University of Liège
Aerospace & Mechanical Engineering

Aircraft Structures
Aircraft Components – Part II

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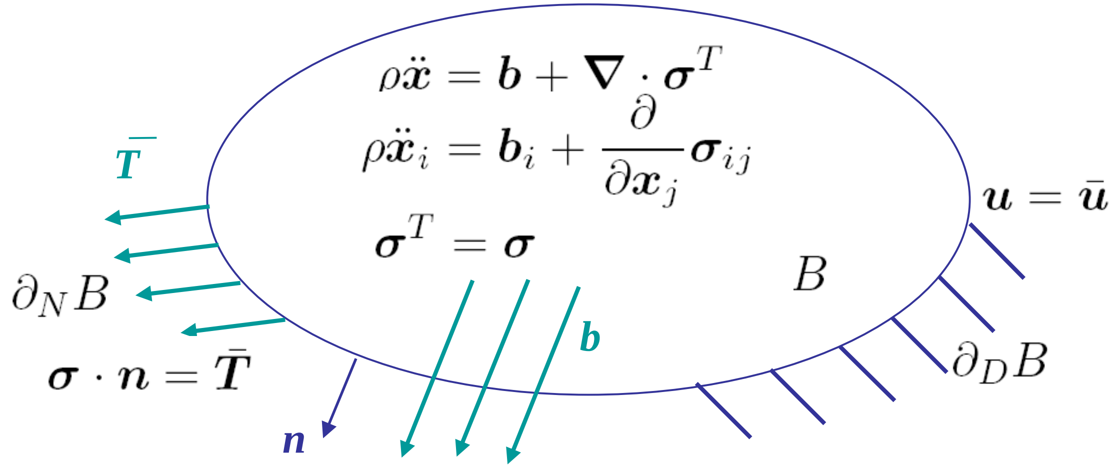
- Balance of body B
  - Momenta balance
    - Linear
    - Angular
  - Boundary conditions
    - Neumann
    - Dirichlet



- Small deformations with linear elastic, homogeneous & isotropic material
  - (Small) Strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2} \left(\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla} \right)$, or $\begin{cases} \boldsymbol{\varepsilon}_{ij} = \frac{1}{2} \left(\frac{\partial}{\partial \boldsymbol{x}_i} \boldsymbol{u}_j + \frac{\partial}{\partial \boldsymbol{x}_j} \boldsymbol{u}_i \right) \\ \boldsymbol{\varepsilon}_{ij} = \frac{1}{2} \left(\boldsymbol{u}_{j,i} + \boldsymbol{u}_{i,j} \right) \end{cases}$
  - Hooke's law $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon}$, or $\boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl} \boldsymbol{\varepsilon}_{kl}$

    with $\mathcal{H}_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)} \delta_{ij}\delta_{kl} + \frac{E}{1+\nu} \left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk} \right)$

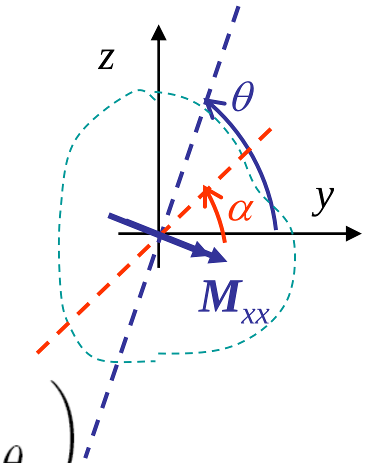
    $\lambda = K - 2\mu/3$ $\quad$ 2μ
  - Inverse law $\boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma}$

    with $\mathcal{G}_{ijkl} = \frac{1+\nu}{E} \left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk} \right) - \frac{\nu}{E} \delta_{ij}\delta_{kl}$

Pure bending: linear elasticity summary

- General expression for unsymmetrical beams
  - Stress $\boldsymbol{\sigma}_{xx} = \kappa E z \cos\alpha - \kappa E y \sin\alpha$

    With $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{\kappa E} \begin{pmatrix} I_{yy} & -I_{yz} \\ -I_{yz} & I_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$

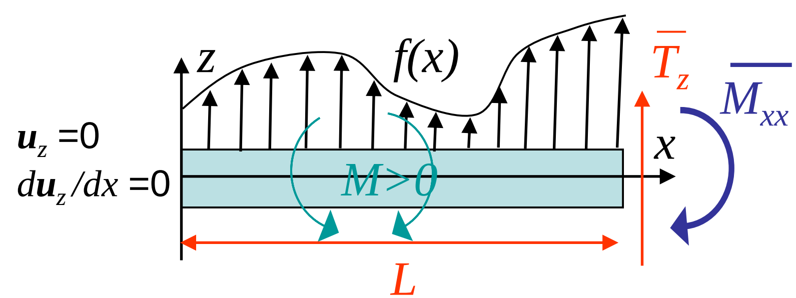


  - Curvature

$$\begin{pmatrix} -\boldsymbol{u}_{z,xx} \\ \boldsymbol{u}_{y,xx} \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{E\left(I_{yy}I_{zz} - I_{yz}I_{yz}\right)} \begin{pmatrix} I_{zz} & I_{yz} \\ I_{yz} & I_{yy} \end{pmatrix} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

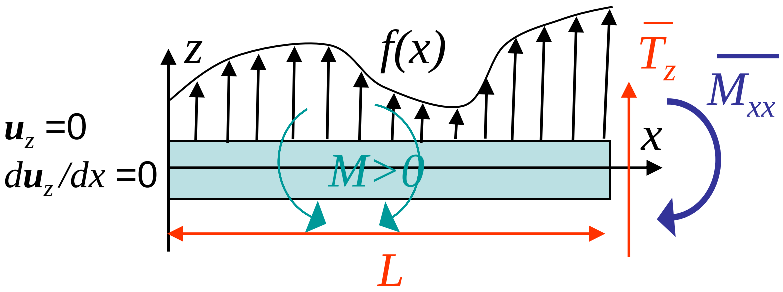
  - In the principal axes $I_{yz} = 0$

- Euler-Bernoulli equation in the principal axis
  - $\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f(x)$ for x in [0 L]
  - BCs $\begin{cases} -\frac{\partial}{\partial x}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\Big|_{0,\,L} = \bar{T}_z\Big|_{0,\,L} \\ -EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\Big|_{0,\,L} = \bar{M}_{xx}\Big|_{0,\,L} \end{cases}$
  - Similar equations for $\boldsymbol{u}_y$

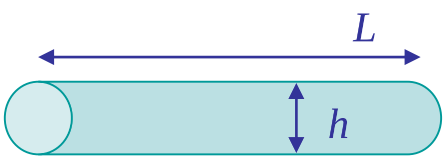


Beam shearing: linear elasticity summary

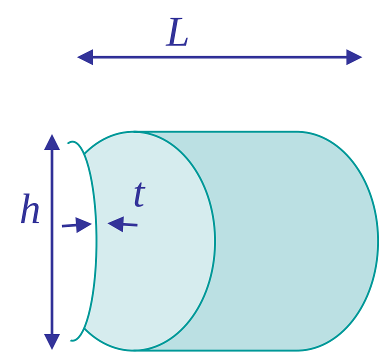
- General relationships
  - $$\begin{cases} f_z(x) = -\partial_x T_z = -\partial_{xx} M_y \\ f_y(x) = -\partial_x T_y = \partial_{xx} M_z \end{cases}$$



- Two problems considered
  - Thick symmetrical section

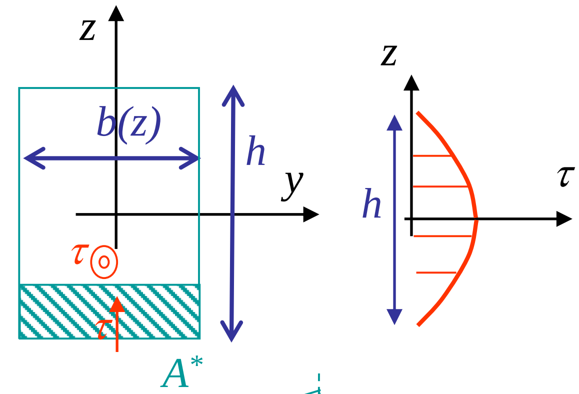


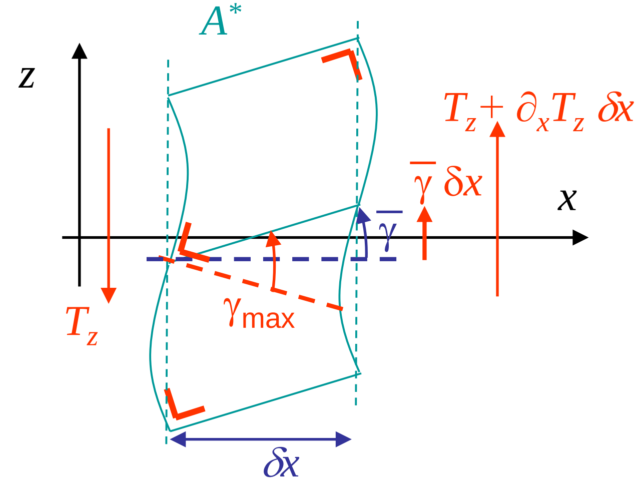
    - Shear stresses are small compared to bending stresses if h/L << 1
  - Thin-walled (unsymmetrical) sections

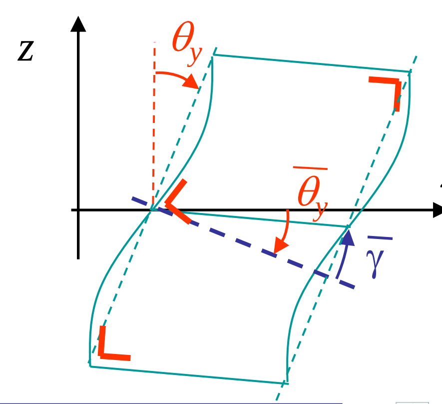


    - Shear stresses are not small compared to bending stresses
    - Deflection mainly results from bending stresses
    - 2 cases
      - Open thin-walled sections
        - » Shear = shearing through the shear center + torque
      - Closed thin-walled sections
        - » Twist due to shear has the same expression as torsion

- Shearing of symmetrical thick-section beams
  - Stress $\boldsymbol{\sigma}_{zx} = -\dfrac{T_z S_n(z)}{I_{yy} b(z)}$
    - With $S_n(z) = \displaystyle\int_{A^*} z\,dA$
    - Accurate only if $h > b$
  - Energetically consistent averaged shear strain
    - $\bar{\gamma} = \dfrac{T_z}{A'\mu}$ with $A' = \dfrac{1}{\int_A \frac{S_n^2}{I_{yy}^2 b^2} dA}$
    - Shear center on symmetry axes
  - Timoshenko equations
    - $\bar{\gamma} = 2\bar{\boldsymbol{\varepsilon}}_{xz} = \dfrac{\partial \boldsymbol{u}_x}{\partial z} + \dfrac{\partial \boldsymbol{u}_z}{\partial x} = \theta_y + \partial_x \boldsymbol{u}_z$ & $\kappa = \dfrac{\partial \theta_y}{\partial x}$
    - On [0 L]: $\begin{cases} \dfrac{\partial}{\partial_x}\left(EI\dfrac{\partial \theta_y}{\partial x}\right) - \mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right) = 0 \\ \dfrac{\partial}{\partial x}\left(\mu A'\left(\theta_y + \partial_x \boldsymbol{u}_z\right)\right) = -f \end{cases}$

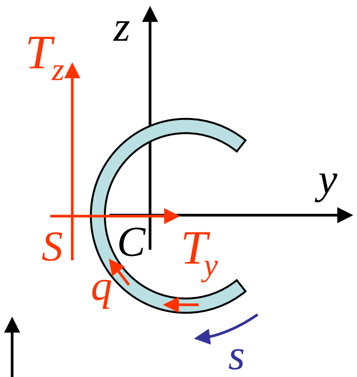


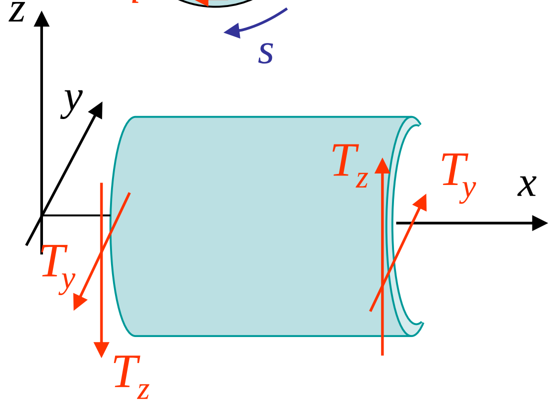


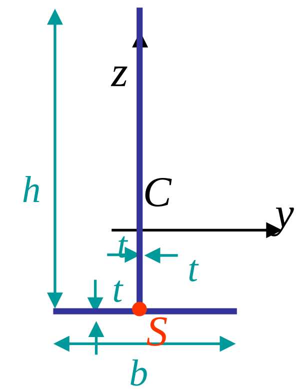


Beam shearing: linear elasticity summary

- Shearing of open thin-walled section beams
  - Shear flow $q = t\tau$
    - $$q(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s tz\,ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s ty\,ds'$$
    - In the principal axes
    $$q(s) = -\frac{T_z}{I_{yy}}\int_0^s tz\,ds' - \frac{T_y}{I_{zz}}\int_0^s ty\,ds'$$
  - Shear center S
    - On symmetry axes
    - At walls intersection
    - Determined by momentum balance
  - Shear loads correspond to
    - Shear loads passing through the shear center &
    - Torque







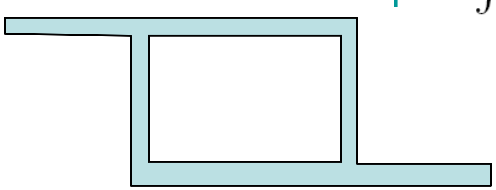
- Shearing of closed thin-walled section beams
  – Shear flow $q = t\tau$
    - $q(s) = q_o(s) + q(0)$
    - Open part (for anticlockwise of q, s)

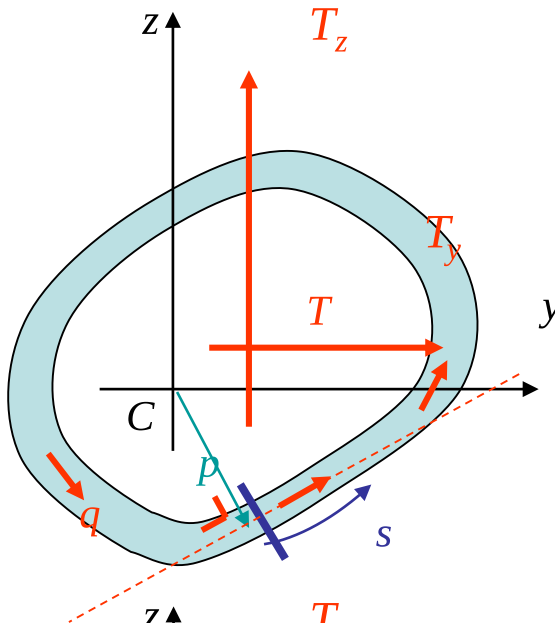
$$q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s t(s')\,z(s')\,ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s t(s')\,y(s')\,ds'$$

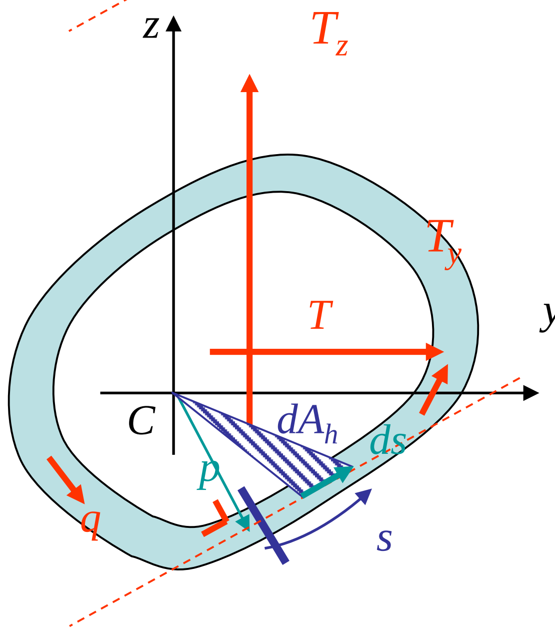
    - Constant twist part

$$q(s=0) = \frac{y_T T_z - z_T T_y - \oint p(s)\,q_o(s)\,ds}{2A_h}$$

    - The $q(0)$ is related to the closed part of the section, but there is a $q_o(s)$ in the open part which should be considered for the shear torque $\oint p(s)\,q_o(s)\,ds$





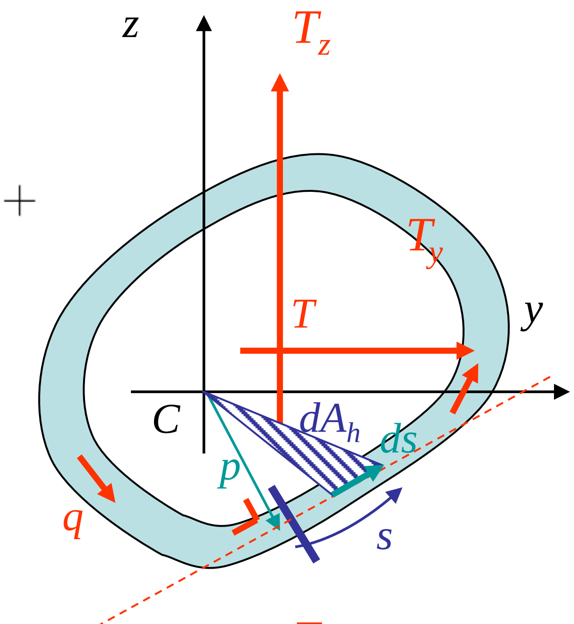


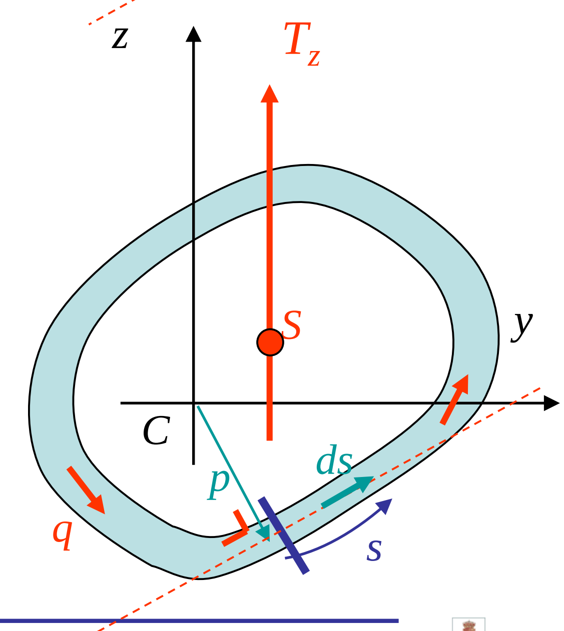
Beam shearing: linear elasticity summary

- Shearing of closed thin-walled section beams
  - Warping around twist center R
    - $$\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \int_0^s \frac{q}{\mu t} ds - \frac{1}{A_h} \oint \frac{q}{\mu t} ds \left\{ A_{Cp}(s) + \frac{z_R\left[y(s) - y(0)\right] - y_R\left[z(s) - z(0)\right]}{2} \right\}$$
    - With $\boldsymbol{u}_x(0) = \dfrac{\oint t \boldsymbol{u}_x(s)\, ds}{\oint t(s)\, ds}$
      - $\boldsymbol{u}_x(0)=0$ for symmetrical section if origin on the symmetry axis
  - Shear center S
    - Compute q for shear passing thought S
    - Use

$$q(s=0) = \frac{y_T T_z - z_T T_y - \oint p(s)\, q_o(s)\, ds}{2A_h}$$

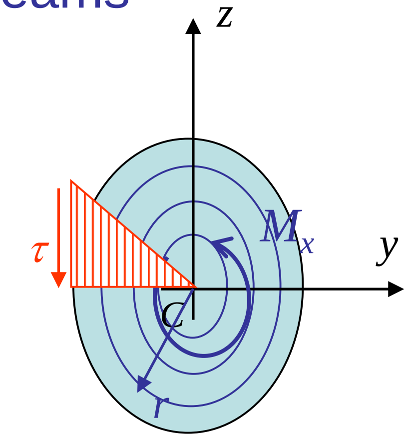
With point S=T

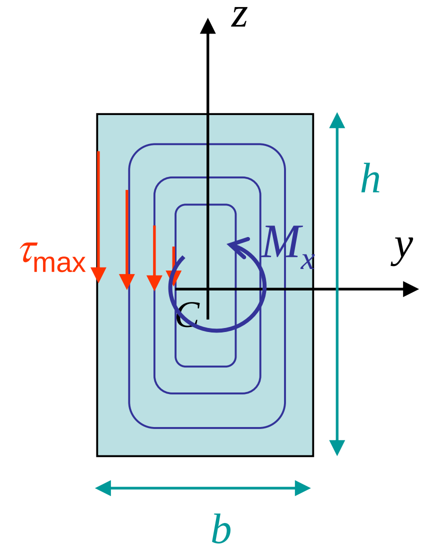




Beam torsion: linear elasticity summary

- Torsion of symmetrical thick-section beams
  - Circular section
    - $\tau = \mu\gamma = r\mu\theta_{,x}$
    - $C = \dfrac{M_x}{\theta_{,x}} = \int_A \mu r^2 dA$
  - Rectangular section
    - $\tau_{\max} = \dfrac{M_x}{\alpha h b^2}$
    - $C = \dfrac{M_x}{\theta_{,x}} = \beta h b^3 \mu$
    - If $h >> b$
      - $\tau_{xy} = 0$ & $\tau_{xz} = 2\mu y\theta_{,x}$
      - $\tau_{\max} = \dfrac{3M_x}{hb^2}$
      - $C = \dfrac{M_x}{\theta_{,x}} = \dfrac{hb^3\mu}{3}$

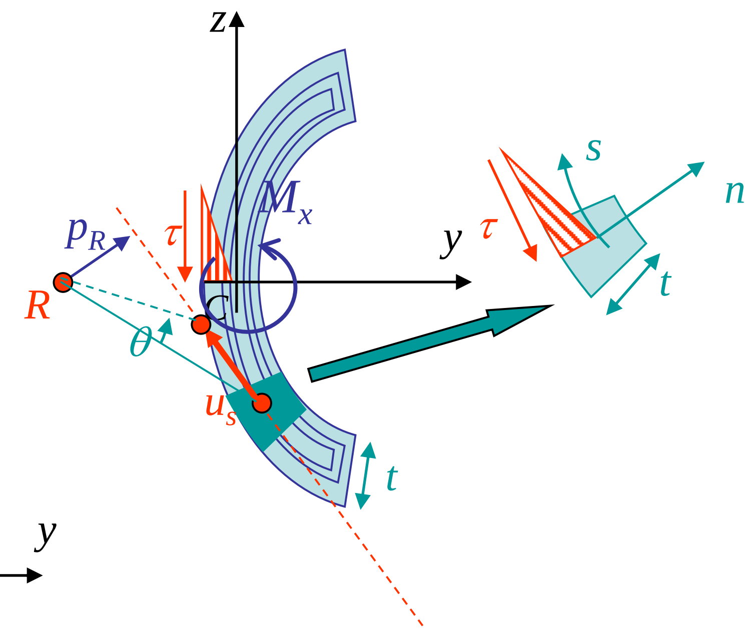




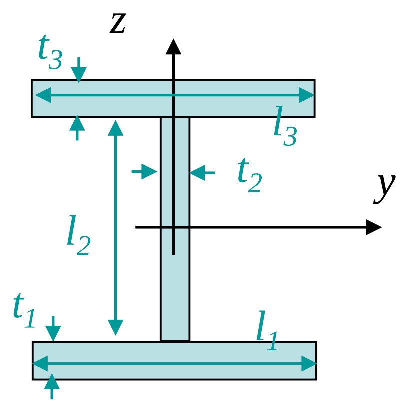
h/b	1	1.5	2	4	∞
α	0.208	0.231	0.246	0.282	1/3
β	0.141	0.196	0.229	0.281	1/3

Beam torsion: linear elasticity summary

- Torsion of open thin-walled section beams
  - Approximated solution for twist rate



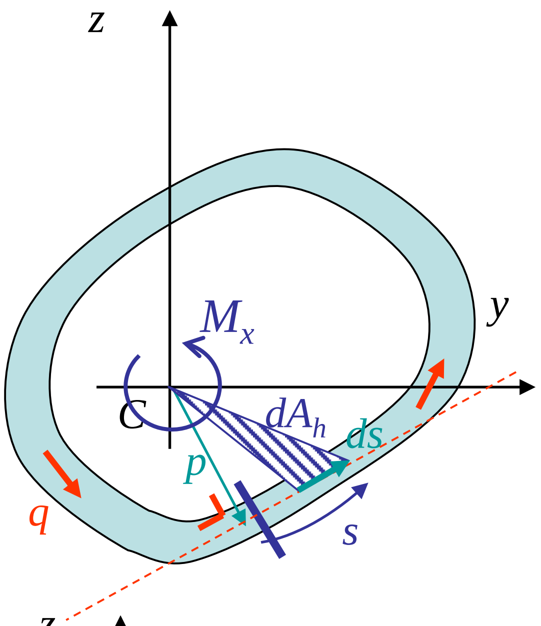
    - Thin curved section
      - $\tau_{xs} = 2\mu n\theta_{,x}$
      - $C = \dfrac{M_x}{\theta_{,x}} = \dfrac{1}{3}\displaystyle\int \mu t^3 ds$
    - Rectangles
      - $\tau_{\max_i} = \mu t_i \theta_{,x}$
      - $\dfrac{M_x}{\theta_{,x}} = \displaystyle\sum_i \dfrac{l_i t_i^3 \mu}{3}$



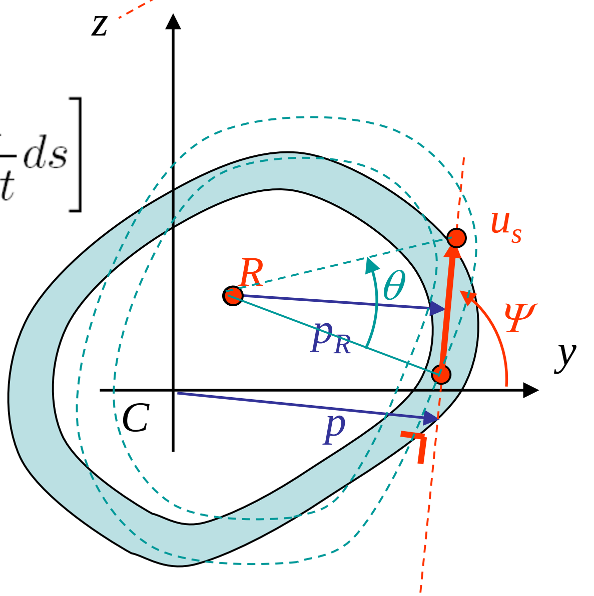
  - Warping of s-axis
    - $\boldsymbol{u}_x^s(s) = \boldsymbol{u}_x^s(0) - \theta_{,x}\displaystyle\int_0^s p_R ds' = \boldsymbol{u}_x^s(0) - 2A_{R_p}(s)\,\theta_{,x}$

- Torsion of closed thin-walled section beams
  - Shear flow due to torsion $M_x = 2A_h q$
  - Rate of twist
    - $\theta_{,x} = \frac{M_x}{4A_h^2} \oint \frac{1}{\mu t} ds$
    - Torsion rigidity for constant μ

$$I_T = \frac{4A_h^2}{\oint \frac{1}{t} ds} \leq I_p = \int_A r^2 dA$$

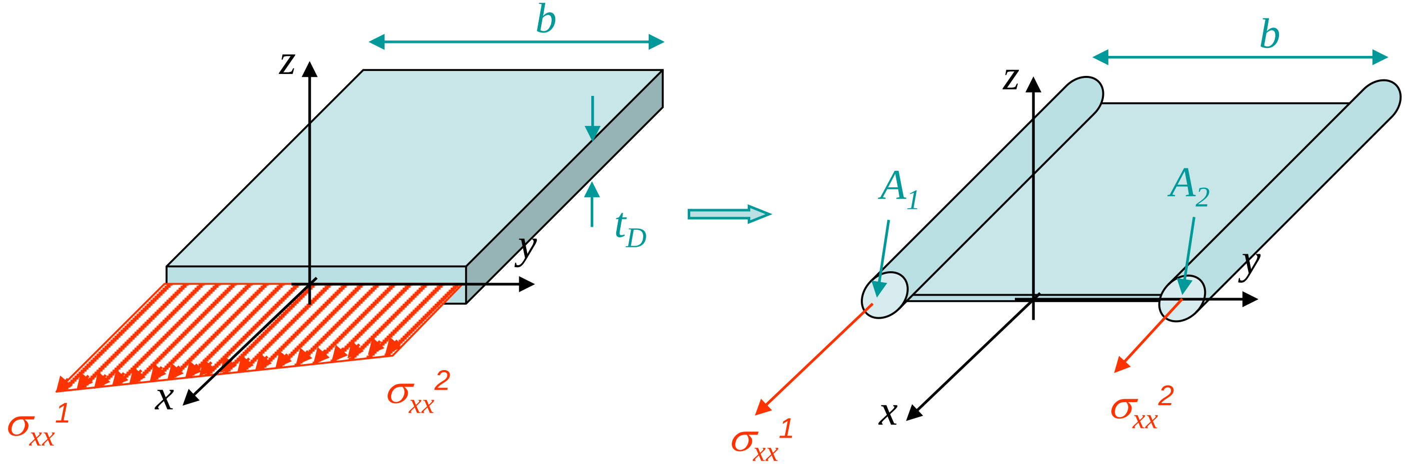


  - Warping due to torsion
    - $\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \frac{M_x}{2A_h}\left[\int_0^s \frac{1}{\mu t} ds - \frac{A_{R_p}(s)}{A_h} \oint \frac{1}{\mu t} ds\right]$
    - A_{Rp} from twist center



- Panel idealization
  - Booms' area **depending on loading**
    - For linear direct stress distribution

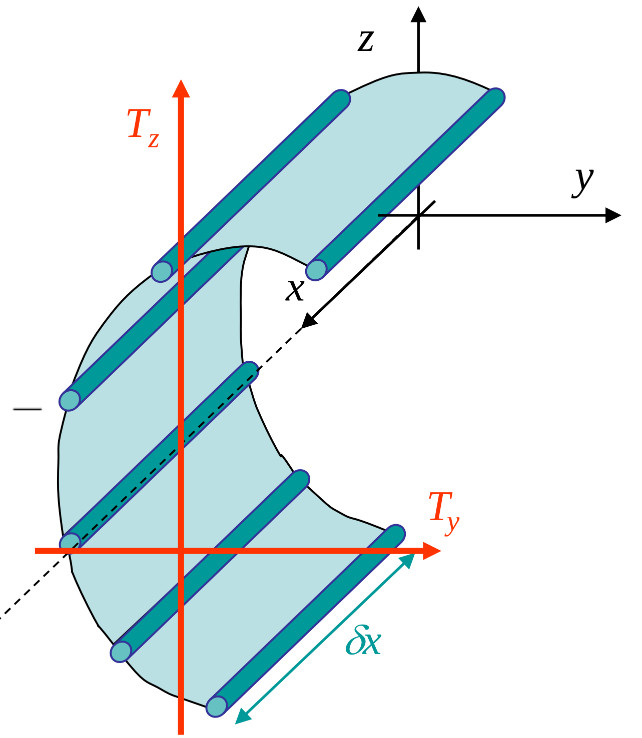
$$\begin{cases} A_1 = \frac{t_D b}{6}\left(2 + \frac{\boldsymbol{\sigma}_{xx}^2}{\boldsymbol{\sigma}_{xx}^1}\right) \\ A_2 = \frac{t_D b}{6}\left(2 + \frac{\boldsymbol{\sigma}_{xx}^1}{\boldsymbol{\sigma}_{xx}^2}\right) \end{cases}$$



Structure idealization summary

- Consequence on bending
  - The position of the neutral axis, and thus the second moments of area
    - Refer to the direct stress carrying area only
    - Depend on the loading case only
- Consequence on shearing
  - Open part of the shear flux
    - Shear flux for open sections

$$q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\sigma} z ds + \sum_{i:\, s_i \le s} z_i A_i\right] - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\sigma} y ds + \sum_{i:\, s_i \le s} y_i A_i\right]$$

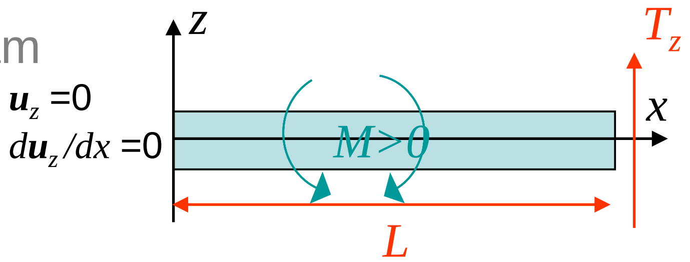


- Consequence on torsion
  - If no axial constraint
    - Torsion analysis does not involve axial stress
    - So torsion is unaffected by the structural idealization

- Virtual displacement
  - In linear elasticity the general formula of virtual displacement reads

$$\int_0^L \int_A \boldsymbol{\sigma}^{(1)} : \boldsymbol{\varepsilon} dA dx = P^{(1)} \Delta_P$$

    - $\boldsymbol{\sigma}^{(1)}$ is the stress distribution corresponding to a (unit) load $P^{(1)}$
    - Δ_P is the energetically conjugated displacement to P in the direction of $P^{(1)}$ that corresponds to the strain distribution $\boldsymbol{\varepsilon}$
  - Example: bending of a semi cantilever beam

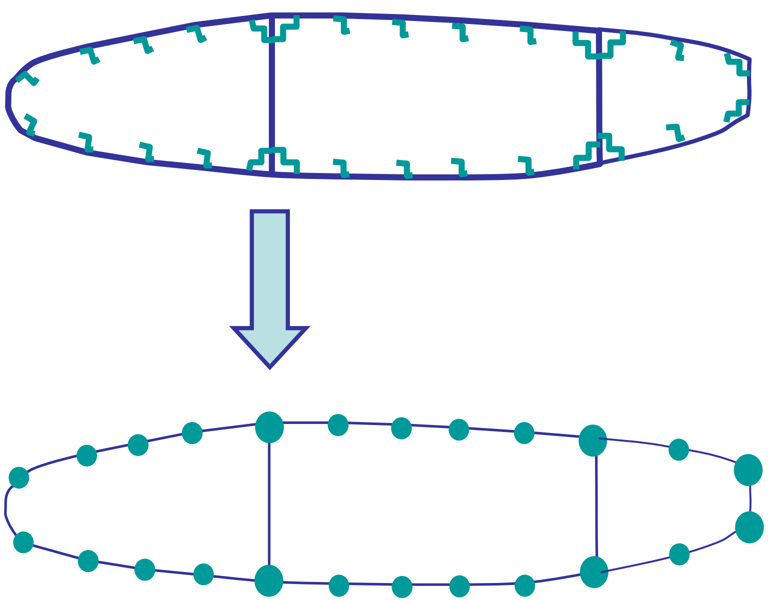


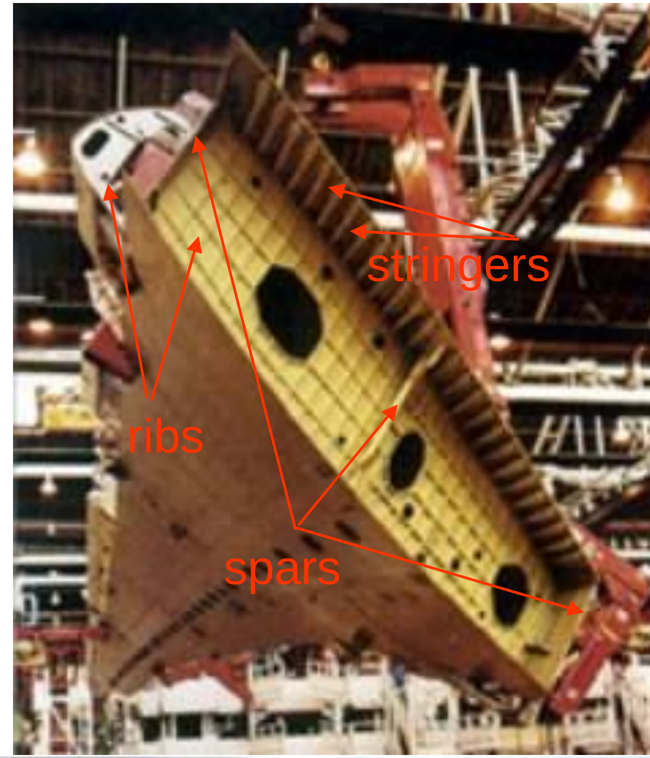
    - $$\int_0^L \int_A \boldsymbol{\sigma}_{xx}^{(1)} \boldsymbol{\varepsilon}_{xx} dA dx = \Delta_P u$$
      - In the principal axes

$$\Delta_P u = \frac{1}{E I_{yy} I_{zz}} \int_0^L \left\{ I_{zz} M_y^{(1)} M_y + I_{yy} M_z^{(1)} M_z \right\} dx$$

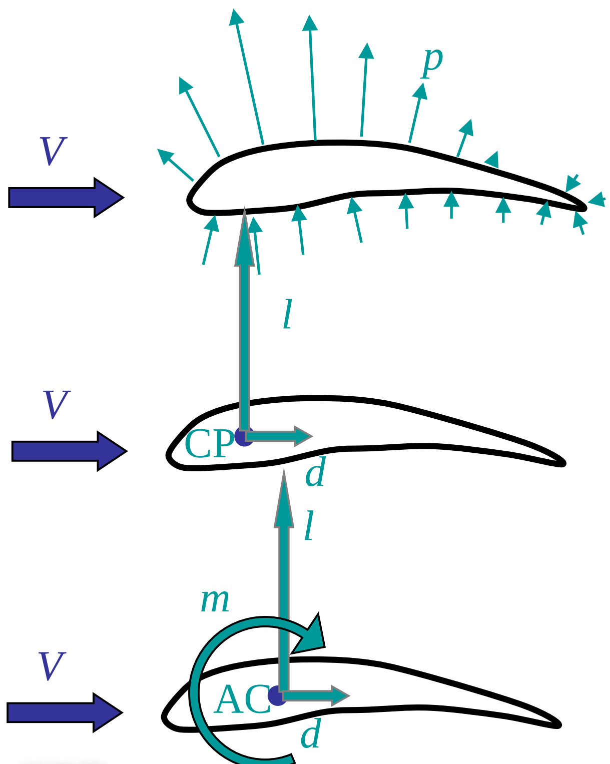
  - Example: shearing of a semi-cantilever beam
    - $$\int_0^L \int_s q^{(1)} \frac{q}{\mu t} ds dx = \boldsymbol{T}^{(1)} \bar{\Delta \boldsymbol{u}} = \Delta_T u$$

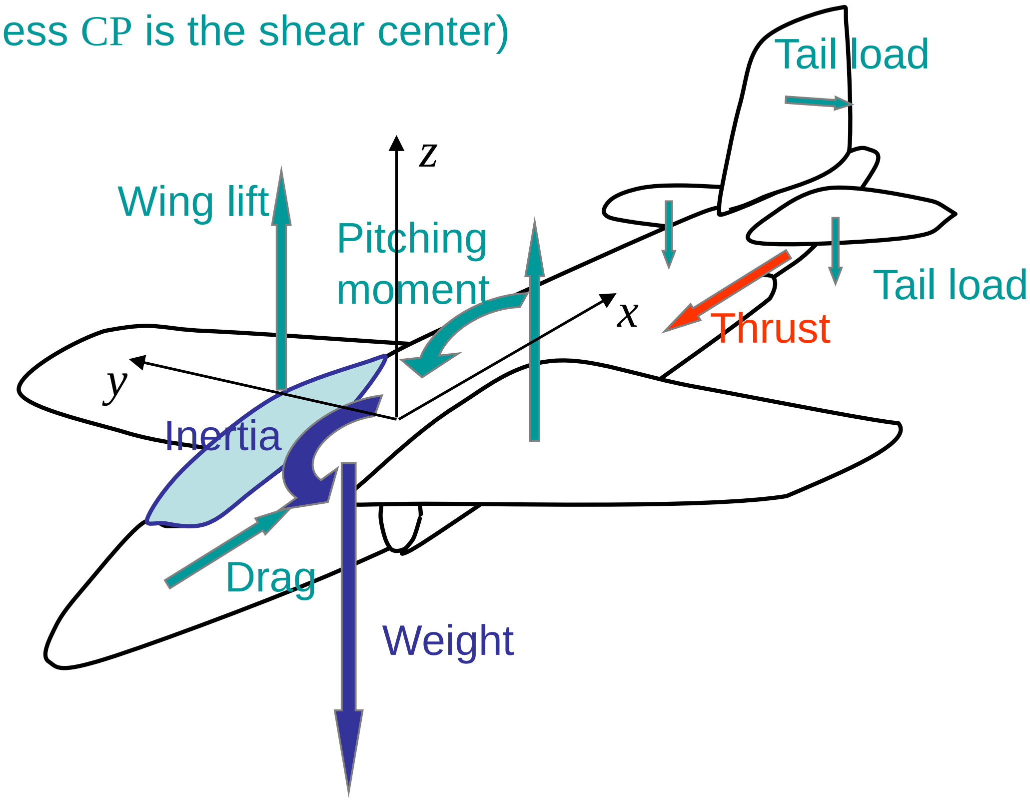
- Real structure
  - One, two or more cells
  - Usually
    - Resistance of stringers to shear stress is generally reduced
    - Distance between stringers is small
      - Shear can be considered constant in the skin between 2 stringers
  - Idealized section approximation holds



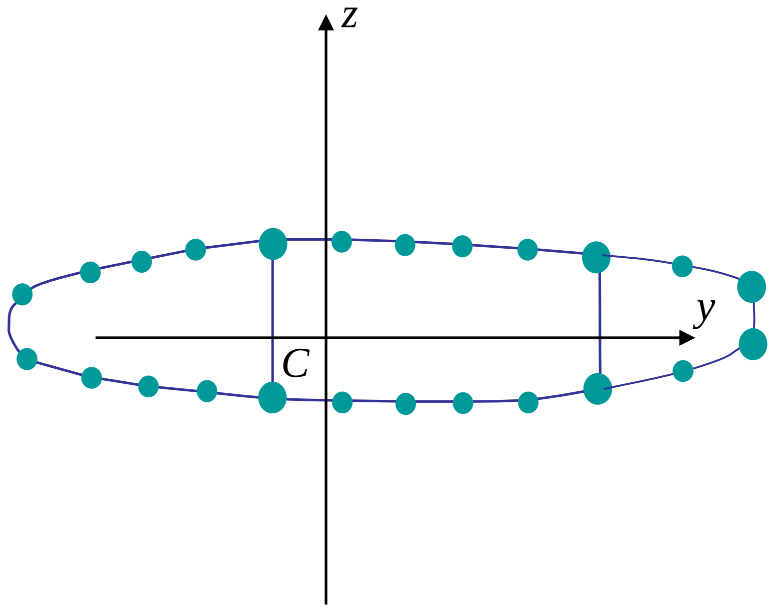


- Wing loading
  - Pressure distribution in airfoil can be substituted by
    - Lift & drag acting through the AC, resulting into
      - Bending
      - Shearing (including twist if AC ≠ Shear center)
    - Pitching moment, resulting into
      - Torsion around AC (unless CP is the shear center)



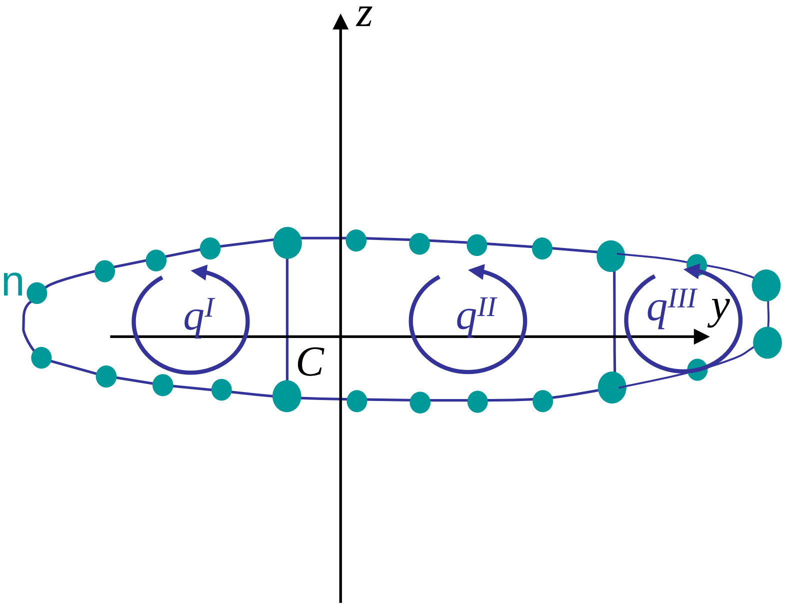


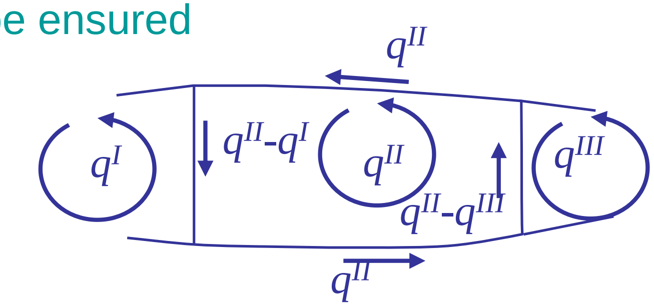
- Bending
  - As before $\boldsymbol{\sigma}_{xx} = \frac{(I_{zz}M_y + I_{yz}M_z)\,z - (I_{yz}M_y + I_{yy}M_z)\,y}{I_{yy}I_{zz} - I_{yz}^2}$
  - Still considering only direct stress carrying structures: *i.e*. the booms



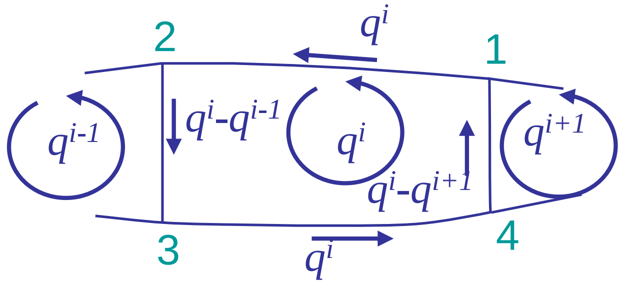
- Torsion
  - Assumptions
    - No axial constraints (warping allowed)
      - This is a severe restriction for wings
      - No axial stresses
    - Shape of the wing unchanged after torsion
  - Methodology
    - No axial stresses ⟹ only shearing
      ⟹ booms can be ignored
    - The N cells carry the torque M_x
    - There is a constant shear flux q_i in each cell i of open area $A_h{}^i$: $M_x = \sum_i 2q^i A_h^i$
    - To compute the fluxes
      - Compatibility of displacements has to be ensured
        ⟹ same rate of twist in each cell
      - For constant μ

$$\theta_{,x} = \frac{1}{2A_h^i \mu} \oint_i \frac{q^i}{t} ds$$





- Torsion



  – Twist rate compatibility

$$\theta_{,x} = \frac{1}{2A_h^i \mu} \oint_i \frac{q^i}{t} ds$$

$$\Longrightarrow \theta_{,x} = \frac{1}{2A_h^i \mu} \left[q^i \bar{l}^{12} + q^i \bar{l}^{34} + \left(q^i - q^{i-1}\right) \bar{l}^{23} + \left(q^i - q^{i+1}\right) \bar{l}^{41} \right]$$

    - With the non dimensional length $\bar{l}^{i\, i+1} = \int_i^{i+1} \frac{ds}{t}$

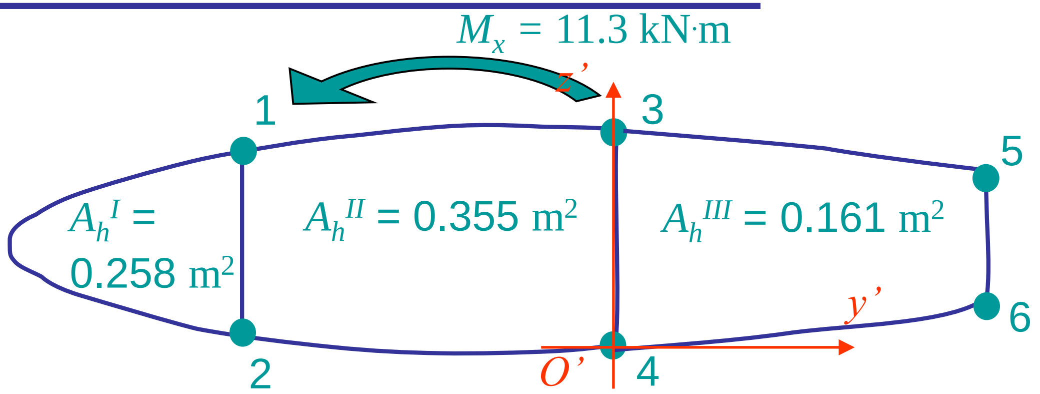
$$\Longrightarrow \theta_{,x} = \frac{1}{2A_h^i \mu} \left[-q^{i-1} \bar{l}_{i-1}^i + q^i \bar{l}^i - q^{i+1} \bar{l}_i^{i+1} \right]$$

    - With the non dimensional lengths $\begin{cases} \bar{l}_i^{i+1} = \int_{\text{cell } i \cap \text{ cell } i+1} \frac{ds}{t} \\ \bar{l}^i = \oint_{\text{cell } i} \frac{ds}{t} \end{cases}$

  – This formula has to be adapted

    - If a cell side is connected to more than one cell
    - If shear modulus is not constant $\Longrightarrow \bar{l}^{i\, i+1} = \int_i^{i+1} \frac{ds}{t \frac{\mu}{\mu_{\text{REF}}}}$

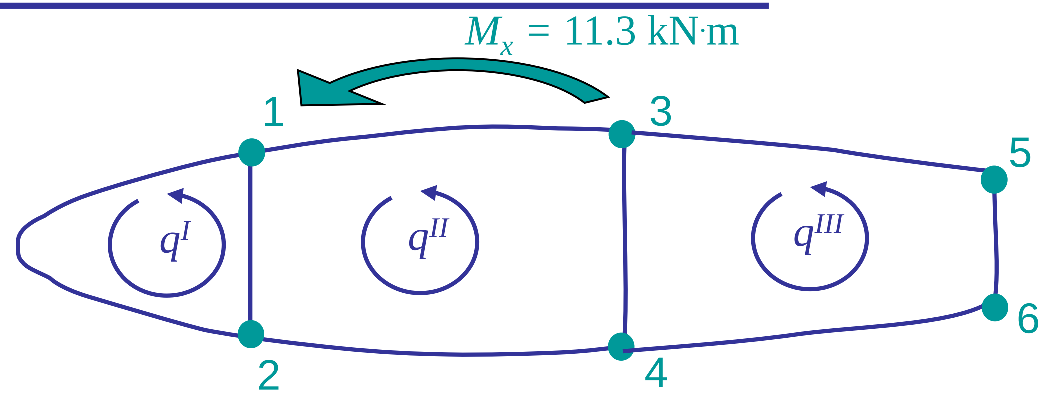
- Example
  - Three-cell section
  - Shear stress ?



Wall	Length (m)	Thickness (mm)	μ (GPa)
12 (	1.650	1.22	24.2
12 |	0.508	2.03	27.6
13 & 24	0.775	1.22	24.2
34	0.380	1.63	27.6
35 & 46	0.508	0.92	20.7
56	0.254	0.92	20.7

Wings

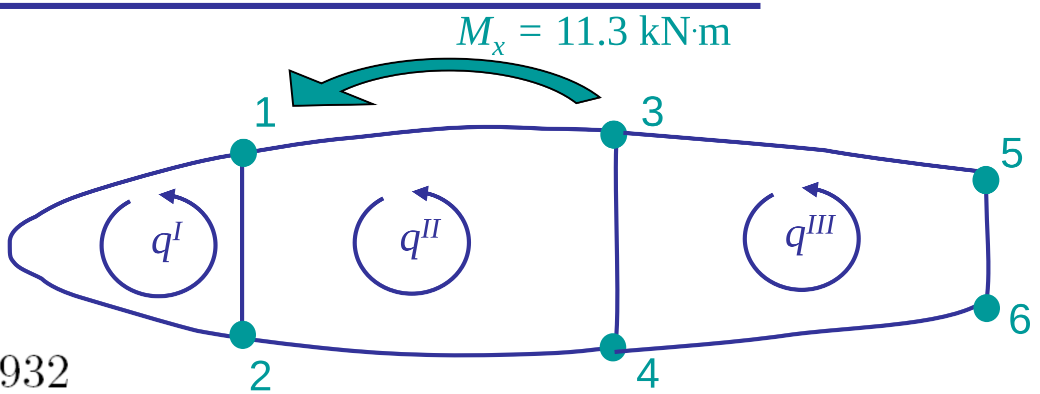
- Non dimensional lengths
  - Take μ_{REF} = 27.6 GPa
  - Side lengths



$$
\begin{cases}
\bar{l}^{1\,2(} = \dfrac{1.65}{1.22\ 10^{-3}\frac{24.2}{27.6}} = 1542 \\
\bar{l}^{1\,2|} = \dfrac{0.508}{2.03\ 10^{-3}} = 250 \\
\bar{l}^{1\,3} = \bar{l}^{2\,4} = \dfrac{0.775}{1.22\ 10^{-3}\frac{24.2}{27.6}} = 724.5 \\
\bar{l}^{3\,4} = \dfrac{0.38}{1.63\ 10^{-3}} = 233 \\
\bar{l}^{3\,5} = \bar{l}^{4\,6} = \dfrac{0.508}{0.92\ 10^{-3}\frac{20.7}{27.6}} = 736 \\
\bar{l}^{5\,6} = \dfrac{0.254}{0.92\ 10^{-3}\frac{20.7}{27.6}} = 368
\end{cases}
$$

Wall	Length (m)	Thickness (mm)	μ (GPa)
12 (	1.650	1.22	24.2
12 |	0.508	2.03	27.6
13 & 24	0.775	1.22	24.2
34	0.380	1.63	27.6
35 & 46	0.508	0.92	20.7
56	0.254	0.92	20.7

- Non dimensional lengths (2)



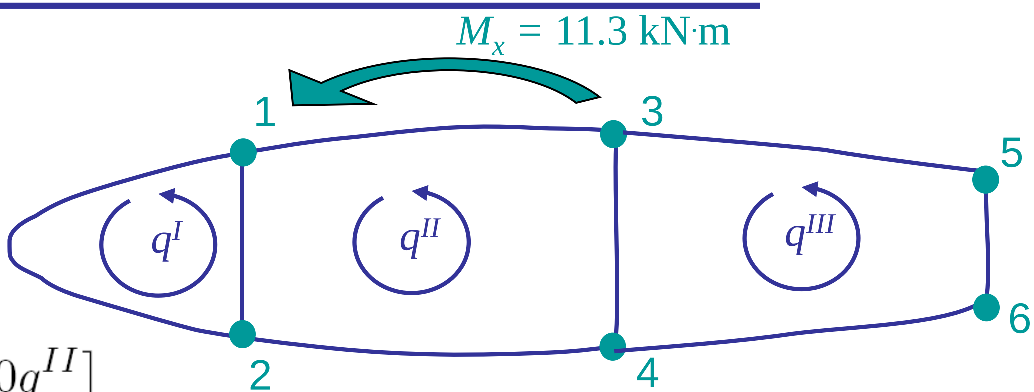
  – Cell lengths

$$\begin{cases} \bar{l}^{1} = \bar{l}^{1\,2(} + \bar{l}^{1\,2|} = 1792 \\ \bar{l}^{2} = \bar{l}^{1\,2|} + \bar{l}^{2\,4} + \bar{l}^{4\,3} + \bar{l}^{3\,1} = 1932 \\ \bar{l}^{3} = \bar{l}^{3\,4} + \bar{l}^{4\,6} + \bar{l}^{6\,5} + \bar{l}^{5\,3} = 2073 \end{cases}$$

  – Intersecting lengths

$$\begin{cases} \bar{l}_{1}^{2} = \bar{l}^{1\,2|} = 250 \\ \bar{l}_{2}^{3} = \bar{l}^{3\,4} = 233 \end{cases}$$

- Twist rates



  - Cell I

$$\theta_{,x} = \frac{1}{2A_h^I \mu_{\text{REF}}} \left[q^I \bar{l}^1 - q^{II} \bar{l}_1^2\right]$$
$$= \frac{1}{2\ 0.258\ 27.6\ 10^9} \left[1792 q^I - 250 q^{II}\right]$$
$$\Longrightarrow \theta_{,x} = 126\ 10^{-9}\ \ \text{N}^{-1}\ q^I - 17.6\ 10^{-9}\ \ \text{N}^{-1}\ q^{II}$$

  - Cell II

$$\theta_{,x} = \frac{1}{2A_h^{II} \mu_{\text{REF}}} \left[-q^I \bar{l}_1^2 + q^{II} \bar{l}^2 - q^{III} \bar{l}_2^3\right] = \frac{1}{2\ 0.355\ 27.6\ 10^9} \left[-250 q^I + 1932 q^{II} - 233 q^{III}\right]$$
$$\Longrightarrow \theta_{,x} = -12.8\ 10^{-9}\ \ \text{N}^{-1}\ q^I + 98.6\ 10^{-9}\ \ \text{N}^{-1}\ q^{II} - 11.9\ 10^{-9}\ \ \text{N}^{-1}\ q^{III}$$

  - Cell III

$$\theta_{,x} = \frac{1}{2A_h^{III} \mu_{\text{REF}}} \left[-q^{II} \bar{l}_2^3 + q^{III} \bar{l}^3\right] = \frac{1}{2\ 0.161\ 27.6\ 10^9} \left[-233 q^{II} + 2073 q^{III}\right]$$
$$\Longrightarrow \theta_{,x} = -26.2\ 10^{-9}\ \ \text{N}^{-1}\ q^{II} + 233\ 10^{-9}\ \ \text{N}^{-1}\ q^{III}$$

- 3 equations for 4 unknowns $\Longrightarrow$ momentum equilibrium

- Shear fluxes
  - Applied torque
    - $M_x = \sum_i 2q^i A_h^i = 2\left[0.258q^I + 0.355q^{II} + 0.161q^{III}\right]\ \text{m}^2 = 11300\ \text{N}\cdot\text{m}$
  - Equations
    - $2\left[0.258q^I + 0.355q^{II} + 0.161q^{III}\right]\ \text{m}^2 = 11300\ \text{N}\cdot\text{m}$
    - $126\ 10^{-9}\ \text{N}^{-1}\ q^I - 17.6\ 10^{-9}\ \text{N}^{-1}\ q^{II}$
      $= -12.8\ 10^{-9}\ \text{N}^{-1}\ q^I + 98.6\ 10^{-9}\ \text{N}^{-1}\ q^{II} - 11.9\ 10^{-9}\ \text{N}^{-1}\ q^{III}$
    - $126\ 10^{-9}\ \text{N}^{-1}\ q^I - 17.6\ 10^{-9}\ \text{N}^{-1}\ q^{II}$
      $= -26.2\ 10^{-9}\ \text{N}^{-1}\ q^{II} + 233\ 10^{-9}\ \text{N}^{-1}\ q^{III}$
  - Resolution

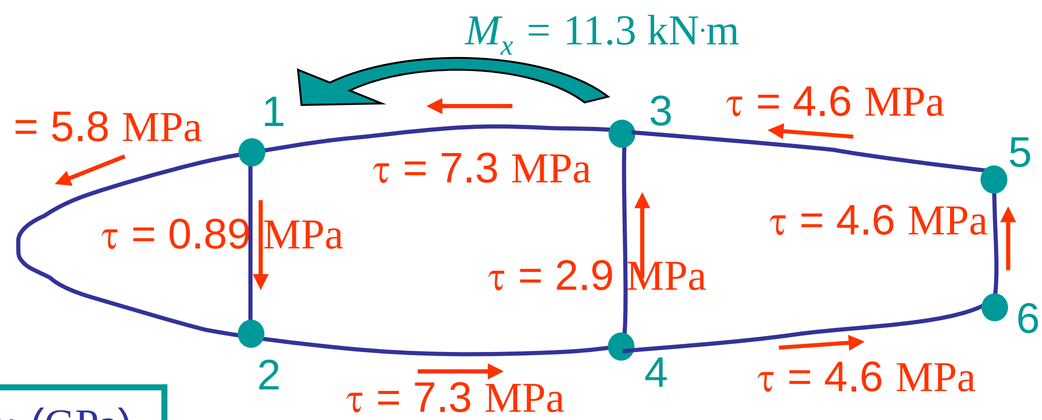
$$q^I = \frac{98.6 + 17.6}{126 + 12.8} q^{II} - \frac{11.9}{126 + 12.8} q^{III} = 0.837q^{II} - 0.086q^{III}$$

$$\Longrightarrow \left\{ \begin{aligned} q^{II} &= \frac{233 + 126\ 0.086}{126\ 0.837 - 17.6 + 26.2} q^{III} = 2.138q^{III} \\ q^I &= \left(0.837\ 2.138 - 0.086\right) q^{III} = 1.7q^{III} \end{aligned} \right.$$

$$\Longrightarrow 2\left[0.258\ 1.7 + 0.355\ 2.138 + 0.161\right]\ \text{m}^2 q^{III} = 11300\ \text{N}\cdot\text{m}$$

- Shear stress
  - Solution

$$\begin{cases} q^{III} = \dfrac{11300}{2.72} = 4.2\ 10^3 \quad \mathrm{N \cdot m^{-1}} \\ q^{II} = 4200\ 2.138 = 8.9\ 10^3 \quad \mathrm{N \cdot m^{-1}} \\ q^{I} = 4200\ 1.7 = 7.1\ 10^3 \quad \mathrm{N \cdot m^{-1}} \end{cases}$$



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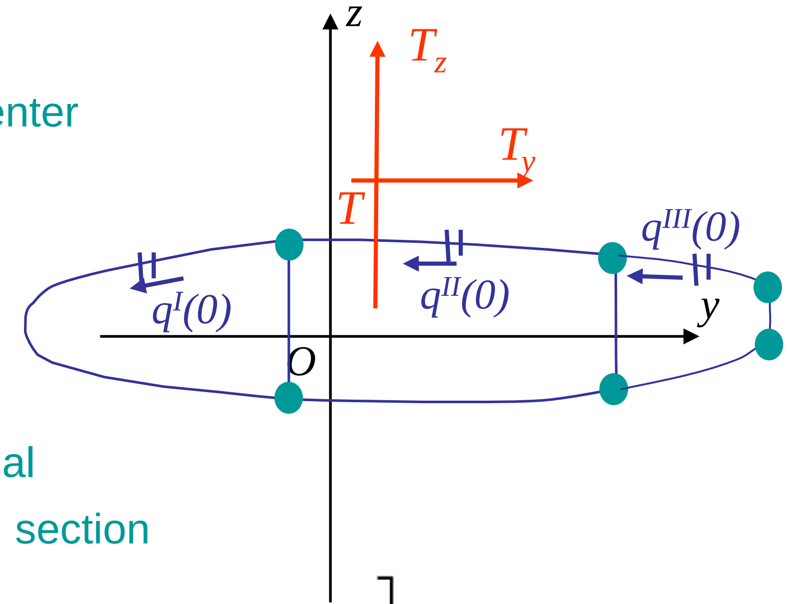
- Shearing
  - Assumptions
    - Shear loads do not pass through shear center
    - Booms carry direct stress only
    - Skins carry shear stress
    - Generally skins also carry direct stress
  - Methodology
    - Cut each cell
      - Cut on top so the correction is minimal
    - Compute the open flux of the whole beam section

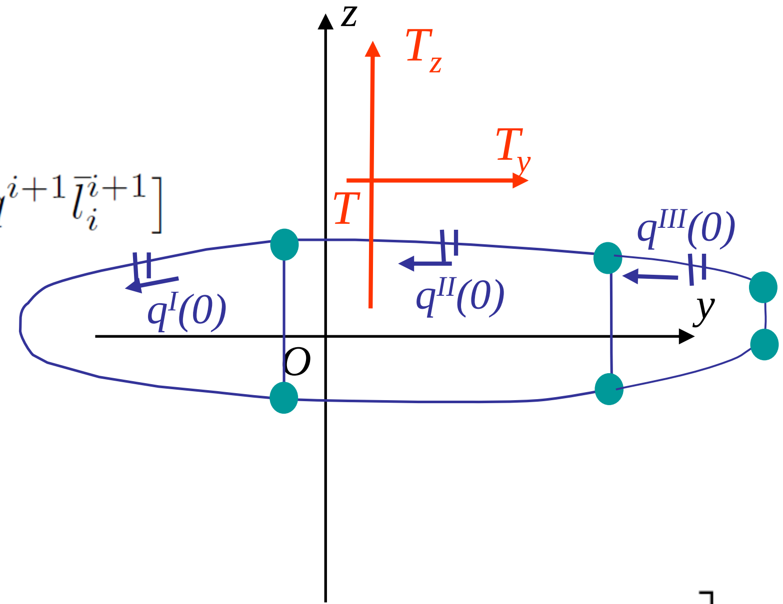


$$q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\boldsymbol{\sigma}} z ds + \sum_{i:\ s_i \leq s} z_i A_i\right] - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\boldsymbol{\sigma}} y ds + \sum_{i:\ s_i \leq s} y_i A_i\right]$$

- Compute the missing $q^i(0)$ at the cuts using compatibility of twist rate

- Shearing (2)
  - Twist rate compatibility
    - For torsion we found

$$\theta_{,x} = \frac{1}{2A_h^i \mu_{\text{REF}}} \left[-q^{i-1}\bar{l}^i_{i-1} + q^i \bar{l}^i - q^{i+1}\bar{l}^{i+1}_i\right]$$



    - But $q^i = q_o + q^i(0)$
      - q_o not dependant on the cell number (computed for the whole section)
    - So twist rate in each cell i is rewritten

$$\theta_{,x} = \frac{1}{2A_h^i \mu_{\text{REF}}} \left[-q^{i-1}(0)\bar{l}^i_{i-1} + q^i(0)\bar{l}^i + \oint_{\text{cell } i} \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}} ds - q^{i+1}(0)\bar{l}^{i+1}_i\right]$$

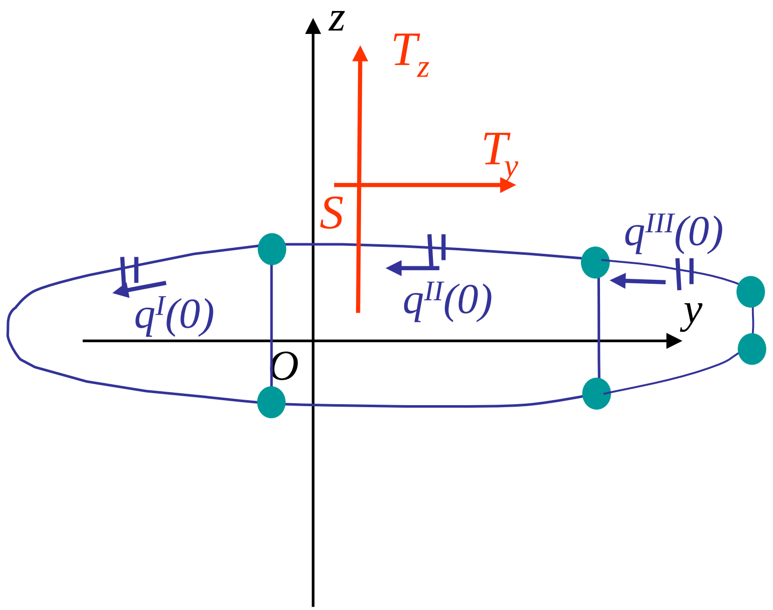
    - Section twist rate computed by momentum equilibrium (anticlockwise q, s)

$$M_x = y_T T_z - z_T T_y = \sum_i \int_{\text{wall } i} q p ds$$

$$\Longrightarrow y_T T_z - z_T T_y = \sum_i \int_{\text{wall } i} q_o p ds + \sum_i \oint_{\text{cell } i} q^i(0) p ds$$

$$\Longrightarrow y_T T_z - z_T T_y = \sum_i \int_{\text{wall } i} q_o p ds + \sum_i 2A_h^i q^i(0)$$

- Shear center
  - Define successively shear loads
    - T_z to determine y_S
    - T_y to determine z_S
  - Compute open shear fluxes
    - As before
  - To compute the fluxes at cut
    - Twist rate is equal to zero
      - As loads pass through shear center
    - $q^i(0)$ are deduced from

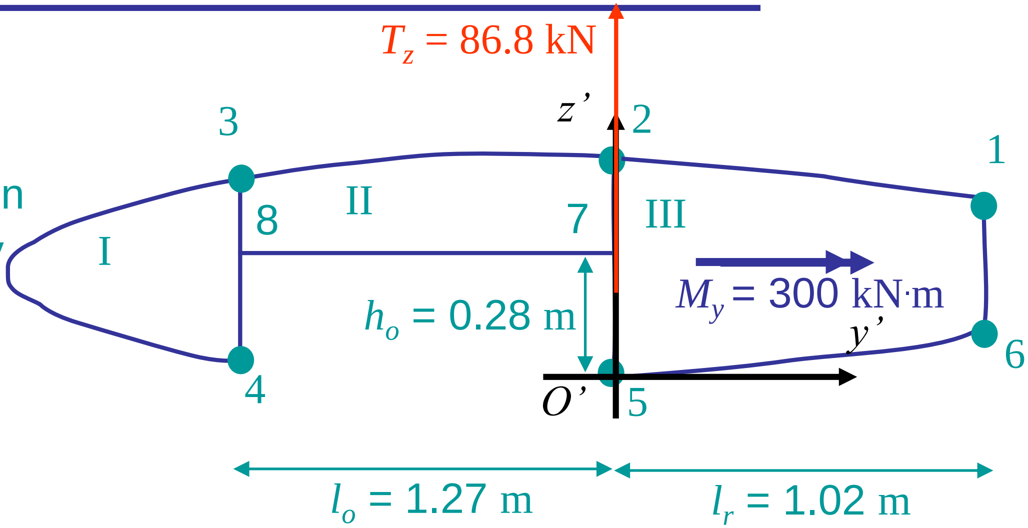


$$0 = \theta_{,x} = \frac{1}{2A_h^i \mu_{\text{REF}}} \left[-q^{i-1}(0)\bar{l}_{i-1}^i + q^i(0)\bar{l}^i + \oint_{\text{cell } i} \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}} ds - q^{i+1}(0)\bar{l}_i^{i+1} \right]$$

  - Shear center position y_S (z_S) is obtained from momentum equilibrium

$$y_T T_z (-z_T T_y) = \sum_i \int_{\text{wall } i} q_o p ds + \sum_i \oint_{\text{cell } i} q^i(0) p ds$$

- Example
  - Idealized wing section
    - Symmetrical booms distribution
    - Booms carry direct stress only
    - Skins carry shear stress only
  - Direct stress due to bending moment?
  - Shear stress?



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12 & 56	1.023	1.22	27.6
23	1.274	1.63	27.6
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483	0.400	2.64	27.6
572	0.460	2.64	27.6
61	0.330	1.63	27.6
78	1.270	1.22	82.8

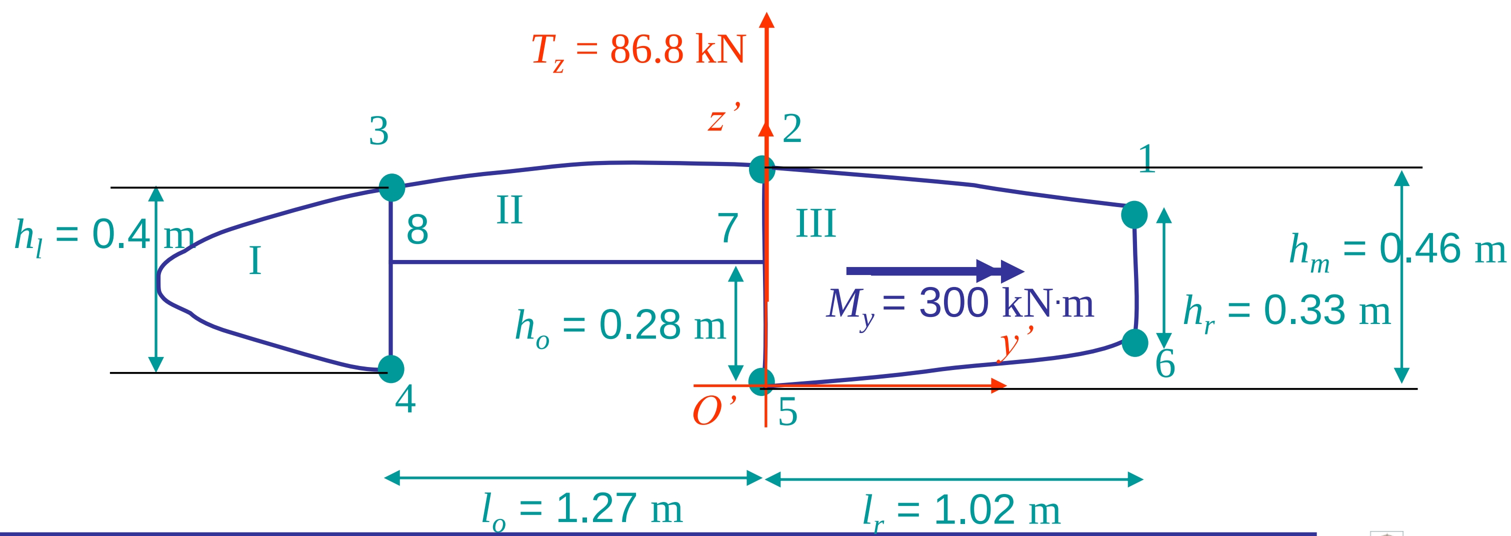
Boom	Area (mm^2)
1, 6	2580
2, 5	3880
3, 4	3230

Cell	Area (m^2)
I	0.265
II	0.213
III	0.413

Wings

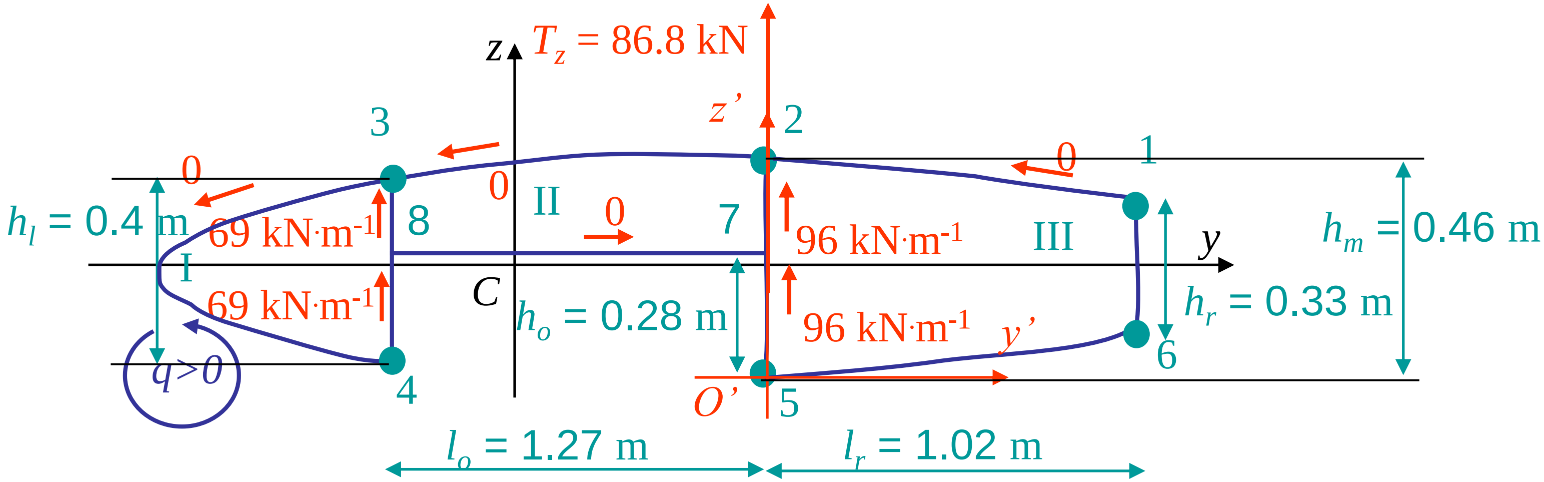
- Bending
  - As boom distribution is symmetric: $z'_C = 0.23$ m
    - $$I_{yy} = 2\left(A_3 {z^3}^2 + A_2 {z^2}^2 + A_1 {z^1}^2\right)$$
      $$= 2\left(0.003230\ 0.2^2 + 0.00388\ 0.23^2 + 0.00258\ 0.165^2\right) = 0.000809\ \ \mathrm{m}^4$$
    - $$\begin{cases} \boldsymbol{\sigma}^1_{xx} = -\boldsymbol{\sigma}^6_{xx} = \dfrac{M_y}{I_{yy}} z^1 = \dfrac{300\ 10^3}{0.000809}\ 0.165 = 61.2\ \ \mathrm{MPa} \\ \boldsymbol{\sigma}^2_{xx} = -\boldsymbol{\sigma}^5_{xx} = \dfrac{M_y}{I_{yy}} z^2 = \dfrac{300\ 10^3}{0.000809}\ 0.23 = 85.3\ \ \mathrm{MPa} \\ \boldsymbol{\sigma}^3_{xx} = -\boldsymbol{\sigma}^4_{xx} = \dfrac{M_y}{I_{yy}} z^3 = \dfrac{300\ 10^3}{0.000809}\ 0.2 = 74.2\ \ \mathrm{MPa} \end{cases}$$

Boom	Area (mm^2)
1, 6	2580
2, 5	3880
3, 4	3230



- Open-cells shearing
  - As only booms are carrying direct stress $q_o(s) = -\frac{T_z}{I_{yy}} \sum_{i:\, s_i \leq s} z_i A_i$
    - Cut: $q_o^{3\,4} = q_o^{2\,3} = q_o^{1\,2} = 0$
    - $q_o^{4\,8} = -\frac{T_z}{I_{yy}} z^4 A_4 = -\frac{86800}{0.809\ 10{-}3}(-0.2)\, 0.00323 = 69\ 10^3 \quad \mathrm{N \cdot m^{-1}}$
    - $-q_o^{8\,3} = q_o^{3\,8} = -\frac{T_z}{I_{yy}} z^3 A_3 = -\frac{86800}{0.809\ 10{-}3}\, 0.2\ 0.00323 = -69\ 10^3 \quad \mathrm{N \cdot m^{-1}}$
    - $q_o^{4\,8} = q_o^{8\,7} + q_o^{8\,3} = 69\ 10^3 \quad \mathrm{N \cdot m^{-1}} \Longrightarrow q_o^{8\,7} = 0$
    - $-q_o^{7\,2} = q_o^{2\,7} = -\frac{T_z}{I_{yy}} z^2 A_2 = -\frac{86800}{0.809\ 10{-}3}\, 0.23\ 0.003880 = -96\ 10^3 \quad \mathrm{N \cdot m^{-1}}$
    - $q_o^{7\,5} = q_o^{2\,7} + q_o^{8\,7} = q_o^{2\,7} = -96\ 10^3 \quad \mathrm{N \cdot m^{-1}}$

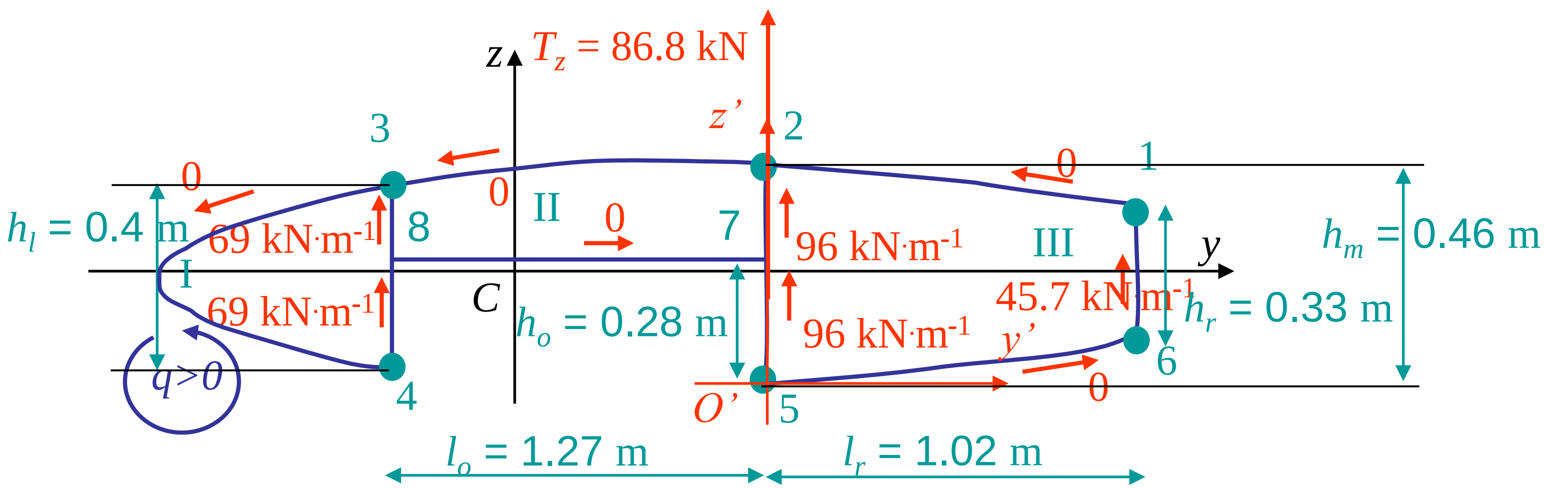
Boom	Area (mm²)
1, 6	2580
2, 5	3880
3, 4	3230



- Open-cells shearing (2)
  - As only booms are carrying direct stress (2)
    - $q_o^{5\,6} = q_o^{7\,5} - \frac{T_z}{I_{yy}} z^5 A_5 = -96\ 10^3 - \frac{86800}{0.809\ 10^{-3}} (-0.23)\, 0.003880 = 0$
    - $q_o^{6\,1} = q_o^{5\,6} - \frac{T_z}{I_{yy}} z^6 A_6 = -\frac{86800}{0.809\ 10^{-3}} (-0.165)\, 0.00258 = 45.7\ \ \mathrm{N \cdot m^{-1}}$
- What remain to be determined are the $q^i(0)$ at the cuts
  - Use of twist rate compatibility

$$\theta_{,x} = \frac{1}{2A_h^i \mu_{\mathrm{REF}}} \left[-q^{i-1}(0)\bar{l}^i_{i-1} + q^i(0)\bar{l}^i + \oint_{\mathrm{cell}\ i} \frac{q_o}{t\frac{\mu}{\mu_{\mathrm{REF}}}} ds - q^{i+1}(0)\bar{l}^{i+1}_i \right]$$

Boom	Area (mm²)
1, 6	2580
2, 5	3880
3, 4	3230



- Non dimensional lengths
  - Take μ_{REF} = 27.6 GPa
  - Side lengths

$$\left\{\begin{array}{l}
\bar{l}^{12} = \bar{l}^{56} = \dfrac{1.023}{0.00122} = 839 \\
\bar{l}^{23} = \dfrac{1.274}{0.00163} = 782 \\
\bar{l}^{34} = \dfrac{2.2}{0.00203} = 1084 \\
\bar{l}^{483} = \dfrac{0.4}{0.00264} = 152 \\
\bar{l}^{572} = \dfrac{0.46}{0.00264} = 174 \\
\bar{l}^{61} = \dfrac{0.33}{0.00163} = 202 \\
\bar{l}^{78} = \dfrac{1.27}{0.00122\frac{82.8}{27.6}} = 347
\end{array}\right.$$

Wall	Length (m)	Thickness (mm)	μ (GPa)
12 & 56	1.023	1.22	27.6
23	1.274	1.63	27.6
34	2.200	2.03	27.6
483	0.400	2.64	27.6
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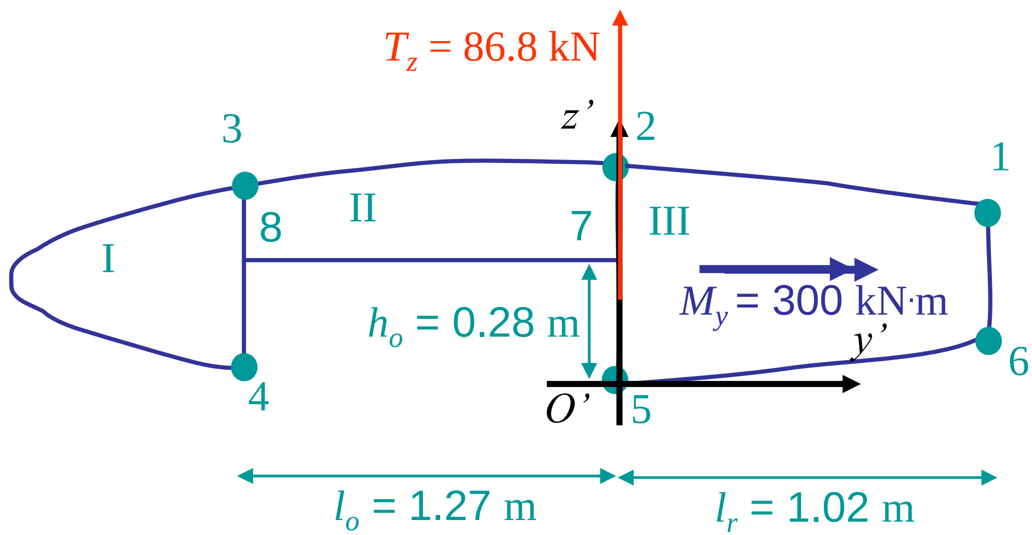
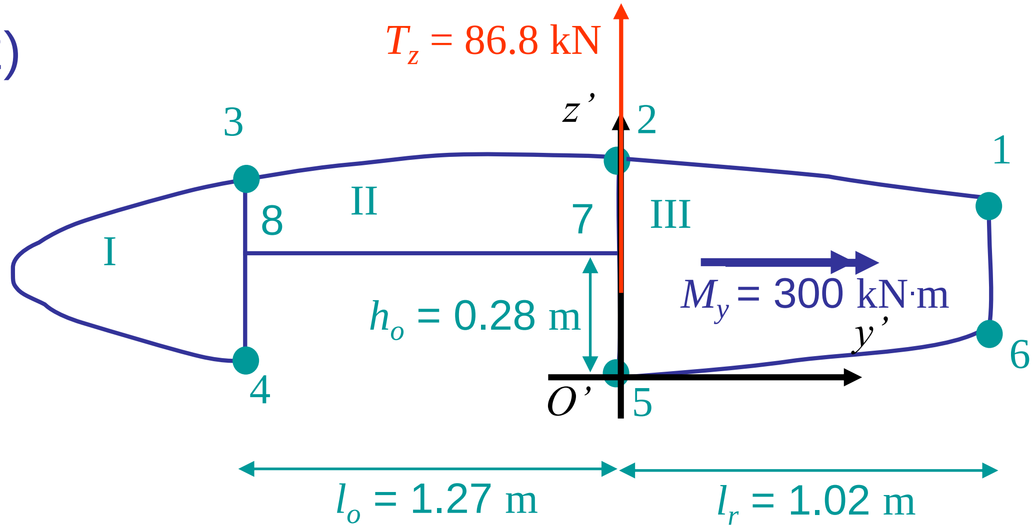


- Non dimensional lengths (2)



  - Cell lengths

$$\begin{cases} \bar{l}^1 = \bar{l}^{3\,4} + \bar{l}^{4\,8\,3} \\ \quad = 1084 + 152 = 1236 \\ \bar{l}^2 = \bar{l}^{2\,3} + \bar{l}^{3\,8} + \bar{l}^{7\,8} + \bar{l}^{2\,7} \\ \quad = 782 + 152\dfrac{0.15}{0.40} + 347 + 174\dfrac{0.18}{0.46} = 1254 \\ \bar{l}^3 = \bar{l}^{1\,2} + \bar{l}^{2\,7\,5} + \bar{l}^{5\,6} + \bar{l}^{1\,6\,5} = 839 + 174 + 839 + 202 = 2054 \end{cases}$$

  - Intersecting lengths

$$\begin{cases} \bar{l}_1^2 = \bar{l}^{38} = 152\dfrac{0.15}{0.4} = 57 \\ \bar{l}_2^3 = \bar{l}^{27} = 174\dfrac{0.18}{0.46} = 68 \end{cases}$$

- Integration of open-cell flux on cells
  - $$\oint_1 \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}} ds = 69\ 10^3 \frac{0.4}{0.00264} = 10.5\ 10^6 \quad \text{N} \cdot \text{m}^{-1}$$
  - $$\oint_2 \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}} ds = -69\ 10^3 \frac{0.15}{0.00264} + 96\ 10^3 \frac{0.18}{0.00264} = 2.63\ 10^6 \quad \text{N} \cdot \text{m}^{-1}$$
  - $$\oint_3 \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}} ds = -96\ 10^3 \frac{0.46}{0.00264} + 45.7\ 10^3 \frac{0.33}{0.00163} = -7.5\ 10^6 \quad \text{N} \cdot \text{m}^{-1}$$

Wall	Length (m)	Thickness (mm)	μ (GPa)
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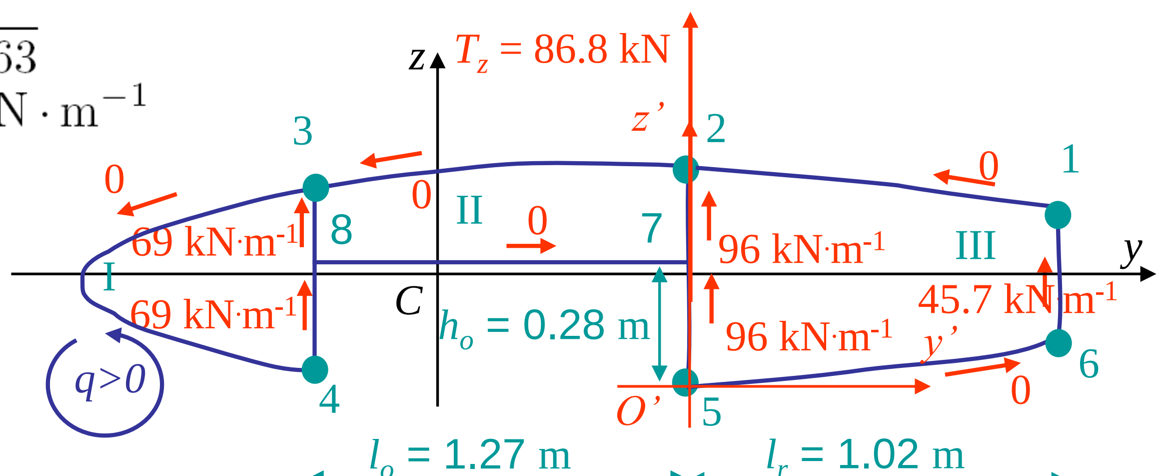


- Twist rate
  - Cell I: $\theta_{,x} = \frac{1}{2\ A_h^I\ \mu_{\text{REF}}} \left[q^I(0)\bar{l}^1 + \oint_1 \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}} ds - q^{II}(0)\bar{l}_1^2 \right]$

$$\Longrightarrow \theta_{,x} = \frac{1}{2\ 0.2652\ 27.6\ 10^9} \left[1236 q^I(0) + 10.5\ 10^6 - 57 q^{II}(0)\right]$$

$$\Longrightarrow \theta_{,x} = 0.00072\ \text{m}^{-1} + 84.4\ 10^{-9}\ \text{N}^{-1}\ q^I(0) - 3.89\ 10^{-9}\ \text{N}^{-1}\ q^{II}(0)$$

  - Cell II: $\theta_{,x} = \frac{1}{2\ A_h^{II}\ \mu_{\text{REF}}} \left[-q^I \bar{l}_1^2 + q^{II}(0)\bar{l}^2 + \oint_2 \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}} ds - q^{III}(0)\bar{l}_2^3 \right]$

$$\Longrightarrow \theta_{,x} = \frac{1}{2\ 0.213\ 27.6\ 10^9} \left[-57 q^I + 1254 q^{II}(0) + 2.63\ 10^6 - 68 q^{III}(0)\right]$$

$$\Longrightarrow \theta_{,x} = 0.000224\ \text{m}^{-1} - 4.85\ 10^{-9}\ \text{N}^{-1}\ q^I(0) + 106.7\ 10^{-9}\ \text{N}^{-1}\ q^{II}(0) - 5.78\ 10^{-9}\ \text{N}^{-1}\ q^{III}(0)$$

  - Cell III: $\theta_{,x} = \frac{1}{2\ A_h^{III}\ \mu_{\text{REF}}} \left[-q^{II} \bar{l}_2^3 + q^{III}(0)\bar{l}^3 + \oint_3 \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}} ds \right]$

$$\Longrightarrow \theta_{,x} = \frac{1}{2\ 0.413\ 27.6\ 10^9} \left[-68 q^{II} + 2054 q^{III}(0) - 7.5\ 10^6\right]$$

$$\Longrightarrow \theta_{,x} = -0.000326\ \text{m}^{-1} - 2.98\ 10^{-9}\ \text{N}^{-1}\ q^{II}(0) + 90.1\ 10^{-9}\ \text{N}^{-1} q^{III}(0)$$

- Momentum balance
  - Equation

$$y_T T_z - z_T T_y = \sum_i \int_{\text{wall } i} q_o p ds + \sum_i 2A_h^i q^i(0)$$

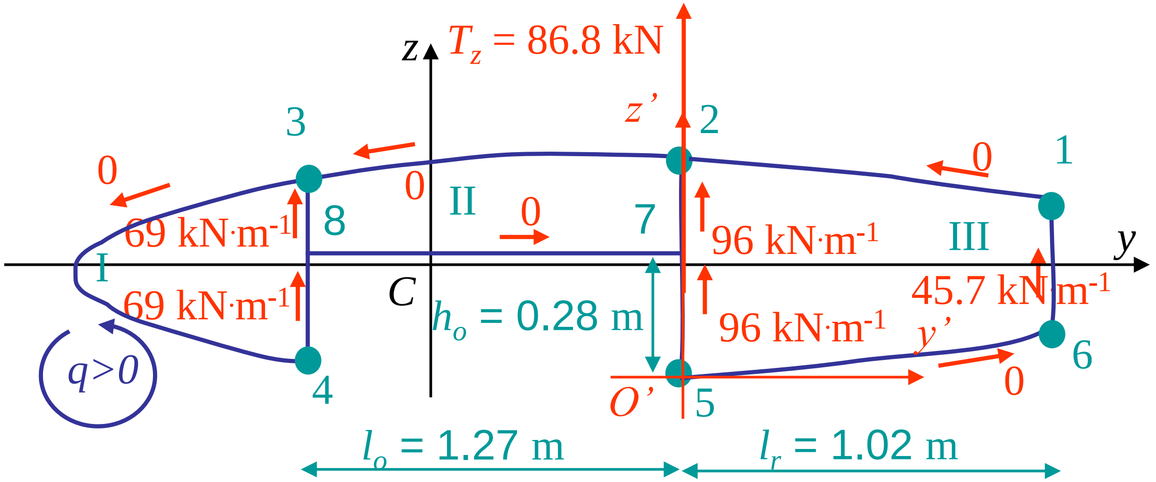
  - Balance around *O*'

$$\int q_o p_{O'} ds = -\left(q^{48} l^{48} + q^{83} l^{83}\right) l_o + q^{61} l^{61} l_r$$
$$= -69\ 10^3\ 0.4\ 1.27 + 45.7\ 10^3\ 0.33\ 1.02 = -19.7\ 10^3\ \ \text{N}\cdot\text{m}$$

$$\Rightarrow\ 0 = -19700\ \text{N}\cdot\text{m} + 2\ 0.265\ \text{m}^2\ q^I(0) + 2\ 0.213\ \text{m}^2\ q^{II}(0) + 2\ 0.413\ \text{m}^2\ q^{III}(0)$$

$$\Rightarrow\ 0.53\ \text{m}^2\ q^I(0) + 0.426\ \text{m}^2\ q^{II}(0) + 0.826\ \text{m}^2\ q^{III}(0) = 19700\ \text{N}\cdot\text{m}$$

Cell	Area (m²)
I	0.265
II	0.213
III	0.413



- Compatibility

$$\begin{cases} 0.00072 \text{ m}^{-1} + 84.4\ 10^{-9} \text{ N}^{-1}\ q^{I}(0) - 3.89\ 10^{-9} \text{ N}^{-1}\ q^{II}(0) \\ \quad = 0.000224 \text{ m}^{-1} - 4.85\ 10^{-9} \text{ N}^{-1}\ q^{I}(0) + 106.7\ 10^{-9} \text{ N}^{-1}\ q^{II}(0) - 5.78\ 10^{-9} \text{ N}^{-1}\ q^{III}(0) \\ 0.00072 \text{ m}^{-1} + 84.4\ 10^{-9} \text{ N}^{-1}\ q^{I}(0) - 3.89\ 10^{-9} \text{ N}^{-1}\ q^{II}(0) \\ \quad = -0.000326 \text{ m}^{-1} - 2.98\ 10^{-9} \text{ N}^{-1}\ q^{II}(0) + 90.1\ 10^{-9} \text{ N}^{-1} q^{III}(0) \\ 0.53 \text{ m}^2\ q^{I}(0) + 0.426 \text{ m}^2\ q^{II}(0) + 0.826 \text{ m}^2\ q^{III}(0) = 19700 \text{ N} \cdot \text{m} \end{cases}$$

$$q^{I}(0) = \frac{0.000224 - 0.00072}{84.4\ 10^{-9} + 4.85\ 10^{-9}} + \frac{106.7 + 3.89}{84.4 + 4.85} q^{II}(0) - \frac{5.78}{84.4 + 4.85} q^{III}$$

$$\Longrightarrow q^{I}(0) = -5557 + 1.239 q^{II}(0) - 0.065 q^{III}$$

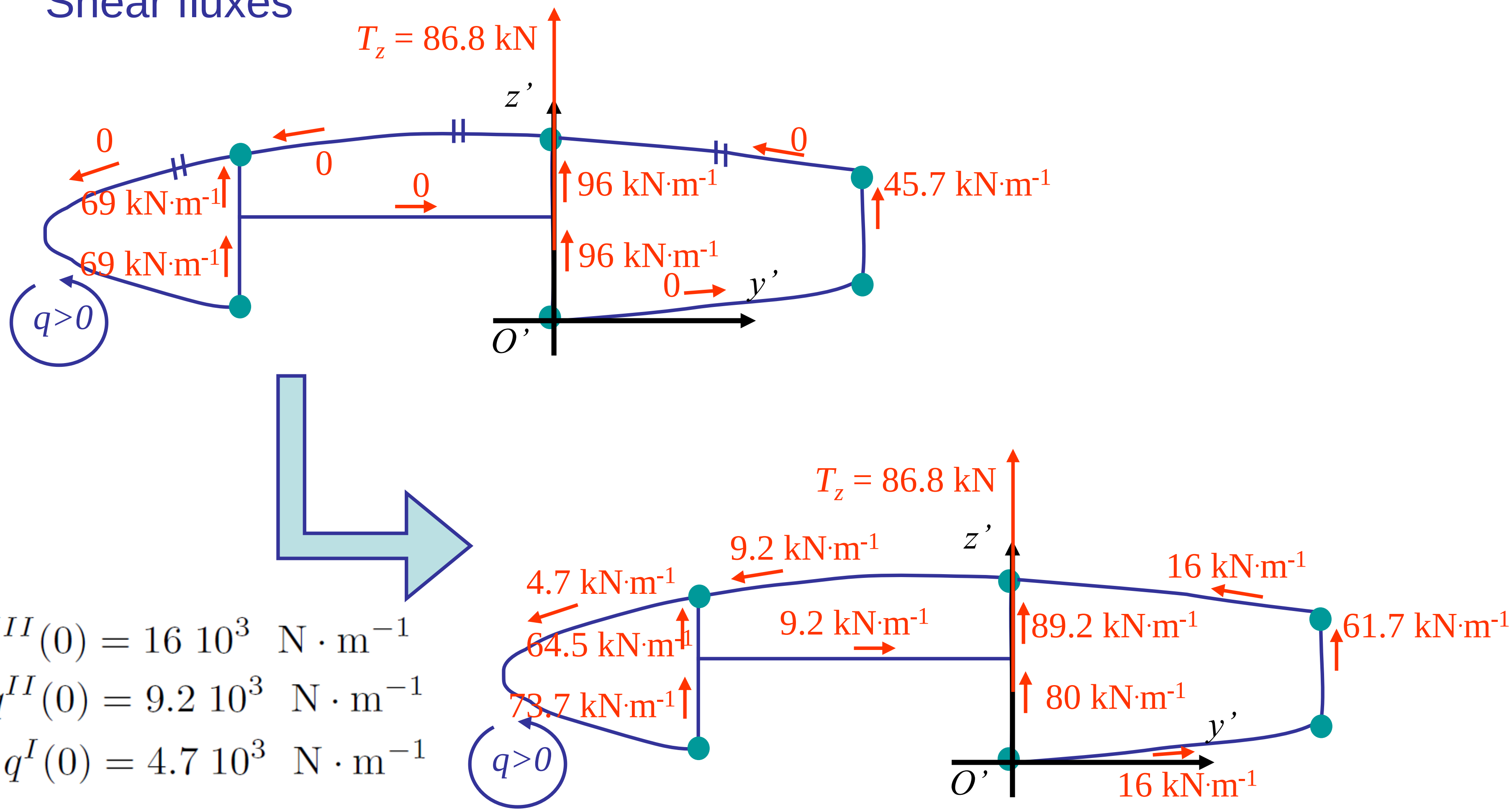
$$0.00072 - 0.00047 + (104.6 - 3.89 + 2.98)\ 10^{-9}\ q^{II}(0) - (5.49 + 90.1)\ 10^{-9}\ q^{III}(0) = -0.000326$$

$$\Longrightarrow q^{II}(0) = 0.92\ q^{III}(0) - 5555 \Longrightarrow q^{I}(0) = -12440 + 1.07 q^{III}$$

$$-6593. + 0.57 q^{III}(0) - 2366 + 0.39\ q^{III}(0) + 0.826\ q^{III}(0) = 19700$$

$$\Longrightarrow \begin{cases} q^{III}(0) = 16\ 10^3 \quad \text{N} \cdot \text{m}^{-1} \\ q^{II}(0) = 9.2\ 10^3 \quad \text{N} \cdot \text{m}^{-1} \\ q^{I}(0) = 4.7\ 10^3 \quad \text{N} \cdot \text{m}^{-1} \end{cases}$$

- Shear fluxes



$$\begin{cases} q^{III}(0) = 16\ 10^3 \quad \mathrm{N \cdot m^{-1}} \\ q^{II}(0) = 9.2\ 10^3 \quad \mathrm{N \cdot m^{-1}} \\ q^{I}(0) = 4.7\ 10^3 \quad \mathrm{N \cdot m^{-1}} \end{cases}$$

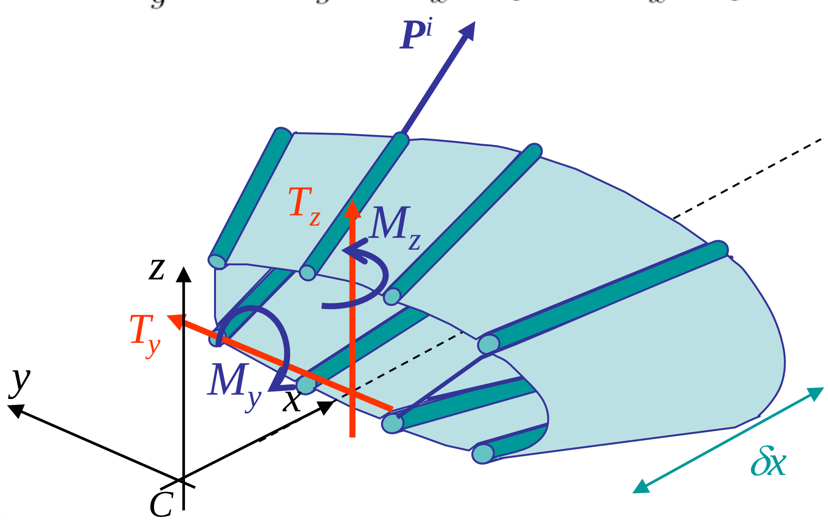
Wings

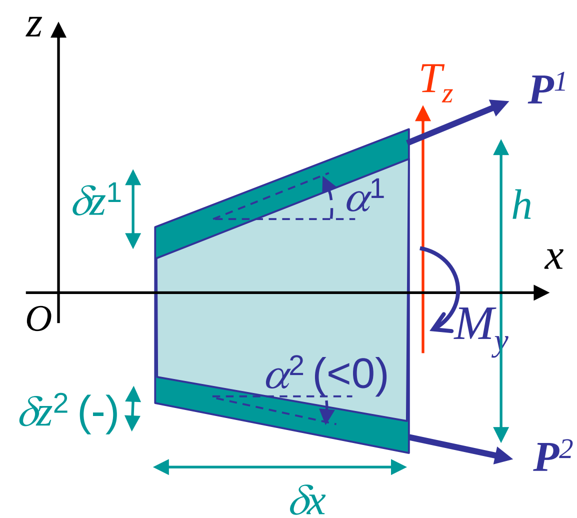
- Tapered wing
  - Effect on a single cell beam: same as for wing spars & box beams
    - Web shear modified by direct loads on adjacent booms

$$q_o(s) = -\frac{I_{zz}T_z^{\text{web}} - I_{yz}T_y^{\text{web}}}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\boldsymbol{\sigma}} z ds + \sum_{i:\ s_i \leq s} z_i A_i\right] - \frac{I_{yy}T_y^{\text{web}} - I_{yz}T_z^{\text{web}}}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\boldsymbol{\sigma}} z ds + \sum_{i:\ s_i \leq s} y_i A_i\right]$$

$$T_z^{\text{web}} = T_z - \boldsymbol{P}_x^1 \partial_x z^1 - \boldsymbol{P}_x^2 \partial_x z^2$$

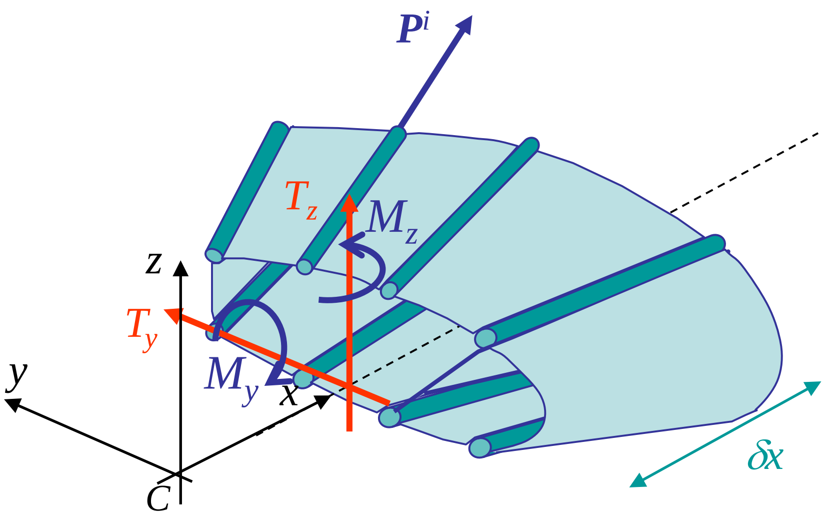
$$T_y^{\text{web}} = T_y - \boldsymbol{P}_x^1 \partial_x y^1 - \boldsymbol{P}_x^2 \partial_x y^2$$

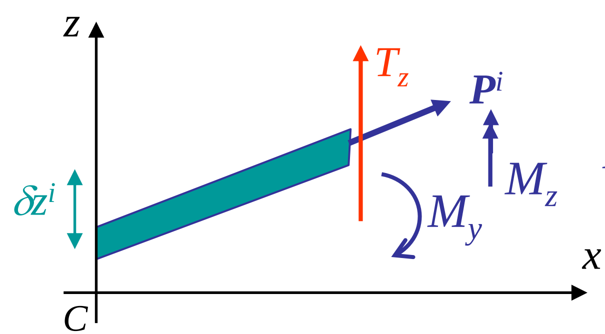


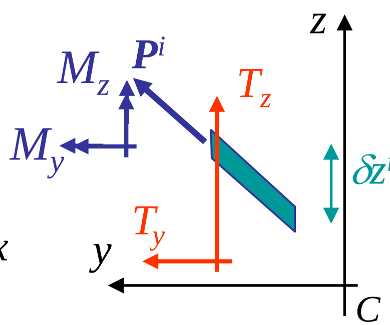


- Tapered wing (2)
  - Effect on $q^i(0)$
    - Direct loads on booms are modifying the momentum balance
    - So equation $y_T T_z - z_T T_y = \sum_i \int_{\text{wall } i} q_o p ds + \sum_i 2 A_h^i q^i(0)$ becomes

$$y_T T_z - z_T T_y = \sum_i \int_{\text{wall } i} q_o p ds + \sum_{\text{cell } c} 2 A_h^c q^c(0) + \sum_{\text{boom } j} y^j \boldsymbol{P}_z^j - \sum_{\text{boom } j} z^j \boldsymbol{P}_y^j$$

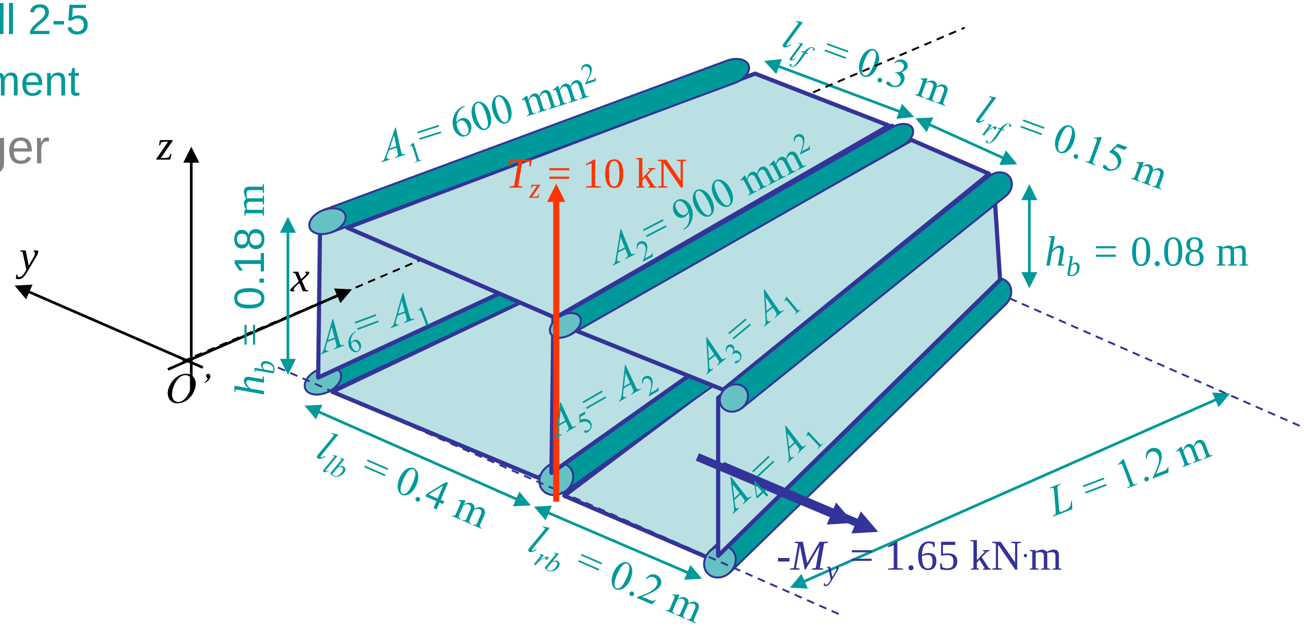






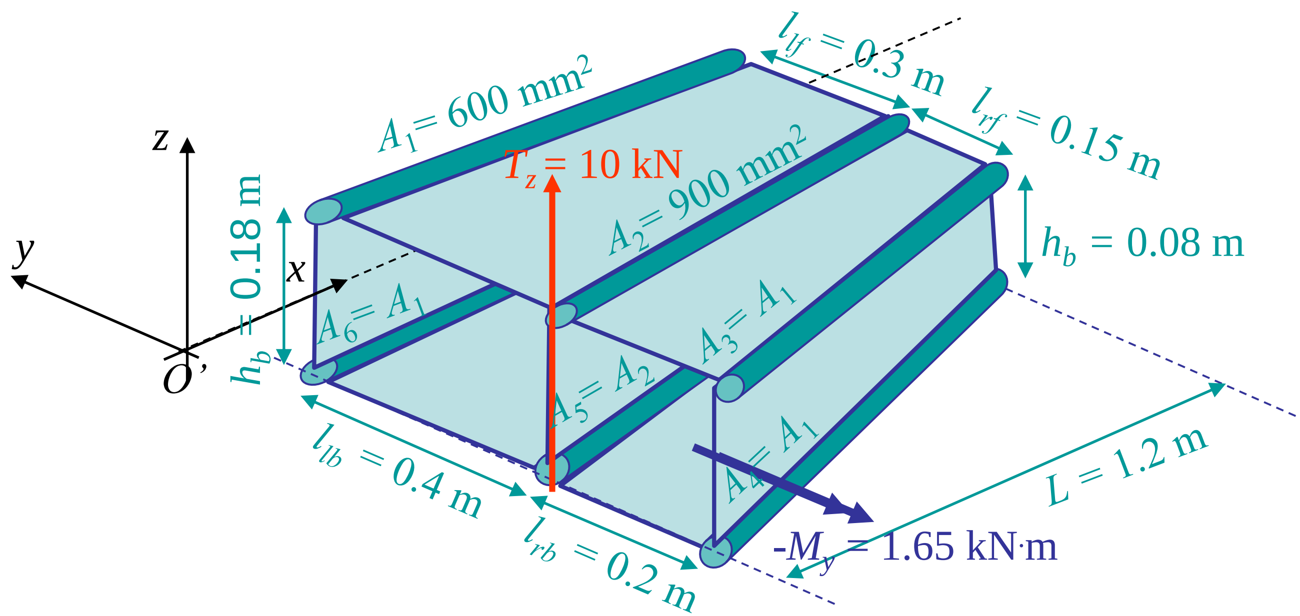
- Example
  - Tapered 2-cell wing
  - Idealized cross-sections
    - Booms carry direct stresses only
    - Skins carry shear stress only
  - Singly symmetrical section
    - Symmetrically tapered in the z-direction
    - Booms 1 & 6 in $O'xz$ plane
  - Loading in larger cross-section known
    - Shear on wall 2-5
    - Bending moment
  - Stresses at larger cross-section?
    - Direct?
    - Shear flux?

Web	Thickness (mm)	μ (GPa)
12	0.8	27.6
23	0.8	27.6
45	0.8	27.6
56	0.8	27.6
16	1	27.6
25	1	27.6
34	1	27.6



- Bending
  - Symmetrical with *y*-axis ⟹ $I_{yz} = 0$
    - $I_{yy} = 2\left(A_1 + A_2 + A_3\right)\frac{h_b^2}{4} = \frac{2\ 0.0006 + 0.0009}{2} 0.18^2 = 34\ 10^{-6} \quad \mathrm{m}^4$
    - $\boldsymbol{P}_x^1 = \boldsymbol{P}_x^3 = -\boldsymbol{P}_x^4 = -\boldsymbol{P}_x^6 = \boldsymbol{\sigma}_{xx}^1 A_1 =$

      $\frac{M_y z^1}{I_{yy}} A_1 = \frac{-1650\ 0.09}{34\ 10^{-6}} 0.0006 = -2.62\ 10^3 \quad \mathrm{N}$
    - $\boldsymbol{P}_x^2 = -\boldsymbol{P}_x^5 = \boldsymbol{\sigma}_{xx}^2 A_2 = \frac{M_y z^2}{I_{yy}} A_2 = \frac{-1650\ 0.09}{34\ 10^{-6}} 0.0009 = -3.93\ 10^3 \quad \mathrm{N}$



- Booms loading
  - As $\boldsymbol{P}_x^1 = \boldsymbol{P}_x^3 = -\boldsymbol{P}_x^4 = -\boldsymbol{P}_x^6 = -2.62\ 10^3\ \ \text{N}$ & $\boldsymbol{P}_x^2 = -\boldsymbol{P}_x^5 = -3.93\ 10^3\ \ \text{N}$

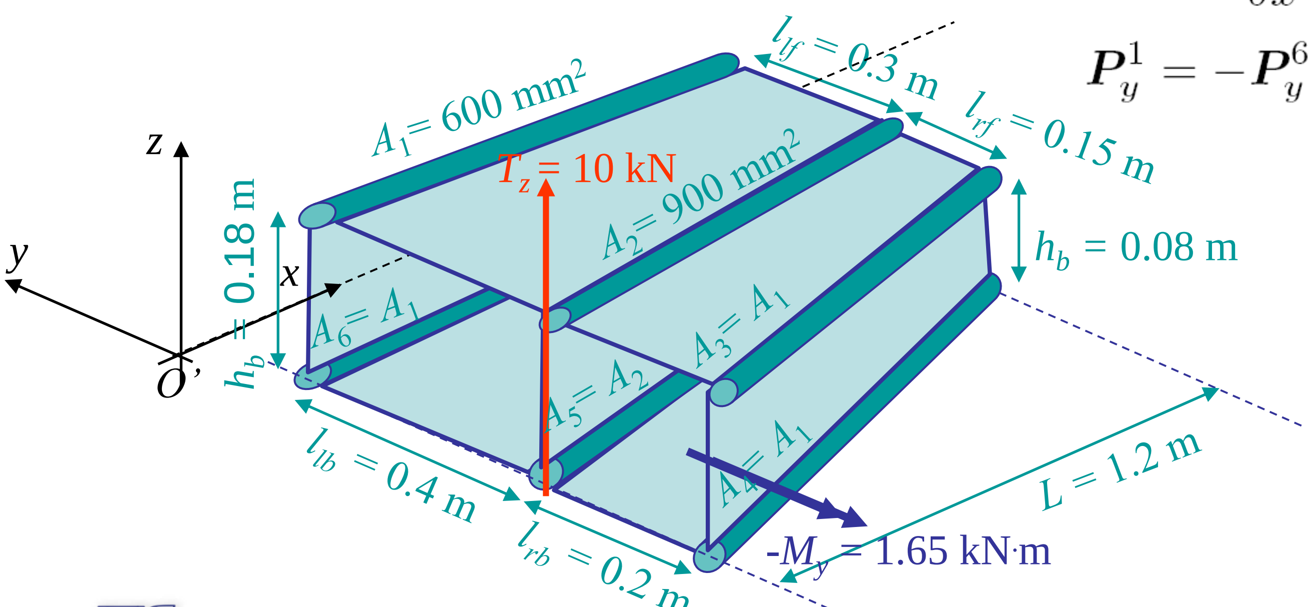
$$\boldsymbol{P}_z^1 = \boldsymbol{P}_z^3 = \boldsymbol{P}_z^4 = \boldsymbol{P}_z^6 = \boldsymbol{P}_x^1 \frac{\delta z^1}{\delta x^1} = -2620\ \frac{-0.05}{1.2} = 109\ \ \text{N}$$

$$\boldsymbol{P}_z^2 = \boldsymbol{P}_z^5 = \boldsymbol{P}_x^2 \frac{\delta z^2}{\delta x^2} = -3930\ \frac{-0.05}{1.2} = 164\ \ \text{N}$$

$$\boldsymbol{P}_y^3 = -\boldsymbol{P}_y^4 = \boldsymbol{P}_x^3 \frac{\delta y^3}{\delta x^3} = -2620\ \frac{0.15}{1.2} = -328\ \ \text{N}$$

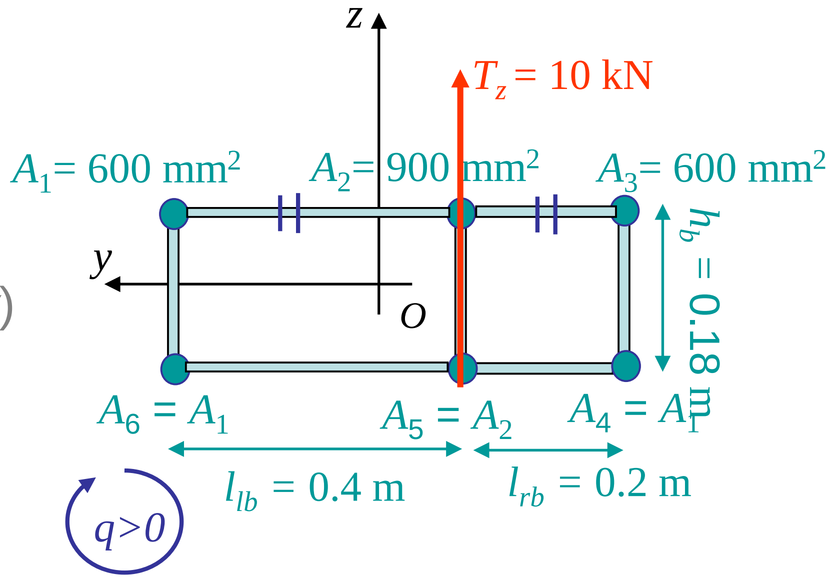
$$\boldsymbol{P}_y^2 = -\boldsymbol{P}_y^5 = \boldsymbol{P}_x^2 \frac{\delta y^2}{\delta x^2} = -3930\ \frac{0.1}{1.2} = -328\ \ \text{N}$$

$$\boldsymbol{P}_y^1 = -\boldsymbol{P}_y^6 = \boldsymbol{P}_x^1 \frac{\delta y^1}{\delta x^1} = -2620\ 0 = 0$$



- Web loading
  - $T_z^{\text{web}} = T_z - \sum_{\text{boom } i} \boldsymbol{P}_z^i = 10000 - 4\,109 - 2\,164 = 9236 \text{ N}$
  - $T_y^{\text{web}} = T_y - \sum_{\text{boom } i} \boldsymbol{P}_y^i = 0$

- Remark
  - q, s are anticlockwise (orientation of Oy)



- Open cells shear flux
  - Cut the cells $\Longrightarrow$ $q_o^{12} = 0$ & $q_o^{23} = 0$
  - $q_o^{25} = q_o^{12} - q_o^{23} - \frac{T_z^{\text{web}}}{I_{yy}} A_2 z^2$

$$\Longrightarrow q_o^{25} = -\frac{9236}{34\ 10^{-6}}\ 0.0009\ 0.09$$
$$= -22\ 10^3 \quad \text{N} \cdot \text{m}^{-1}$$

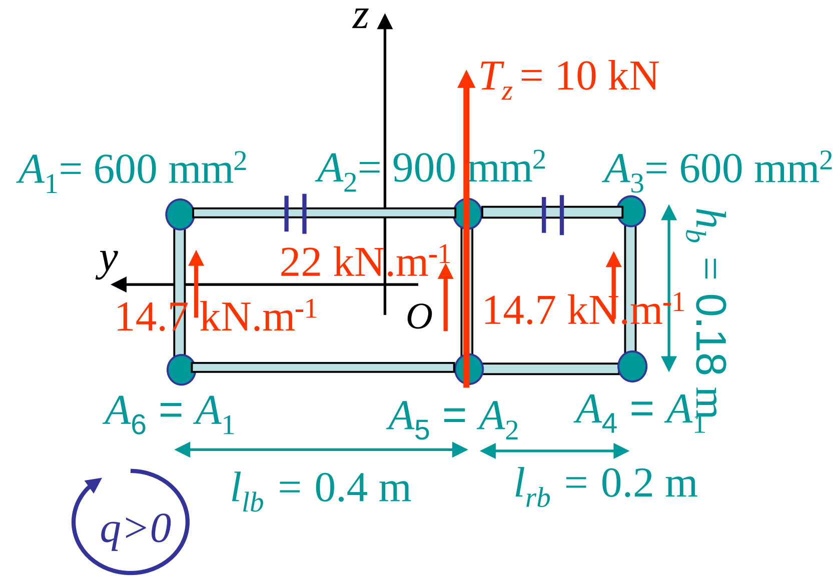
  - $q_o^{34} = q_o^{23} - \frac{T_z^{\text{web}}}{I_{yy}} A_3 z^3$

$$\Longrightarrow q_o^{34} = -\frac{9236}{34\ 10^{-6}}\ 0.0006\ 0.09 = -14.7\ 10^3 \quad \text{N} \cdot \text{m}^{-1}$$

  - $q_o^{45} = q_o^{34} - \frac{T_z^{\text{web}}}{I_{yy}} A_4 z^4 \Longrightarrow q_o^{45} = -14.7\ 10^3 - \frac{9236}{34\ 10^{-6}}\ 0.0006\ (-0.09) = 0$
  - $q_o^{56} = q_o^{25} + q_o^{45} - \frac{T_z^{\text{web}}}{I_{yy}} A_5 z^5$

$$\Longrightarrow q_o^{56} = -22\ 10^3 - \frac{9236}{34\ 10^{-6}}\ 0.0009\ (-0.09) = 0$$

  - $q_o^{61} = q_o^{56} - \frac{T_z^{\text{web}}}{I_{yy}} A_6 z^6 \Longrightarrow q_o^{61} = -\frac{9236}{34\ 10^{-6}}\ 0.0006\ (-0.06) = 14.7\ 10^3 \quad \text{N} \cdot \text{m}^{-1}$



- Non dimensional lengths
  - Take $\mu_{\text{REF}} = 27.6$ GPa
  - Side lengths

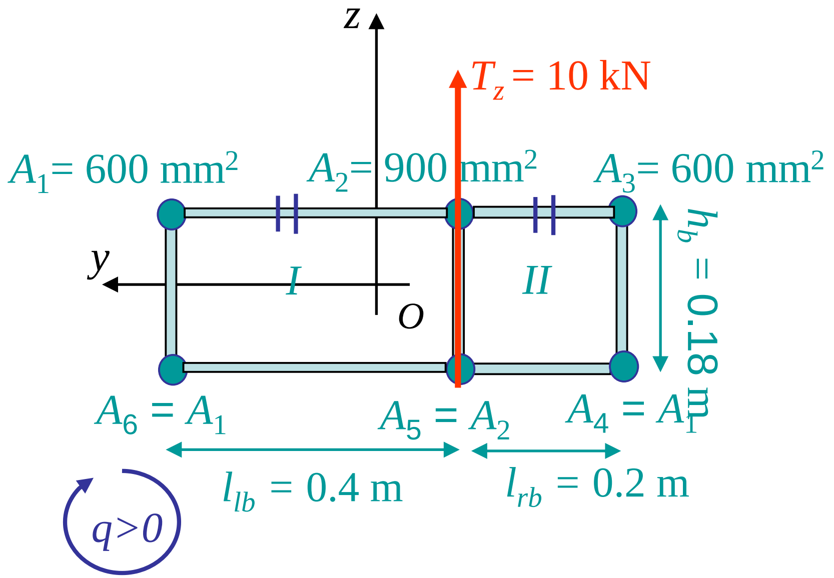
$$\begin{cases} \bar{l}^{1\,2} = \bar{l}^{5\,6} = \dfrac{0.4}{0.0008} = 500 \\ \bar{l}^{2\,3} = \bar{l}^{4\,5} = \dfrac{0.2}{0.0008} = 250 \\ \bar{l}^{1\,6} = \bar{l}^{2\,5} = \bar{l}^{3\,4} = \dfrac{0.18}{0.001} = 180 \end{cases}$$

  - Cell lengths

$$\begin{cases} \bar{l}^{1} = \bar{l}^{12} + \bar{l}^{25} + \bar{l}^{56} + \bar{l}^{61} = 1360 \\ \bar{l}^{2} = \bar{l}^{23} + \bar{l}^{34} + \bar{l}^{45} + \bar{l}^{52} = 860 \end{cases}$$

  - Intersecting length

$$\bar{l}_1^2 = \bar{l}^{25} = 180$$



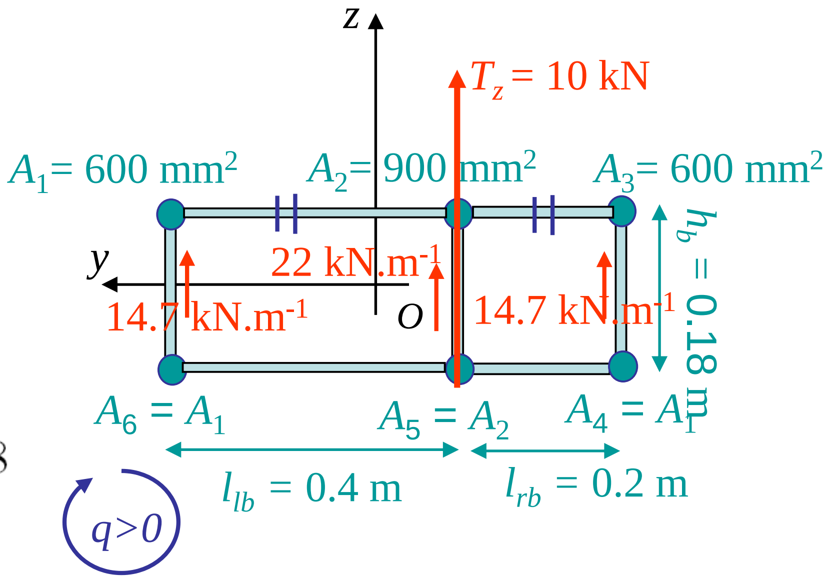
Web	Thickness (mm)	μ (GPa)
12	0.8	27.6
23	0.8	27.6
45	0.8	27.6
56	0.8	27.6
16	1	27.6
25	1	27.6
34	1	27.6

- Integration of open shear flux on cells
  - Cell *I*

$$\oint_1 \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}} ds = \frac{14700}{0.001} 0.18 - \frac{22000}{0.001} 0.18$$
$$= -1.31\ 10^6 \quad \text{N} \cdot \text{m}^{-1}$$

  - Cell *II*

$$\oint_2 \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}} ds = -\frac{14700}{0.001} 0.18 + \frac{22000}{0.001} 0.18$$
$$= 1.31\ 10^6 \quad \text{N} \cdot \text{m}^{-1}$$



- **Twist Rate**
  - Cell I

$$\theta_{,x} = \frac{1}{2\ A_h^I\ \mu_{\mathrm{REF}}}\left[q^I(0)\bar{l}^1 + \oint_1 \frac{q_o}{t\frac{\mu}{\mu_{\mathrm{REF}}}}ds - q^{II}(0)\bar{l}_1^2\right]$$

$$\Longrightarrow \theta_{,x} = \frac{1}{2\ 0.4\ 0.18\ 27.6\ 10^9}\left[1360\ q^I(0) - 1.31\ 10^6 - 180\ q^{II}(0)\right]$$

$$\Longrightarrow \theta_{,x} = -0.0003296\ \ \mathrm{m}^{-1} + 342\ 10^{-9}\ \mathrm{N}^{-1} q^I(0) - 45.3\ 10^{-9}\ \mathrm{N}^{-1} q^{II}(0)$$

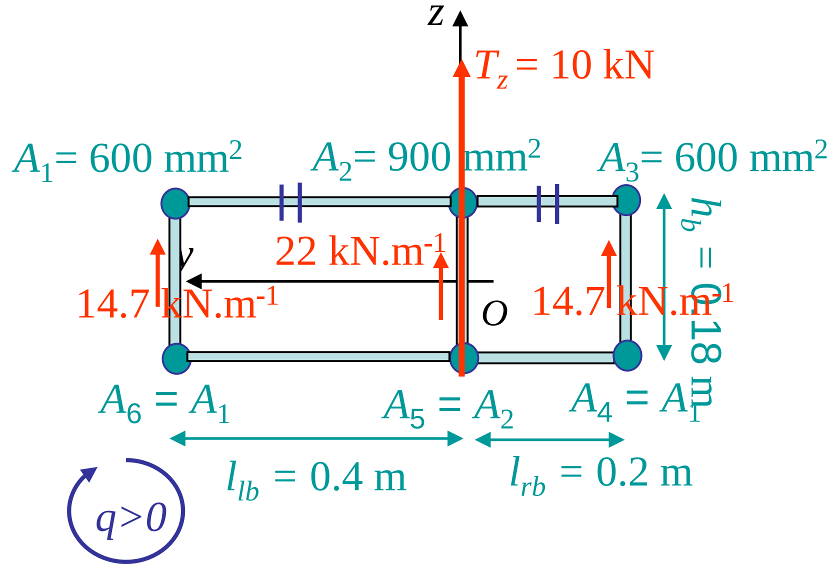
  - Cell II

$$\theta_{,x} = \frac{1}{2\ A_h^{II}\ \mu_{\mathrm{REF}}}\left[q^{II}(0)\bar{l}^2 + \oint_2 \frac{q_o}{t\frac{\mu}{\mu_{\mathrm{REF}}}}ds - q^I(0)\bar{l}_1^2\right]$$

$$\Longrightarrow \theta_{,x} = \frac{1}{2\ 0.2\ 0.18\ 27.6\ 10^9}\left[860\ q^{II}(0) + 1.31\ 10^6 - 180\ q^I(0)\right]$$

$$\Longrightarrow \theta_{,x} = 0.000659\ \ \mathrm{m}^{-1} - 90.6\ 10^{-9}\ \mathrm{N}^{-1} q^I(0) + 432.\ 10^{-9}\ \mathrm{N}^{-1} q^{II}(0)$$

- Momentum balance
  - Move origin to web 2-5



$$y_T T_z = \sum_i \int_{\text{wall } i} q_o p ds + \sum_{\text{cell } c} 2A_h^c q^c(0) + \sum_{\text{boom } j} y^j \boldsymbol{P}_z^j - \sum_{\text{boom } j} z^j \boldsymbol{P}_y^j$$

- With

$$\left\{\begin{array}{l} \displaystyle\sum_i \int_{\text{wall } i} q_o p ds = q_o^{61} l_{lb} h_b + q_o^{34} l_{rb} h_b \\ \quad = 14700\,(0.4\ 0.18 - 0.2\ 0.18) = 529.2 \quad \text{N}\cdot\text{m} \\ \displaystyle\sum_{\text{boom } j} y^j \boldsymbol{P}_z^j = 2\ 109\ 0.4 + 2\ 109\ (-0.2) = 43.8 \quad \text{N}\cdot\text{m} \\ \displaystyle\sum_{\text{boom } j} z^j \boldsymbol{P}_y^j = 2\ (-328)\ 0.09 + 2\ 328\ (-0.09) = -118 \quad \text{N}\cdot\text{m} \end{array}\right.$$

$$\Longrightarrow 0 = 529.2 + 2\ 0.4\ 0.18\ q^I(0) + 2\ 0.2\ 0.18\ q^{II}(0) + 43.8 + 118$$

$$\Longrightarrow 0 = 691 \quad \text{N}\cdot\text{m} + 0.144\ \text{m}^2\ q^I(0) + 0.072\ \text{m}^2\ q^{II}(0)$$

- Compatibility

$$- \left\{ \begin{array}{l} -0.0003296 \;\; \mathrm{m}^{-1} + 342\ 10^{-9}\ \mathrm{N}^{-1} q^{I}(0) - 45.3\ 10^{-9}\ \mathrm{N}^{-1} q^{II}(0) \\ \quad = 0.000659 \;\; \mathrm{m}^{-1} - 90.6\ 10^{-9}\ \mathrm{N}^{-1} q^{I}(0) + 432.\ 10^{-9}\ \mathrm{N}^{-1} q^{II}(0) \\ \\ 0 = 691 \;\; \mathrm{N}\cdot\mathrm{m} + 0.144\ \mathrm{m}^2\ q^{I}(0) + 0.072\ \mathrm{m}^2\ q^{II}(0) \end{array} \right.$$

$$\Longrightarrow q^{I}(0) = \frac{0.000659 + 0.0003296}{342\ 10^{-9} + 90.6\ 10^{-9}} + \frac{432 + 45.3}{342 + 90.6} q^{II}(0)$$

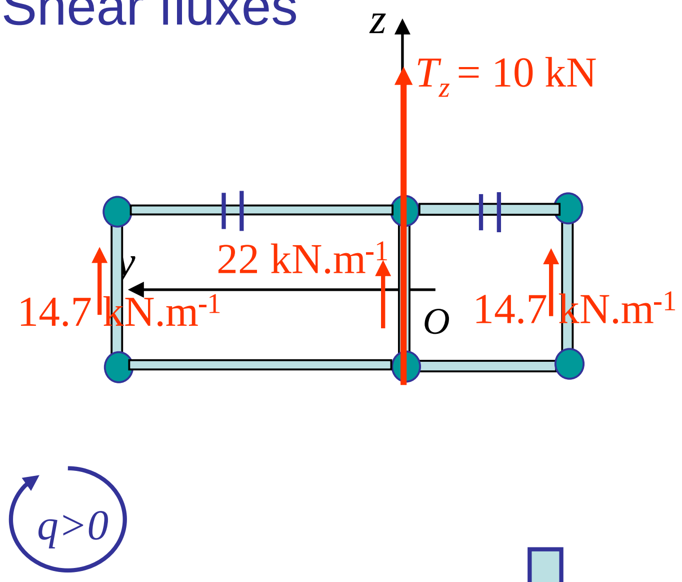
$$\Longrightarrow q^{I}(0) = 2285 \;\; \mathrm{N}\cdot\mathrm{m}^{-1} + 1.03 q^{II}(0)$$

$$\Longrightarrow 0 = 691 + 329 + (0.072 + 0.159)\, q^{II}(0)$$

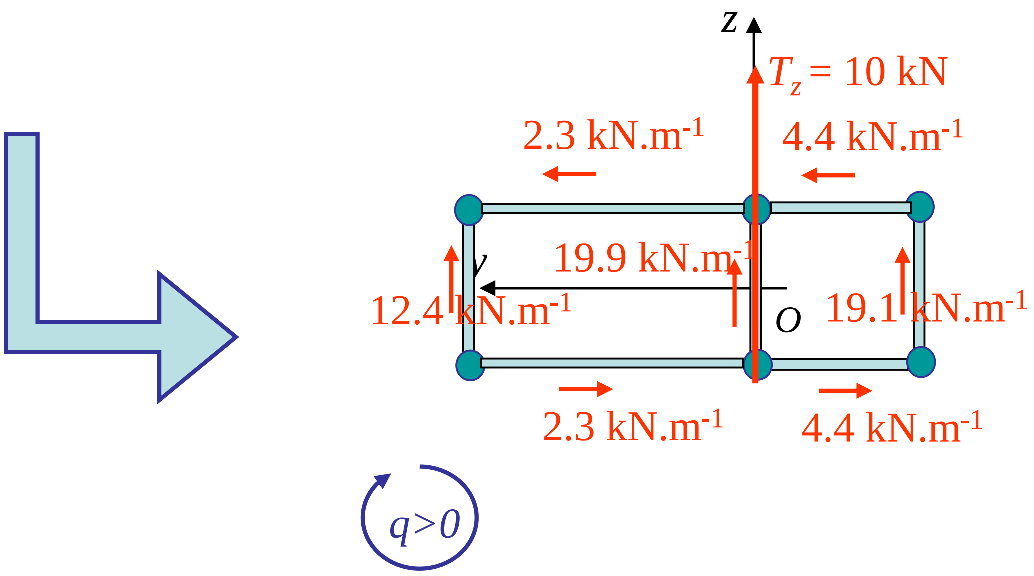
$$\Longrightarrow q^{II}(0) = -4.4\ 10^{3}\ \mathrm{N}\cdot\mathrm{m}^{-1}$$

$$\Longrightarrow q^{I}(0) = -2.3\ 10^{3}\ \mathrm{N}\cdot\mathrm{m}^{-1}$$

- Shear fluxes

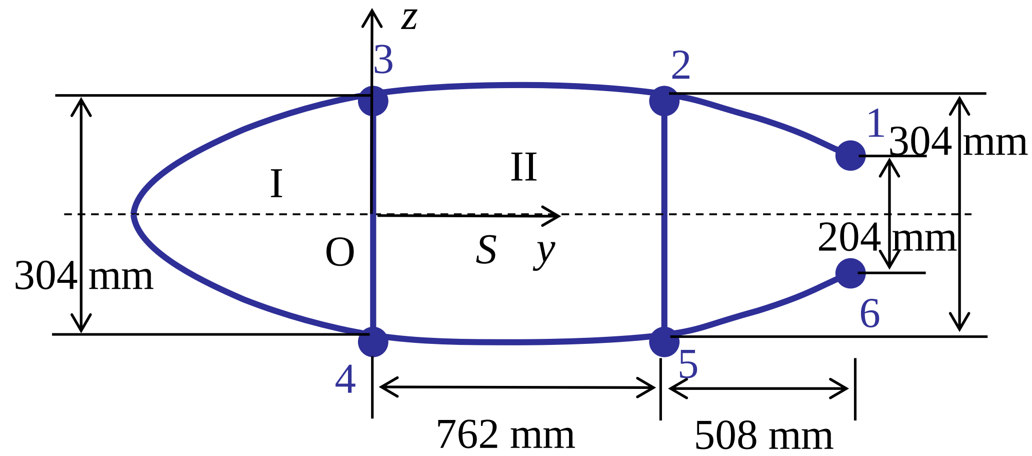


$$\begin{cases} q^I(0) = -2.3\ 10^3\ \mathrm{N}\cdot\mathrm{m}^{-1} \\ q^{II}(0) = -4.4\ 10^3\ \mathrm{N}\cdot\mathrm{m}^{-1} \end{cases}$$



Exercise 1: Wing

- Symmetric wing section
  - 2 closed cells
  - 1 open cell
- Idealized section
  - Walls carry shear stress
    - Constant shear modulus
    - 12 & 56 are assumed straight
  - Booms carry direct stress
- Shear center?

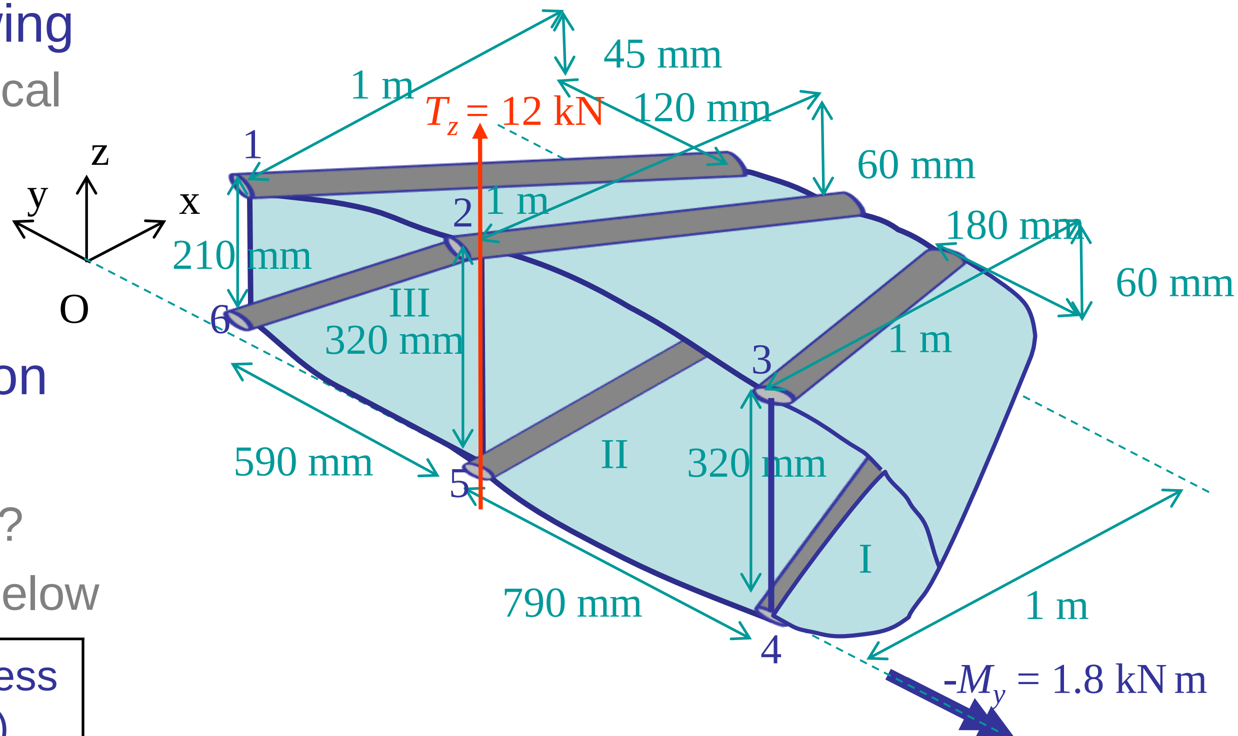


Wall	Length (mm)	Thickness (mm)
12, 56	510	0.559
23, 45	765	0.915
$34^{(}$	1015	0.559
$34^{\vert}$	304	2.030
25	304	1.625

Boom	Area (mm^2)
1, 6	645
2, 5	1290
3, 4	1935

Cell	Area (mm^2)
I	93 000
II	258 000

- Tapered three-cell wing
  - Idealized symmetrical cross-sections
    - Constant shear modulus
- At larger cross-section
  - Load in booms?
  - Shear flow in walls?
  - Dimensions here below



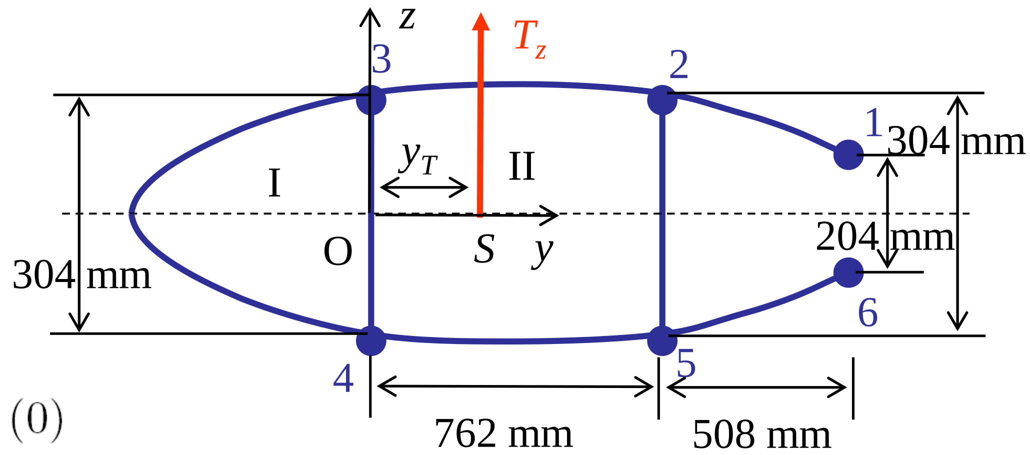
Wall	Length (mm)	Thickness (mm)
12, 56	600	1.0
23, 45	800	1.0
$34^{(}$	1200	0.6
$34^{\vert}$	320	2.0
25	320	2.0
16	210	1.5

Boom	Area (mm^2)
1, 6	600
2, 5	800
3, 4	800

Cell	Area (mm^2)
I	100 000
II	260 000
III	180 000

References

- Lecture notes
  - Aircraft Structures for engineering students, T. H. G. Megson, Butterworth-Heinemann, An imprint of Elsevier Science, 2003, ISBN 0 340 70588 4
- Other references
  - Books
    - Mécanique des matériaux, C. Massonet & S. Cescotto, De boek Université, 1994, ISBN 2-8041-2021-X



- Symmetric wing section
  - Shear center lies on Oy
    - Consider T_z only
  - $I_{yz} = 0$
  - Idealized section

$$\Rightarrow q(s) = -\frac{T_z}{I_{yy}} \sum_{i:\, s_i \leq s} z_i A_i + q(0)$$

- With

$$I_{yy} = \sum_{i=1}^{6} A_i z_i^2$$
$$= 2 \times 645 \times 102^2 + 2 \times 1290 \times 152^2 + 2 \times 1935 \times 152^2 = 162.4 \times 10^6 \; mm^4$$

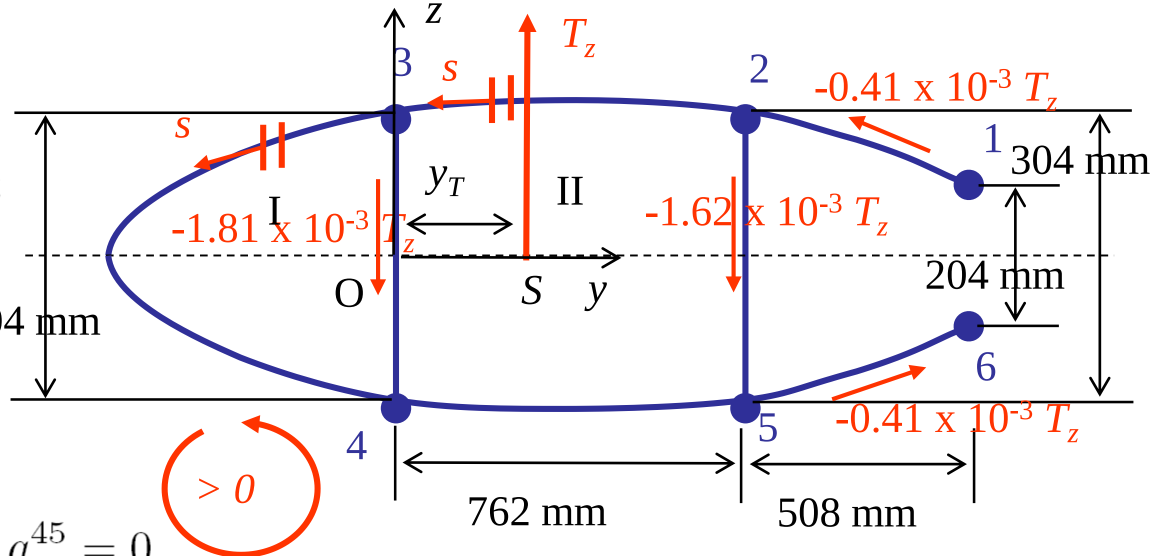
Boom	Area (mm^2)
1, 6	645
2, 5	1290
3, 4	1935

- Open shear flow

$$q_o(s) = -\frac{T_z}{I_{yy}} \sum_{i:\, s_i \leq s} z_i A_i \Leftrightarrow q_o(s) = -6.16 \times 10^{-9} T_z \sum_{i:\, s_i \leq s} A_i z_i$$

- Open shear flow

$$q_o(s) = -6.16 \times 10^{-9} T_z \sum_{i:\, s_i \leq s} A_i z_i$$



  – Cut cells I & II

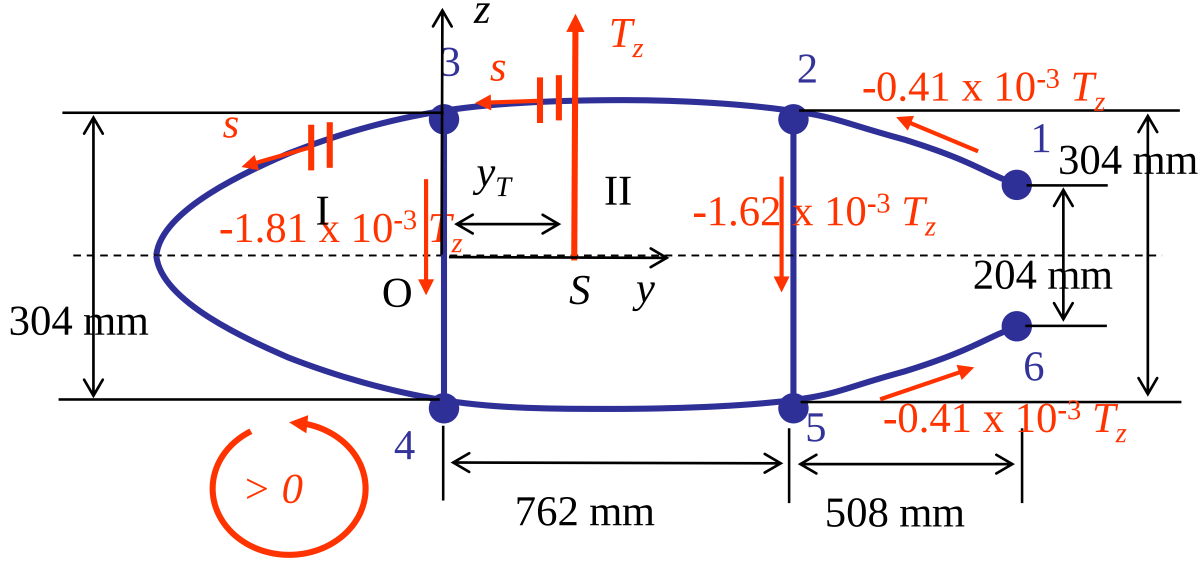
    - $q_o^{34(} = q_o^{23} = 0$
    - From symmetry $q_o^{23} = q_o^{45} = 0$

$$\Rightarrow \begin{cases} q_o^{34|} = -6.16 \times 10^{-9} T_z \times 1935 \times 152 = -1.81 \times 10^{-3} T_z\ N/mm \\ q_o^{65} = -6.16 \times 10^{-9} T_z \times 645 \times (-102) = 0.41 \times 10^{-3} T_z\ N/mm \\ q_o^{52} = 0.41 \times 10^{-3} T_z - 6.16 \times 10^{-9} T_z \times 1290 \times (-152) = 1.62 \times 10^{-3} T_z\ N/mm \end{cases}$$

- From symmetry $q_o^{65} = q_o^{21} = 0.41 \times 10^{-3} T_z\ N/mm$

Boom	Area (mm^2)
1, 6	645
2, 5	1290
3, 4	1935

- Compatibility
  - Rate of twist in each cell
    - Should be the same (by compatibility)
    - Should be zero (no torsion)

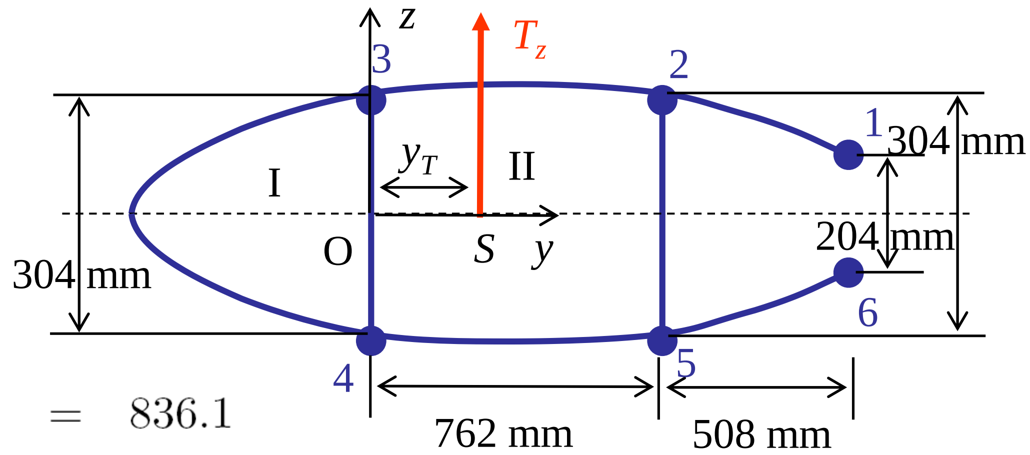


  - Use formula

$$\theta_{,x} = \frac{1}{2A_h^i \mu_{\text{REF}}} \left[-q^{i-1}(0)\bar{l}^i_{i-1} + q^i(0)\bar{l}^i + \oint_{\text{cell } i} \frac{q_o}{t \frac{\mu}{\mu_{\text{REF}}}} ds - q^{i+1}(0)\bar{l}^{i+1}_i \right]$$

- Requires non dimensional lengths and integration of open shear flux

- Non dimensional lengths
  - Constant shear modulus



  - Sides length

$$
\left\{
\begin{array}{lllll}
\bar{l}^{23} & = & \bar{l}^{45} & = \dfrac{765}{0.915} & = 836.1 \\
\bar{l}^{34|} & = & \dfrac{304}{2.03} & = 149.8 & \\
\bar{l}^{34(} & = & \dfrac{1015}{0.559} & = 1815.7 & \\
\bar{l}^{25} & = & \dfrac{304}{1.625} & = 187.1 &
\end{array}
\right.
$$

Wall	Length (mm)	Thickness (mm)
12, 56	510	0.559
23, 45	765	0.915
34(	1015	0.559
34|	304	2.030
25	304	1.625

  - Intercepting length

$$\bar{l}_1^2 = \bar{l}^{34|} = 149.8$$

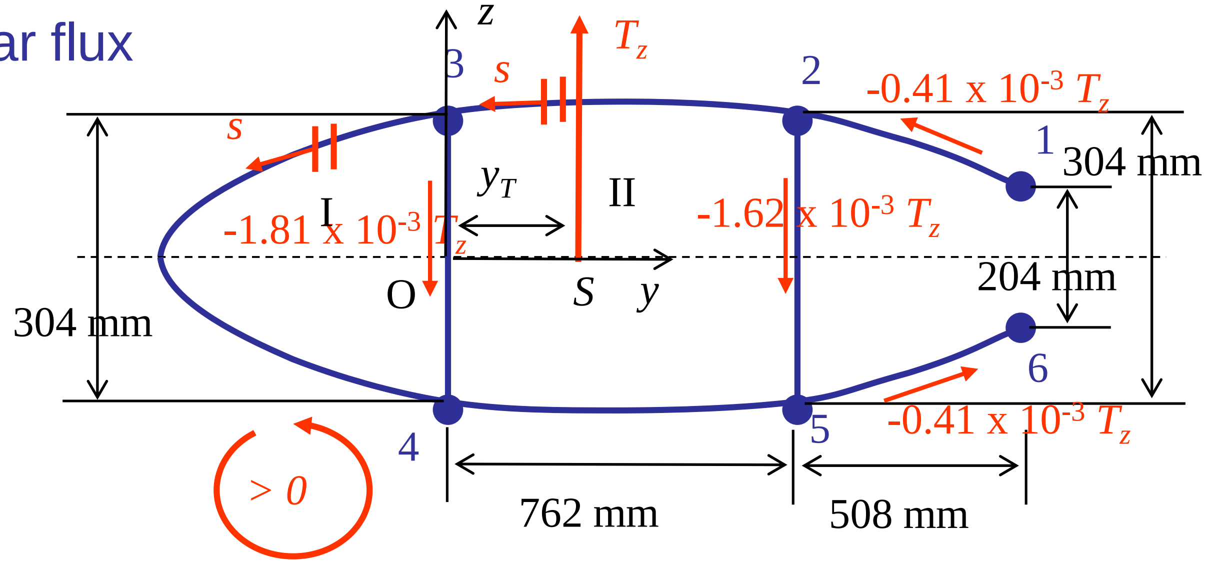
  - Cells length

$$
\left\{
\begin{array}{lll}
\bar{l}^{1} & = \bar{l}^{34(} + \bar{l}^{34|} & = 1965.5 \\
\bar{l}^{2} & = \bar{l}^{23} + \bar{l}^{34|} + \bar{l}^{45} + \bar{l}^{52} & = 2009.1
\end{array}
\right.
$$

Exercise 1: Wing

- Integration of open shear flux

Wall	Length (mm)	Thickness (mm)
12, 56	510	0.559
23, 45	765	0.915
$34^{(}$	1015	0.559
$34^{\vert}$	304	2.030
25	304	1.625



– Cell I

$$\oint_1 \frac{q_o}{t\frac{\mu}{\mu_{REF}}}\,ds = \frac{q_0^{43^|}\times l^{43^|}}{t^{43^|}} = \frac{1.81\times 10^{-3}T_z\times 304}{2.03} = 0.271T_z$$

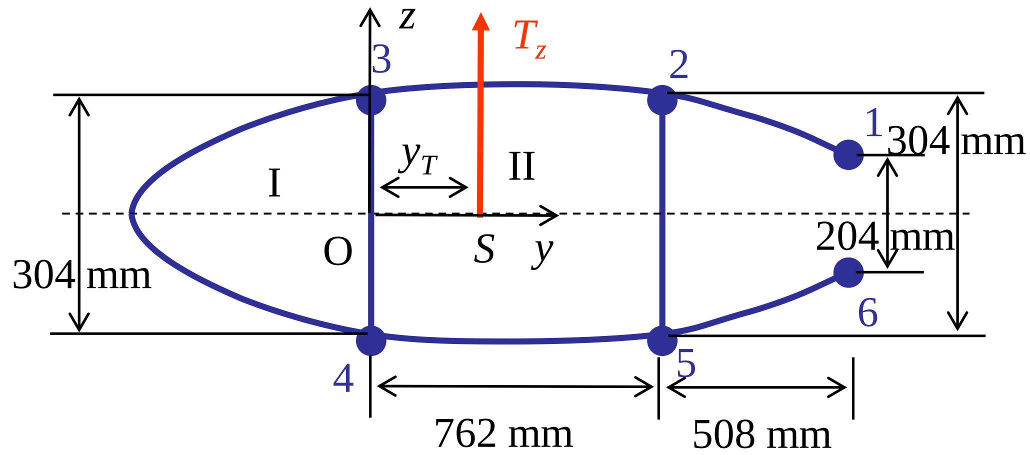
– Cell II

$$\oint_2 \frac{q_o}{t\frac{\mu}{\mu_{REF}}}\,ds = \frac{q_0^{34^|}\times l^{34^|}}{t^{34^|}} + \frac{q_0^{52}\times l^{52}}{t^{52}}$$

$$= \frac{-1.81\times 10^{-3}T_z\times 304}{2.03} + \frac{1.62\times 10^{-3}T_z\times 304}{1.625} = 319.64\times 10^{-4}T_z$$

- Twist rates

Cell	Area (mm^2)
I	93 000
II	258 000



- Cell I

$$\theta_{,x} = \frac{1}{2\ A_h^I\ \mu_{\text{REF}}}\left[q^I(0)\bar{l}^1 + \oint_1 \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}}ds - q^{II}(0)\bar{l}_1^2\right]$$

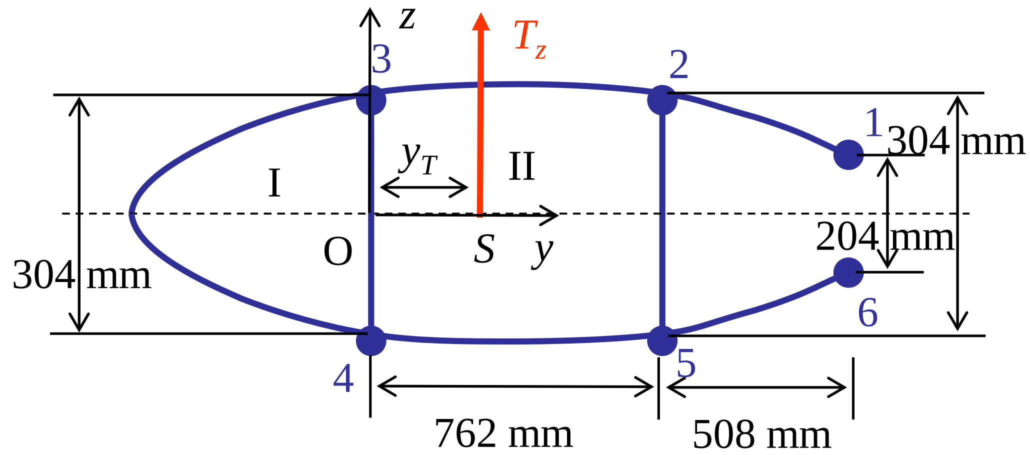
$$\theta_{,x} = \frac{1}{2\times 93000\mu}\left(1965.5q^I\left(0\right) - 149.8q^{II}\left(0\right) + 0.271T_z\right)$$

- Cell II

$$\theta_{,x} = \frac{1}{2\ A_h^{II}\ \mu_{\text{REF}}}\left[-q^I\left(0\right)\bar{l}_1^2 + q^{II}\left(0\right)\bar{l}^2 + \oint_2 \frac{q_o}{t\frac{\mu}{\mu_{REF}}}\right]$$

$$\theta_{,x} = \frac{1}{2\times 258000\mu}\left(-149.8q^I\left(0\right) + 2009.1q^{II}\left(0\right) + 319.64\times 10^{-4}T_z\right)$$

- Constant shear flows
  - As there is no torsion twist rates should be equal to 0



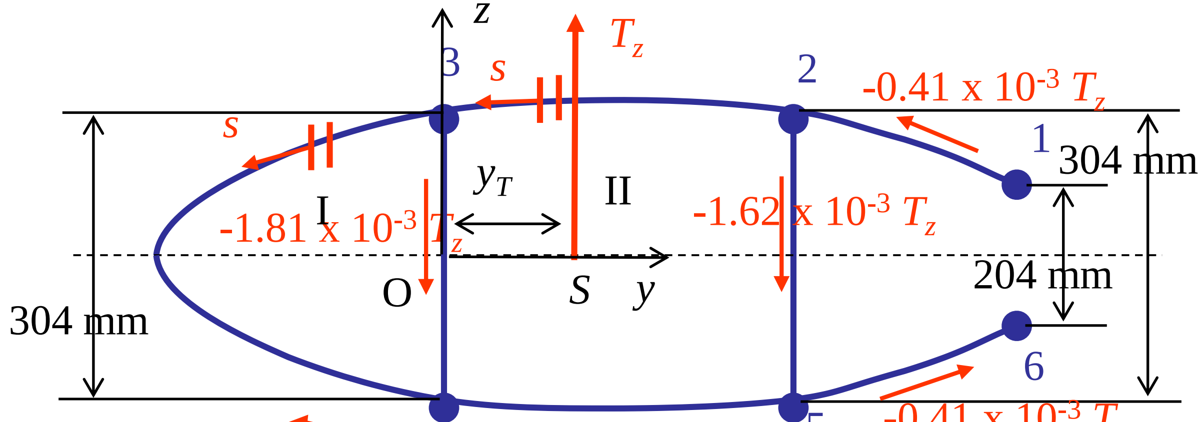
$$\begin{cases} 1965.5q^{I}(0) - 149.8q^{II}(0) + 0.271T_z &= 0 \\ -149.8q^{I}(0) + 2009.1q^{II}(0) + 319.64\times10^{-4}T_z &= 0 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} 1965.5 & -149.8 \\ -149.8 & 2009.1 \end{pmatrix} * \begin{bmatrix} q^{I}(0) \\ q^{II}(0) \end{bmatrix} = \begin{bmatrix} -0.271T_z \\ -319.64\times10^{-4}T_z \end{bmatrix}$$

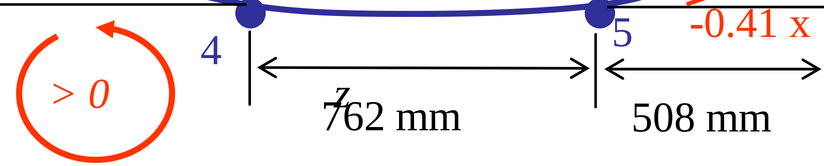
$$\Leftrightarrow \begin{bmatrix} q^{I}(0) \\ q^{II}(0) \end{bmatrix} = \begin{bmatrix} -0.139\times10^{-3}T_z \\ -0.026\times10^{-3}T_z \end{bmatrix}$$

- Shear center

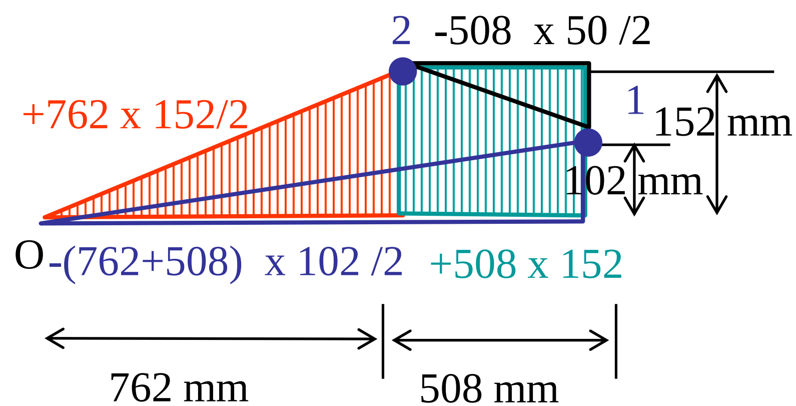
Cell	Area (mm^2)
I	93 000
II	258 000



$$\begin{bmatrix} q^I(0) \\ q^{II}(0) \end{bmatrix} = \begin{bmatrix} -0.139 \times 10^{-3} T_z \\ -0.026 \times 10^{-3} T_z \end{bmatrix}$$



– Moment about the mid-point of web 43
  - Open shear flows in the wall 12 and 65 (assumed straight) are accounted for by computing the swept areas



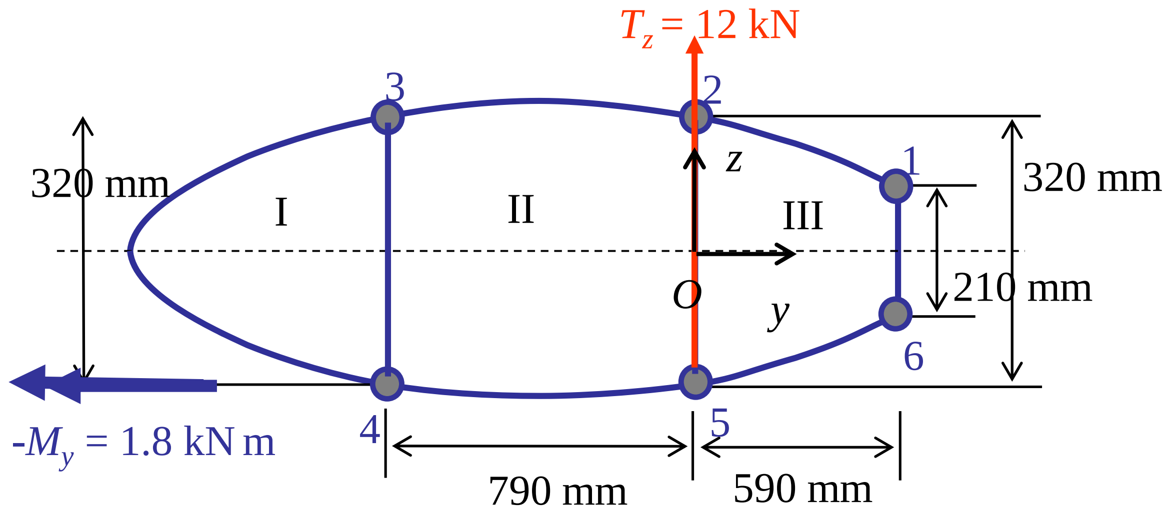
$$y_T T_z = \sum_i \int_{\text{wall } i} q_o p \, ds + \sum_{\text{cell } c} 2 A_h^c q_c(0)$$

$$y_T T_z = -2 \times q_o^{21} \times 508 \times 152 - 2 \times q_o^{21} \times 50 \times 762 + q_o^{52} \times 304 \times 762 + 2 \times 258000 \, q^{II}(0) + 2 \times 93000 \, q^{I}(0)$$

$$y_T = 241.4 \; mm$$

Exercise 2: Tapered wing

- Bending
  - By symmetry
    - Centroid on Oy
    - $I_{yz} = 0$
  - Only idealized booms are carrying direct stress



Boom	Area (mm²)
1, 6	600
2, 5	800
3, 4	800

- $$I_{yy} = \sum_{i=1}^{6} A_i z_i^2 = 2 \times 600 \times 105^2 + 4 \times 800 \times 160^2$$
  $$= 95.2 \times 10^6 \; mm^4$$
- Loading in r^{th} boom

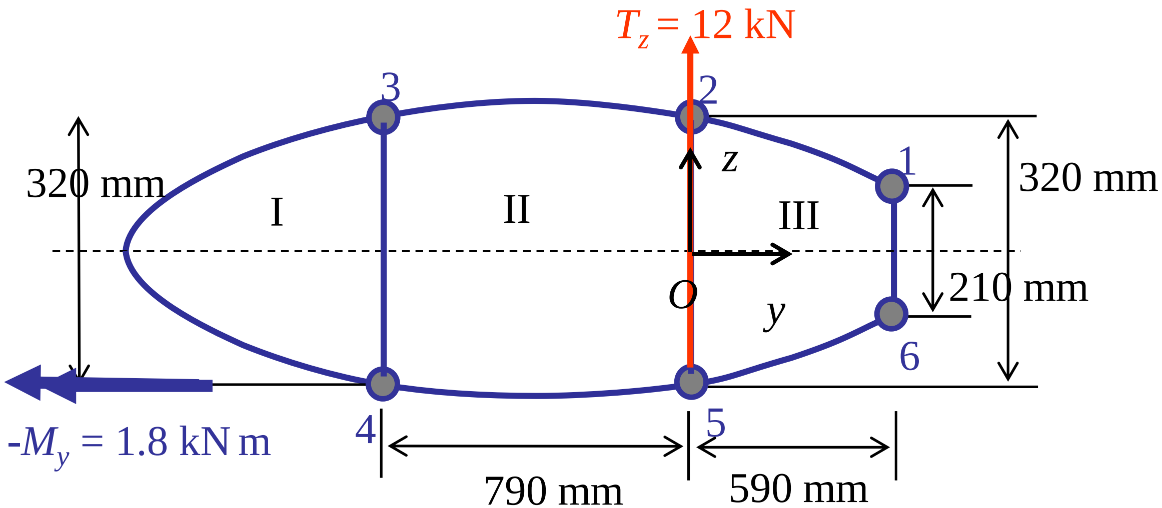
$$\mathbf{P}_x^r = \sigma_{xx}^r \times A_r = \frac{M_{yy} A_r z_r}{I_{yy}}$$

$$\Leftrightarrow \mathbf{P}_x^r = \frac{-1800 \times 10^3}{95.2 \times 10^6} A_r z_r$$

$$\Leftrightarrow \mathbf{P}_x^r = -1.89 \times 10^{-2} A_r z_r$$

- Bending (2)

Boom	Area (mm²)
1, 6	600
2, 5	800
3, 4	800



  – Booms loading

$$\mathbf{P}_x^r = -1.89 \times 10^{-2} A_r z_r$$

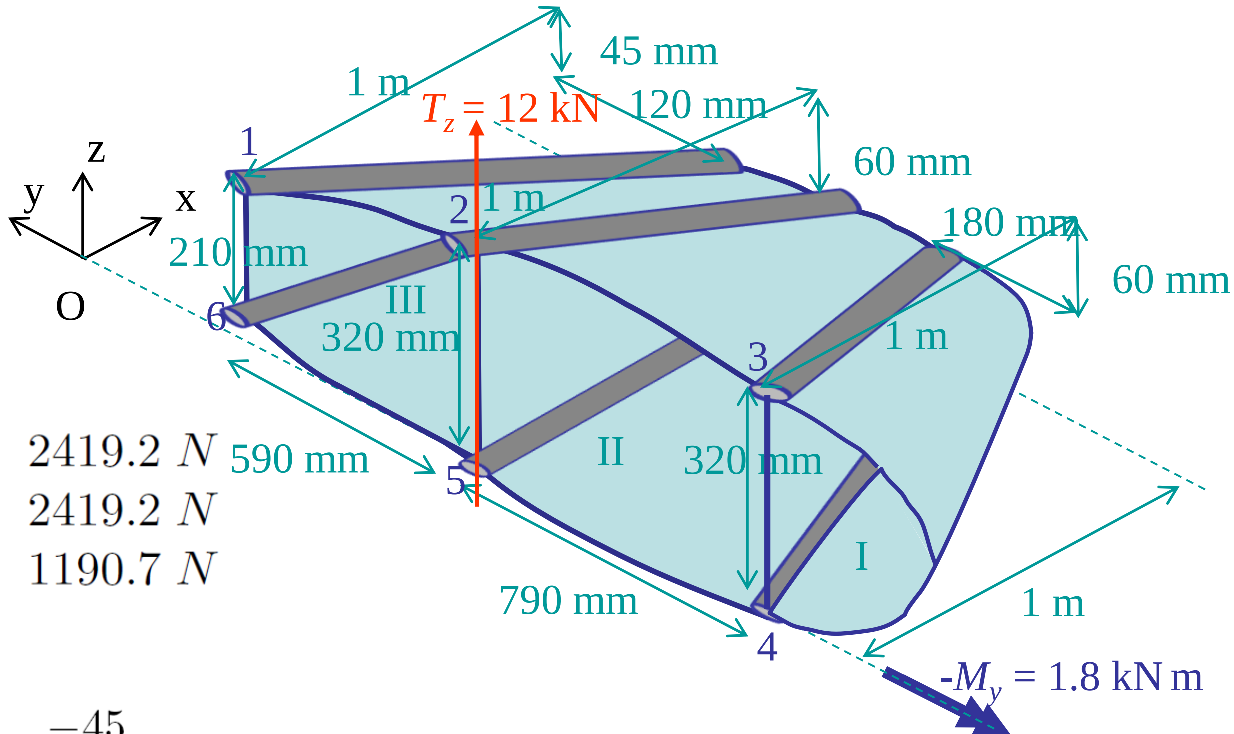
$$\Leftrightarrow \begin{cases} \mathbf{P}_x^1 &= -1.89 \times 10^{-2} \times 600 \times 105 &= -1190.7\ N \\ \mathbf{P}_x^2 &= -1.89 \times 10^{-2} \times 800 \times 160 &= -2419.2\ N \\ \mathbf{P}_x^3 &= \mathbf{P}_x^2 &= -2419.2\ N \end{cases}$$

$$\Leftrightarrow \begin{cases} \mathbf{P}_x^4 &= -\mathbf{P}_x^3 &= 2419.2\ N \\ \mathbf{P}_x^5 &= -\mathbf{P}_x^2 &= 2419.2\ N \\ \mathbf{P}_x^6 &= -\mathbf{P}_x^1 &= 1190.7\ N \end{cases}$$

- Bending (3)
  - Due to taper

$$\begin{cases} \mathbf{P}_y^r = \mathbf{P}_x^r \dfrac{\delta y}{\delta x} \\ \mathbf{P}_z^r = \mathbf{P}_x^r \dfrac{\delta z}{\delta x} \end{cases}$$

$$\begin{cases} \mathbf{P}_x^4 & = & -\mathbf{P}_x^3 & = & 2419.2\ N \\ \mathbf{P}_x^5 & = & -\mathbf{P}_x^2 & = & 2419.2\ N \\ \mathbf{P}_x^6 & = & -\mathbf{P}_x^1 & = & 1190.7\ N \end{cases}$$



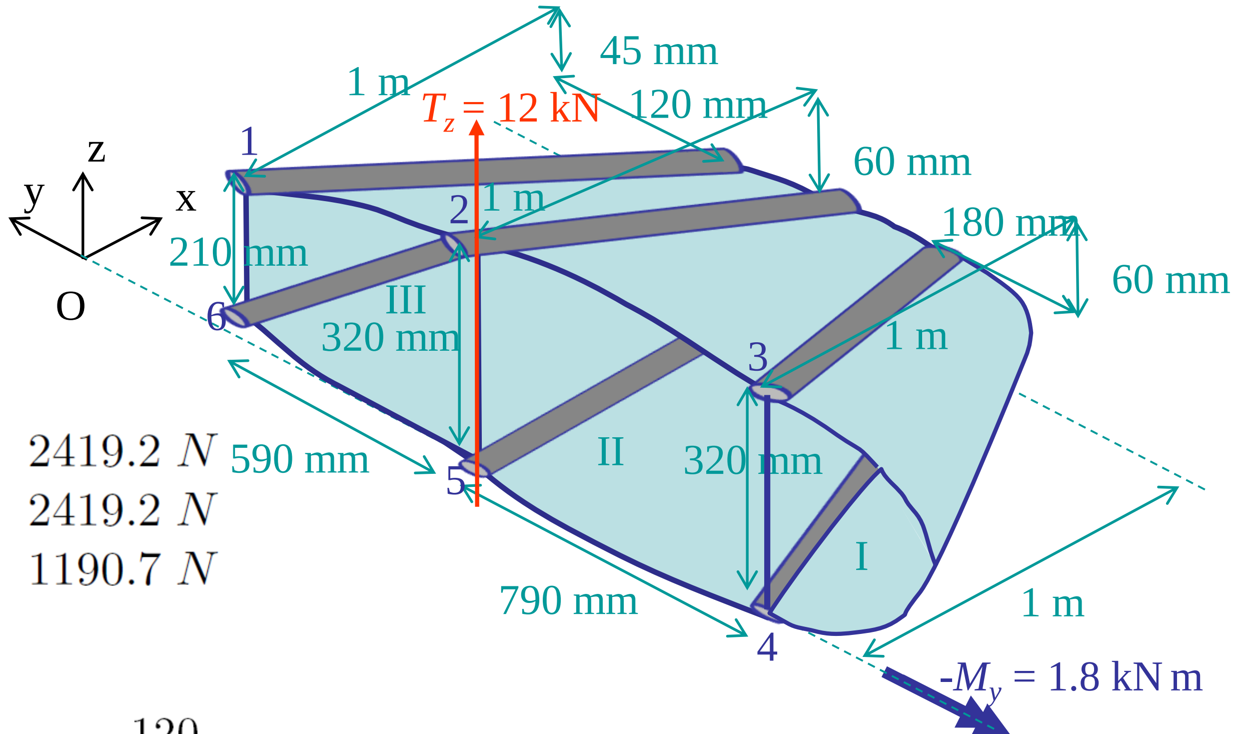
  - Booms loading (2)

$$\begin{cases} \mathbf{P}_z^1 & = & -1190.7 \times \dfrac{-45}{1000} & = & 53.6\ N \\ \mathbf{P}_z^2 & = & -2419.2 \times \dfrac{-60}{1000} & = & 145.2\ N \\ \mathbf{P}_z^3 & = & -2419.2 \times \dfrac{-60}{1000} & = & 145.2\ N \end{cases}$$

$$\begin{cases} \mathbf{P}_z^4 & = & 2419.2 \times \dfrac{60}{1000} & = & 145.2\ N \\ \mathbf{P}_z^5 & = & 2419.2 \times \dfrac{60}{1000} & = & 145.2\ N \\ \mathbf{P}_z^6 & = & 1190.7 \times \dfrac{45}{1000} & = & 53.6\ N \end{cases}$$

- Bending (4)
  - Due to taper (2)

$$\left\{\begin{array}{l} \mathbf{P}_y^r = \mathbf{P}_x^r \dfrac{\delta y}{\delta x} \\ \mathbf{P}_z^r = \mathbf{P}_x^r \dfrac{\delta z}{\delta x} \end{array}\right.$$



$$\left\{\begin{array}{lllll} \mathbf{P}_x^4 & = & -\mathbf{P}_x^3 & = & 2419.2\ N \\ \mathbf{P}_x^5 & = & -\mathbf{P}_x^2 & = & 2419.2\ N \\ \mathbf{P}_x^6 & = & -\mathbf{P}_x^1 & = & 1190.7\ N \end{array}\right.$$

  - Booms loading (3)

$$\left\{\begin{array}{lllll} \mathbf{P}_y^1 & = & -1190.7 \times \dfrac{-120}{1000} & = & 142.9\ N \\ \mathbf{P}_y^2 & = & -2419.2 \times \dfrac{0}{1000} & = & 0\ N \\ \mathbf{P}_y^3 & = & -2419.2 \times \dfrac{180}{1000} & = & -435.5\ N \end{array}\right.$$

$$\left\{\begin{array}{lllll} \mathbf{P}_y^4 & = & -\mathbf{P}_y^3 & = & 435.5\ N \\ \mathbf{P}_y^5 & = & -\mathbf{P}_y^2 & = & 0\ N \\ \mathbf{P}_y^6 & = & -\mathbf{P}_y^1 & = & -142.9\ N \end{array}\right.$$

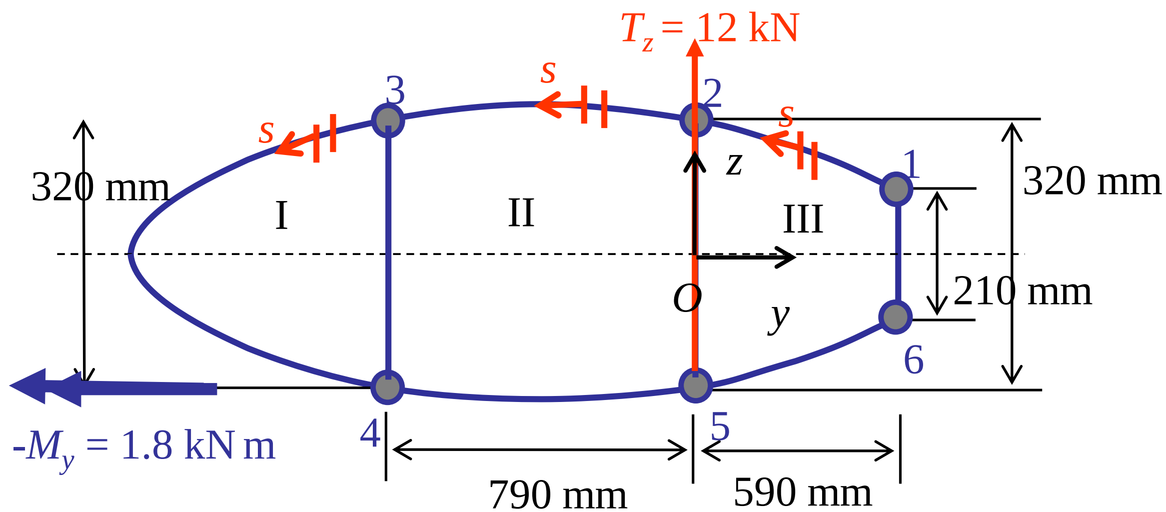
- Web shearing
  - Due to taper

$$T_y^{web} = T_y - \sum_{i=1}^{6} \mathbf{P}_y^i = 0$$

$$T_z^{web} = T_z - \sum_{i=1}^{6} \mathbf{P}_z^i = 12000 - 2 \times (53.6 + 145.2 + 145.2) = 11312 \ N$$

- Open shear flow
  - As $T_y^{web} = 0$

$$q_o = -\frac{T_z^{web}}{I_{yy}} \sum_{i: s_i \leq s} A_i z_i$$

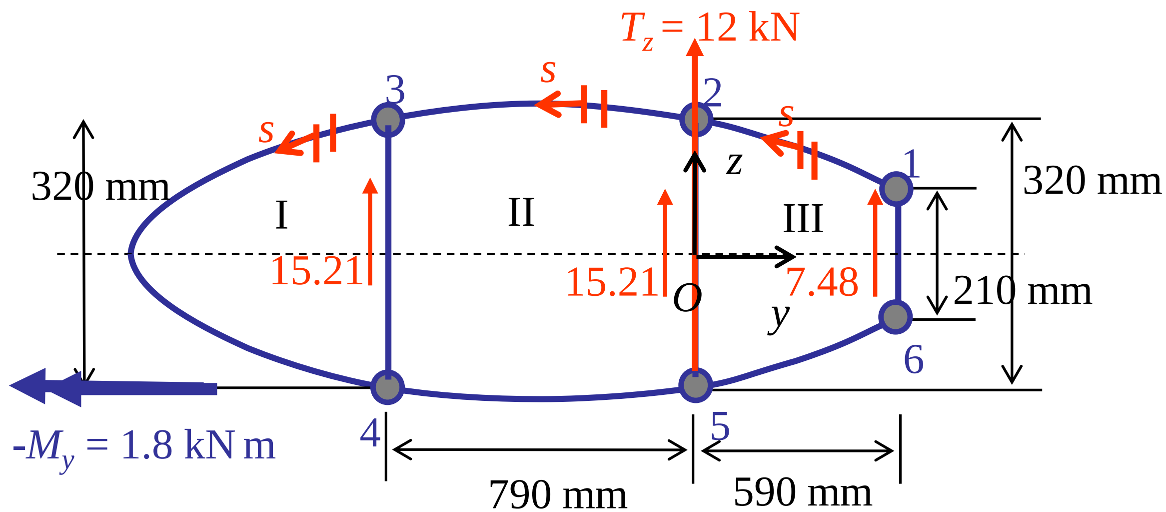


$$\Leftrightarrow \quad q_o(s) = -\frac{11312}{95.2 \times 10^6} \sum_{i: s_i \leq s} A_i z_i$$

$$\Leftrightarrow \quad q_o(s) = -1.188 \times 10^{-4} \sum_{i: s_i \leq s} A_i z_i$$

- Open shear flow (2)
  - Cut each cell

$$q_o^{12} = q_o^{23} = q_o^{34(} = 0$$



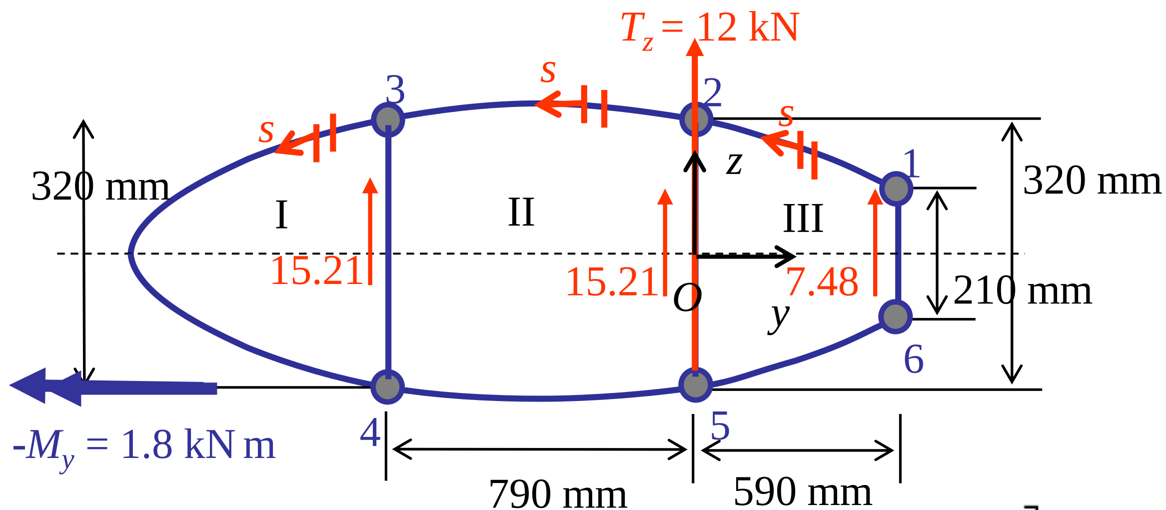
  - Use

$$q_o(s) = -1.188 \times 10^{-4} \sum_{i:s_i \leq s} A_i z_i$$

Boom	Area (mm²)
1, 6	600
2, 5	800
3, 4	800

- $q_o^{34|} = -1.188 \times 10^{-4} \times 800 \times 160 = -15.21\ N/mm$
- Symmetry: $q_o^{45} = q_o^{23} = 0$
- $q_o^{25} = -1.188 \times 10^{-4} \times 800 \times 160 = -15.21\ N/mm$
- Symmetry: $q_o^{12} = q_o^{56} = 0$
- $q_o^{16} = -1.188 \times 10^{-4} \times 600 \times 105 = -7.48\ N/mm$

- Compatibility
  - Rate of twist in each cell
    - Should be the same (by compatibility)
  - Use formula
    - $$\theta_{,x} = \frac{1}{2A_h^i \mu_{\text{REF}}}\left[-q^{i-1}(0)\bar{l}_{i-1}^{i} + q^{i}(0)\bar{l}^{i} + \oint_{\text{cell } i} \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}} ds - q^{i+1}(0)\bar{l}_{i}^{i+1}\right]$$
    - Requires non dimensional lengths and integration of open shear flux



Exercise 2: Tapered wing

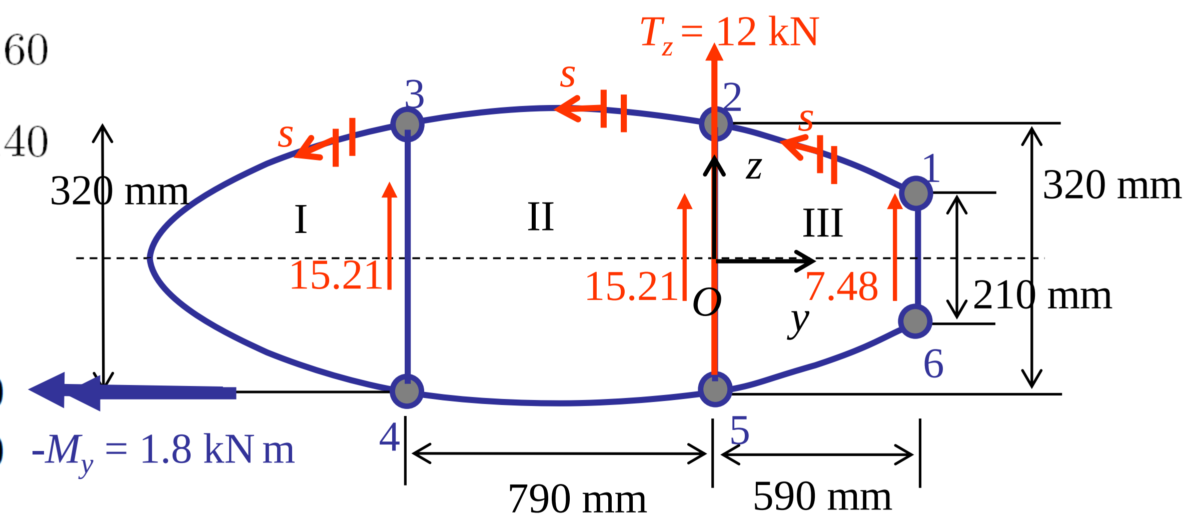
- Non dimensional lengths
  - Constant shear modulus
  - Sides length

$$\begin{cases} \bar{l}^{12} = \bar{l}^{56} = \frac{600}{1} = 600 \\ \bar{l}^{23} = \bar{l}^{45} = \frac{800}{1} = 800 \\ \bar{l}^{34|} = \frac{320}{2} = 160 \\ \bar{l}^{34(} = \frac{1200}{0.6} = 2000 \\ \bar{l}^{25} = \frac{320}{2} = 160 \\ \bar{l}^{61} = \frac{210}{1.5} = 140 \end{cases}$$

Wall	Length (mm)	Thickness (mm)
12, 56	600	1.0
23, 45	800	1.0
34(	1200	0.6
34|	320	2.0
25	320	2.0
16	210	1.5

  - Intercepting length

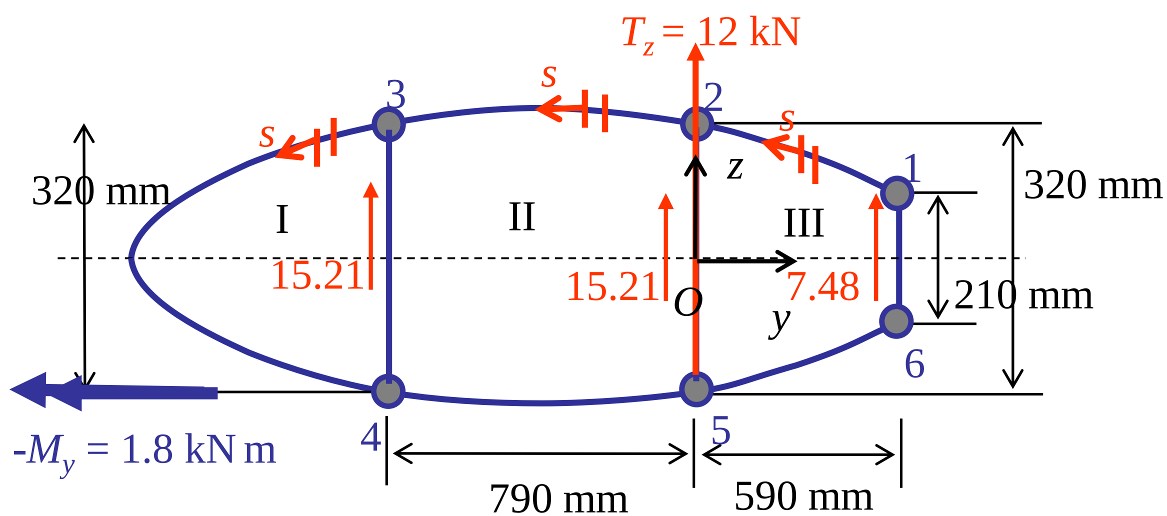
$$\begin{cases} \bar{l}_1^2 = \bar{l}^{34|} = 160 \\ \bar{l}_2^3 = \bar{l}^{25} = 160 \end{cases}$$



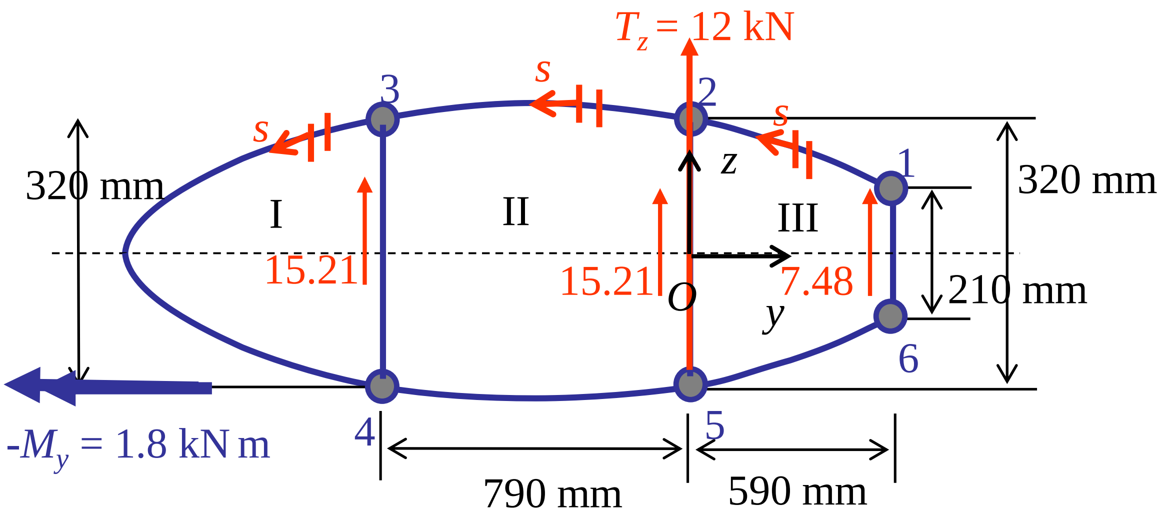
- Non dimensional lengths (2)
  - Cells length

$$\begin{cases} \bar{l}^{1} & = & \bar{l}^{34(} + \bar{l}^{34|} & = & 2160 \\ \bar{l}^{2} & = & \bar{l}^{23} + \bar{l}^{34|} + \bar{l}^{45} + \bar{l}^{52} & = & 1920 \\ \bar{l}^{3} & = & \bar{l}^{12} + \bar{l}^{25} + \bar{l}^{56} + \bar{l}^{61} & = & 1500 \end{cases}$$

Wall	Length (mm)	Thickness (mm)
12, 56	600	1.0
23, 45	800	1.0
34(	1200	0.6
34|	320	2.0
25	320	2.0
16	210	1.5



- Integration of open shear flux



Wall	Length (mm)	Thickness (mm)
12, 56	600	1.0
23, 45	800	1.0
34(	1200	0.6
34|	320	2.0
25	320	2.0
16	210	1.5

– Cell I, II & III

$$\oint_1 \frac{q_o}{t\frac{\mu}{\mu_{REF}}} ds = \frac{q_0^{43|} \times l^{43|}}{t^{43|}} = \frac{15.21 \times 320}{2} = 2433.6$$

$$\oint_2 \frac{q_o}{t\frac{\mu}{\mu_{REF}}} ds = \frac{q_0^{34|} \times l^{34|}}{t^{34|}} + \frac{q_0^{52} \times l^{52}}{t^{52}} = \frac{(-15.21) \times 320}{2} + \frac{15.21 \times 320}{2} = 0$$

$$\oint_3 \frac{q_o}{t\frac{\mu}{\mu_{REF}}} ds = \frac{q_0^{25} \times l^{25}}{t^{25}} + \frac{q_0^{61} \times l^{61}}{t^{61}} = \frac{(-15.21) \times 320}{2} + \frac{7.48 \times 210}{1.5} = -1384.8$$

Exercise 2: Tapered wing

- Twist rate
  - Cell I

$$\theta_{,x} = \frac{1}{2\ A_h^I\ \mu_{\text{REF}}} \left[q^I(0)\bar{l}^1 + \oint_1 \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}} ds - q^{II}(0)\bar{l}_1^2 \right]$$

$$\Rightarrow \theta_{,x} = \frac{1}{2 \times 100000\mu} \left(2160 q^I(0) - 160 q^{II}(0) + 2433.6 \right)$$

Cell	Area (mm^2)
I	100 000
II	260 000
III	180 000

  - Cell II

$$\theta_{,x} = \frac{1}{2\ A_h^{II}\ \mu_{\text{REF}}} \left[-q^I(0)\,\bar{l}_1^2 + q^{II}(0)\,\bar{l}^2 + \oint_2 \frac{q_o}{t\frac{\mu}{\mu_{REF}}} - q^{III}(0)\,\bar{l}_2^3 \right]$$

$$\Rightarrow \theta_{,x} = \frac{1}{2 \times 260000\mu} \left(-160 q^I(0) + 1920 q^{II}(0) - 160 q^{III}(0) \right)$$

  - Cell III

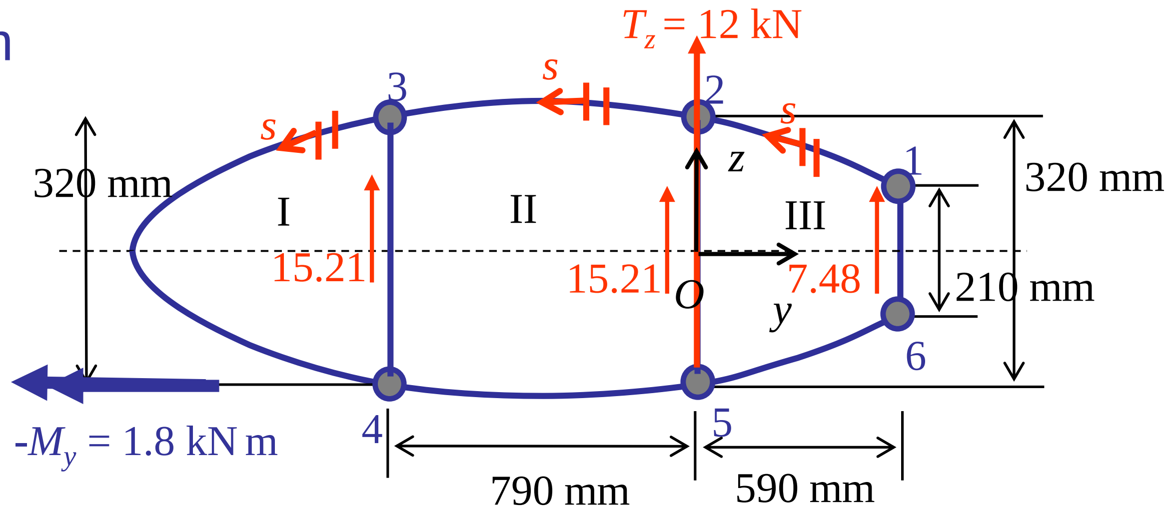
$$\theta_{,x} = \frac{1}{2\ A_h^{III}\ \mu_{\text{REF}}} \left[-q^{II}(0)\,\bar{l}_2^3 + q^{III}(0)\,\bar{l}^3 + \oint_3 \frac{q_o}{t\frac{\mu}{\mu_{REF}}} \right]$$

$$\Rightarrow \theta_{,x} = \frac{1}{2 \times 180000\mu} \left(-160 q^{II}(0) + 1500 q^{III}(0) - 1384.8 \right)$$

  - Three equations and 4 unknowns $\Rightarrow$ momentum equilibrium

- Momentum equilibrium

Cell	Area (mm^2)
I	100 000
II	260 000
III	180 000



- Equation for tapered wing

$$y_T T_z = \sum_i \int_{\text{wall } i} q_o p ds + \sum_{\text{cell } c} 2A_h^c q^c(0) + \sum_{\text{boom } j} y^j \boldsymbol{P}_z^j - \sum_{\text{boom } j} z^j \boldsymbol{P}_y^j$$

  - With

$$\sum_{\text{boom } j} y^j \mathbf{P}_z^j = y^1 \mathbf{P}_z^1 + y^3 \mathbf{P}_z^3 + y^4 \mathbf{P}_z^4 + y^6 \mathbf{P}_z^6$$

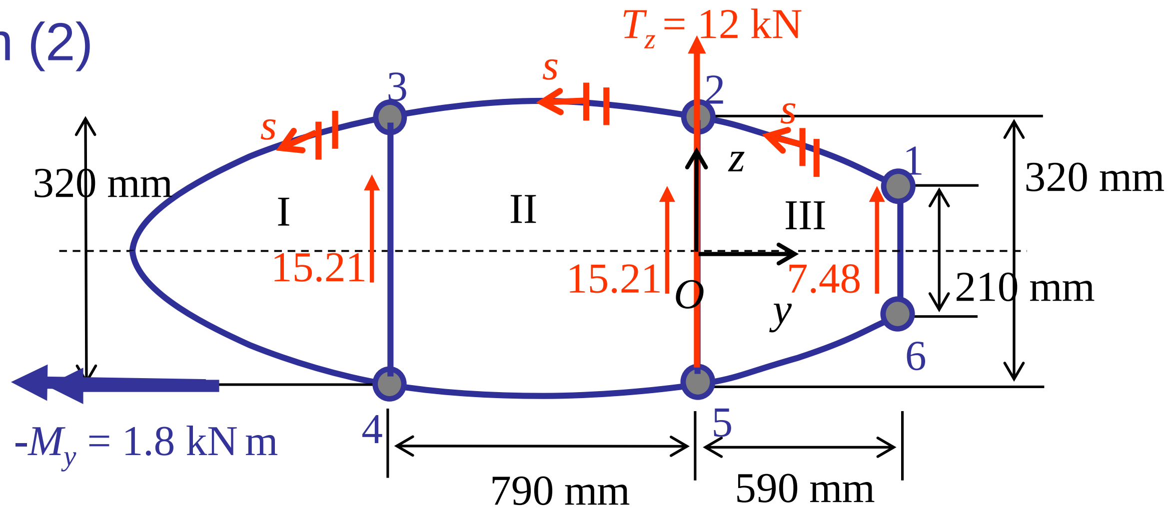
$$= 590 \times 53.6 + (-790) \times 145.2 + (-790) \times 145.2 + 590 \times 53.6 = -166170 \; N.mm$$

$$\sum_{\text{boom } j} z^j \mathbf{P}_y^j = z^1 \mathbf{P}_y^1 + z^3 \mathbf{P}_y^3 + z^4 \mathbf{P}_y^4 + z^6 \mathbf{P}_y^6$$

$$= 105 \times 142.9 + 160 \times (-435.5) + (-160) \times 435.5 + (-105) \times (-142.9) = -109351 \; N.mm$$

- Momentum equilibrium (2)

Cell	Area (mm²)
I	100 000
II	260 000
III	180 000



– Equation for tapered wing (2)

$$y_T T_z = \sum_i \int_{\text{wall } i} q_o p ds + \sum_{\text{cell } c} 2A_h^c q^c(0) + \sum_{\text{boom } j} y^j \boldsymbol{P}_z^j - \sum_{\text{boom } j} z^j \boldsymbol{P}_y^j$$

- With

$$\sum_i \int_{\text{wall } i} q_o p \, ds = q_o^{61} \times l^{61} \times 590 - q_o^{43^|} \times l^{43^|} \times 790$$

$$= 7.48 \times 210 \times 590 - 15.21 \times 320 \times 790 = -2918000 \; N.mm$$

- Leading to

$$0 = -2918000 + 2 \times 100000 q^I(0) + 2 \times 260000 q^{II}(0) + 2 \times 180000 q^{III}(0) + (-166170) - (-109351)$$

$$\Leftrightarrow \quad q^I(0) + 2.6 q^{II}(0) + 1.8 q^{III}(0) - 14.88 = 0$$

Exercise 2: Tapered wing

- System of equations

$$\begin{cases} \frac{1}{2\times 1000\mu}\left(21.60q^I(0) - 1.60q^{II}(0) + 24.34\right) = \frac{1}{2\times 1000\mu}\left(-0.615q^I(0) + 7.385q^{II}(0) - 0.615q^{III}(0)\right) \\ \frac{1}{2\times 1000\mu}\left(21.60q^I(0) - 1.60q^{II}(0) + 24.34\right) = \frac{1}{2\times 1000\mu}\left(-0.889q^{II}(0) + 8.333q^{III}(0) - 7.691\right) \\ q^I(0) + 2.6q^{II}(0) + 1.8q^{III}(0) - 14.88 = 0 \end{cases}$$

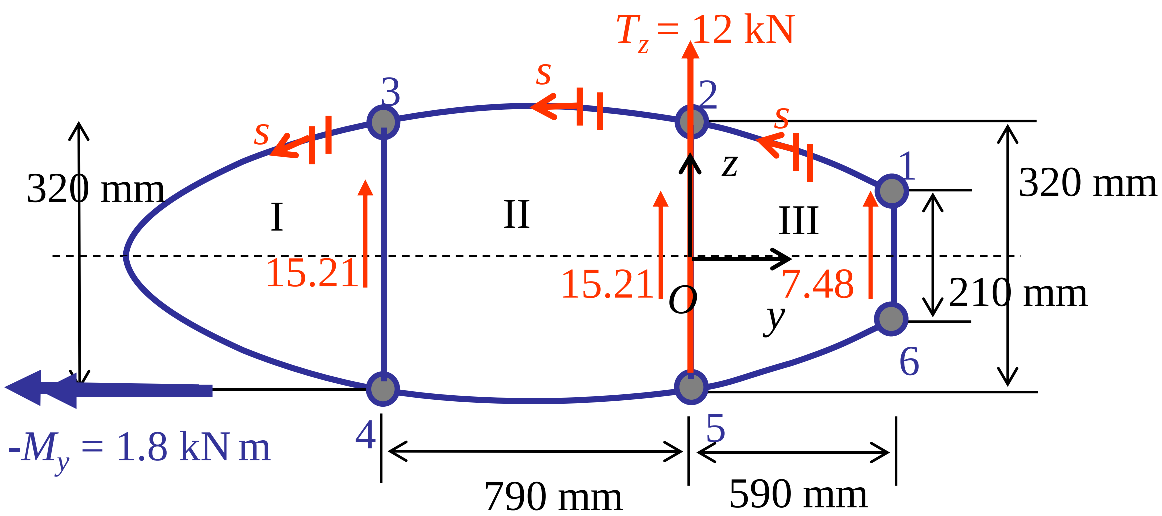
$$\Leftrightarrow \begin{cases} 22.215q^I(0) - 8.985q^{II}(0) + 0.615q^{III}(0) + 24.34 = 0 \\ 21.60q^I(0) - 0.711q^{II}(0) - 8.333q^{III}(0) + 32.031 = 0 \\ q^I(0) + 2.6q^{II}(0) + 1.8q^{III}(0) - 14.88 = 0 \end{cases}$$

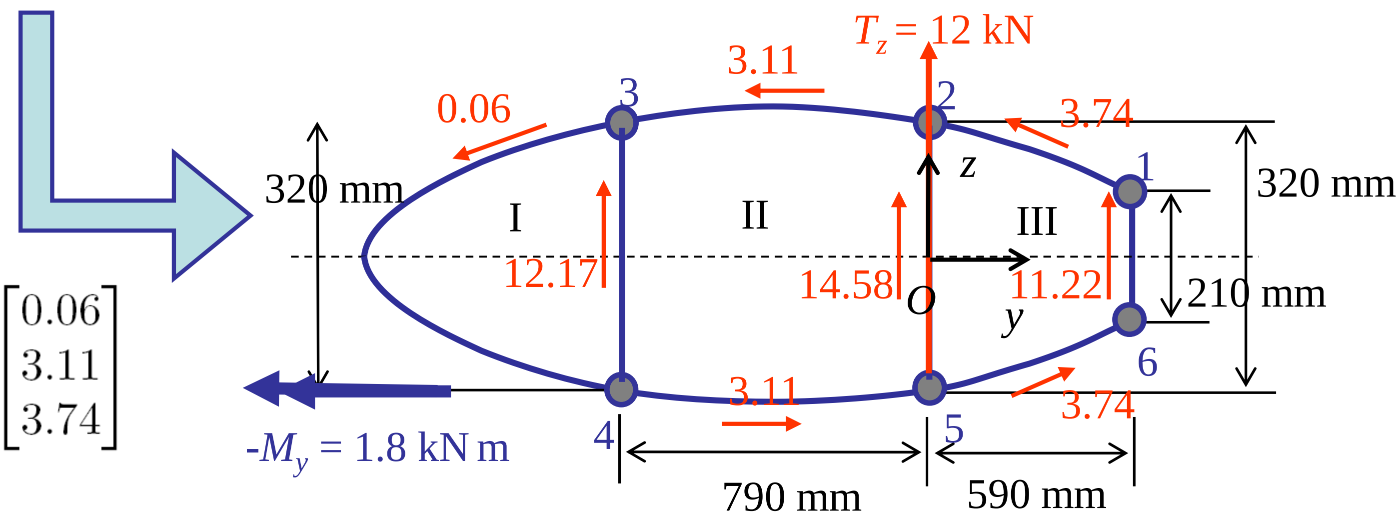
- Constant shear flows (in $\mathrm{N/mm}$)

$$\begin{pmatrix} 22.215 & -8.985 & 0.615 \\ 21.6 & -0.711 & -8.333 \\ 1 & 2.6 & 1.8 \end{pmatrix} * \begin{bmatrix} q^I(0) \\ q^{II}(0) \\ q^{III}(0) \end{bmatrix} = \begin{bmatrix} -24.34 \\ -32.031 \\ 14.88 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} q^I(0) \\ q^{II}(0) \\ q^{III}(0) \end{bmatrix} = \begin{bmatrix} 0.06 \\ 3.11 \\ 3.74 \end{bmatrix}$$

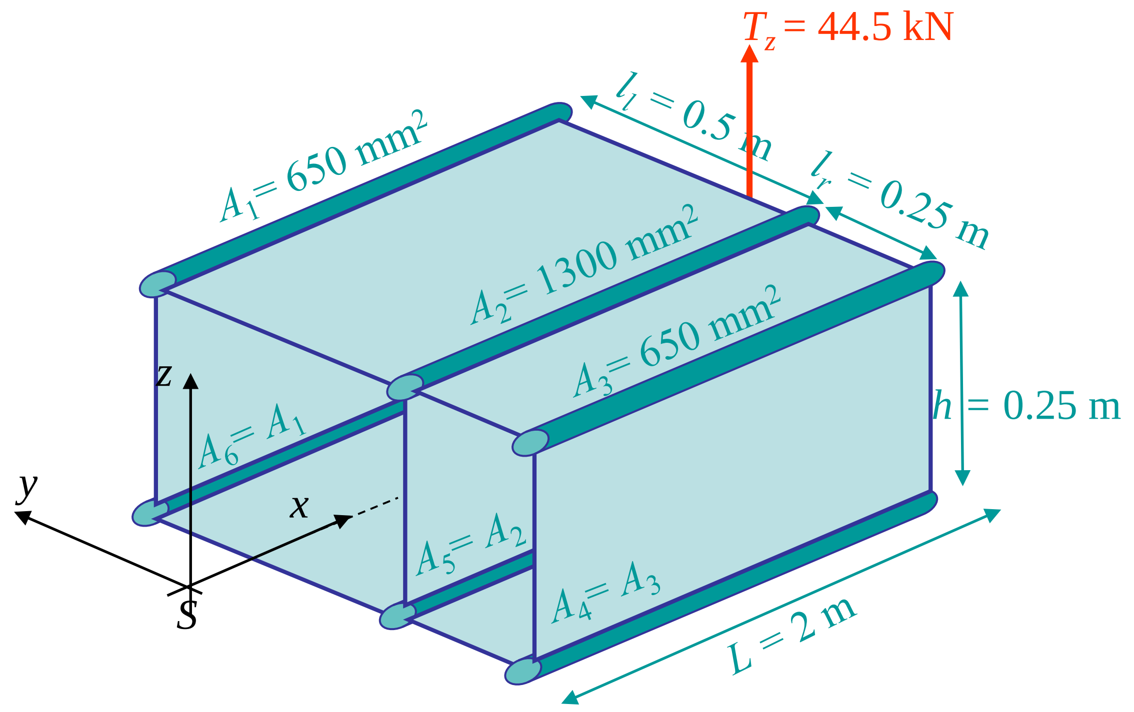
- Total shear flow





$$\begin{bmatrix} q^{I}(0) \\ q^{II}(0) \\ q^{III}(0) \end{bmatrix} = \begin{bmatrix} 0.06 \\ 3.11 \\ 3.74 \end{bmatrix}$$

- Example
  - Straight wing box
  - Idealized cross-sections
    - Booms carry direct stresses only
    - Skins carry shear stress only
      - 2-mm thick
      - E = 69 GPa, μ = 25.9 GPa
    - Singly symmetrical section
  - Load
    - At the free surface
    - Through shear center
  - Deflection
    - Due to direct stress?
    - Due to shear flux?

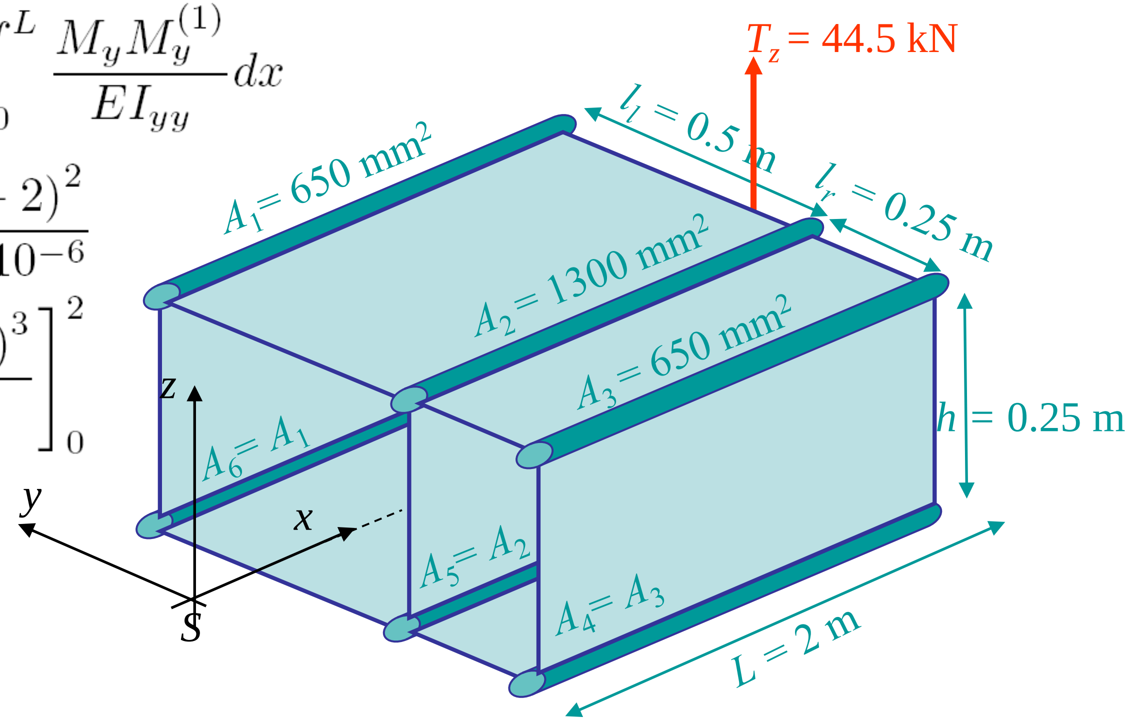


- Bending
  - As boom distribution is symmetric
    - $I_{yy} = \sum_i A_i {z^i}^2 = 4\ 0.00065\ 0.125^2 + 2\ 0.0013\ 0.125^2 = 81.25\ 10^{-6}\ \ \mathrm{m}^4$
  - Moment
    - For T_z = 44.5 kN: $M_y = -44.5\ 10^3\ \ (2-x) = 44.5\ 10^3\ x\ \ \mathrm{N} - 89\ 10^3\ \ \mathrm{N}\cdot\mathrm{m}$
    - For T_z = 1 N: $M_y^{(1)} = -1\ \ (2-x) = x\ \ \mathrm{N} - 2\ \ \mathrm{N}\cdot\mathrm{m}$
  - Deflection due to bending
    - As M_Z = 0: $\Delta_P u = \int_0^L \frac{M_y M_y^{(1)}}{EI_{yy}} dx$

$$\Rightarrow \Delta_P u = \int_0^2 \frac{44.5\ 10^3\ \ (x-2)^2}{69\ 10^9\ 81.25\ 10^{-6}}$$

$$= 7.94\ 10^{-3} \left[\frac{(x-2)^3}{3}\right]_0^2$$

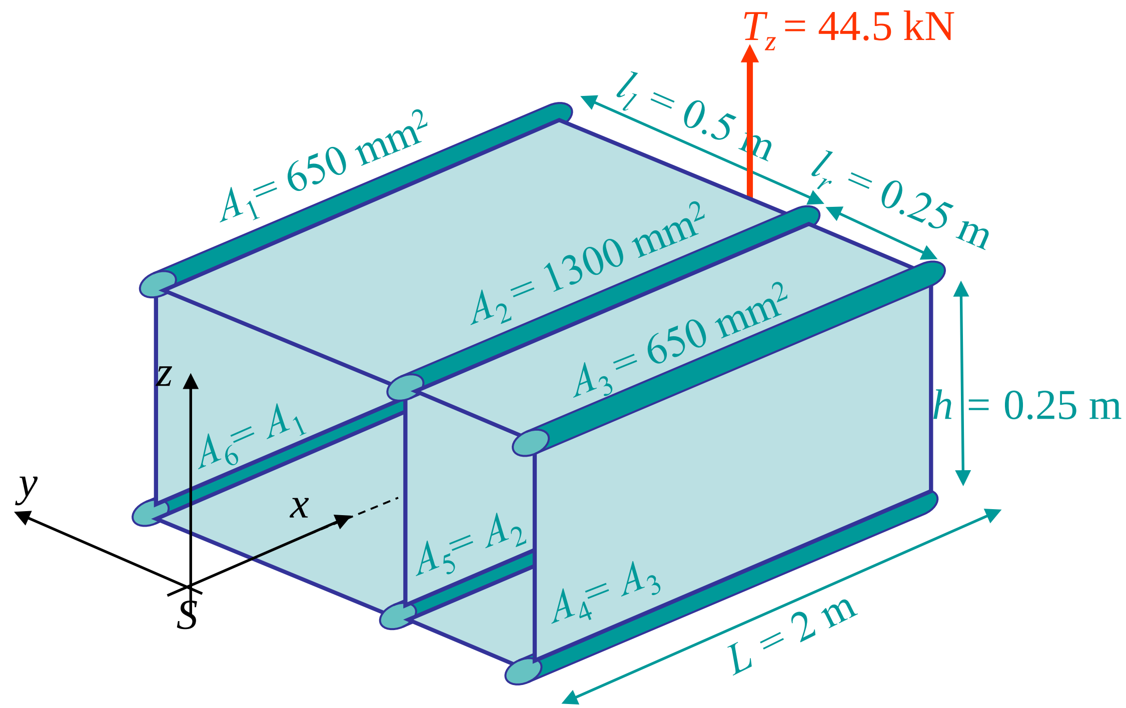
$$= 0.021\ \ \mathrm{m}$$



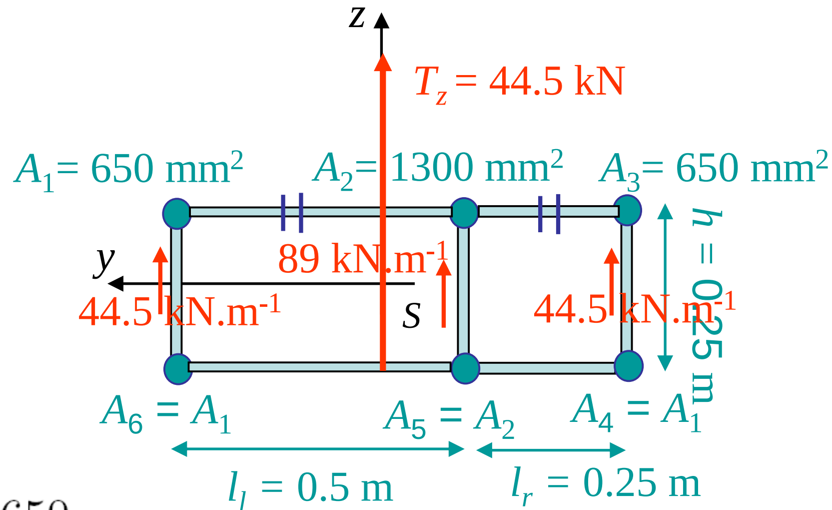
- Shearing
  - As the wing is not tapered, the shear flux is constant with respect to *x*

$$q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\boldsymbol{\sigma}} z ds + \sum_{i:\, s_i \leq s} z_i A_i\right] -$$

$$\frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\boldsymbol{\sigma}} y ds + \sum_{i:\, s_i \leq s} y_i A_i\right]$$



- Open cell shear flux
  - As skins carry only shear stress
    - $q_o\left(s\right) = -\frac{T_z}{I_{yy}} \sum_{i:s_i<s} z^i A_i$
  - For T_z = 44.5 kN



$$\begin{cases} q_o^{21} = q_o^{23} = 0 \\ q_o^{16} = q_o^{21} - \frac{44.5\ 10^3}{81.25\ 10^{-6}}\ 0.125\ 0.000650 \\ \quad = -44.5\ 10^3\ \ \mathrm{N}\cdot\mathrm{m}^{-1} \\ q_o^{65} = q_o^{16} - \frac{44.5\ 10^3}{81.25\ 10^{-6}}\ (-0.125)\ \ 0.000650 = 0 \\ q_o^{34} = q_o^{23} - \frac{44.5\ 10^3}{81.25\ 10^{-6}}\ 0.125\ 0.000650 = -44.5\ 10^3\ \ \mathrm{N}\cdot\mathrm{m}^{-1} \\ q_o^{45} = q_o^{34} - \frac{44.5\ 10^3}{81.25\ 10^{-6}}\ (-0.125)\ \ 0.000650 = 0 \\ q_o^{25} = q_o^{12} + q_o^{32} - \frac{44.5\ 10^3}{81.25\ 10^{-6}}\ 0.125\ 0.0013 = -89\ 10^3\ \ \mathrm{N}\cdot\mathrm{m}^{-1} \end{cases}$$

- Non dimensional lengths
  - Take μ_{REF} = 25.9 GPa
  - Sides length

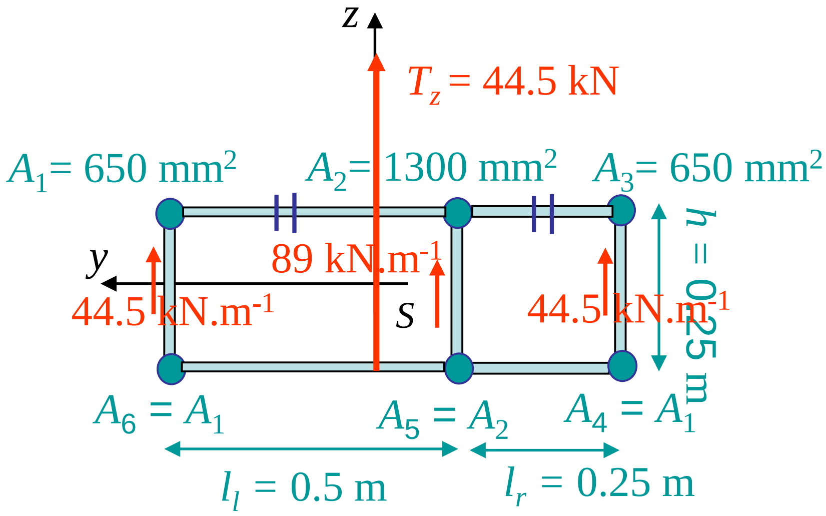
$$\begin{cases} \bar{l}^{12} = \bar{l}^{56} = \dfrac{0.5}{0.002} = 250 \\ \bar{l}^{23} = \bar{l}^{45} = \dfrac{0.25}{0.002} = 125 \\ \bar{l}^{16} = \bar{l}^{25} = \bar{l}^{34} = \dfrac{0.25}{0.002} = 125 \end{cases}$$

  - Cells length

$$\begin{cases} \bar{l}^{1} = \bar{l}^{12} + \bar{l}^{25} + \bar{l}^{56} + \bar{l}^{61} = 750 \\ \bar{l}^{2} = \bar{l}^{23} + \bar{l}^{34} + \bar{l}^{45} + \bar{l}^{25} = 500 \end{cases}$$

  - Intersecting length

$$\bar{l}^{2}_{1} = \bar{l}^{25} = 125$$

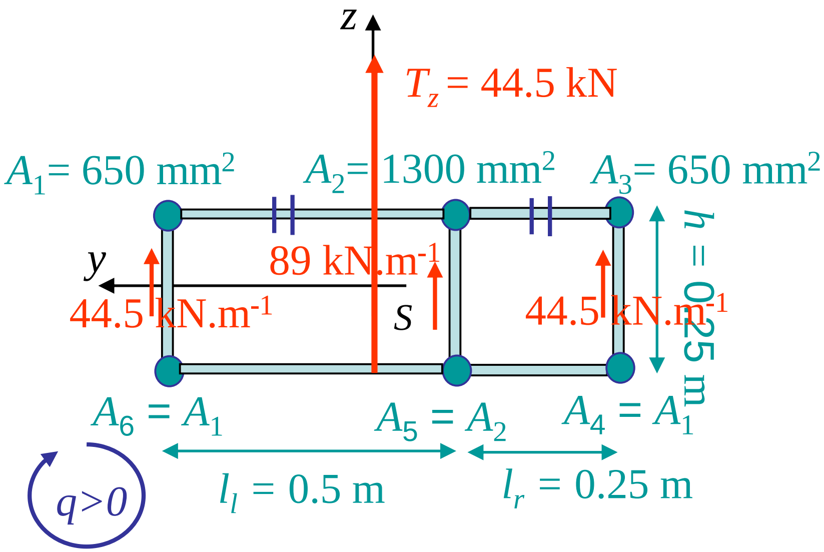


- Integration of open shear flux on cells
  - Cell *I*

$$\oint_1 \frac{q_o}{t\frac{\mu}{\mu_{\mathrm{REF}}}} ds = -\frac{89000}{0.002} 0.25 + \frac{44500}{0.002} 0.25 = -5.56\ 10^6 \mathrm{N} \cdot \mathrm{m}^{-1}$$

  - Cell *II*

$$\oint_2 \frac{q_o}{t\frac{\mu}{\mu_{\mathrm{REF}}}} ds = -\frac{44500}{0.002} 0.25 + \frac{89000}{0.002} 0.25 = 5.56\ 10^6 \mathrm{N} \cdot \mathrm{m}^{-1}$$



- Twist rate
  - Is equal to zero as the loading passes through the shear center
  - Cell I

$$0 = \theta_{,x} = \frac{1}{2\ A_h^I\ \mu_{\text{REF}}}\left[q^I(0)\bar{l}^1 + \oint_1 \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}}ds - q^{II}(0)\bar{l}_1^2\right]$$

$$\Longrightarrow q^I(0)\bar{l}^1 + \oint_1 \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}}ds - q^{II}(0)\bar{l}_1^2 = 0$$

$$\Longrightarrow 750\ q^I(0) - 5.56\ 10^6\ \text{N} \cdot \text{m}^{-1} - 125\ q^{II}(0) = 0$$

  - Cell II

$$0 = \theta_{,x} = \frac{1}{2\ A_h^{II}\ \mu_{\text{REF}}}\left[q^{II}(0)\bar{l}^2 + \oint_2 \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}}ds - q^I(0)\bar{l}_1^2\right]$$

$$\Longrightarrow q^{II}(0)\bar{l}^2 + \oint_2 \frac{q_o}{t\frac{\mu}{\mu_{\text{REF}}}}ds - q^I(0)\bar{l}_1^2 = 0$$

$$\Longrightarrow 500\ q^{II}(0) + 5.56\ 10^6\ \text{N} \cdot \text{m}^{-1} - 125\ q^I(0) = 0$$

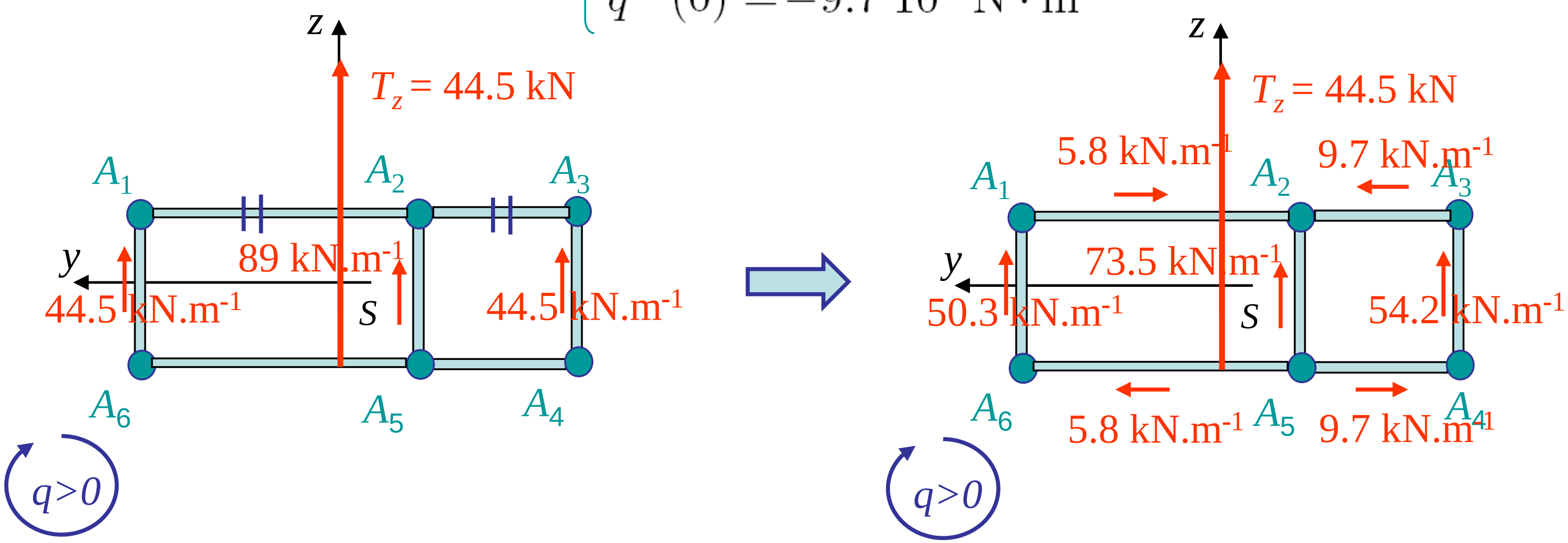
  - Solution

$$-3\ 5.56\ 10^6\ \text{N} \cdot \text{m}^{-1} + (4\ 750 - 125)\ \ q^I(0) = 0 \Longrightarrow q^I(0) = 5.8\ 10^3\ \text{N} \cdot \text{m}^{-1}$$

$$\Longrightarrow q^{II}(0) = \frac{125 q^I(0) - 5.56\ 10^6}{500} = -9.7\ 10^3\ \text{N} \cdot \text{m}^{-1}$$

- Shear flux
  - For T_z = 44.5 kN

$$\begin{cases} q^{I}(0) = 5.8\ 10^{3}\ \mathrm{N}\cdot\mathrm{m}^{-1} \\ q^{II}(0) = -9.7\ 10^{3}\ \mathrm{N}\cdot\mathrm{m}^{-1} \end{cases}$$

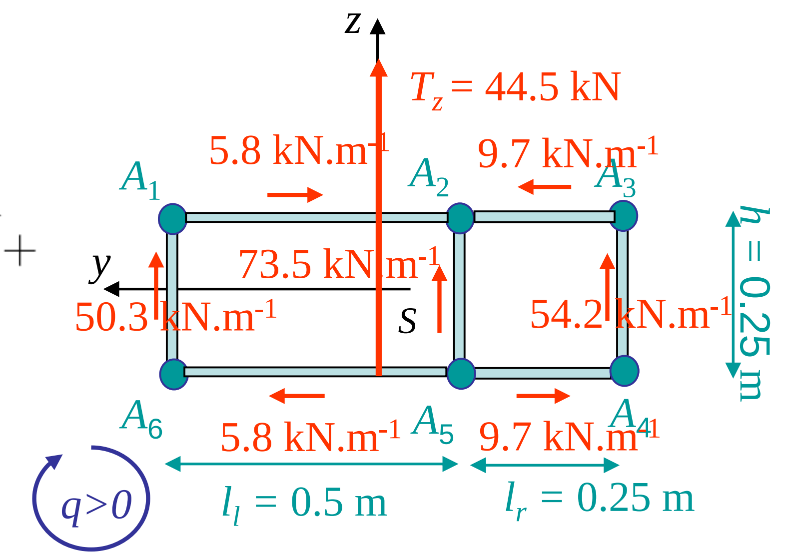


  - For T_z = 1 kN
    - By linearity, $q^{(1)} = q$ / 44500

- Deflection due to shearing

$$\Delta_T u = \int_0^L \int_s \frac{q^{(1)} q}{\mu t} ds dx = \int_0^2 \int_s \frac{q^2}{44.5\ 10^3\ 0.002\ 25.9\ 10^9} ds dx$$
$$= 0.8676\ 10^{-12} \int_s q^2 ds$$

$$\Rightarrow \Delta_T u = 0.8676\ 10^{-12}\ 10^6\ \left[5.8^2\ 0.5\ 2 + 9.7^2\ 0.25\ 2 + 50.3^2\ 0.25 + 73.5^2\ 0.25 + 54.2^2\ 0.25\right] = 0.00243 \text{ m}$$



- Total deflection
  - As displacement components are both oriented toward z

$$\Delta u = \Delta_T u + \Delta_P = 0.002 + 0.021 = 0.023 \text{ m}$$