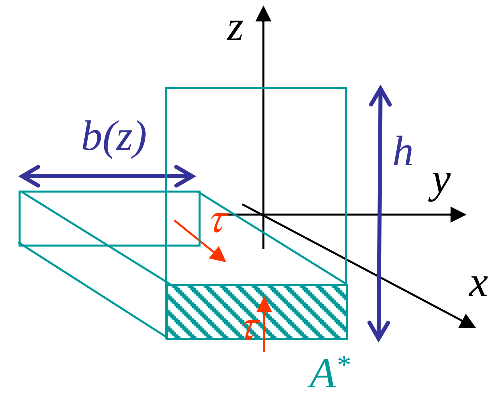
    - Maximum shear stress reached at y=z=0

$$\tau_{\max} = \alpha \frac{3T_z}{2bh}$$

h/b	2	1	1/2
α	1.033	1.126	1.396

    - Shear stress is not uniform on b, but this is a correct approximation for $h>b$

- Example: shear strain for a rectangular section



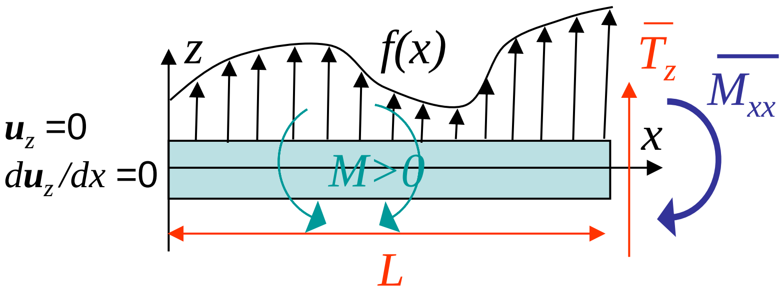
  - Reduced moment of area $S_n(z) = \frac{bh^2}{8}\left[\left(\frac{2z}{h}\right)^2 - 1\right]$
  - Corrected area $A' = \frac{1}{\int_A \frac{S_n^2}{I_{yy}^2 b^2} dA}$

$$\Rightarrow \frac{1}{A'} = \int_A \frac{S_n^2}{I_{yy}^2 b^2} dA = \frac{144}{b^3 h^6} \int_{-\frac{h}{2}}^{\frac{h}{2}} S_n^2 dz = \frac{144}{b^3 h^6} \int_{-\frac{h}{2}}^{\frac{h}{2}} b^2 \frac{\left(z^2 - \frac{h^2}{4}\right)^2}{4} dz$$

$$\Rightarrow \frac{1}{A'} = \frac{36}{bh^6} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(z^4 - \frac{z^2 h^2}{2} + \frac{h^4}{16}\right) dz = \frac{36}{bh^6}\left[\frac{h^5}{80} - \frac{h^5}{24} + \frac{h^5}{16}\right] = \frac{36}{30bh} = \frac{6}{5bh} = \frac{6}{5A}$$

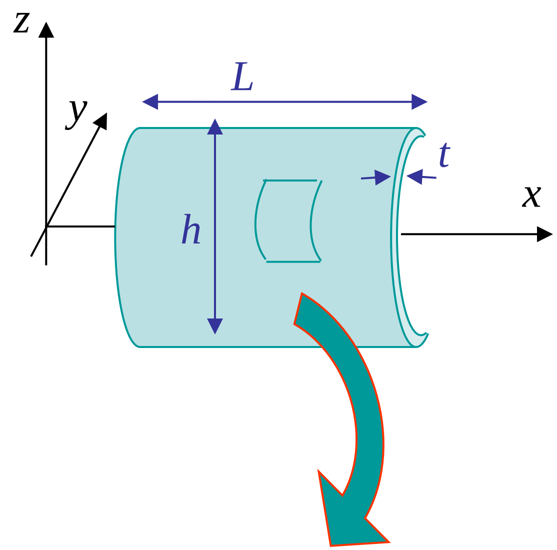
  - So if shear stress is uniform on *b* (ok for *h*>*b*) $A' = \frac{5}{6}A$

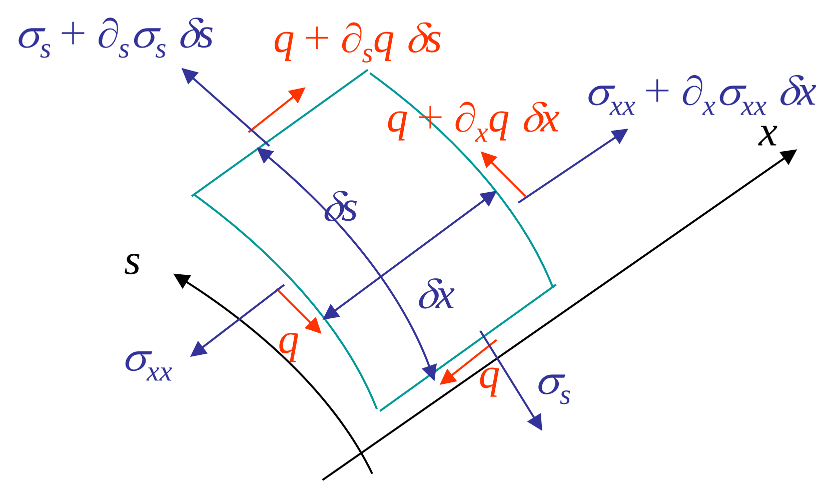
- Extending these equations to unsymmetrical beams
  - The equations remain valid
    - If we are in principal axes as $I_{yz} = 0$ as everything happens as if uncoupled
    - In linear elasticity, as superposition principle holds, the same equations can be derived for $\boldsymbol{u}_y$



  - But if the loads do not pass through a point called the shear center S there will be a twist of the beam
    - This point is difficult to obtained for unsymmetrical non-thin-walled sections
      - This problem is not relevant for light aeronautic structures based on thin-walled sections
    - Only thin-walled sections will be considered

- General relationships
  - Consider thin-walled sections
    - Symmetrical or not
    - Closed or open
  - Assumption
    - Shear is uniform on the thickness t
      - ⟹ Definition of the shear flow $q = t\tau$
  - Balance equations
    - $$\left(\boldsymbol{\sigma}_{xx} + \partial_x \boldsymbol{\sigma}_{xx}\delta x\right) t\delta s - \boldsymbol{\sigma}_{xx} t\delta s + \left(q + \partial_s q\delta s\right)\delta_x - q\delta x = 0$$
      ⟹ $t\partial_x \boldsymbol{\sigma}_{xx} + \partial_s q = 0$
    - $$\left(\boldsymbol{\sigma}_{s} + \partial_s \boldsymbol{\sigma}_{s}\delta s\right) t\delta x - \boldsymbol{\sigma}_{s} t\delta x + \left(q + \partial_x q\delta x\right)\delta_s - q\delta s = 0$$
      ⟹ $t\partial_s \boldsymbol{\sigma}_{s} + \partial_x q = 0$





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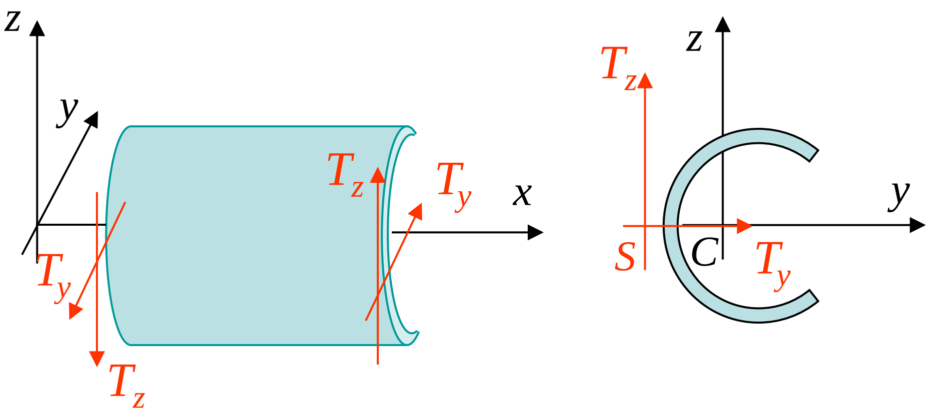
- Shear flow
  - Assumption
    - No twisting of the beam
      ⇒ Shear loads pass through a particular point called shear center S
    - If this is not the case there is a torsion combined to the shearing

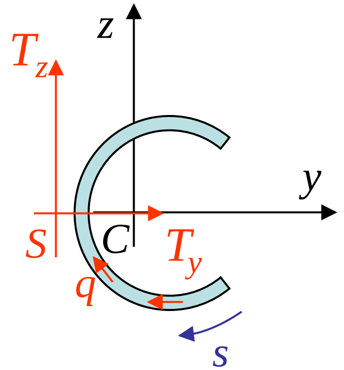


  - Equation $t\partial_x \boldsymbol{\sigma}_{xx} + \partial_s q = 0$
    - Direct bending stress obtained from
    - ! We use pure bending theory: section assumed to remain planar (not true)

$$\left\{\begin{array}{l} \boldsymbol{\sigma}_{xx} = \kappa E z \cos\alpha - \kappa E y \sin\alpha \\ \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \dfrac{\|\boldsymbol{M}_{xx}\|}{\kappa E} \begin{pmatrix} I_{yy} & -I_{yz} \\ -I_{yz} & I_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \end{array}\right.$$

$$\Rightarrow \boldsymbol{\sigma}_{xx} = \kappa E \begin{pmatrix} z & -y \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{1}{I_{yy}I_{zz} - I_{yz}^2} \begin{pmatrix} z & -y \end{pmatrix} \begin{pmatrix} I_{zz} & I_{yz} \\ I_{yz} & I_{yy} \end{pmatrix} \begin{pmatrix} M_y \\ M_z \end{pmatrix}$$

- Shear flow (2)
  - Equations (2)
    - $$\boldsymbol{\sigma}_{xx} = \frac{1}{I_{yy}I_{zz} - I_{yz}^2} \begin{pmatrix} z & -y \end{pmatrix} \begin{pmatrix} I_{zz} & I_{yz} \\ I_{yz} & I_{yy} \end{pmatrix} \begin{pmatrix} M_y \\ M_z \end{pmatrix}$$



$$\Longrightarrow \boldsymbol{\sigma}_{xx} = \frac{\left(I_{zz}M_y + I_{yz}M_z\right) z - \left(I_{yz}M_y + I_{yy}M_z\right) y}{I_{yy}I_{zz} - I_{yz}^2}$$

  - Assuming the variation of moment is larger than the variation of section

$$\partial_x \boldsymbol{\sigma}_{xx} = \frac{\left(I_{zz}M_{y,x} + I_{yz}M_{z,x}\right) z - \left(I_{yz}M_{y,x} + I_{yy}M_{z,x}\right) y}{I_{yy}I_{zz} - I_{yz}^2}$$

- But $t\partial_x \boldsymbol{\sigma}_{xx} + \partial_s q = 0$, $T_z = \partial_x M_y$ & $T_y = -\partial_x M_z$

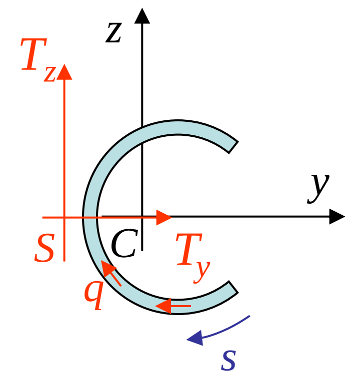
$$\Longrightarrow \partial_s q = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2} tz - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2} ty$$

- Eventually

$$q\left(s\right) - q\left(0\right) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2} \int_0^s t\left(s'\right) z\left(s'\right) ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2} \int_0^s t\left(s'\right) y\left(s'\right) ds'$$

- Shear flow for an open section beam
  - For an open section $q(s) = 0$ as

    on the boundary $\sigma_{xz}(0) = \sigma_{zx}(0) = 0$

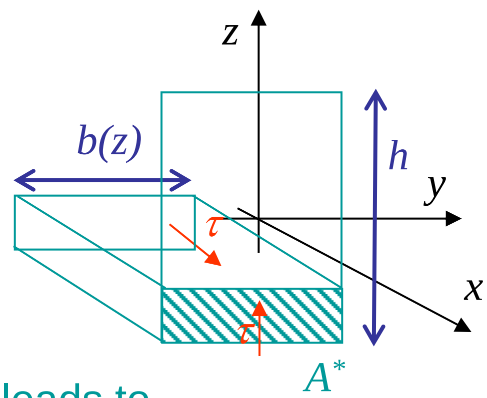


$$\Rightarrow \quad q\left(s\right) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s tz ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s ty ds'$$

- In the principal axes

$$q\left(s\right) = -\frac{T_z}{I_{yy}}\int_0^s tz ds' - \frac{T_y}{I_{zz}}\int_0^s ty ds'$$

- To be compared to the thick cross section approximation



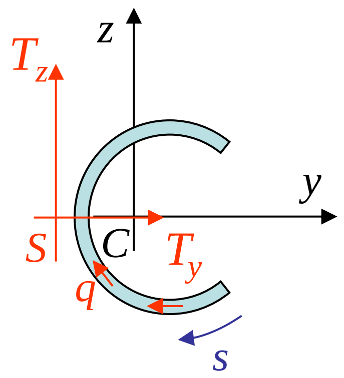
  - $\boldsymbol{\sigma}_{zx} = -\frac{T_z S_n\left(z\right)}{I_{yy} b\left(z\right)}$
  - Has been obtained for symmetrical beam

    and assuming constant stress on b
  - For unsymmetrical beams, the same approximation leads to

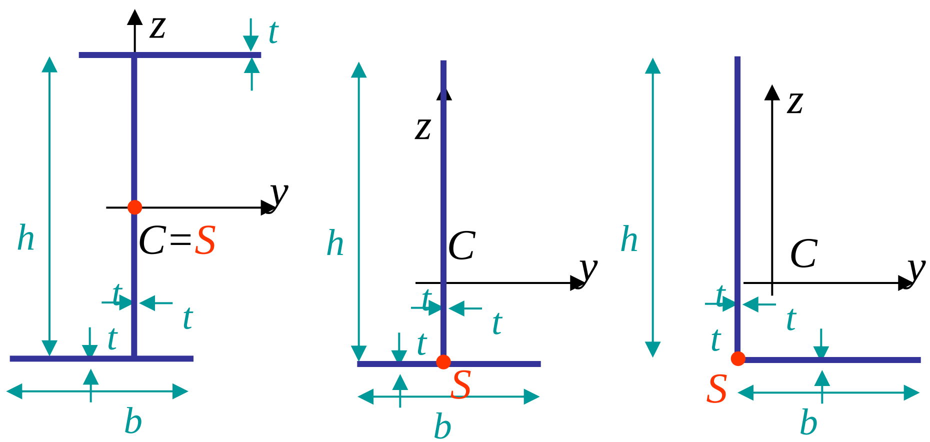
$$b\left(s\right)\tau\left(s\right) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s bz ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s by ds'$$

  - Accurate only for $h/b > 1$

- Shear center
  - Definition:
    - No twisting of the beam if shear loads pass through a particular point called shear center S



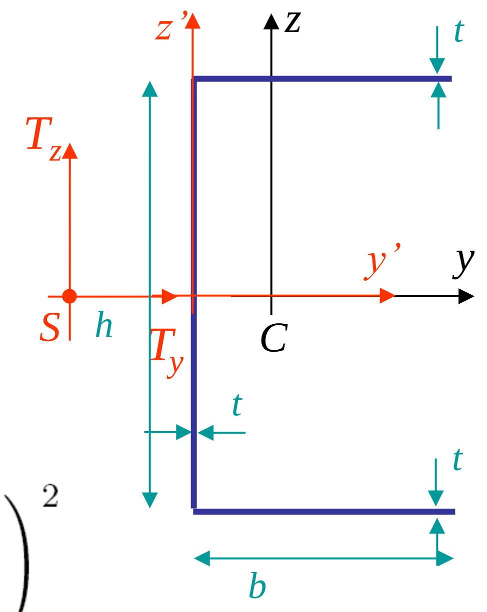
  - Practically
    - Shear loads are represented by loads passing through S and by a torque
  - Location
    - If axis of symmetry: S lies on it
    - If cruciform or angle sections, as q is along these section, S is at the intersection



    - Generally speaking: obtained by considering the moment equilibrium

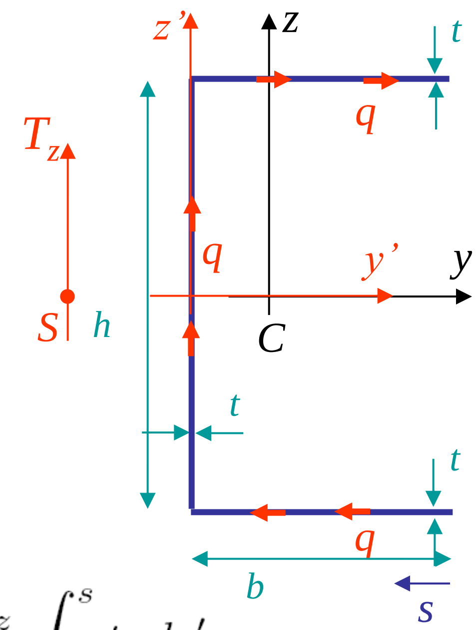
- Example
  - U-thin walled cross section
    - From previous exercise

$$\begin{cases} y'_C = \dfrac{b}{2 + \frac{h}{b}} \\ I_{yy} = \dfrac{th^3}{12}\left[1 + \dfrac{6b}{h}\right] \\ I_{zz} = \dfrac{2tb^3}{12} + 2tb\left(\dfrac{b}{2} - \dfrac{b}{2 + \frac{h}{b}}\right)^2 + ht\left(\dfrac{b}{2 + \frac{h}{b}}\right)^2 \\ I_{yz} = 0 \end{cases}$$



  - Location of the shear center S?

- Location of the shear center
  - By symmetry:
    - On Cy
  - Horizontal location?
    - Consider a shearing T_z (T_y is useless as we know the vertical location)
    - Ensure moment equilibrium
  - Shear flow



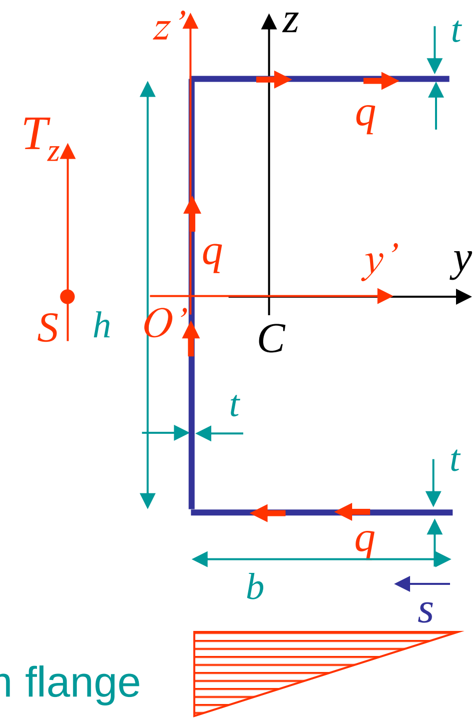
- $q(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s tz ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s ty ds'$
- As $I_{yz} = 0 \Longrightarrow q(s) = -\frac{T_z}{I_{yy}}\int_0^s tz ds'$
- With $I_{yy} = \frac{th^3}{12}\left[1 + \frac{6b}{h}\right]$

$$\Longrightarrow q(s) = -\frac{12T_z}{h^3\left(1+\frac{6b}{h}\right)}\int_0^s z ds'$$

- Shear flow
  - Shear flow on bottom flange
    - As $q(s) = -\frac{12T_z}{h^3\left(1+\frac{6b}{h}\right)} \int_0^s z ds'$

$$\Longrightarrow \quad q(s) = -\frac{12T_z}{h^3\left(1+\frac{6b}{h}\right)} \left(-\frac{h}{2}\right)(b-y')$$

$$\Longrightarrow \quad q(y') = \frac{6T_z}{h^2\left(1+\frac{6b}{h}\right)}(b-y')$$



- Maximum shear flow $q(y'=0) = \frac{6T_z b}{h^2\left(1+\frac{6b}{h}\right)}$
- Moment around O' induced by the shearing of bottom flange

$$+M_{O'} = -\frac{h}{2}\frac{q(y'=0)\,b}{2} = -\frac{3T_z b^2}{2h\left(1+\frac{6b}{h}\right)}$$

- Shear flow (2)
  - Shear flow on the web
    - As $q(s) = -\frac{12T_z}{h^3\left(1+\frac{6b}{h}\right)}\int_0^s z ds'$

$$\Rightarrow q(z') = \frac{6T_z b}{h^2\left(1+\frac{6b}{h}\right)} - \frac{12T_z}{h^3\left(1+\frac{6b}{h}\right)}\int_{-\frac{h}{2}}^{z'} z'' dz''$$

$$\Rightarrow q(z') = \frac{6T_z b}{h^2\left(1+\frac{6b}{h}\right)} - \frac{6T_z}{h^3\left(1+\frac{6b}{h}\right)}\left[z'^2 - \frac{h^2}{4}\right]$$

    - "Boundary" shear flow

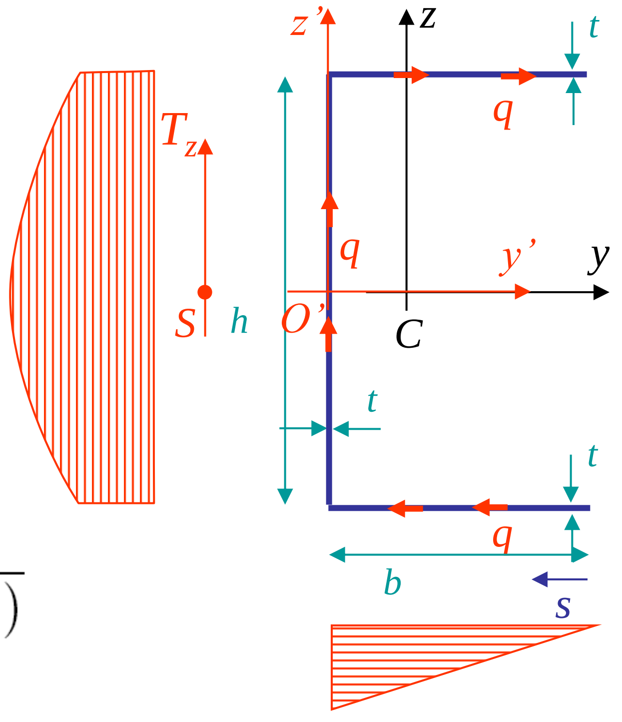
$$q\left(z' = -\frac{h}{2}\right) = q\left(z' = \frac{h}{2}\right) = \frac{6T_z b}{h^2\left(1+\frac{6b}{h}\right)}$$

    - Maximum shear flow

$$q(z'=0) = \frac{6T_z b}{h^2\left(1+\frac{6b}{h}\right)} + \frac{3T_z}{2h\left(1+\frac{6b}{h}\right)}$$

    - Moment around O' induced by the shearing of the web

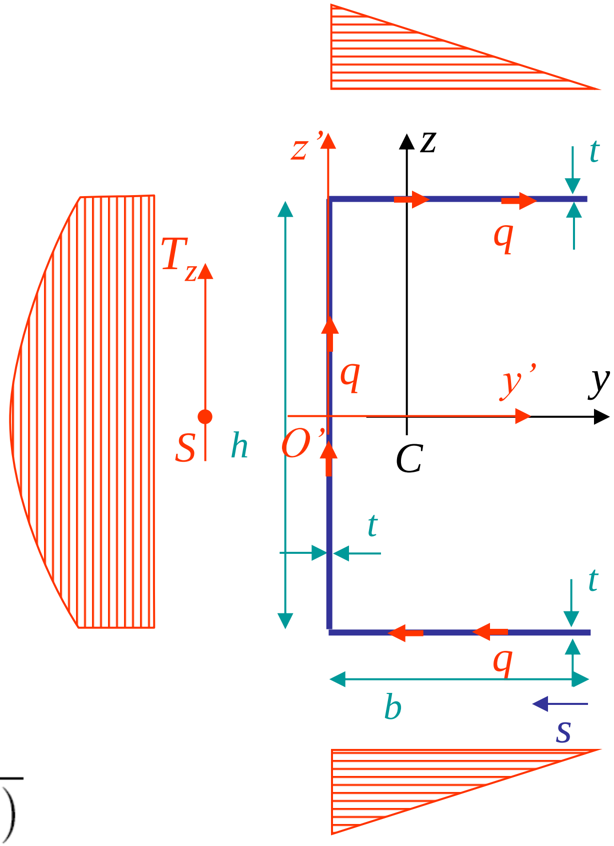
$$+\circlearrowright M_{O'} = 0$$



- Shear flow (3)
  - Shear flow on top flange
    - As $q(s) = -\frac{12T_z}{h^3\left(1+\frac{6b}{h}\right)}\int_0^s z ds'$

$$\Rightarrow q(y') = \frac{6T_z b}{h^2\left(1+\frac{6b}{h}\right)} - \frac{12T_z}{h^3\left(1+\frac{6b}{h}\right)}\int_0^{y'} \frac{h}{2} dy''$$

$$\Rightarrow q(y') = \frac{6T_z}{h^2\left(1+\frac{6b}{h}\right)}(b-y)$$



    - Moment around O' induced by the shearing of the top flange

$$+M_{O'} = \frac{h}{2}\left(-\frac{q(y'=0)\,b}{2}\right) = -\frac{3T_z b^2}{2h\left(1+\frac{6b}{h}\right)}$$

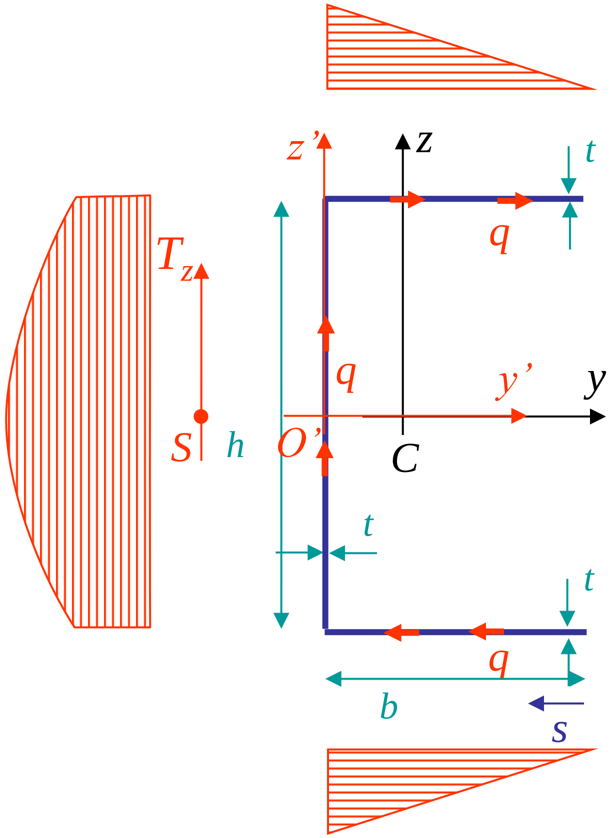
- Shear center
  - Moment due to the shear flow:

$$+M_{O'} = -\frac{3T_z b^2}{h\left(1+\frac{6b}{h}\right)}$$

  - It has to be balanced by the shearing

$$\Rightarrow T_z y'_S = -\frac{3T_z b^2}{h\left(1+\frac{6b}{h}\right)}$$

$$\Rightarrow y'_S = -\frac{3b^2}{h\left(1+\frac{6b}{h}\right)}$$



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