University of Liège
Aerospace & Mechanical Engineering

Aircraft Structures
Aircraft Components – Part I

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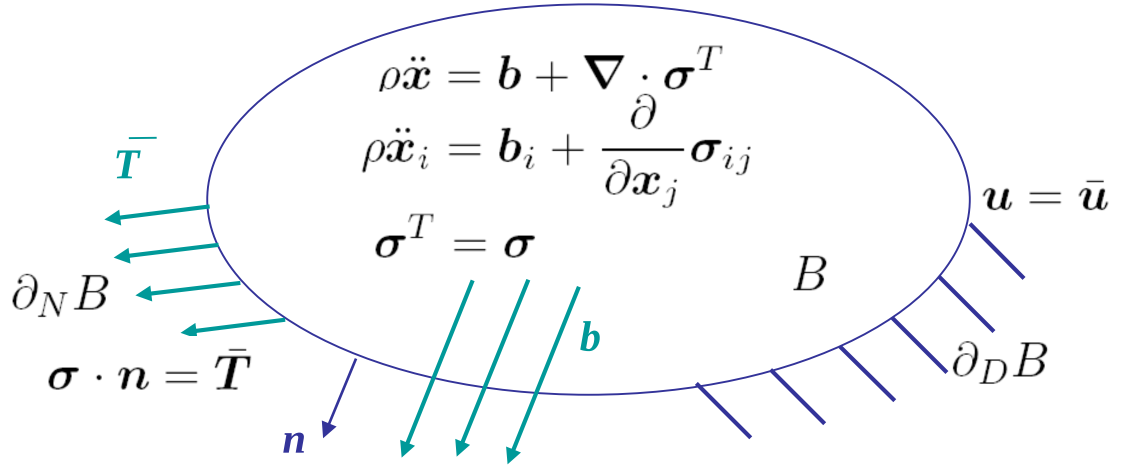
- Balance of body B
  - Momenta balance
    - Linear
    - Angular
  - Boundary conditions
    - Neumann
    - Dirichlet

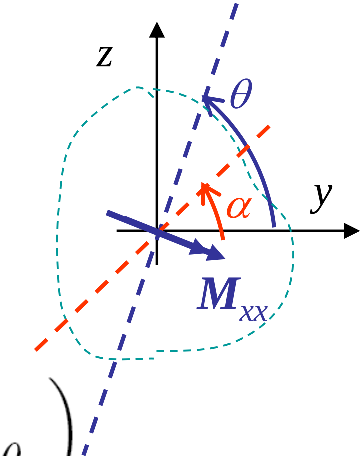


- Small deformations with linear elastic, homogeneous & isotropic material
  - (Small) Strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{\nabla} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{\nabla}\right)$, or $\begin{cases} \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{x}_i}\boldsymbol{u}_j + \frac{\partial}{\partial \boldsymbol{x}_j}\boldsymbol{u}_i\right) \\ \boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\boldsymbol{u}_{j,i} + \boldsymbol{u}_{i,j}\right) \end{cases}$
  - Hooke's law $\boldsymbol{\sigma} = \mathcal{H} : \boldsymbol{\varepsilon}$, or $\boldsymbol{\sigma}_{ij} = \mathcal{H}_{ijkl}\boldsymbol{\varepsilon}_{kl}$

    with $$\mathcal{H}_{ijkl} = \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda = K - 2\mu/3}\delta_{ij}\delta_{kl} + \underbrace{\frac{E}{1+\nu}}_{2\mu}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right)$$
  - Inverse law $\boldsymbol{\varepsilon} = \mathcal{G} : \boldsymbol{\sigma}$

    with $$\mathcal{G}_{ijkl} = \frac{1+\nu}{E}\left(\frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{il}\delta_{jk}\right) - \frac{\nu}{E}\delta_{ij}\delta_{kl}$$

Pure bending: linear elasticity summary

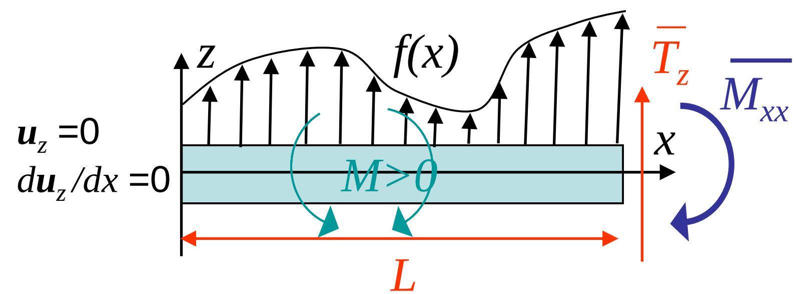


- General expression for unsymmetrical beams
  - Stress $\boldsymbol{\sigma}_{xx} = \kappa E z \cos\alpha - \kappa E y \sin\alpha$

    With $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{\kappa E} \begin{pmatrix} I_{yy} & -I_{yz} \\ -I_{yz} & I_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$
  - Curvature

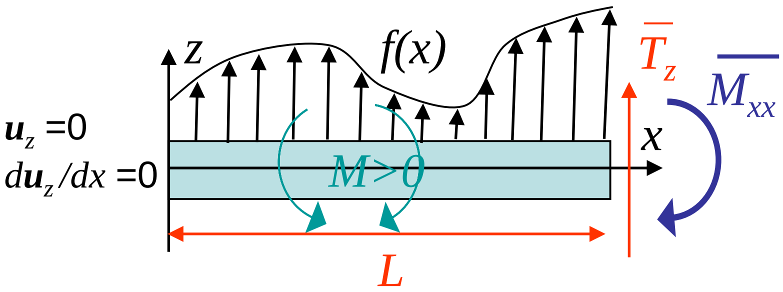
    $$\begin{pmatrix} -\boldsymbol{u}_{z,xx} \\ \boldsymbol{u}_{y,xx} \end{pmatrix} = \frac{\|\boldsymbol{M}_{xx}\|}{E\left(I_{yy}I_{zz} - I_{yz}I_{yz}\right)} \begin{pmatrix} I_{zz} & I_{yz} \\ I_{yz} & I_{yy} \end{pmatrix} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$
  - In the principal axes $I_{yz} = 0$

- Euler-Bernoulli equation in the principal axis
  - $\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right) = f(x)$ for x in [0 L]
  - BCs $\begin{cases} -\frac{\partial}{\partial x}\left(EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\right)\Big|_{0,\,L} = \bar{T}_z\big|_{0,\,L} \\ -EI\frac{\partial^2 \boldsymbol{u}_z}{\partial x^2}\Big|_{0,\,L} = \bar{M}_{xx}\big|_{0,\,L} \end{cases}$
  - Similar equations for $\boldsymbol{u}_y$

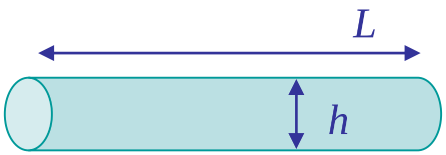


Beam shearing: linear elasticity summary

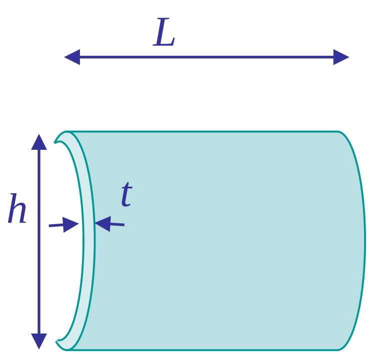
- General relationships
  - $$\begin{cases} f_z(x) = -\partial_x T_z = -\partial_{xx} M_y \\ f_y(x) = -\partial_x T_y = \partial_{xx} M_z \end{cases}$$



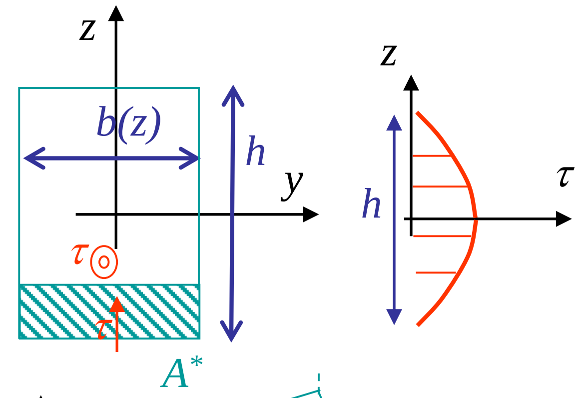
- Two problems considered
  - Thick symmetrical section



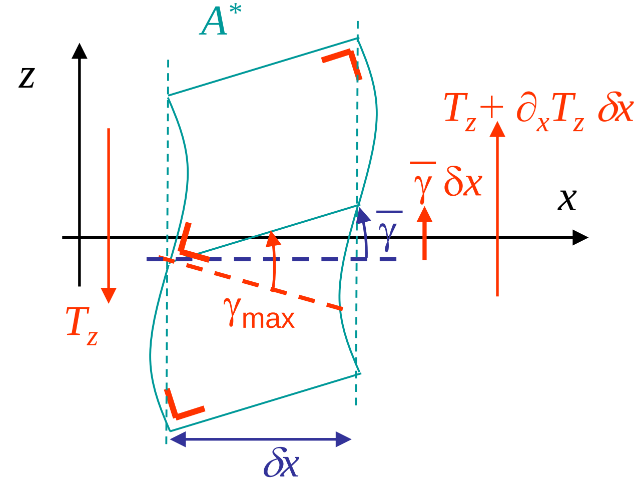
    - Shear stresses are small compared to bending stresses if h/L << 1
  - Thin-walled (unsymmetrical) sections
    - Shear stresses are not small compared to bending stresses
    - Deflection mainly results from bending stresses
    - 2 cases
      - Open thin-walled sections
        - » Shear = shearing through the shear center + torque
      - Closed thin-walled sections
        - » Twist due to shear has the same expression as torsion



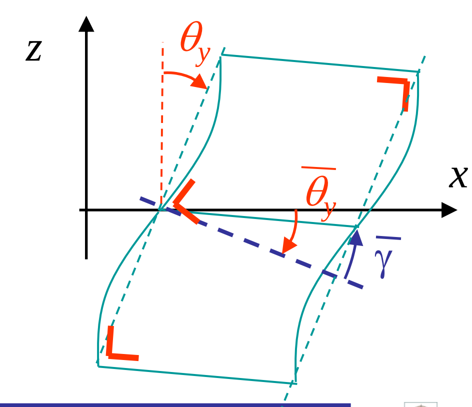
- Shearing of symmetrical thick-section beams



  - Stress $\boldsymbol{\sigma}_{zx} = -\frac{T_z S_n(z)}{I_{yy} b(z)}$
    - With $S_n(z) = \int_{A^*} z dA$
    - Accurate only if $h > b$
  - Energetically consistent averaged shear strain



    - $\bar{\gamma} = \frac{T_z}{A'\mu}$ with $A' = \frac{1}{\int_A \frac{S_n^2}{I_{yy}^2 b^2} dA}$
    - Shear center on symmetry axes
  - Timoshenko equations
    - $\bar{\gamma} = 2\bar{\boldsymbol{\varepsilon}}_{xz} = \frac{\partial \boldsymbol{u}_x}{\partial z} + \frac{\partial \boldsymbol{u}_z}{\partial x} = \theta_y + \partial_x \boldsymbol{u}_z$ & $\kappa = \frac{\partial \theta_y}{\partial x}$



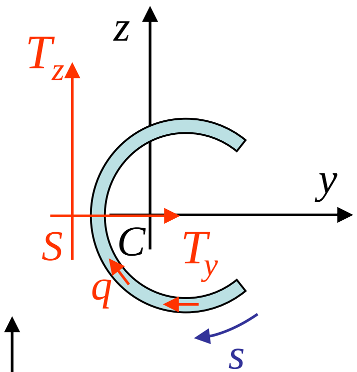
    - On [0 L]: $\begin{cases} \frac{\partial}{\partial_x}\left(EI\frac{\partial \theta_y}{\partial x}\right) - \mu A'(\theta_y + \partial_x \boldsymbol{u}_z) = 0 \\ \frac{\partial}{\partial x}\left(\mu A'(\theta_y + \partial_x \boldsymbol{u}_z)\right) = -f \end{cases}$

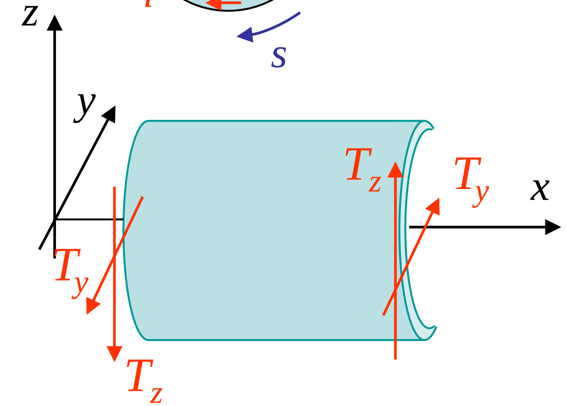
Beam shearing: linear elasticity summary

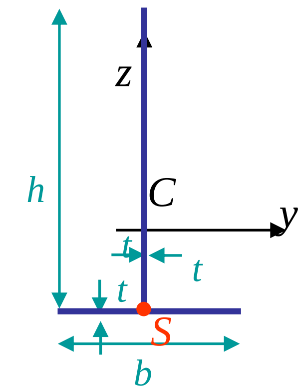
- Shearing of open thin-walled section beams
  - Shear flow $q = t\tau$
    - $$q(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s tz\,ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s ty\,ds'$$
    - In the principal axes

$$q(s) = -\frac{T_z}{I_{yy}}\int_0^s tz\,ds' - \frac{T_y}{I_{zz}}\int_0^s ty\,ds'$$

  - Shear center S
    - On symmetry axes
    - At walls intersection
    - Determined by momentum balance
  - Shear loads correspond to
    - Shear loads passing through the shear center &
    - Torque







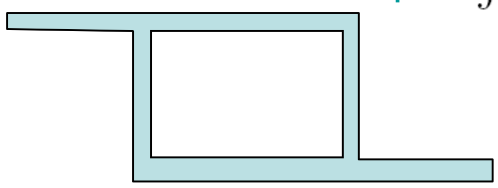
- Shearing of closed thin-walled section beams
  - Shear flow $q = t\tau$
    - $q(s) = q_o(s) + q(0)$
    - Open part (for anticlockwise of q, s)

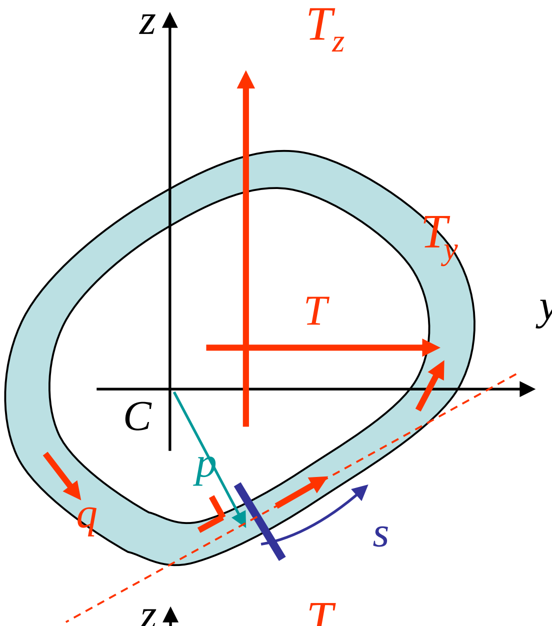
$$q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s t(s')\,z(s')\,ds' - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\int_0^s t(s')\,y(s')\,ds'$$

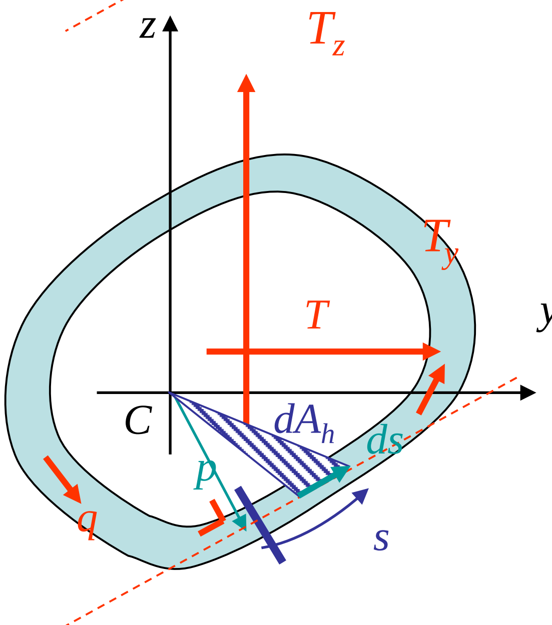
    - Constant twist part

$$q(s=0) = \frac{y_T T_z - z_T T_y - \oint p(s)\,q_o(s)\,ds}{2A_h}$$

    - The $q(0)$ is related to the closed part of the section, but there is a $q_o(s)$ in the open part which should be considered for the shear torque $\oint p(s)\,q_o(s)\,ds$



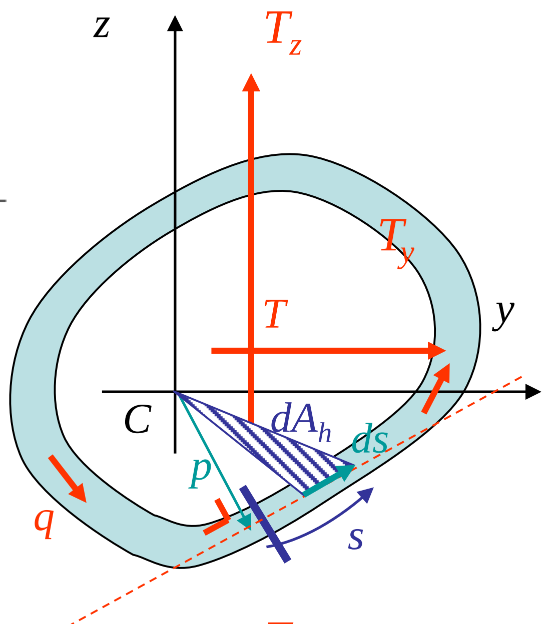


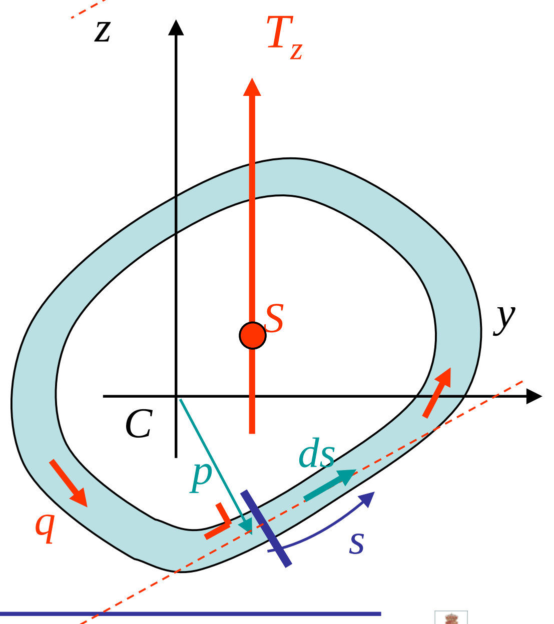


- Shearing of closed thin-walled section beams
  - Warping around twist center R
    - $$\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \int_0^s \frac{q}{\mu t} ds - \frac{1}{A_h} \oint \frac{q}{\mu t} ds \left\{ A_{Cp}(s) + \frac{z_R\left[y(s) - y(0)\right] - y_R\left[z(s) - z(0)\right]}{2} \right\}$$
    - With $\boldsymbol{u}_x(0) = \frac{\oint t \boldsymbol{u}_x(s)\, ds}{\oint t(s)\, ds}$
      - $\boldsymbol{u}_x(0)=0$ for symmetrical section if origin on the symmetry axis
  - Shear center S
    - Compute q for shear passing thought S
    - Use

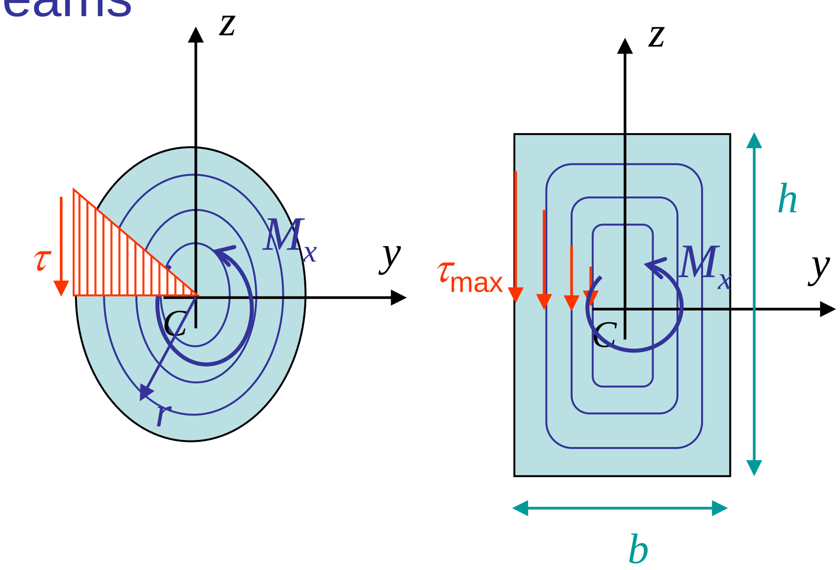
$$q(s=0) = \frac{y_T T_z - z_T T_y - \oint p(s)\, q_o(s)\, ds}{2A_h}$$

With point S=T



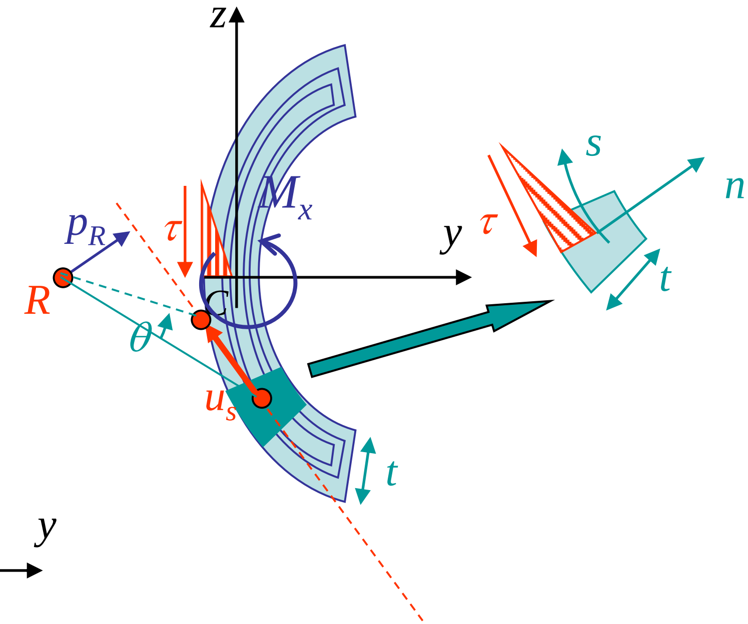


- Torsion of symmetrical thick-section beams
  - Circular section
    - $\tau = \mu\gamma = r\mu\theta_{,x}$
    - $C = \frac{M_x}{\theta_{,x}} = \int_A \mu r^2 dA$
  - Rectangular section
    - $\tau_{\text{max}} = \frac{M_x}{\alpha h b^2}$
    - $C = \frac{M_x}{\theta_{,x}} = \beta h b^3 \mu$
    - If $h >> b$
      - $\tau_{xy} = 0$ & $\tau_{xz} = 2\mu y \theta_{,x}$
      - $\tau_{\text{max}} = \frac{3M_x}{hb^2}$
      - $C = \frac{M_x}{\theta_{,x}} = \frac{hb^3\mu}{3}$

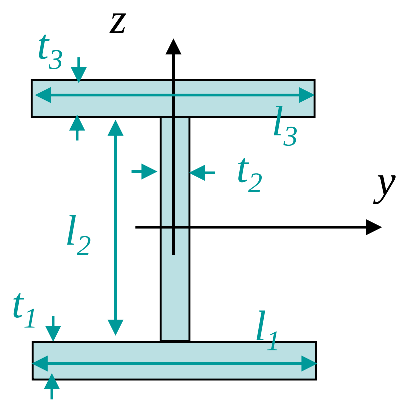


h/b	1	1.5	2	4	∞
α	0.208	0.231	0.246	0.282	1/3
β	0.141	0.196	0.229	0.281	1/3

- Torsion of open thin-walled section beams
  - Approximated solution for twist rate



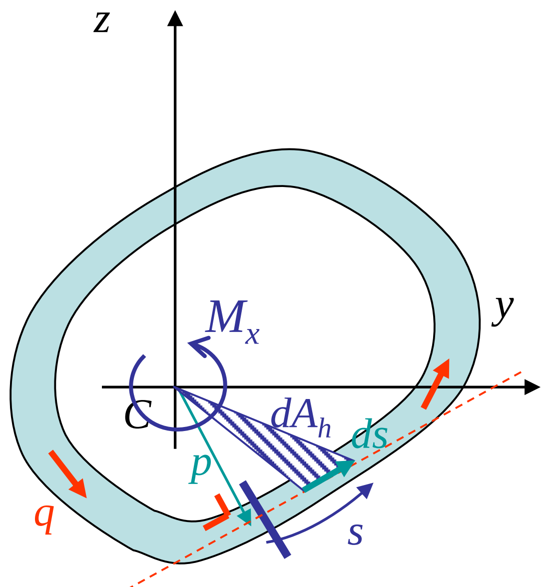
- Thin curved section
  - $\tau_{xs} = 2\mu n \theta_{,x}$
  - $C = \dfrac{M_x}{\theta_{,x}} = \dfrac{1}{3}\displaystyle\int \mu t^3 ds$
- Rectangles
  - $\tau_{\max_i} = \mu t_i \theta_{,x}$
  - $\dfrac{M_x}{\theta_{,x}} = \displaystyle\sum_i \dfrac{l_i t_i^3 \mu}{3}$



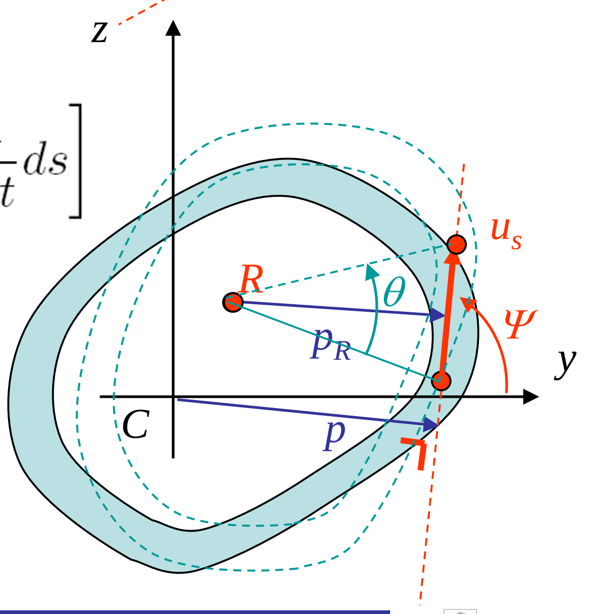
  - Warping of s-axis
    - $\boldsymbol{u}_x^s(s) = \boldsymbol{u}_x^s(0) - \theta_{,x}\displaystyle\int_0^s p_R ds' = \boldsymbol{u}_x^s(0) - 2A_{R_p}(s)\,\theta_{,x}$

- Torsion of closed thin-walled section beams
  - Shear flow due to torsion $M_x = 2A_h q$
  - Rate of twist
    - $\theta_{,x} = \frac{M_x}{4A_h^2} \oint \frac{1}{\mu t} ds$
    - Torsion rigidity for constant μ

$$I_T = \frac{4A_h^2}{\oint \frac{1}{t} ds} \leq I_p = \int_A r^2 dA$$

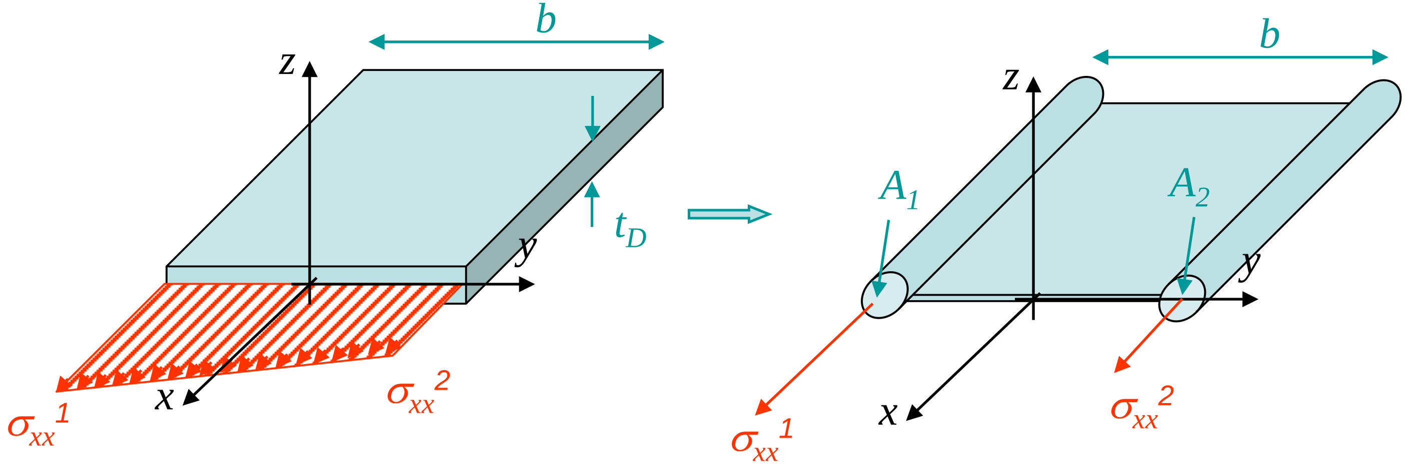


  - Warping due to torsion
    - $\boldsymbol{u}_x(s) = \boldsymbol{u}_x(0) + \frac{M_x}{2A_h}\left[\int_0^s \frac{1}{\mu t} ds - \frac{A_{R_p}(s)}{A_h} \oint \frac{1}{\mu t} ds\right]$
    - A_{Rp} from twist center



- Panel idealization
  - Booms' area **depending on loading**
    - For linear direct stress distribution

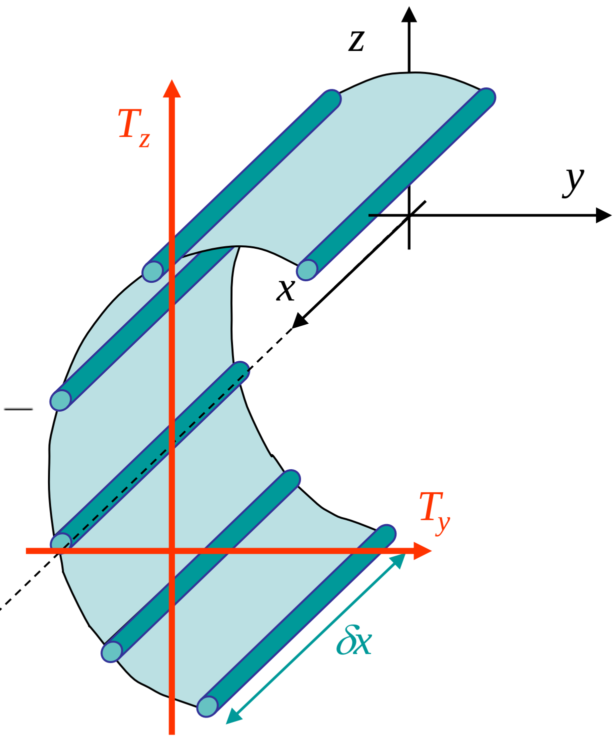
$$\begin{cases} A_1 = \frac{t_D b}{6}\left(2 + \frac{\boldsymbol{\sigma}_{xx}^2}{\boldsymbol{\sigma}_{xx}^1}\right) \\ A_2 = \frac{t_D b}{6}\left(2 + \frac{\boldsymbol{\sigma}_{xx}^1}{\boldsymbol{\sigma}_{xx}^2}\right) \end{cases}$$



Structure idealization summary

- Consequence on bending
  - The position of the neutral axis, and thus the second moments of area
    - Refer to the direct stress carrying area only
    - Depend on the loading case only
- Consequence on shearing
  - Open part of the shear flux
    - Shear flux for open sections

$$q_o\left(s\right) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\boldsymbol{\sigma}} z ds + \sum_{i:\ s_i \leq s} z_i A_i\right] - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\boldsymbol{\sigma}} y ds + \sum_{i:\ s_i \leq s} y_i A_i\right]$$



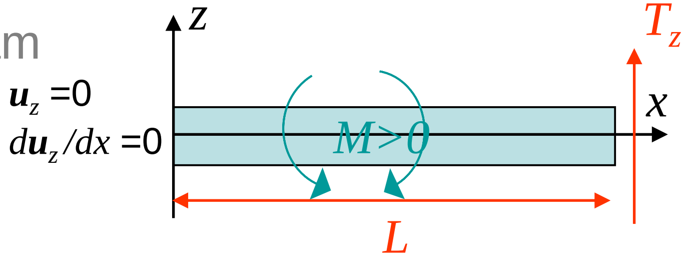
- Consequence on torsion
  - If no axial constraint
    - Torsion analysis does not involve axial stress
    - So torsion is unaffected by the structural idealization

- Virtual displacement (see annex of beams for demonstrations)
  - In linear elasticity the general formula of virtual displacement reads

$$\int_0^L \int_A \boldsymbol{\sigma}^{(1)} : \boldsymbol{\varepsilon} dA dx = P^{(1)} \Delta_P$$

    - $\boldsymbol{\sigma}^{(1)}$ is the stress distribution corresponding to a (unit) load $P^{(1)}$
    - Δ_P is the energetically conjugated displacement to P in the direction of $P^{(1)}$ that corresponds to the strain distribution $\boldsymbol{\varepsilon}$

  - Example: bending of a semi cantilever beam



    - $$\int_0^L \int_A \boldsymbol{\sigma}_{xx}^{(1)} \boldsymbol{\varepsilon}_{xx} dA dx = \Delta_P u$$

      - In the principal axes

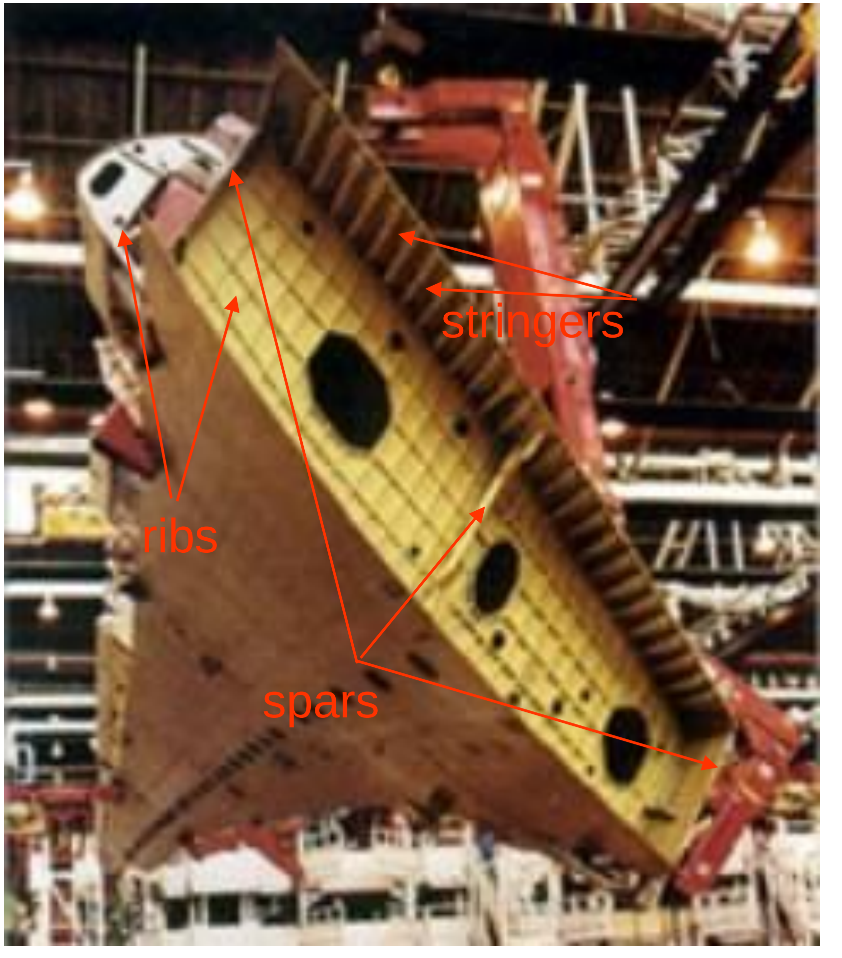
$$\Delta_P u \quad = \quad \frac{1}{E I_{yy} I_{zz}} \int_0^L \left\{ I_{zz} M_y^{(1)} M_y + I_{yy} M_z^{(1)} M_z \right\} dx$$

  - Example: shearing of a semi-cantilever beam

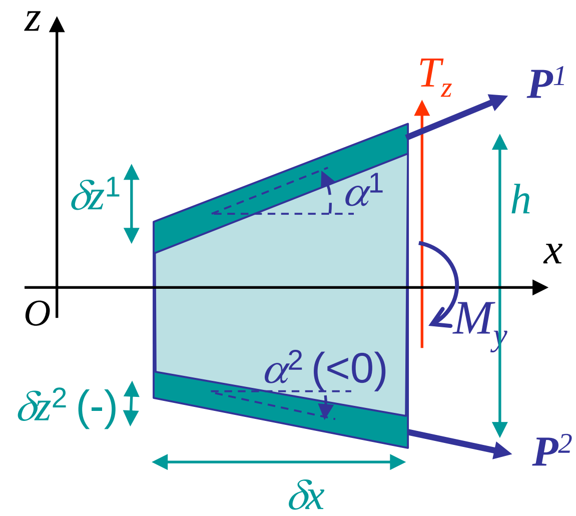
    - $$\int_0^L \int_s q^{(1)} \frac{q}{\mu t} ds dx = \boldsymbol{T}^{(1)} \bar{\Delta} \boldsymbol{u} = \Delta_T u$$

Wing spars & box beams

- Real wing
  - Complex sheet/stringer arrangements
  - Highly hyperstatic
  - Warping restraint
  - Load discontinuities
  - …
- For high accuracy
  - Requires FE simulations
    - Can be slow
    - Not well suited for defining the structural arrangement (number of stringers, spars, …)
- In preliminary design
  - A fastest method is required
    - To select a structural configuration
    - To have a starting point for the FE
    - To get insight on physical behaviors
  - This requires idealizations
    - Reduce the accuracy
    - All formula previously developed were assuming uniform section
    - For low changes in section shapes, Saint-Venant principle holds

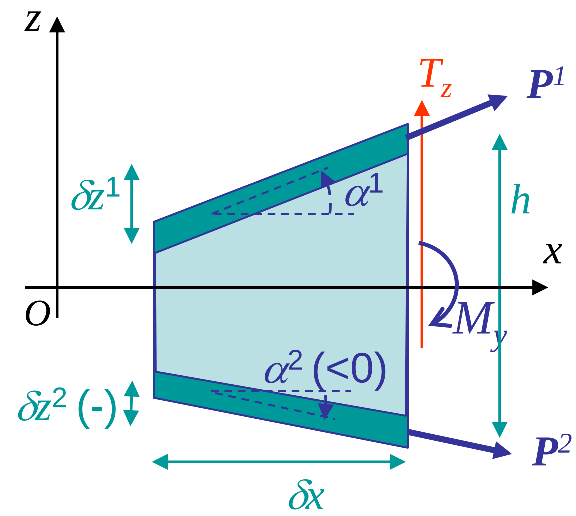


- Tapered wing spar
  - One web & 2 flanges
  - Flanges loading
    - If flanges are assumed to resist all the direct stress
      - $\boldsymbol{P}_x^1 = \frac{M_y}{h}$ & $\boldsymbol{P}_x^2 = -\frac{M_y}{h}$
    - If web is effective in carrying direct stress
      - $\boldsymbol{P}_x^1 = \boldsymbol{\sigma}_{xx}^1 A_1$ & $\boldsymbol{P}_x^2 = \boldsymbol{\sigma}_{xx}^2 A_2$
      - With $\boldsymbol{\sigma}_{xx} = \frac{(I_{zz}M_y + I_{yz}M_z)\, z - (I_{yz}M_y + I_{yy}M_z)\, y}{I_{yy}I_{zz} - I_{yz}^2}$
    - These formula assume a low variation of the section (Saint-Venant) $\Rightarrow$ low taper
    - Assuming the direct load has the direction of the flanges
      - $\left\|\boldsymbol{P}^1\right\| = \frac{\boldsymbol{P}_x^1}{\cos\alpha^1}$ & $\left\|\boldsymbol{P}^2\right\| = \frac{\boldsymbol{P}_x^2}{\cos\alpha^2}$
      - $\boldsymbol{P}_z^1 = \boldsymbol{P}_x^1 \frac{\delta z^1}{\delta x}$ & $\boldsymbol{P}_z^2 = \boldsymbol{P}_x^2 \frac{\delta z^2}{\delta x}$



- Tapered wing spar (2)
  - Web loading
    - 2 contributions to shearing
      - Shear stress in the web
      - z-component of flange loads
      - $T_z = T_z^{\text{web}} + \boldsymbol{P}_z^1 + \boldsymbol{P}_z^2$

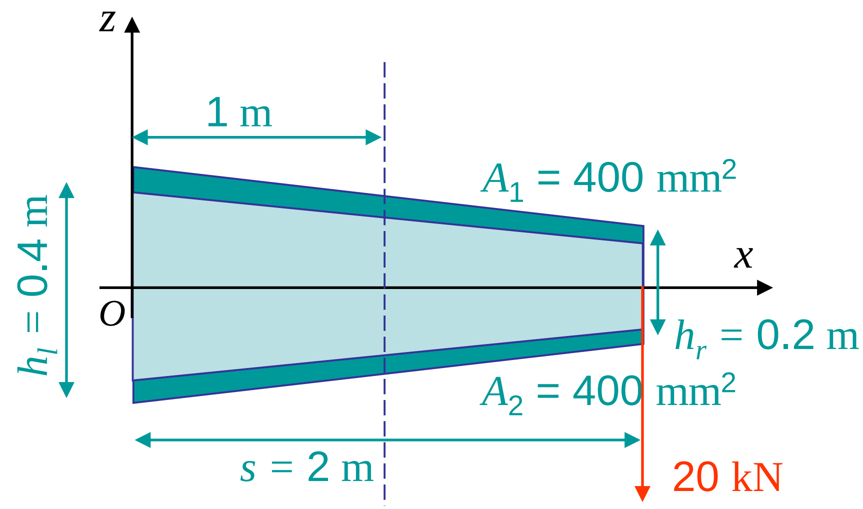
$$\Rightarrow T_z^{\text{web}} = T_z - \boldsymbol{P}_x^1 \frac{\delta z^1}{\delta x} - \boldsymbol{P}_x^2 \frac{\delta z^2}{\delta x}$$

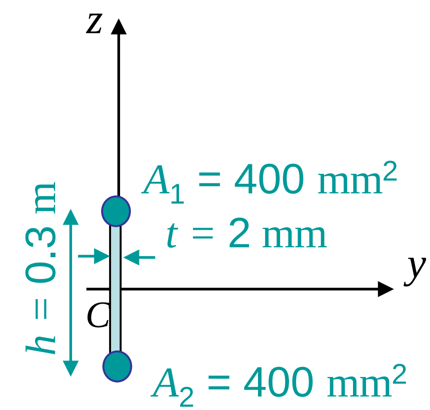


    - Shear stress
      - Flange contributions is removed leading to a new expression of shear flux
      - As web in Oxz plane & as it corresponds to an open section

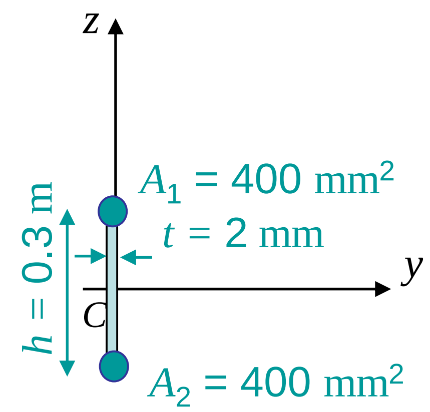
$$q(s) = -\frac{T_z^{\text{web}}}{I_{yy}} \left[\int_0^s t_{\text{direct}\ \boldsymbol{\sigma}} z ds + \sum_{i:\ s_i \leq s} z_i A_i \right]$$

- Example
  - Tapered spar
    - Simply symmetrical
    - Web fully effective in resisting direct stress
  - Shear distribution at $x = 1$ m?





- Direct stress
  - Momentum at $x = 1$ m
    - $M_y = 20\ 10^3 \quad \text{N} \cdot \text{m}$
  - As web is carrying direct stress
    - Second moment of area



$$- \quad I_{yy} = A_1 \frac{h^2}{4} + A_2 \frac{h^2}{4} + \frac{th^3}{12}$$

$$= 2\ 0.0004\ 0.15^2 + \frac{0.002\ 0.3^3}{12} = 22.5\ 10^{-6} \quad \text{m}^4$$

- Flange loads

$$- \quad \boldsymbol{\sigma}_{xx}^1 = \frac{M_y h}{2I_{yy}} = \frac{20\ 10^3\ 0.3}{2\ 22.5\ 10^{-6}} = 133 \quad \text{MPa}$$

$$\Longrightarrow \quad \boldsymbol{P}_x^1 = \boldsymbol{\sigma}_{xx}^1 A_1 = 133\ 10^6\ 0.0004 = 53.33\ 10^3 \quad \text{N}$$

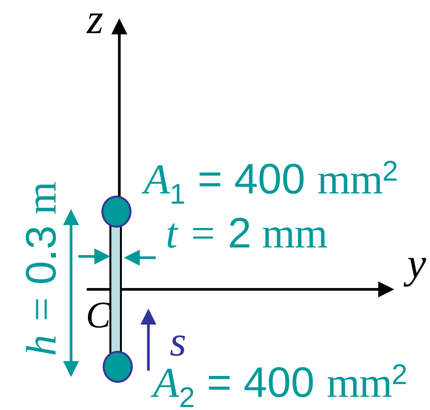
$$- \quad \boldsymbol{\sigma}_{xx}^2 = -\frac{M_y h}{2I_{yy}} = -\frac{20\ 10^3\ 0.3}{2\ 22.5\ 10^{-6}} = -133 \quad \text{MPa}$$

$$\Longrightarrow \quad \boldsymbol{P}_x^2 = \boldsymbol{\sigma}_{xx}^2 A_2 = -133\ 10^6\ 0.0004 = -53.33\ 10^3 \quad \text{N}$$

- Web direct stress

$$- \quad \boldsymbol{\sigma}_{xx}^{\text{web}} = \frac{M_y z}{I_{yy}} = \frac{20\ 10^3}{22.5\ 10^{-6}} z = 888\ \text{MPa} \cdot m^{-1}\ z$$

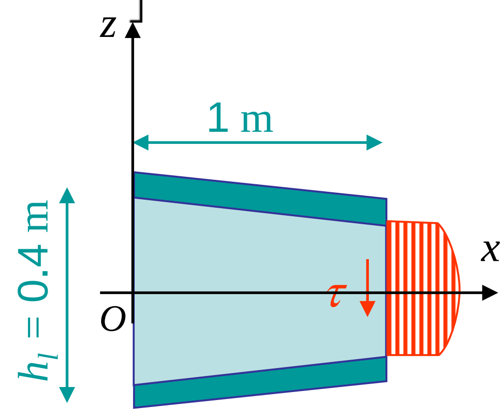
- Shear stress
  - Web shear load
    - $$T_z^{\text{web}} = T_z - \boldsymbol{P}_x^1 \partial_x z^1 - \boldsymbol{P}_x^2 \partial_x z^2$$
      $$= -20\ 10^3 - 53.33\ 10^3\ \frac{-0.1}{2} + 53.33\ 10^3\ \frac{0.1}{2}$$
      $$= -14.67\ 10^3\ \ \text{N}$$
  - Shear flux
    - $$q(s) = -\frac{T_z^{\text{web}}}{I_{yy}} \left[\int_0^s tzds + A_2 z_2 \right]$$
      $$= -\frac{T_z^{\text{web}}}{I_{yy}} \left[\int_{-\frac{h}{2}}^{z} tz'dz' + A_2 z_2 \right] = -\frac{T_z^{\text{web}}}{I_{yy}} \left[t \frac{z^2 - \frac{h^2}{4}}{2} + A_2 z_2 \right]$$



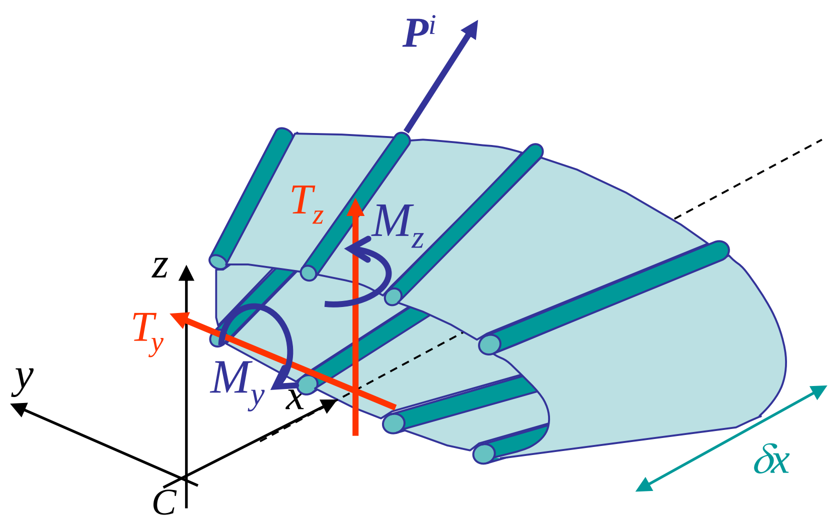
$$\Rightarrow q(z) = -\frac{-14.67\ 10^3}{22.5\ 10^{-6}} \left[\frac{0.002}{2} \left(z^2 - 0.15^2 \right) - 0.15\ 0.0004 \right]$$

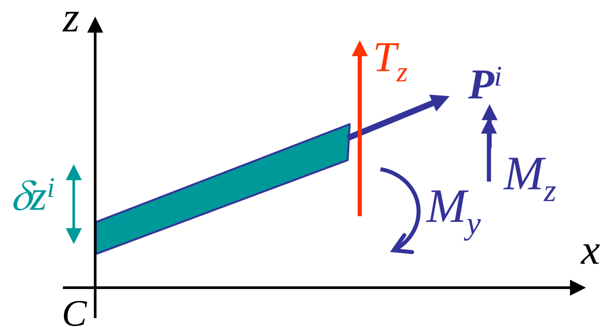
$$= 652\ 10^3\ \ \text{N}\cdot\text{m}^{-3} z^2 - 53.79\ \ \text{N}\cdot\text{m}^{-1}$$

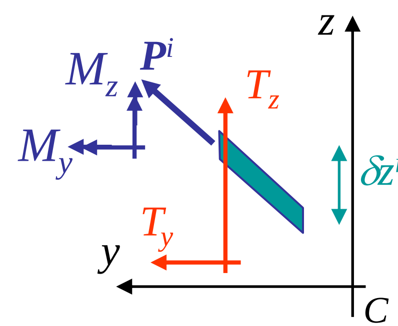
$$\Rightarrow q\left(-\frac{h}{2}\right) = q\left(\frac{h}{2}\right) = -39.12\ \ \text{N}\cdot\text{m}^{-1}$$



- Open & closed section beams
  - Beam tapered in two directions
  - Booms & skin panels
  - Examples: fuselages & wings
  - Assumption for open sections
    - Shear loads pass through the shear center (no torsion)
  - Equations for i th boom
    - If $\boldsymbol{\sigma}_{xx}^{i}$ is the direct stress due to flexion in the i th boom
      - $\boldsymbol{P}_x^i = \boldsymbol{\sigma}_{xx}^i A_i$
    - Assuming the load follows the boom's direction
      - $\boldsymbol{P}_y^i = \boldsymbol{P}_x^i \dfrac{\delta y^i}{\delta x}$ & $\boldsymbol{P}_z^i = \boldsymbol{P}_x^i \dfrac{\delta z^i}{\delta x}$
      - $\left\|\boldsymbol{P}^i\right\| = \boldsymbol{P}_x^i \dfrac{\sqrt{\delta {z^i}^2 + \delta {y^i}^2 + \delta x^2}}{\delta x}$

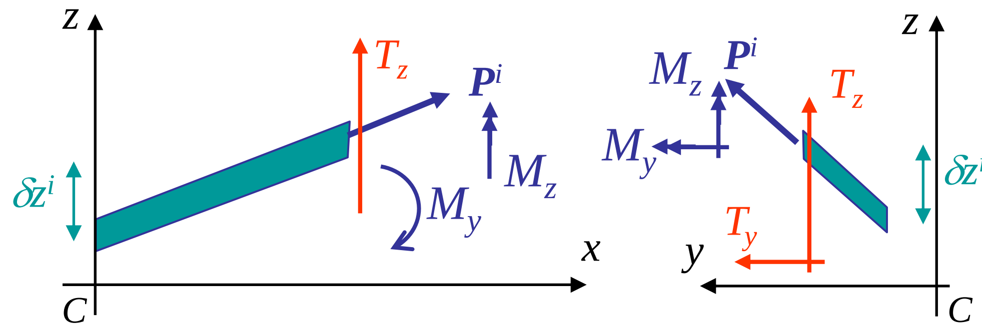






- Open & closed section beams (2)
  - Shear flow
    - Shear loads acting on the skin

$$\begin{cases} T_y^{\text{web}} = T_y - \sum_i \boldsymbol{P}_x^i \dfrac{\delta y^i}{\delta x} \\ T_z^{\text{web}} = T_z - \sum_i \boldsymbol{P}_x^i \dfrac{\delta z^i}{\delta x} \end{cases}$$



    - Shear flux

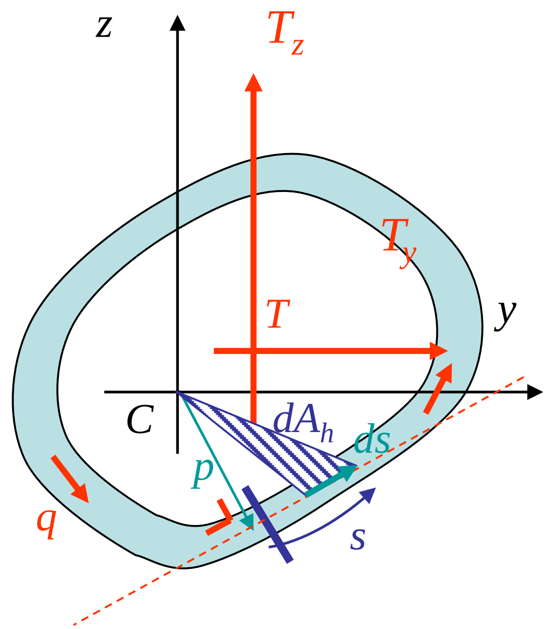
$$-q(s) = -\frac{I_{zz}T_z^{\text{web}} - I_{yz}T_y^{\text{web}}}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct}\,\boldsymbol{\sigma}} z ds + \sum_{i:\ s_i \leq s} z_i A_i\right] - \frac{I_{yy}T_y^{\text{web}} - I_{yz}T_z^{\text{web}}}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct}\,\boldsymbol{\sigma}} y ds + \sum_{i:\ s_i \leq s} y_i A_i\right] + \{q(0)\}$$

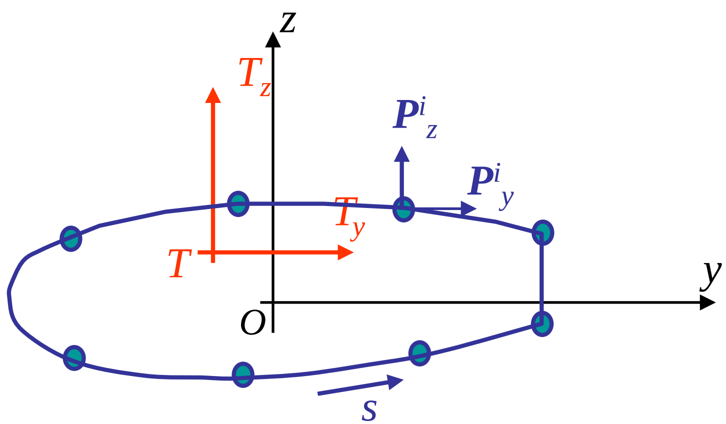
    - $q(0)$?
      - Open section $q(0) = 0$
      - Closed section: correction of previously derived expression to account for the moment induced by the booms' loading

- Open & closed section beams (3)
  - $q(0)$ for closed section
    - Equation (q, s anticlockwise)

$$q\left(s=0\right)=\frac{y_T T_z - z_T T_y - \oint p\left(s\right) q_o\left(s\right) ds}{2A_h}$$

has to be modified to account for booms' loading





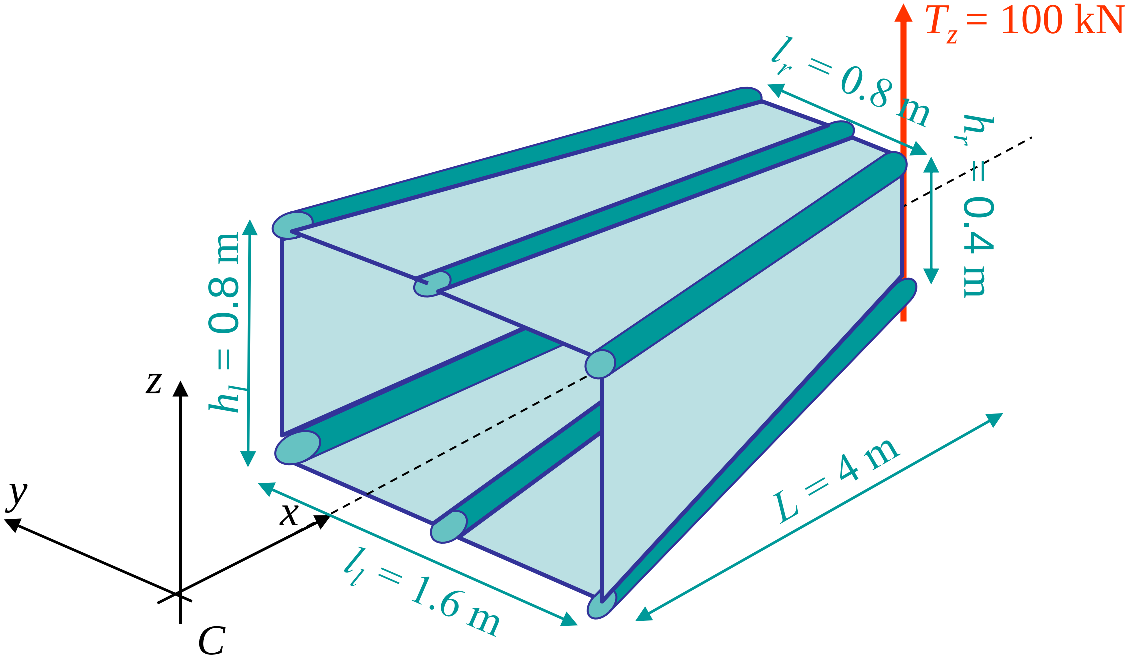
- Moment around O (q, s anticlockwise)

$$y_T T_z - z_T T_y = 2A_h q\left(s=0\right) + \oint p\left(s\right) q_o\left(s\right) ds + \sum_i y^i \boldsymbol{P}^i_z - \sum_i z^i \boldsymbol{P}^i_y$$

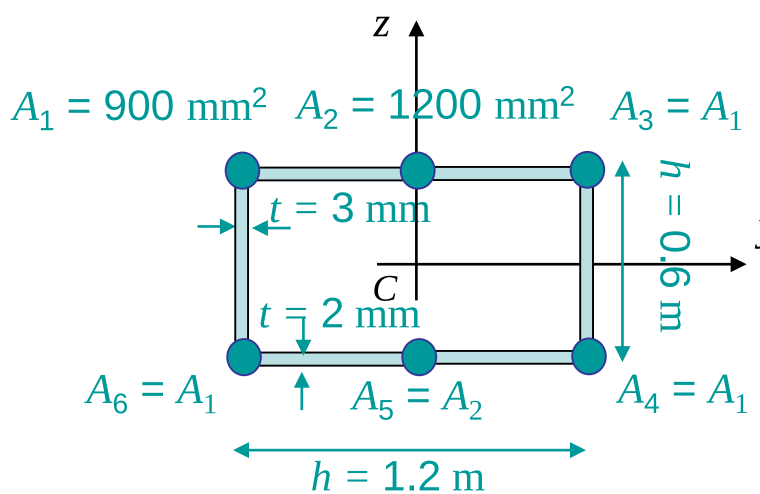
$$\Rightarrow \quad q\left(s=0\right)=\frac{y_T T_z - z_T T_y - \sum_i y^i \boldsymbol{P}^i_z + \sum_i z^i \boldsymbol{P}^i_y - \oint p\left(s\right) q_o\left(s\right) ds}{2A_h}$$

- Open & closed section beams (4)
  - Direct stress
    - As before $$\boldsymbol{\sigma}_{xx} = \frac{\left(I_{zz}M_y + I_{yz}M_z\right)z - \left(I_{yz}M_y + I_{yy}M_z\right)y}{I_{yy}I_{zz} - I_{yz}^2}$$
    - Still considering only direct stress carrying structures

- Example
  - Uniformly tapered beam
  - Idealized structure
    - Booms carry only direct stress
    - Walls carry only shear stress
  - Shear load at free end

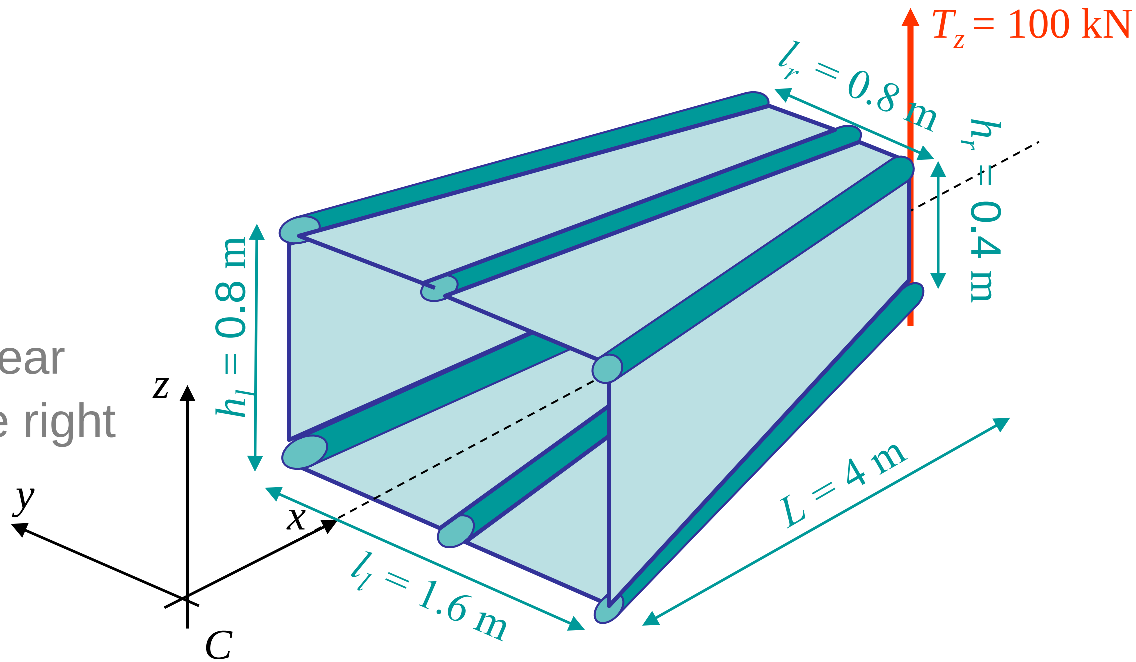


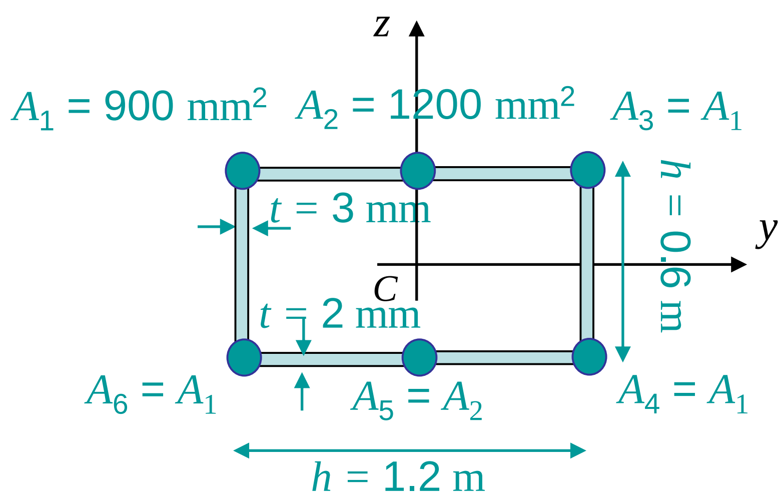
  - Shear distribution at mid section ?



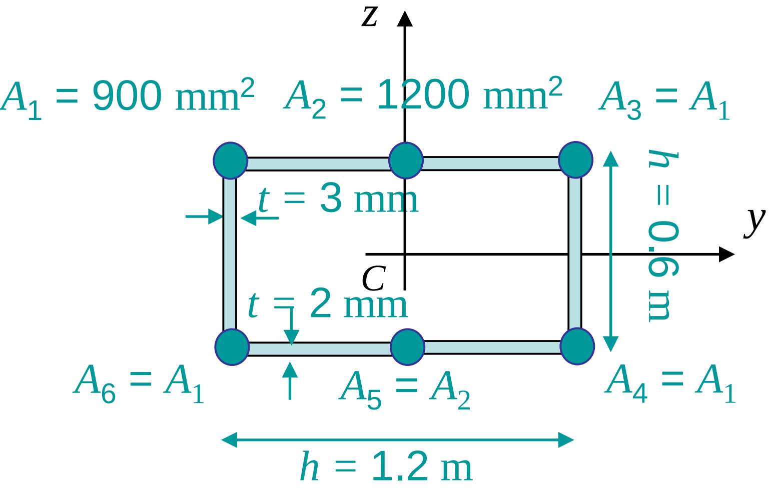
Wing spars & box beams

- Loads at mid section
  - T_y = 0
  - T_z = 100 kN
  - M_y = - 200 kN·m
  - M_z = 0
  - It is assumed that the shear load is transmitted by the right web: y_T = -0.6 m
- Second moment of area
  - Only booms are carrying direct stress
  - Double symmetrical
    - I_{yz} = 0
  - $$I_{yy} = 2\left(2A_1 + A_2\right){z^1}^2 \\ = 2\ 0.003\ 0.3^3 = 0.54\ 10^{-3}\ \ \mathrm{m}^4$$



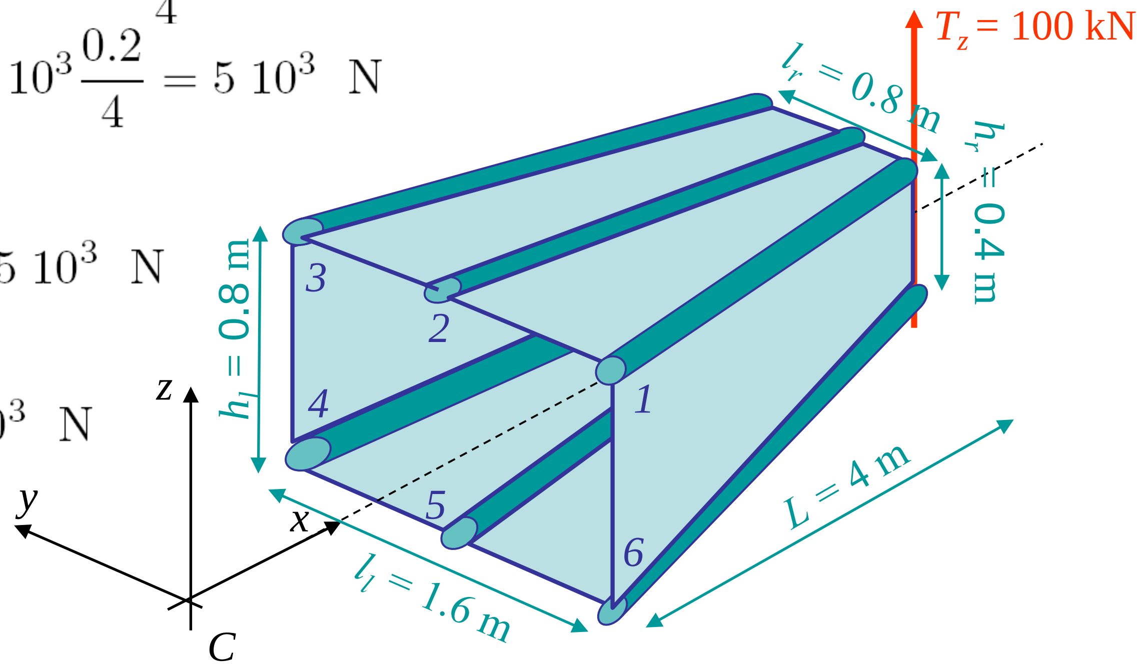


- Direct stress in booms
  - $$\boldsymbol{\sigma}_{xx}^{1} = \boldsymbol{\sigma}_{xx}^{2} = \boldsymbol{\sigma}_{xx}^{3} = \frac{M_y}{I_{yy}} z^1 = \frac{-200\ 10^3}{0.54\ 10^{-3}}\ 0.3 = -111\ \ \text{MPa}$$
  - $$\boldsymbol{\sigma}_{xx}^{4} = \boldsymbol{\sigma}_{xx}^{5} = \boldsymbol{\sigma}_{xx}^{6} = \frac{M_y}{I_{yy}} z^4 = \frac{-200\ 10^3}{0.54\ 10^{-3}}\ (-0.3) = 111\ \ \text{MPa}$$



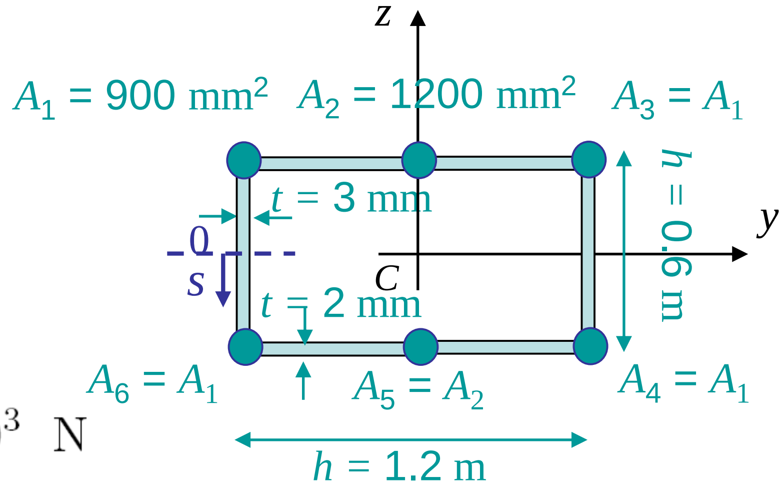
- Loading in booms
  - $$\boldsymbol{P}_x^1 = \boldsymbol{P}_x^3 = A_1 \boldsymbol{\sigma}_{xx}^1 = -0.0009\ 111.\ 10^6 = -100\ 10^3\ \ \text{N}$$
  - $$\boldsymbol{P}_x^2 = A_2 \boldsymbol{\sigma}_{xx}^2 = -0.0012\ 111.\ 10^6 = -133\ 10^3\ \ \text{N}$$
  - $$\boldsymbol{P}_x^4 = \boldsymbol{P}_x^6 = A_4 \boldsymbol{\sigma}_{xx}^4 = 0.0009\ 111.\ 10^6 = 100\ 10^3\ \ \text{N}$$
  - $$\boldsymbol{P}_x^5 = A_5 \boldsymbol{\sigma}_{xx}^5 = 0.0012\ 111.\ 10^6 = 133\ 10^3\ \ \text{N}$$

- Loading in booms (2)
  - $\boldsymbol{P}_y^1 = \boldsymbol{P}_x^1 \partial_x y^1 = -100\ 10^3 \dfrac{0.4}{4} = -10^4 \ \text{ N}, \quad \boldsymbol{P}_y^2 = \boldsymbol{P}_x^2 \partial_x y^2 = 0$
  - $\boldsymbol{P}_y^3 = \boldsymbol{P}_x^3 \partial_x y^3 = -100\ 10^3 \dfrac{-0.4}{4} = 10^4 \ \text{ N}$
  - $\boldsymbol{P}_y^4 = \boldsymbol{P}_x^4 \partial_x y^4 = 100\ 10^3 \dfrac{-0.4}{4} = -10^4 \ \text{ N}, \quad \boldsymbol{P}_y^5 = \boldsymbol{P}_x^5 \partial_x y^5 = 0$
  - $\boldsymbol{P}_y^6 = \boldsymbol{P}_x^6 \partial_x y^6 = 100\ 10^3 \dfrac{0.4}{4} = 10^4 \ \text{ N}$
  - $\boldsymbol{P}_z^1 = \boldsymbol{P}_z^3 = \boldsymbol{P}_x^1 \partial_x z^1 = -100\ 10^3 \dfrac{-0.2}{4} = 5\ 10^3 \ \text{ N}$
  - $\boldsymbol{P}_z^4 = \boldsymbol{P}_z^6 = \boldsymbol{P}_x^4 \partial_x z^4 = 100\ 10^3 \dfrac{0.2}{4} = 5\ 10^3 \ \text{ N}$
  - $\boldsymbol{P}_z^2 = \boldsymbol{P}_x^2 \partial_x z^2$
    $= -133\ 10^3 \dfrac{-0.2}{4} = 6.65\ 10^3 \ \text{ N}$
  - $\boldsymbol{P}_z^5 = \boldsymbol{P}_x^5 \partial_x z^5$
    $= 133\ 10^3 \dfrac{0.2}{4} = 6.65\ 10^3 \ \text{ N}$



- Open shear flux
  - Loading in the web
    - $T_y^{\text{web}} = T_y - \sum_i \boldsymbol{P}_y^i = 0$
    - $T_z^{\text{web}} = T_z - \sum_i \boldsymbol{P}_z^i$

$$= 100\ 10^3 \ - 33.3\ 10^3 = 66.6\ 10^3 \quad \text{N}$$



  - Shear flux
    - As only booms are carrying direct stress $q_o(s) = -\frac{T_z^{\text{web}}}{I_{yy}} \sum_{i:\ s_i \leq s} z_i A_i$
    - Origin between booms 1 & 6

$$q_o^{65} = q^6 - \frac{T_z^{\text{web}}}{I_{yy}} A_6 z_6 = -\frac{66.6\ 10^3}{0.54\ 10^{-3}} 0.0009\ (-0.3) = 33.3\ 10^3 \quad \text{N} \cdot \text{m}^{-1}$$

$$q_o^{54} = q^5 - \frac{T_z^{\text{web}}}{I_{yy}} A_5 z_5 = 33.3\ 10^3 - \frac{66.6\ 10^3}{0.54\ 10^{-3}} 0.0012\ (-0.3) = 77.8\ 10^3 \quad \text{N} \cdot \text{m}^{-1}$$

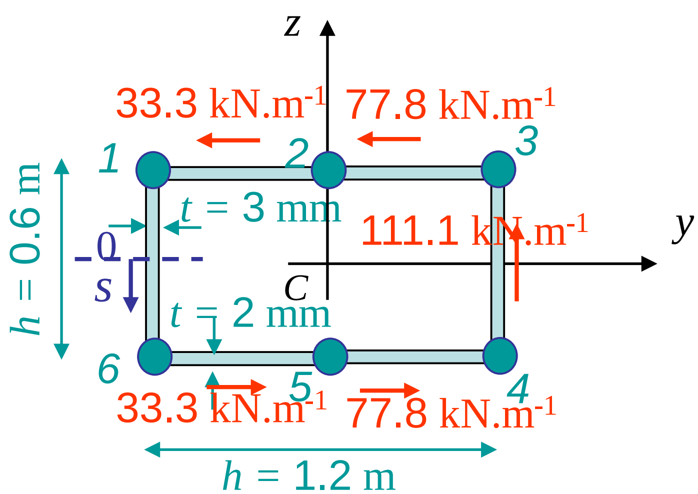
$$q_o^{43} = q^4 - \frac{T_z^{\text{web}}}{I_{yy}} A_4 z_4 = 77.8\ 10^3 - \frac{66.6\ 10^3}{0.54\ 10^{-3}} 0.0009\ (-0.3) = 111.1\ 10^3 \quad \text{N} \cdot \text{m}^{-1}$$

$$q_o^{32} = q^3 - \frac{T_z^{\text{web}}}{I_{yy}} A_3 z_3 = 111.1\ 10^3 - \frac{66.6\ 10^3}{0.54\ 10^{-3}} 0.0009\ 0.3 = 77.8\ 10^3 \quad \text{N} \cdot \text{m}^{-1}$$

- Open shear flux (2)
  - Shear flux (2)

$$q_o^{21} = q^2 - \frac{T_z^{\text{web}}}{I_{yy}} A_2 z_2 = 77.8\ 10^3 - \frac{66.6\ 10^3}{0.54\ 10^{-3}} 0.0012\ 0.3 = 33.3\ 10^3 \quad \text{N}\cdot\text{m}^{-1}$$

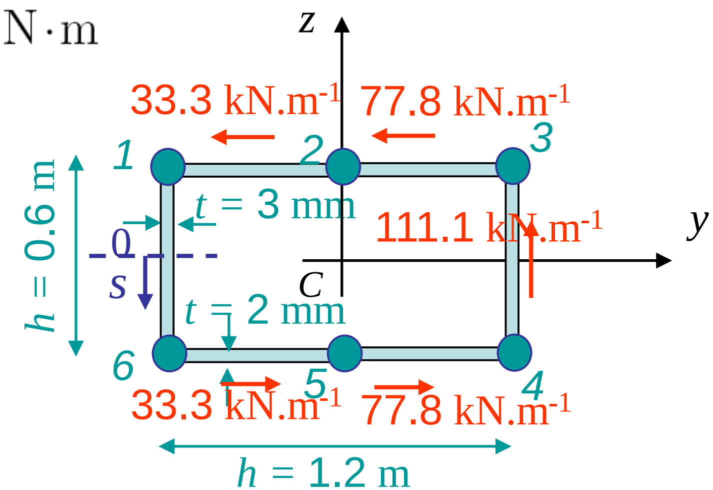
$$q_o^{10} = q^1 - \frac{T_z^{\text{web}}}{I_{yy}} A_1 z_1 = 33.3\ 10^3 - \frac{66.6\ 10^3}{0.54\ 10^{-3}} 0.0009\ 0.3 = 0$$



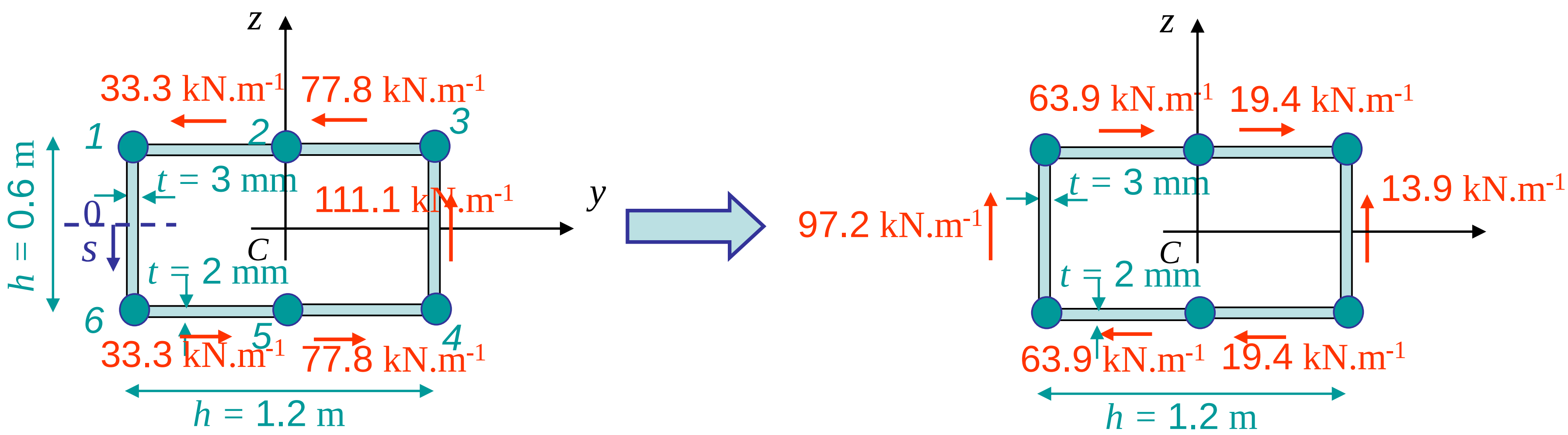
- $q(0)$
  - $$q\left(s=0\right) = \frac{y_T T_z - \sum_i y^i \boldsymbol{P}_z^i + \sum_i z^i \boldsymbol{P}_y^i - \oint p\left(s\right) q_o\left(s\right) ds}{2A_h}$$
  - $$A_h = 1.2\ 0.6 = 0.72 \quad \mathrm{m}^2$$
  - $$\sum_i y^i \boldsymbol{P}_z^i = 5\ 10^3\ \left(-0.6\right) + 5\ 10^3\ 0.6 + 5\ 10^3\ \left(-0.6\right) + 5\ 10^3\ 0.6 = 0$$
  - $$\sum_i z^i \boldsymbol{P}_y^i = -10^4\ 0.3 + 10^4\ 0.3 - 10^4\ \left(-0.3\right) + 10^4\ \left(-0.3\right) = 0$$
  - $$\oint p q_o ds = \left|z^6\right| \left(q_o^{65} l^{65} + q_o^{54} l^{54} + q_o^{32} l^{32} + q_o^{21} l^{21}\right) + \left|y_4\right| q_o^{43} l^{43}$$

$$\Rightarrow \oint p q_o ds = 0.3\ 0.6\ 10^3\ \left(2\ 33.3 + 2\ 77.8\right) + 0.6\ 0.6\ 111.1\ 10^3 = 80\ 10^3 \quad \mathrm{N \cdot m}$$

$$\Rightarrow q\left(s=0\right) = \frac{-0.6\ 100\ 10^3 - 80\ 10^3}{2\ 0.72} = -97.2\ 10^3 \quad \mathrm{N \cdot m^{-1}}$$



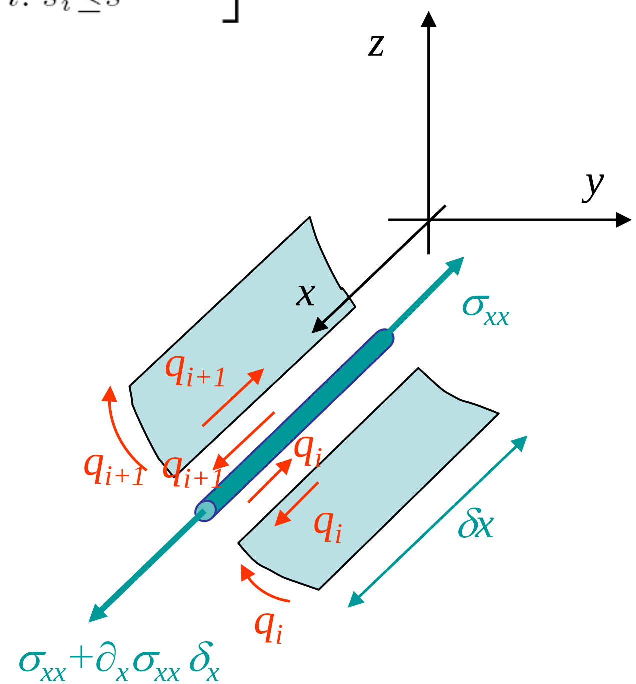
- Shear flux
  - As $q\left(s=0\right) = -97.2\ 10^{3}\ \ \mathrm{N}\cdot\mathrm{m}^{-1}$



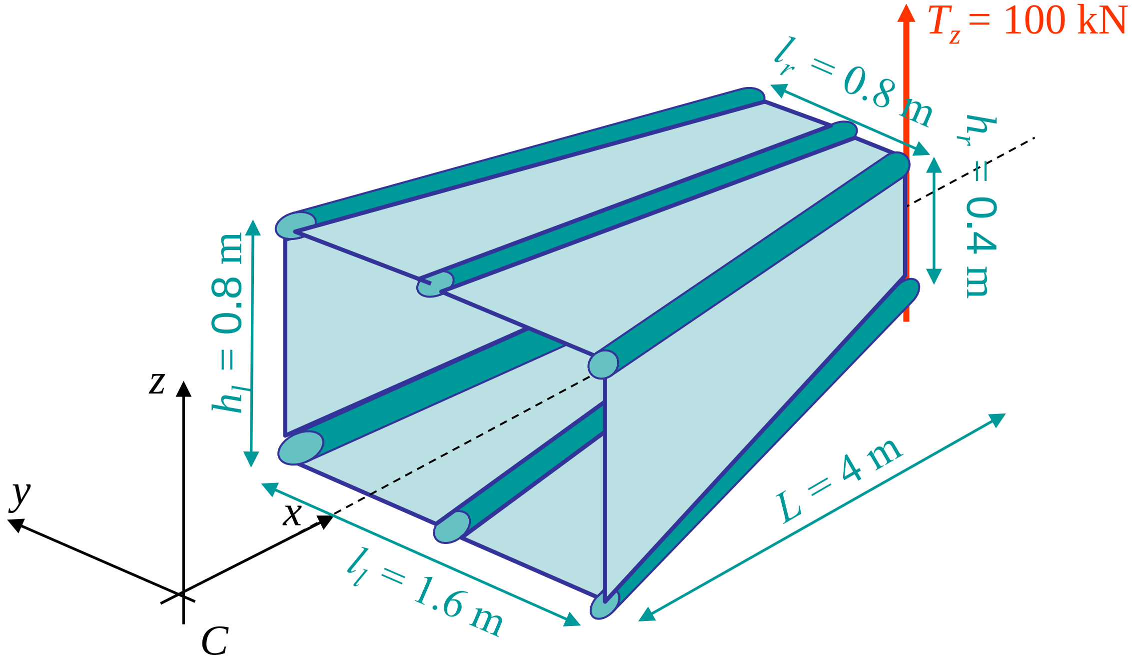
- Variable stringer areas
  - Stringer areas may vary in the spanwise direction
  - When establishing
    - $$q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct}\ \boldsymbol{\sigma}} z ds + \sum_{i:\ s_i \le s} z_i A_i\right] - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct}\ \boldsymbol{\sigma}} y ds + \sum_{i:\ s_i \le s} y_i A_i\right]$$
    - We have used $q_{i+1} - q_i = -\partial_x \boldsymbol{\sigma}_{xx} A_i$

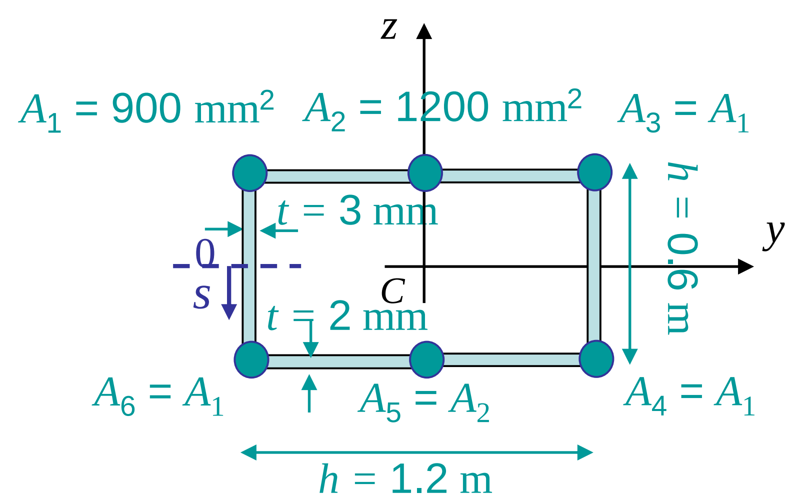
      which assumes a constant boom area
    - If booms area change we have to consider

$$q_{i+1} - q_i = -\frac{\Delta \boldsymbol{P}_x^i}{\Delta x}$$

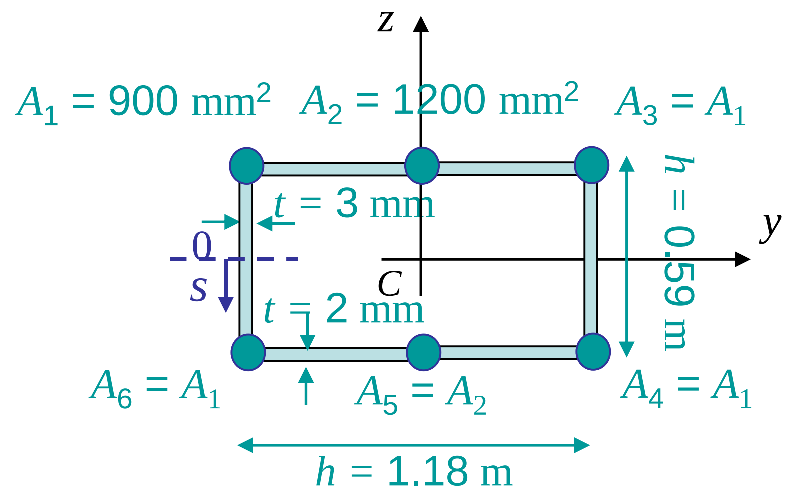


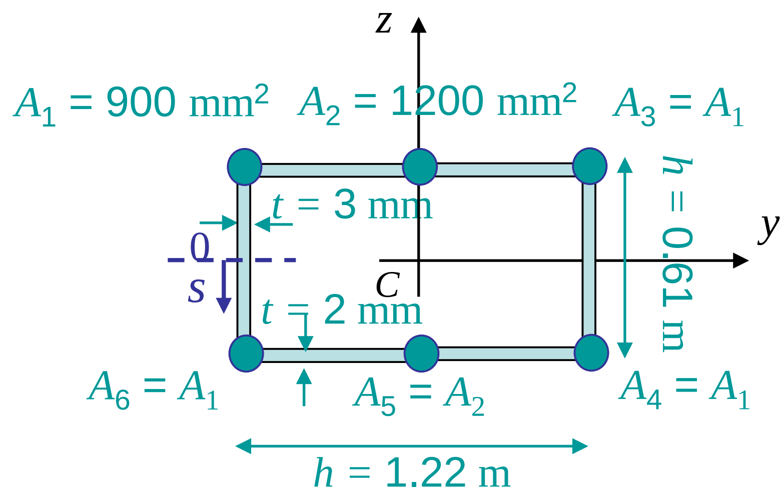
- Example
  - Uniformly tapered beam
  - Idealized structure
    - Booms carry only direct stress
    - Walls carry only shear stress
  - Shear load at free end
  - Shear distribution at mid section by using the method suitable for variable stringer area (even if this is not the case)?





- Loads at mid section
  - Values at mid section ± Δx
    - Example at x = 2.1 m
      - T_y = 0, M_z = 0
      - T_z = 100 kN
      - M_y = - 190 kN·m
    - Example at x = 1.9 m
      - T_y = 0, M_z = 0
      - T_z = 100 kN
      - M_y = - 210 kN·m





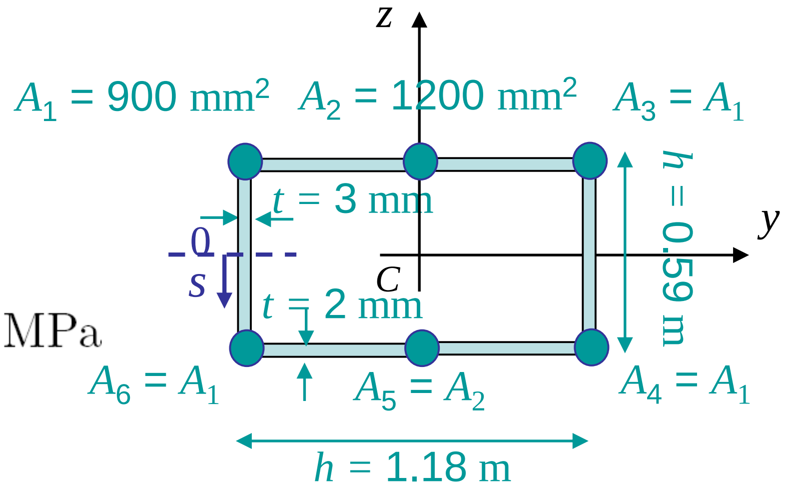
- Second moment of area
  - Only boom are carrying direct stress
  - Double symmetrical
    - I_{yz} = 0
    - At x = 2.1 m

$$I_{yy} = 2\left(2A_1 + A_2\right) {z^1}^2 = 2\,0.003\,0.295^2 = 0.522\;10^{-3}\quad \mathrm{m}^4$$

    - At x = 1.9 m

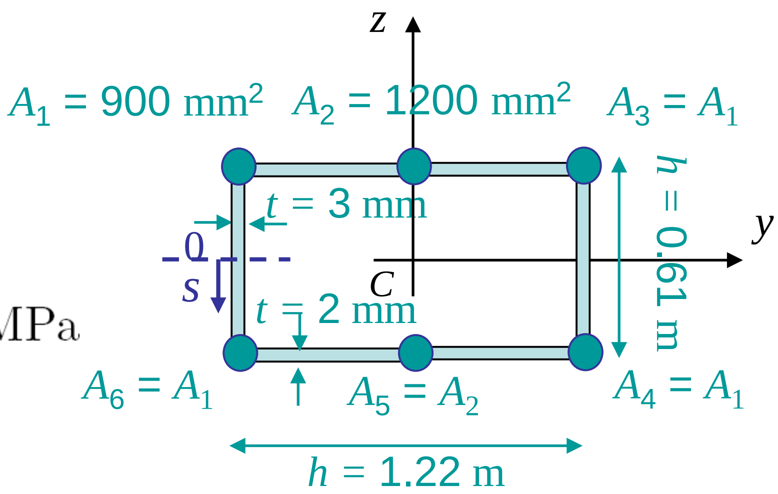
$$I_{yy} = 2\left(2A_1 + A_2\right) {z^1}^2 = 2\,0.003\,0.305^2 = 0.558\;10^{-3}\quad \mathrm{m}^4$$

- Loading in booms
  - At x = 2.1 m
    - $\boldsymbol{\sigma}_{xx}^{1-3} = -\boldsymbol{\sigma}_{xx}^{4-6} = \frac{M_y}{I_{yy}} z^1$
      $= \frac{-190\ 10^3}{0.522\ 10^{-3}} 0.295 = -107.4 \quad \text{MPa}$



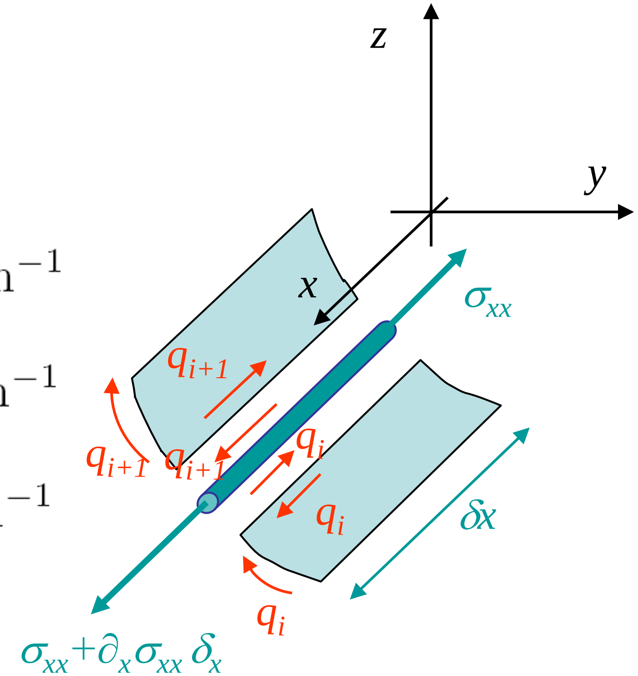
    - $\boldsymbol{P}_x^1 = \boldsymbol{P}_x^3 = -\boldsymbol{P}_x^4 = -\boldsymbol{P}_x^6 = A_1 \boldsymbol{\sigma}_{xx}^1$
      $= -0.0009\ 107.4\ 10^6 = -96.7\ 10^3 \quad \text{N}$
    - $\boldsymbol{P}_x^2 = -\boldsymbol{P}_x^5 = A_2 \boldsymbol{\sigma}_{xx}^2 = -0.0012\ 107.4\ 10^6 = -128.9\ 10^3 \quad \text{N}$

- Loading in booms (2)
  - At x = 1.9 m
    - $$\boldsymbol{\sigma}_{xx}^{1-3} = -\boldsymbol{\sigma}_{xx}^{4-6} = \frac{M_y}{I_{yy}} z^1 = \frac{-210\ 10^3}{0.558\ 10^{-3}}\ 0.305 = -114.8\ \text{ MPa}$$
    - $$\boldsymbol{P}_x^1 = \boldsymbol{P}_x^3 = -\boldsymbol{P}_x^4 = -\boldsymbol{P}_x^6 = A_1 \boldsymbol{\sigma}_{xx}^1 = -0.0009\ 114.8\ 10^6 = -103.3\ 10^3\ \text{ N}$$
    - $$\boldsymbol{P}_x^2 = -\boldsymbol{P}_x^5 = A_2 \boldsymbol{\sigma}_{xx}^2 = -0.0012\ 114.8\ 10^6 = -137.8\ 10^3\ \text{ N}$$

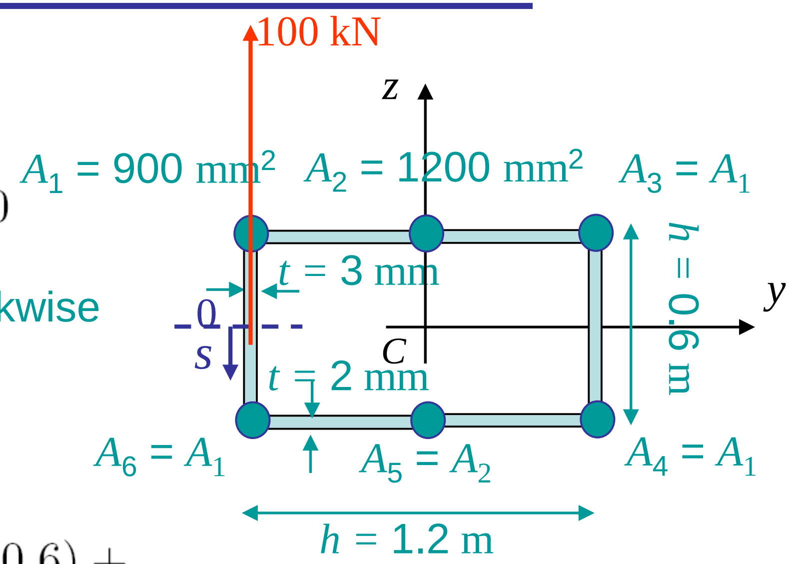


- Change in compressive loads
  - $$\frac{\Delta \boldsymbol{P}_x^1}{\Delta x} = \frac{\Delta \boldsymbol{P}_x^3}{\Delta x} = -\frac{\Delta \boldsymbol{P}_x^4}{\Delta x} = -\frac{\Delta \boldsymbol{P}_x^6}{\Delta x} = \frac{-96.7 + 103.3}{0.2} 10^3 = 33\ 10^3\ \mathrm{N} \cdot \mathrm{m}^{-1}$$
  - $$\frac{\Delta \boldsymbol{P}_x^2}{\Delta x} = -\frac{\Delta \boldsymbol{P}_x^5}{\Delta x} = \frac{-128.9 + 137.8}{0.2} 10^3 = 44.5\ 10^3\ \mathrm{N} \cdot \mathrm{m}^{-1}$$
  - Remark: by trigonometry $\dfrac{\Delta P}{\Delta l} = \dfrac{\Delta \boldsymbol{P}_x}{\Delta x}$
- Shear flux relationships
  - As $q_{i+1} - q_i = -\dfrac{\Delta \boldsymbol{P}_x^i}{\Delta x}$

$$\begin{cases} q^{23} + \dfrac{\Delta \boldsymbol{P}_x^2}{\Delta x} = q^{12} \Longrightarrow q^{23} = q^{12} - 44.5\ 10^3\ \mathrm{N} \cdot \mathrm{m}^{-1} \\ q^{34} + \dfrac{\Delta \boldsymbol{P}_x^3}{\Delta x} = q^{23} \Longrightarrow q^{34} = q^{12} - 77.5\ 10^3\ \mathrm{N} \cdot \mathrm{m}^{-1} \\ q^{45} + \dfrac{\Delta \boldsymbol{P}_x^4}{\Delta x} = q^{34} \Longrightarrow q^{45} = q^{12} - 44.5\ 10^3\ \mathrm{N} \cdot \mathrm{m}^{-1} \\ q^{56} + \dfrac{\Delta \boldsymbol{P}_x^5}{\Delta x} = q^{45} \Longrightarrow q^{56} = q^{12} \\ q^{61} + \dfrac{\Delta \boldsymbol{P}_x^6}{\Delta x} = q^{56} \Longrightarrow q^{61} = q^{12} + 33\ 10^3\ \mathrm{N} \cdot \mathrm{m}^{-1} \end{cases}$$



- Shear flux
  - Moment at x = 2 m
    - We found $\sum_i y^i \boldsymbol{P}_z^i = 0$ & $\sum_i z^i \boldsymbol{P}_y^i = 0$
    - Direction of shear flux $q^{i\,i+1}$ is positive clockwise

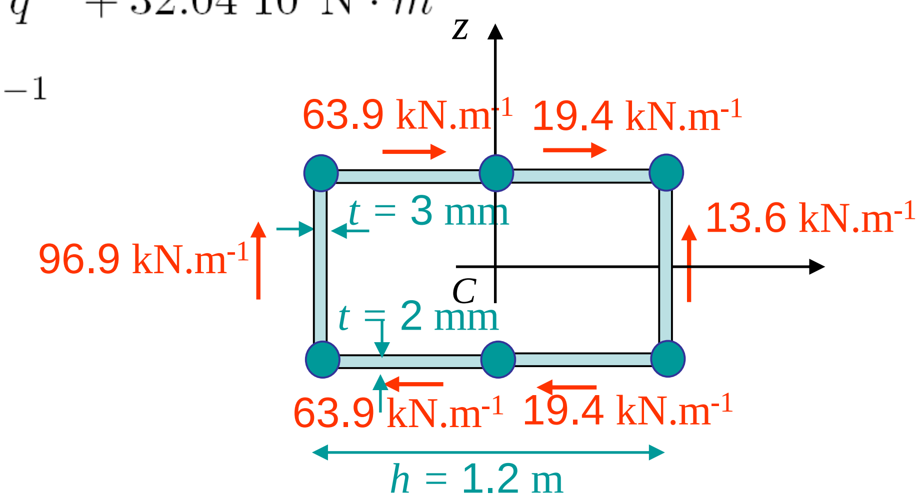


$$\Rightarrow \sum_i -q^{i\,i+1} p_{i\,i+1} l^{i\,i+1} = T_z y_T$$

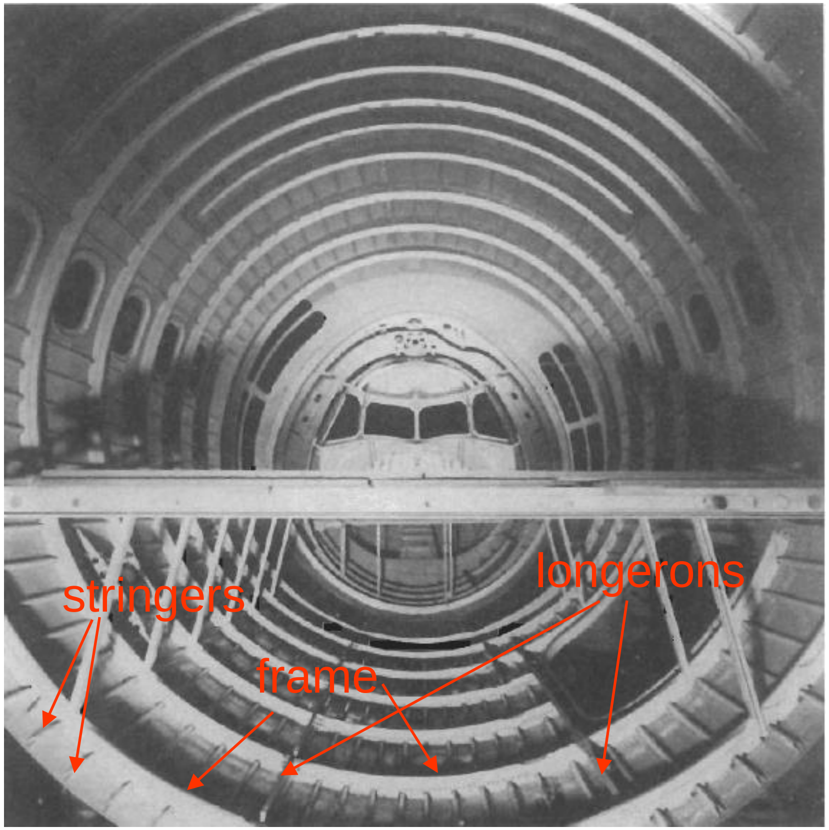
$$\Rightarrow -0.6\ 100\ 10^3 \quad = \quad -q^{12}\,(4\ 0.3\ 0.6 + 2\ 0.6\ 0.6) +$$
$$44.5\ 10^3\ 0.3\ 0.6 + 77.5\ 10^3\ 0.6^2 + 44.5\ 10^3\ 0.3\ 0.6 - 33\ 10^3\ 0.6^2$$

$$\Rightarrow -60\ 10^3\ \mathrm{N}\cdot m = -1.44\ \mathrm{m}^2 q^{12} + 32.04\ 10^3 \mathrm{N}\cdot m$$

$$\Rightarrow q^{12} = 63.9\ 10^3 \mathrm{N}\cdot m^{-1}$$



- Real fuselage
  - Structure
    - Longitudinal stringers (& longerons)
    - Transverse frames
  - Carries
    - Bending
    - Shear
    - Torsion
- Simplification for section
  - Usually
    - Resistance of stringers to shear stress is generally reduced
    - Distance between stringers is small
      - Shear can be considered constant in the skin between 2 stringers
  - Idealized section approximation holds
    - Direct stress carried by booms only or by booms and skin
    - In the first case: boom area is increased to account for skin direct stress carrying capacity
    - Constant shear flow carried by skin only



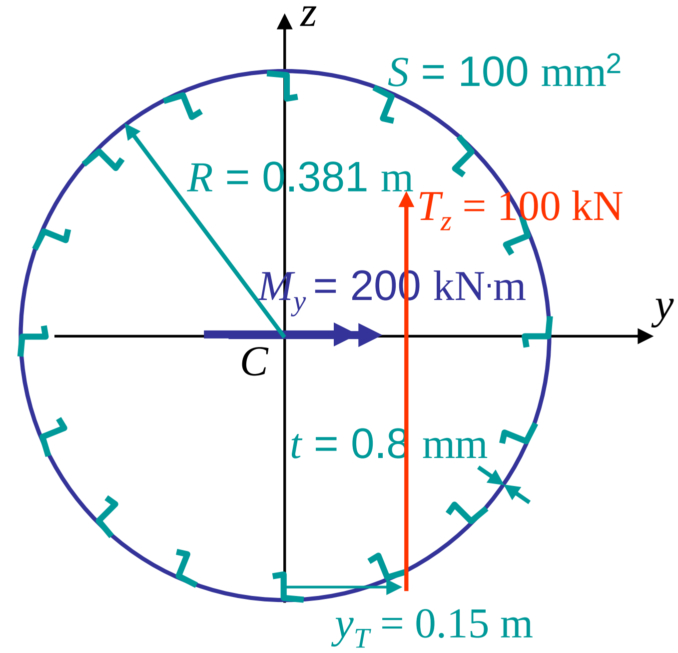
Fuselages

- Bending
  - As before $\boldsymbol{\sigma}_{xx} = \frac{(I_{zz}M_y + I_{yz}M_z)\,z - (I_{yz}M_y + I_{yy}M_z)\,y}{I_{yy}I_{zz} - I_{yz}^2}$
  - Still considering only direct stress carrying structures: *i.e*. the booms
- Shearing
  - Direct stress are carried out by the booms only
  - Fuselages have closed sections

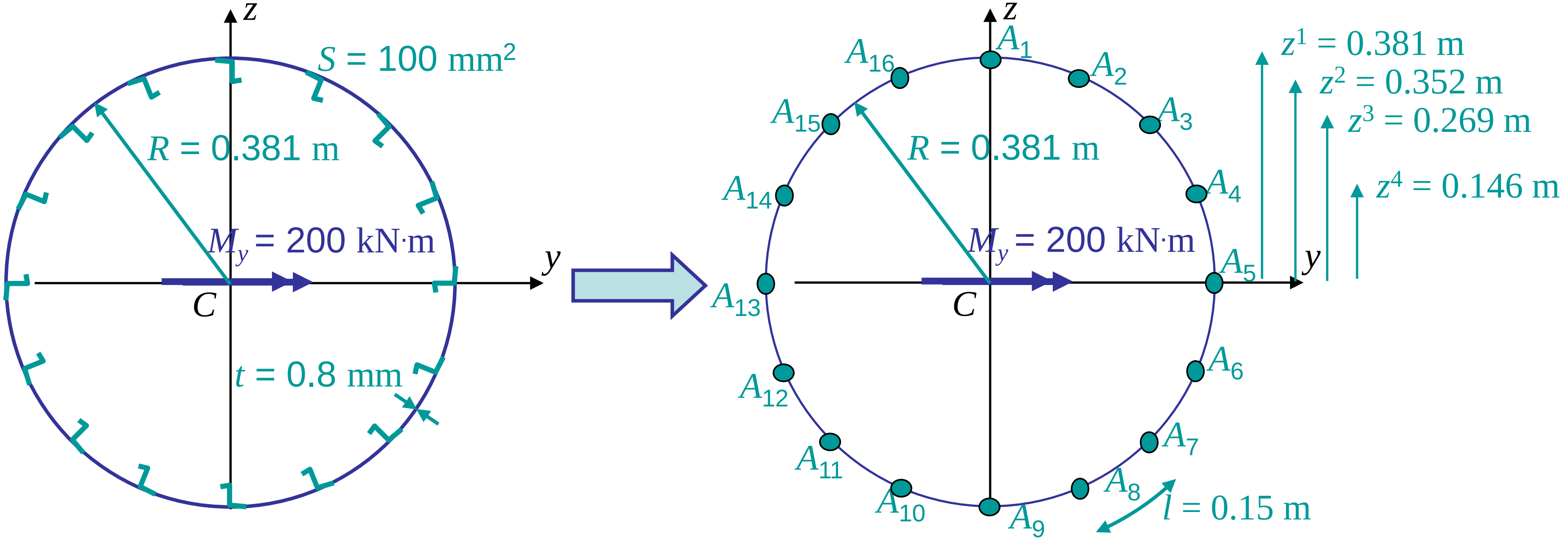
$$q\,(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\sum_{i:\ s_i \leq s} z_i A_i - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\sum_{i:\ s_i \leq s} y_i A_i + q(0)$$

  - Gives the shear flux value for shear loads not necessarily passing through the shear center (in which case the flux resulting from the torque is included)
- Torsion
  - As before $M_x = 2A_h q$
    - Leads to constant q on the skin panels
  - Booms are assumed not to carry shear stress
- Tapered fuselage
  - Adapt formula as for the tapered wing

- Example
  - Small tourism aircraft
  - Circular cross section
  - Each stringer has a 100 mm²-area
  - Direct stress distribution of the idealized section?
  - Shear flux of the idealized section?
  - What is the torsion part of the shear flux?

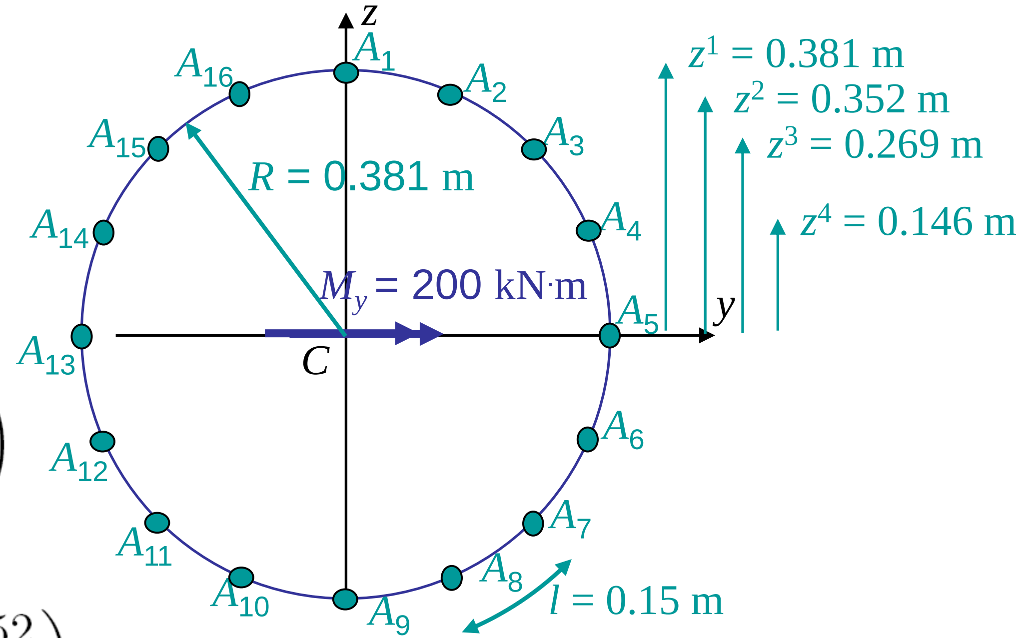


- Section idealization
  - Elementary geometry



  - Idealized booms area
    - As bending results in z-linear stress distribution
    - Let us approximate the skin between booms as being flat (ok if enough booms)
    - So one can use the idealized formula $A_1 = \frac{t_D b}{6}\left(2 + \frac{\boldsymbol{\sigma}_{xx}^2}{\boldsymbol{\sigma}_{xx}^1}\right)$

- Section idealization (2)
  - Idealized booms area (2)
    - Oy principal & bending axis
    - A_5 and A_{13} are meaningless
    - Other ones



$$A_1 = S + \frac{tl}{6}\left(2 + \frac{z^2}{z^1}\right) + \frac{tl}{6}\left(2 + \frac{z^{16}}{z^1}\right)$$

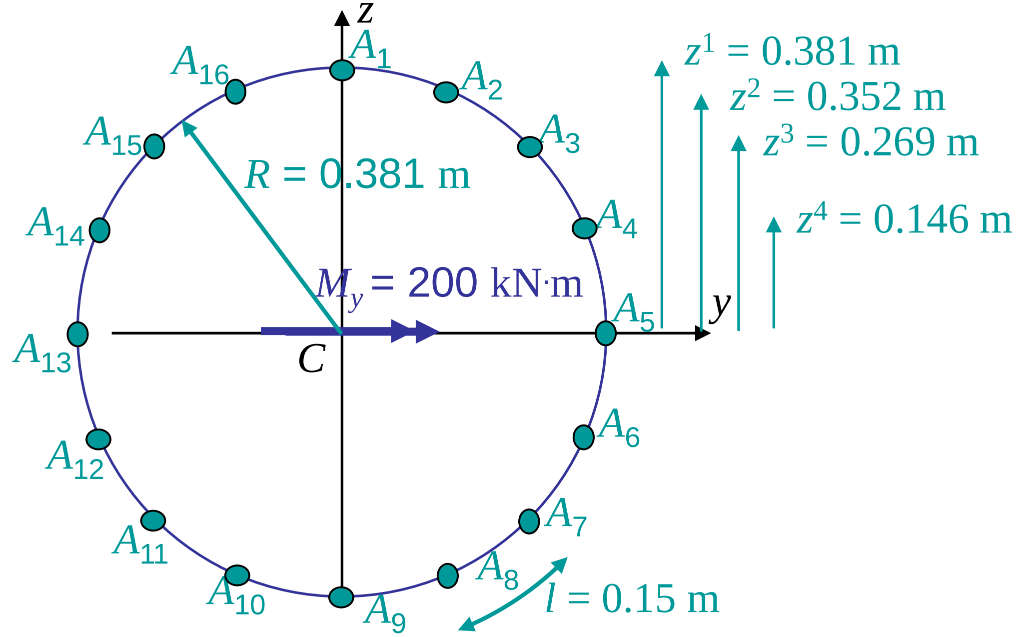
$$\Rightarrow A_1 = A_9 = 100\ 10^{-6} + 2\,\frac{0.0008\ 0.15}{6}\left(2 + \frac{0.352}{0.381}\right) = 217\ 10^{-6}\quad \mathrm{m}^2$$

$$A_2 = A_{16} = A_8 = A_{10} = 100\ 10^{-6} + \frac{0.0008\ 0.15}{6}\left(4 + \frac{0.381}{0.352} + \frac{0.269}{0.352}\right) = 217\ 10^{-6}\quad \mathrm{m}^2$$

$$A_3 = A_{15} = A_7 = A_{11} = 100\ 10^{-6} + \frac{0.0008\ 0.15}{6}\left(4 + \frac{0.352}{0.269} + \frac{0.146}{0.269}\right) = 217\ 10^{-6}\quad \mathrm{m}^2$$

$$A_4 = A_{14} = A_6 = A_{12} = 100\ 10^{-6} + \frac{0.0008\ 0.15}{6}\left(4 + \frac{0.269}{0.146} + \frac{0.}{0.146}\right) = 217\ 10^{-6}\quad \mathrm{m}^2$$

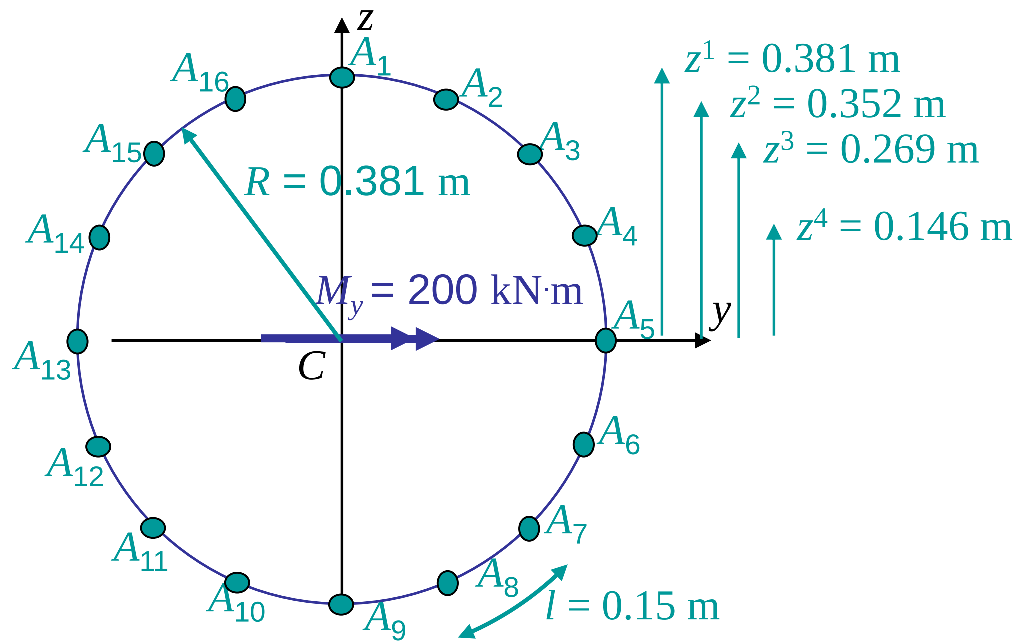
- Bending
  - Second moment of area
    - Doubly symmetrical section
    - I_{yz}=0
    - Only idealized booms are carrying direct stress



$$I_{yy} = \sum_i A_i {z^i}^2$$

$$\Longrightarrow I_{yy} = 217\ 10^{-6}\ \left(2.\ 0.381^2 + 4\ 0.352^2 + 4\ 0.269^2 + 4\ 0.146^2\right) = 252\ 10^{-6}\ \ \mathrm{m}^4$$

- Bending (2)
  - Direct stress
    - As we are in the principal axes

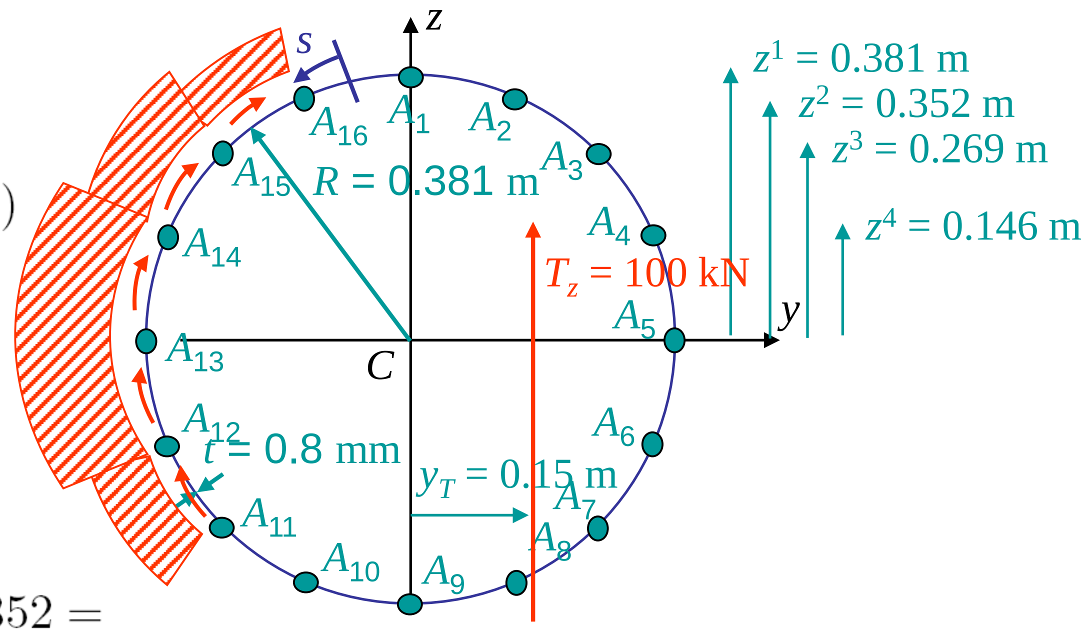


$$\boldsymbol{\sigma}_{xx}^{i} = \frac{M_y}{I_{yy}} z^i$$

$$\Rightarrow \left\{ \begin{aligned}
&\boldsymbol{\sigma}_{xx}^{1} = -\boldsymbol{\sigma}_{xx}^{9} = \frac{200\ 10^3}{252\ 10^{-6}}\ 0.381 = 302\ \ \text{MPa} \\
&\boldsymbol{\sigma}_{xx}^{2} = \boldsymbol{\sigma}_{xx}^{16} = -\boldsymbol{\sigma}_{xx}^{8} = -\boldsymbol{\sigma}_{xx}^{10} = \frac{200\ 10^3}{252\ 10^{-6}}\ 0.352 = 279\ \ \text{MPa} \\
&\boldsymbol{\sigma}_{xx}^{3} = \boldsymbol{\sigma}_{xx}^{15} = -\boldsymbol{\sigma}_{xx}^{7} = -\boldsymbol{\sigma}_{xx}^{11} = \frac{200\ 10^3}{252\ 10^{-6}}\ 0.269 = 213\ \ \text{MPa} \\
&\boldsymbol{\sigma}_{xx}^{4} = \boldsymbol{\sigma}_{xx}^{14} = -\boldsymbol{\sigma}_{xx}^{6} = -\boldsymbol{\sigma}_{xx}^{12} = \frac{200\ 10^3}{252\ 10^{-6}}\ 0.146 = 116\ \ \text{MPa} \\
&\boldsymbol{\sigma}_{xx}^{5} = \boldsymbol{\sigma}_{xx}^{13} = 0
\end{aligned} \right.$$

- Shearing
  - Open shear flux

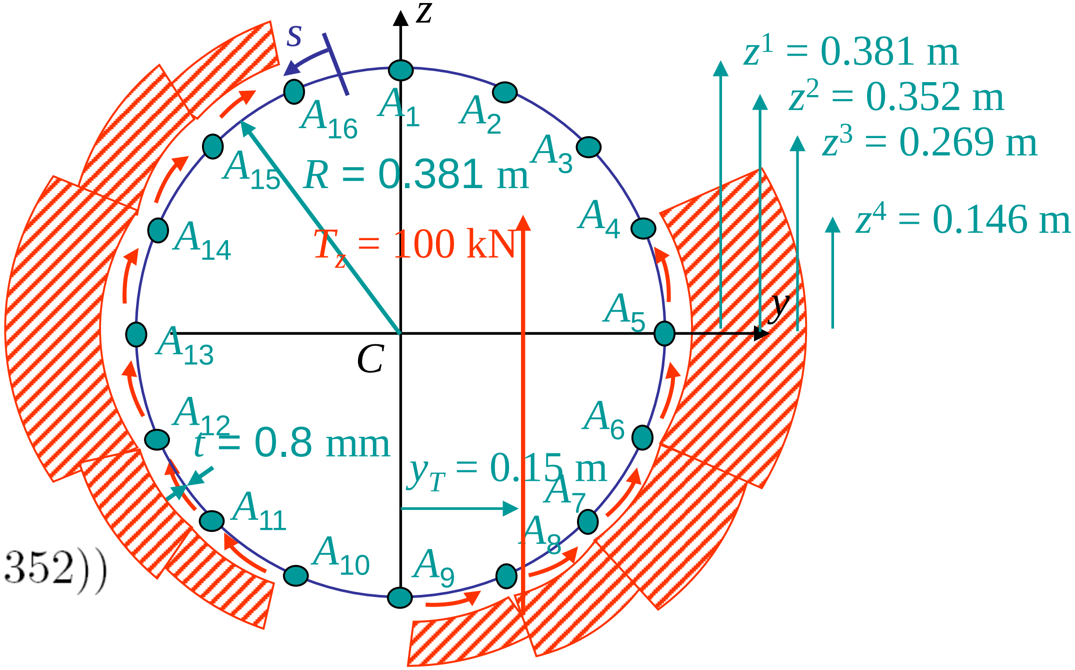
$$q(s) = -\frac{T_z}{I_{yy}} \sum_{i:\, s_i \leq s} z_i A_i + q(0)$$



- Origin between 1 & 16

$$\left\{\begin{array}{l}
q_o^{1\ 16} = 0 \\
q_o^{16\ 15} = -\frac{T_z}{I_{yy}} A_{16} z^{16} = \\
\qquad -\frac{100\ 10^3}{252\ 10^{-6}} 217\ 10^{-6}\ 0.352 = \\
\qquad\qquad -30.3\ 10^3 \quad \mathrm{N \cdot m^{-1}} \\
q_o^{15\ 14} = -\frac{100\ 10^3}{252\ 10^{-6}} 10^{-6} (217\ 0.352 + 217\ 0.268) = -53.5\ 10^3 \quad \mathrm{N} \\
q_o^{14\ 13} = -\frac{100\ 10^3}{252\ 10^{-6}} 10^{-6} (217\ 0.352 + 217\ 0.268 + 217\ 0.146) = -66.0\ 10^3 \quad \mathrm{N{\cdot}m^{-1}} \\
q_o^{13\ 12} = -\frac{100\ 10^3}{252\ 10^{-6}} 10^{-6} (217\ 0.352 + \ldots + 217\ 0) = -66.0\ 10^3 \quad \mathrm{N \cdot m^{-1}} \\
q_o^{12\ 11} = -\frac{100\ 10^3}{252\ 10^{-6}} 10^{-6} (217\ 0.352 + \ldots + 217\,(-0.146)) = -53.5\ 10^3 \quad \mathrm{N \cdot m^{-1}}
\end{array}\right.$$

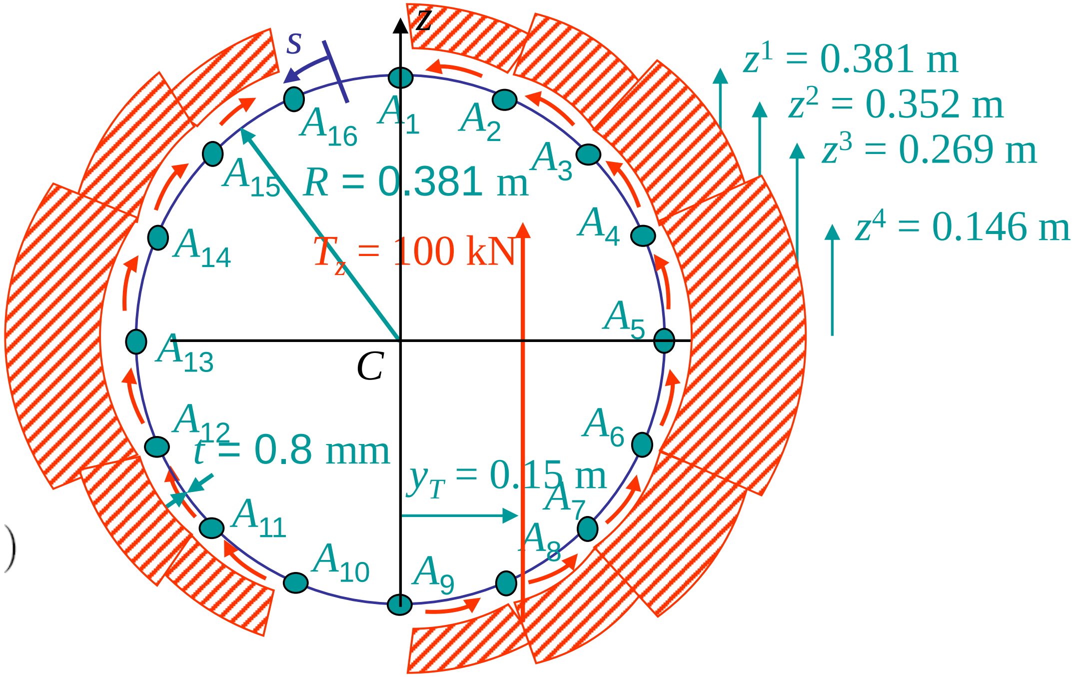
- Shearing (2)
  - Open shear flux (2)



$$\left\{\begin{array}{l}
q_o^{11\ 10} = -\dfrac{100\ 10^3}{252\ 10^{-6}}\ 10^{-6} \\
\quad (217\ 0.352 + \dots \\
\quad\quad + 217\,(-0.269)) = \\
\quad\quad -30.3\ 10^3\ \ \mathrm{N}\cdot\mathrm{m}^{-1} \\
q_o^{10\ 9} = -\dfrac{100\ 10^3}{252\ 10^{-6}}\ 10^{-6} \\
\quad (217\ 0.352 + \dots + 217\,(-0.352)) \\
\quad = 0 \\
q_o^{9\ 8} = -\dfrac{100\ 10^3}{252\ 10^{-6}}\ 10^{-6}\,(217\ 0.352 + \dots + 217\,(-0.381)) = 32.8\ 10^3\ \ \mathrm{N}\cdot\mathrm{m}^{-1} \\
q_o^{8\ 7} = -\dfrac{100\ 10^3}{252\ 10^{-6}}\ 10^{-6}\,(217\ 0.352 + \dots + 217\,(-0.352)) = 63.1\ 10^3\ \ \mathrm{N}\cdot\mathrm{m}^{-1} \\
q_o^{7\ 6} = -\dfrac{100\ 10^3}{252\ 10^{-6}}\ 10^{-6}\,(217\ 0.352 + \dots + 217\,(-0.269)) = 86.3\ 10^3\ \ \mathrm{N}\cdot\mathrm{m}^{-1} \\
q_o^{6\ 5} = -\dfrac{100\ 10^3}{252\ 10^{-6}}\ 10^{-6}\,(217\ 0.352 + \dots + 217\,(-0.146)) = 98.9\ 10^3\ \ \mathrm{N}\cdot\mathrm{m}^{-1} \\
q_o^{5\ 4} = -\dfrac{100\ 10^3}{252\ 10^{-6}}\ 10^{-6}\,(217\ 0.352 + \dots + 217\ 0.) = 98.9\ 10^3\ \ \mathrm{N}\cdot\mathrm{m}^{-1}
\end{array}\right.$$

- Shearing (3)
  - Open shear flux (3)

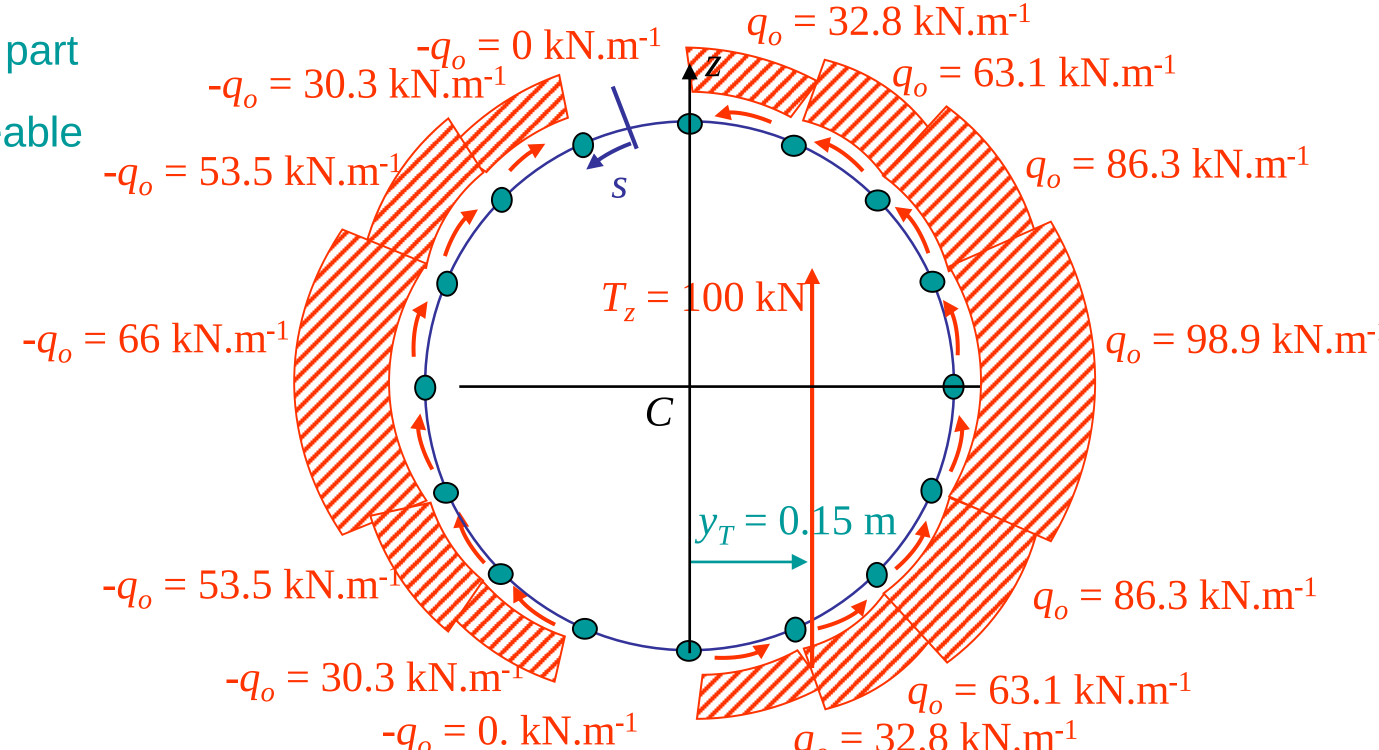
$$\left\{\begin{array}{l} q_o^{4\ 3} = -\dfrac{100\ 10^3}{252\ 10^{-6}}\ 10^{-6} \\ (217\ 0.352 + \ldots + 217\ 0.146) \\ \quad = 86.3\ 10^3\ \ \mathrm{N\cdot m^{-1}} \\ q_o^{3\ 2} = -\dfrac{100\ 10^3}{252\ 10^{-6}}\ 10^{-6} \\ \quad (217\ 0.352 + \ldots + 217\ 0.269) \\ \quad = 63.1\ 10^3\ \ \mathrm{N\cdot m^{-1}} \\ q_o^{2\ 1} = -\dfrac{100\ 10^3}{252\ 10^{-6}}\ 10^{-6}\,(217\ 0.352 + \ldots \\ \quad + 217\ 0.352) = 32.1\ 10^3\ \ \mathrm{N\cdot m^{-1}} \end{array}\right.$$



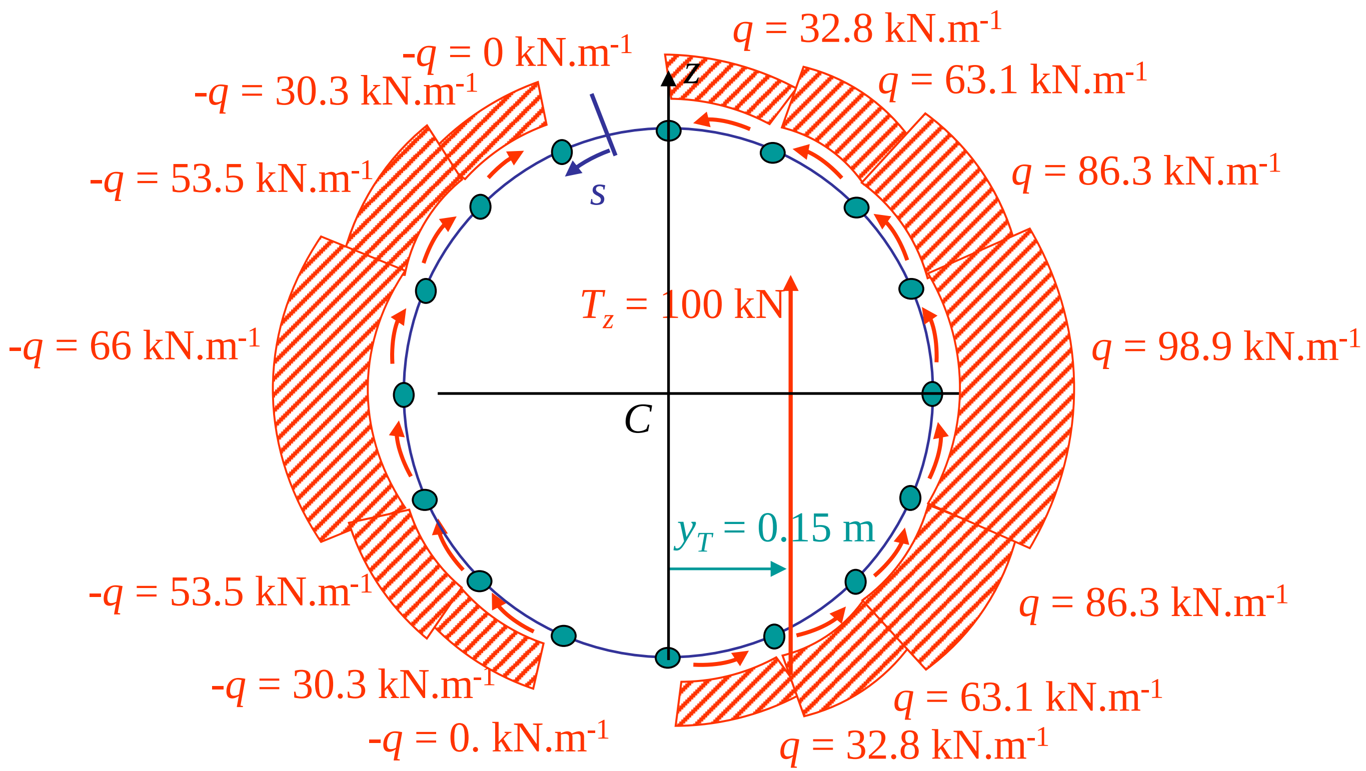
- Shearing (4)
  - Shear flux
    - $A_h = \pi 0.381^2 = 0.456 \quad \mathrm{m}^2$
    - $q(0) = \dfrac{y_T T_z - \oint p q_o ds}{2A_h} = \dfrac{0.15\ 100\ 10^3 - 0.381\ 0.15 \sum_i q_o^{i+1\ i}}{2\ 0.456}$

$$\Rightarrow q(0) = \frac{0.15\ 100\ 10^3 - 0.381\ 0.15\ 262.8\ 10^3}{2\ 0.456} = -20 \quad \mathrm{N \cdot m^{-1}}$$

    - Constant part is negligeable $q \sim q_o$

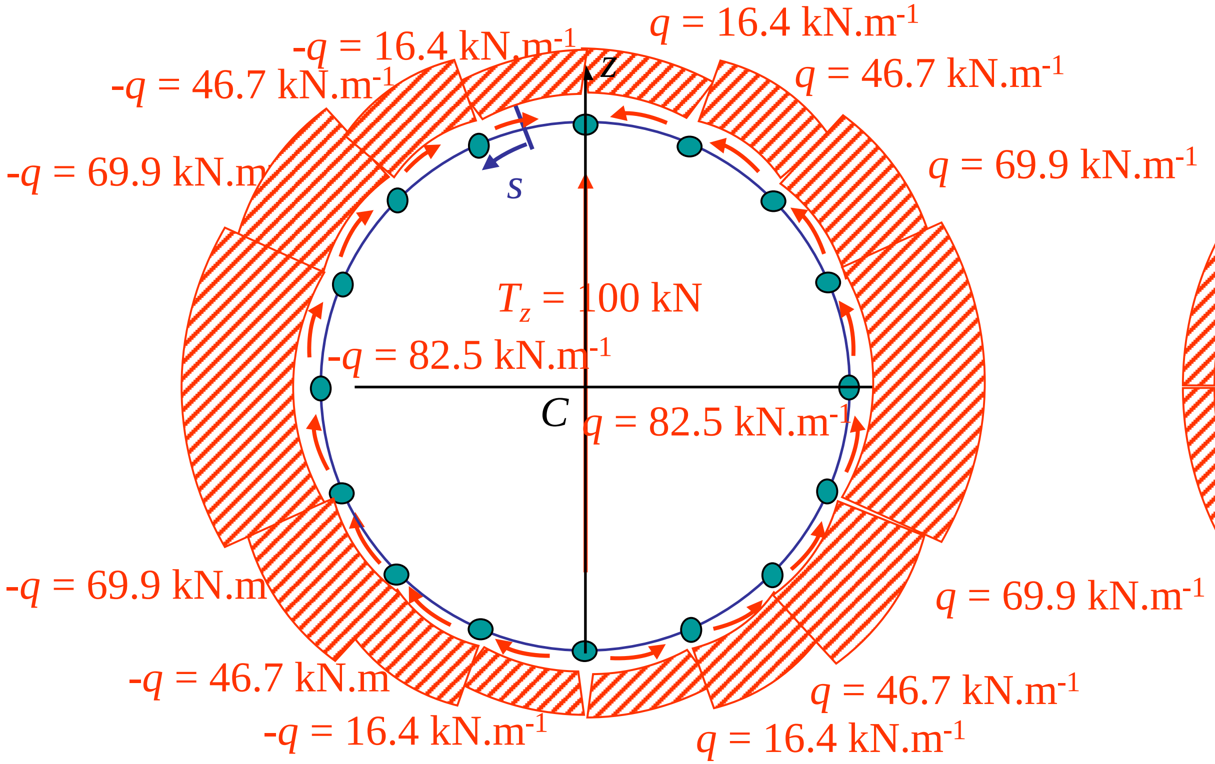


- Torsion
  - Shear center corresponds to the centroid
  - Twist part of the shear load is $M_x = T_z y_T = 100\ 10^3\ 0.15 = 15\ 10^3 \quad \mathrm{N \cdot m}$
  - Twist part of the shear flux (constant) $q_{\text{twist}} = \dfrac{M_x}{2A_h} = 16.4 \quad \mathrm{kN \cdot m^{-1}}$
  - Another method
    - Because of the section symmetry $q_{\text{twist}} = \dfrac{-66000 + 98900}{2} = 16.4 \quad \mathrm{kN \cdot m^{-1}}$

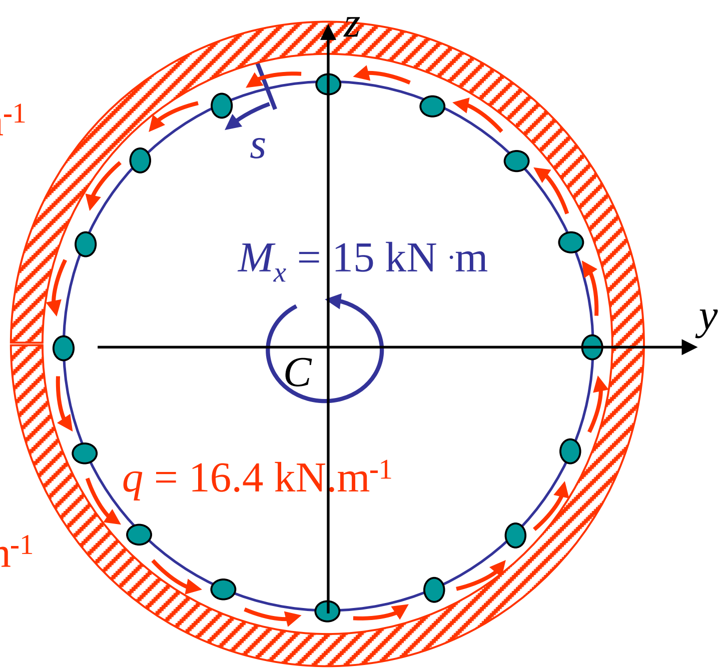


- Shear load components

Shearing

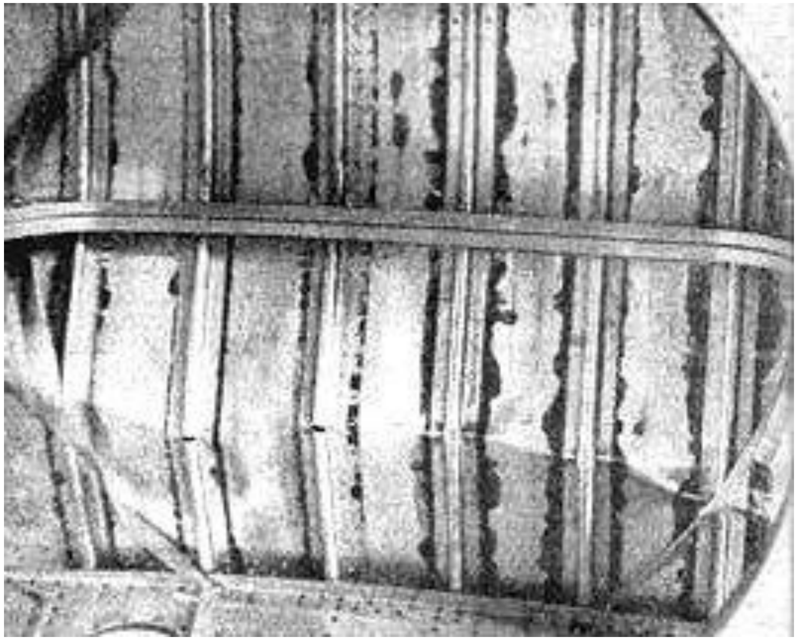


Twist



Stiffener/web construction

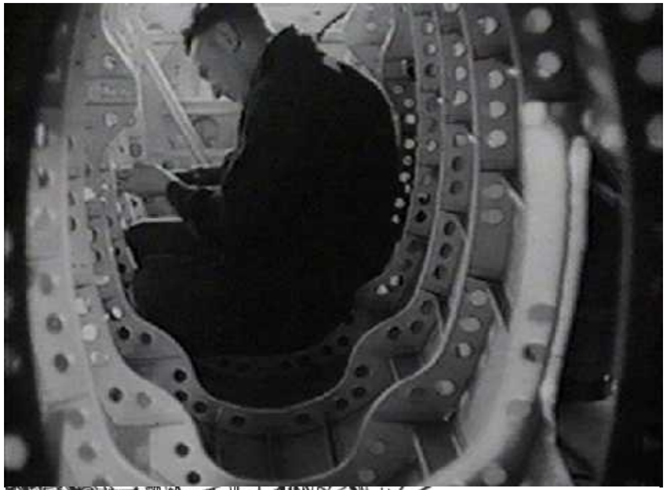
- Thin metal skins
  - Resist in-plane tension
  - Subject to buckling for in-plane compressive loads
- Stringers
  - Resist in plane compression
  - Resist low loadings normal to the skin

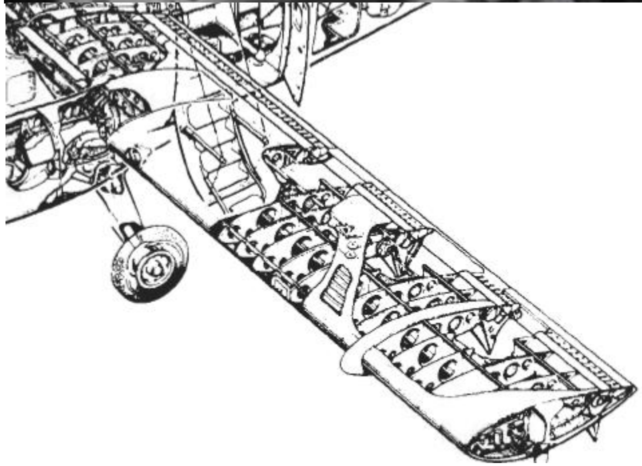


Dove wing

Stiffener/web construction

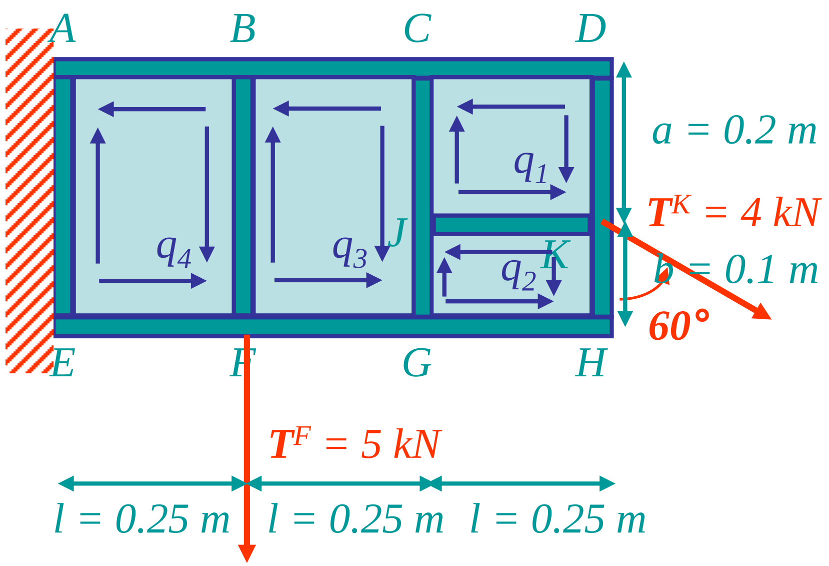
- Fuselage
  - Effective length of stringers
    - Reduced by the frames (or by bulkheads)
- Wings
  - Effective length of stringers
    - Reduced by the ribs





- Frames & ribs
  - Resist concentrated loads and transmit them to stringers
  - Also fabricated from thin sheets of metal
    - Require stiffeners to transmit loads
    - Stiffener/web construction

- Example
  - Stiffener/web construction
  - Web panels
    - Active only in shearing
    - Constant shear flow
  - Stiffener JK required as a load is applied horizontally at K
    - Better than using HD in bending
  - Stiffener loads?
  - Shear flow in webs?
  - Load distribution in flanges?



- Shear flow in webs
  - Stiffener *JK*

$$(q_1 - q_2)\ l = \boldsymbol{T}^K \sin 60\,\mathrm{deg}$$

$$\Rightarrow (q_1 - q_2) = \frac{4000\ \sin 60\,\mathrm{deg}}{0.25}$$

$$= 13.86\,10^3\ \ \mathrm{N}\cdot\ \mathrm{m}^{-1}$$

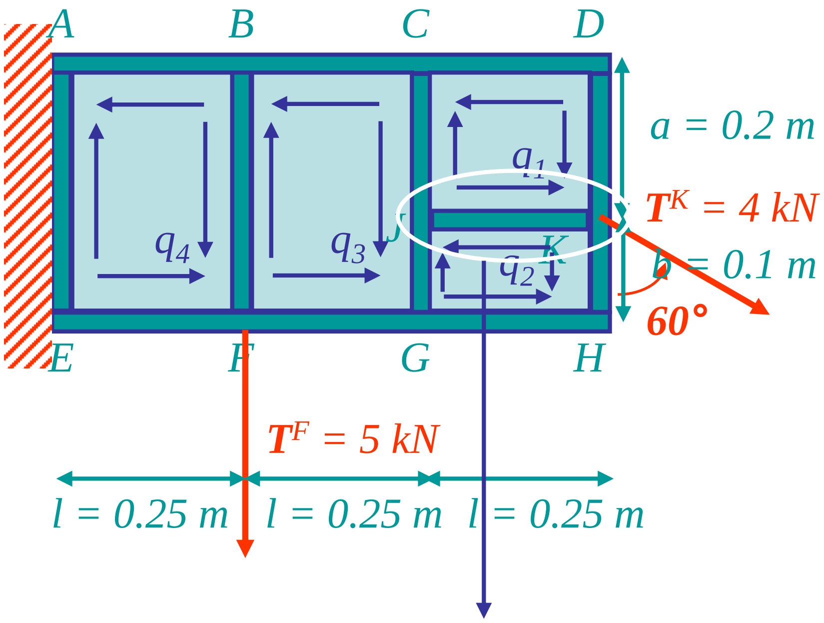
  - Stiffener *HKD*

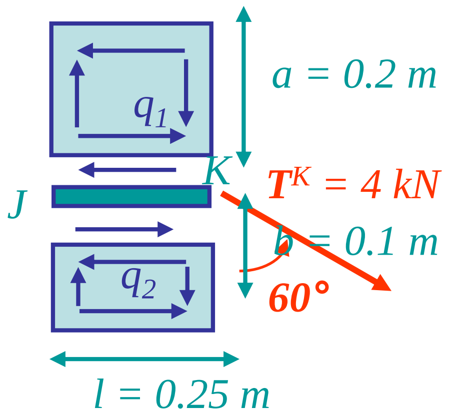
$$a\,q_1 + b\,q_2 = \boldsymbol{T}^K \cos 60\,\mathrm{deg}$$

$$\Rightarrow 2q_1 + q_2 = \frac{4000\ \cos 60\,\mathrm{deg}}{0.1}$$

$$= 20\,10^3\ \ \mathrm{N}\cdot\ \mathrm{m}^{-1}$$

$$\Rightarrow \begin{cases} q_1 = 11.3\,10^3\ \ \mathrm{N}\cdot\ \mathrm{m}^{-1} \\ q_2 = -2.6\,10^3\ \ \mathrm{N}\cdot\ \mathrm{m}^{-1} \end{cases}$$





- Shear flow in webs (2)
  - Stiffener *CJG*

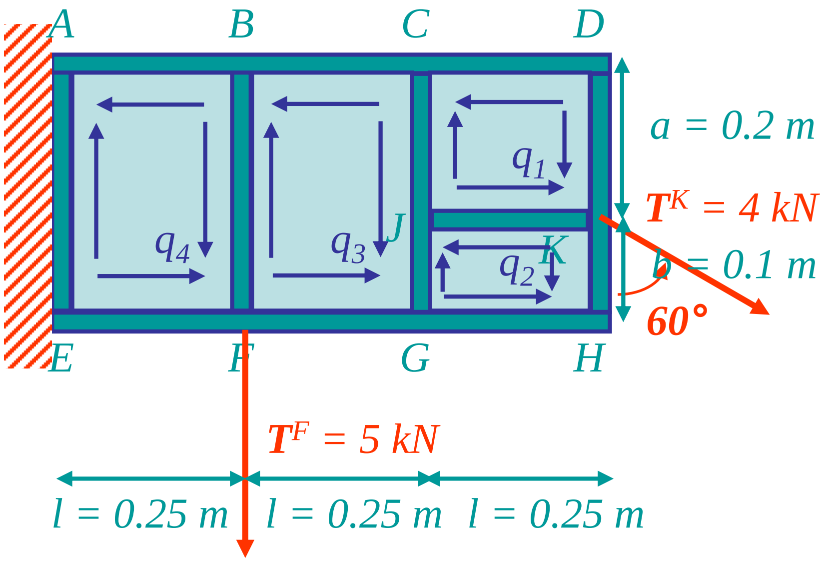
$$(a+b)\,q_3 = a\,q_1 + b\,q_2 = \boldsymbol{T}^K \cos 60\,\mathrm{deg}$$

$$\Rightarrow q_3 = \frac{4000\ \ \cos 60\,\mathrm{deg}}{0.3} = 6.7\,10^3\ \ \mathrm{N}\cdot\ \mathrm{m}^{-1}$$

  - Stiffener *BF*

$$(a+b)\,q_4 = (a+b)\,q_3 + \boldsymbol{T}^F$$

$$\Rightarrow q_4 = 6.7\,10^3 + \frac{5000}{0.3} = 23.3\,10^3\ \ \mathrm{N}\cdot\ \mathrm{m}^{-1}$$



- Load distribution in flanges
  - Top flange (>0 in traction)

$$P^A = l\, q_4 + l\, q_3 + l\, q_1$$

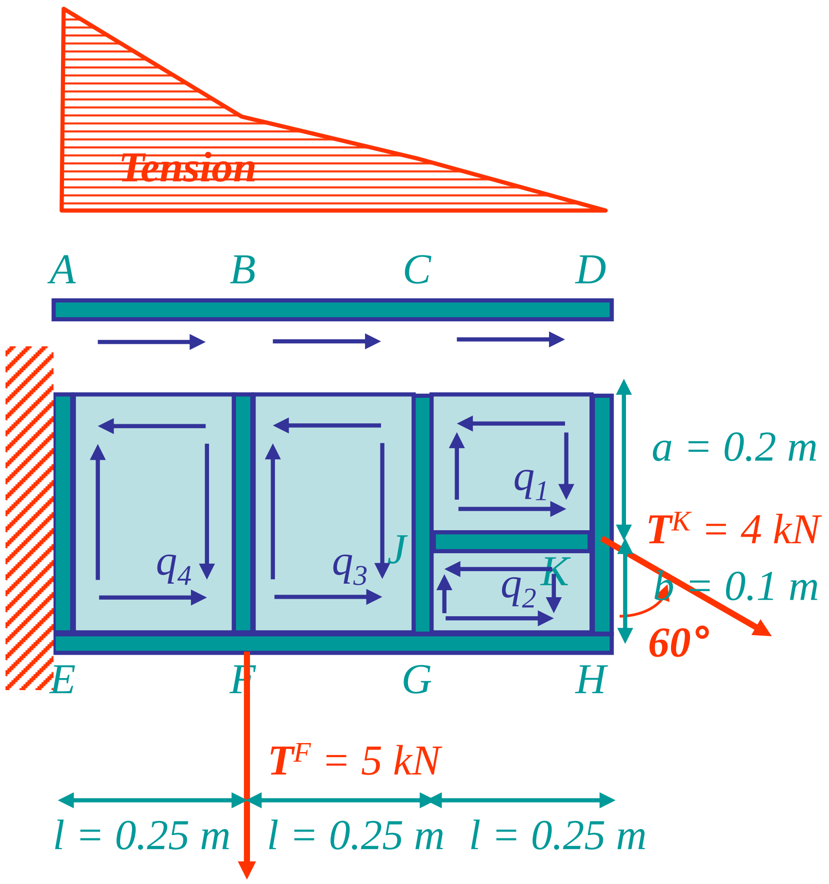
$$\Longrightarrow P^A = 0.25\,(23.3 + 6.7 + 11.3)\ 10^3 = 10.3\,10^3\ \ \mathrm{N}$$

$$P^B = l\, q_3 + l\, q_1$$

$$\Longrightarrow P^B = 0.25\,(6.7 + 11.3)\ 10^3 = 4.5\,10^3\ \ \mathrm{N}$$

$$P^C = l\, q_1$$

$$\Longrightarrow P^C = 0.25\,11.3\,10^3 = 2.8\,10^3\ \ \mathrm{N}$$



- Load distribution in flanges (2)
  - Lower flange (>0 in traction)

$$P^E = -l\, q_4 - l\, q_3 - l\, q_2$$

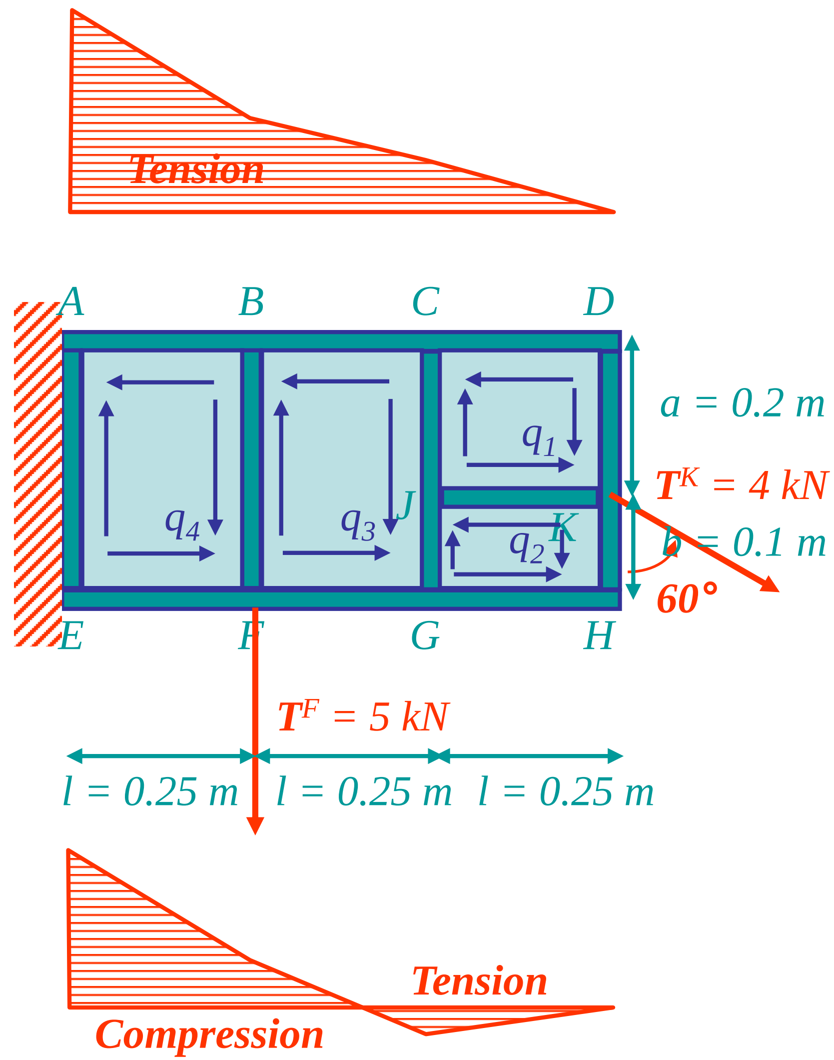
$$\Longrightarrow P^E = 0.25\,(-23.3 - 6.7 + 2.6)\ 10^3$$
$$= -6.9\,10^3 \quad \text{N}$$

$$P^F = -l\, q_3 - l\, q_2$$

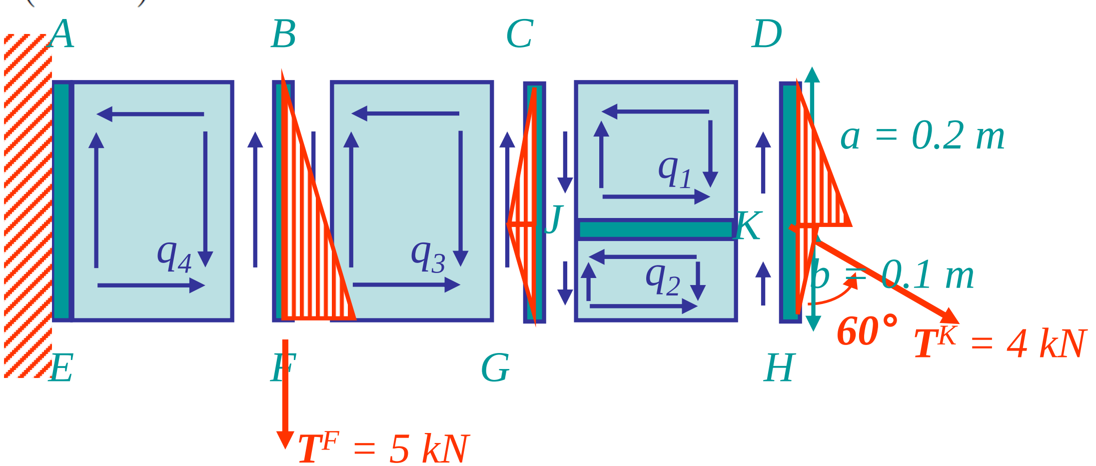
$$\Longrightarrow P^F = 0.25\,(-6.7 + 2.6)\ 10^3$$
$$= -1\,10^3 \quad \text{N}$$

$$P^G = -l\, q_2$$

$$\Longrightarrow P^G = 0.25\,2.6\,10^3$$
$$= 0.7\,10^3 \quad \text{N}$$

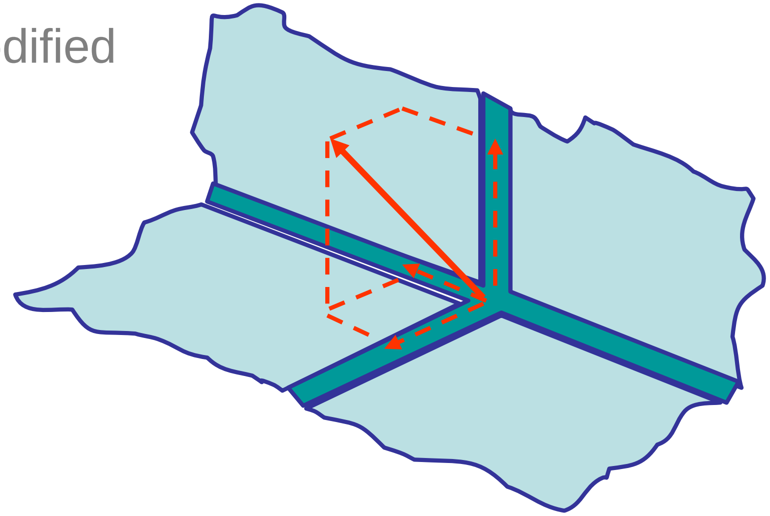


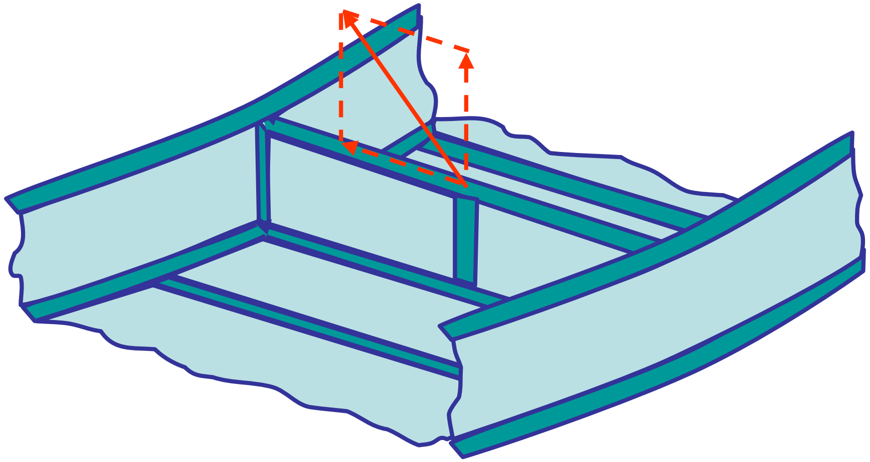
- Load distribution in stiffeners
  - Stiffener *BF* (>0 traction)
    - $P^{BF;B} = 0$ &
    - $P^{BF;F} = (q_4 - q_3)(a+b) = (23.3 - 6.7)\ 10^3\ (0.2 + 0.1) = 5\,10^3\ \ \text{N}$
  - Stiffener *CJG* (>0 traction)
    - $P^{CG;C} = P^{CG;G} = 0$ &
    - $P^{CG;J} = (q_3 - q_1)\,a = (6.7 - 11.3)\ 10^3\ 0.2 = -0.92\,10^3\ \ \text{N}$
  - Stiffener *DKH* (>0 traction)
    - $P^{DK;D} = 0$ & $P^{KH;H} = 0$
    - $P^{DK;K} = q_1 a = 11.3\ 10^3\ 0.2 = 2.26\,10^3\ \ \text{N}$
    - $P^{KH;K} = -q_2 b = -(-2.6)\ 10^3\ 0.1 = 0.26\,10^3\ \ \text{N}$



Stiffener/web construction

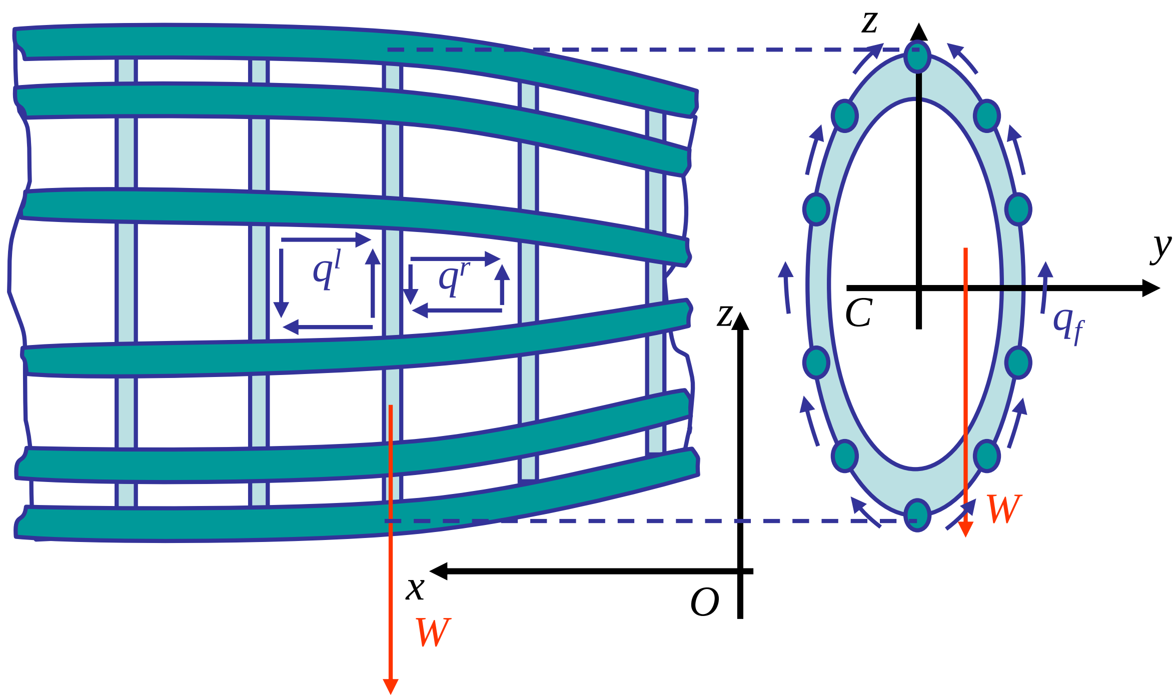
- In practice
  - Out of plane loading
  - Stiffener/web constructions should be modified
  - First construction
    - Two out of plane webs meeting at the load application point
    - Not always possible
  - Second construction
    - Add a web
      - Between adjacent frames/ribs
      - In plane with the loading
  - Design rule
    - Avoid loading normal to a web





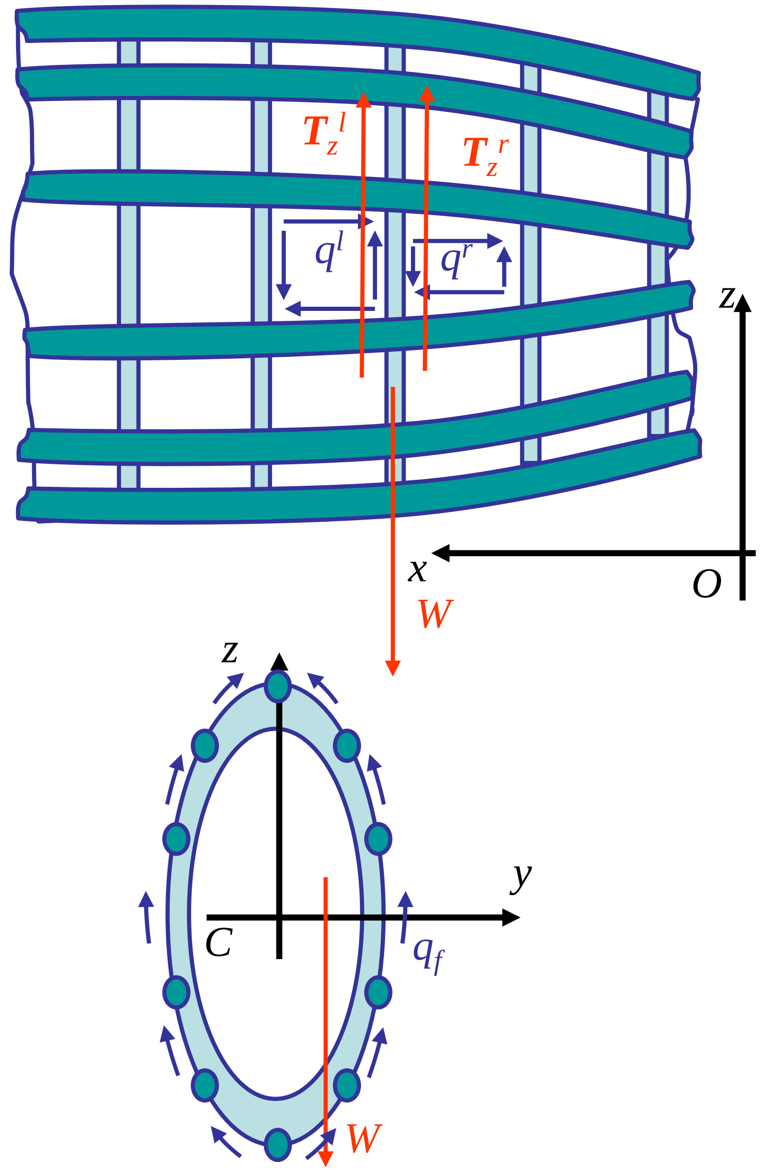
- Fuselage frames
  - Purpose
    - Transfer loads to fuselage shell
    - Provide column support for stringers
  - Structure
    - Open ring

- Fuselage frames (2)
  - Consider a frame
    - Z-Symmetrical
    - In equilibrium
      - Loading
      - Shearing from shells
    - Idealized shells/stringers
      - Shells effective in shearing only



- Fuselage frames (3)
  - Shear force in fuselage
    - Shear force in fuselage at the left of the frame is $\boldsymbol{T}_z^{\,l}$
    - Shear force in fuselage at the right of the frame is $\boldsymbol{T}_z^{\,r}$
    - $\boldsymbol{T}_z^l - W = \boldsymbol{T}_z^r$
  - Shear flow in shells
    - Varies on the fuselage circumference but constant between 2 stringers
    - Obtained from shear force
      - See previous slides
    - Shear flow in the frame can be obtained from the shell shear flows

$$q_f = q^l - q^r$$



- Fuselage frames (4)
  - As
    - Shear flow on either side of the frame (q^l & q^r) are obtained using linear forms as

$$q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\boldsymbol{\sigma}} z ds + \sum_{i:\ s_i \leq s} z_i A_i\right] - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\boldsymbol{\sigma}} y ds + \sum_{i:\ s_i \leq s} y_i A_i\right]$$

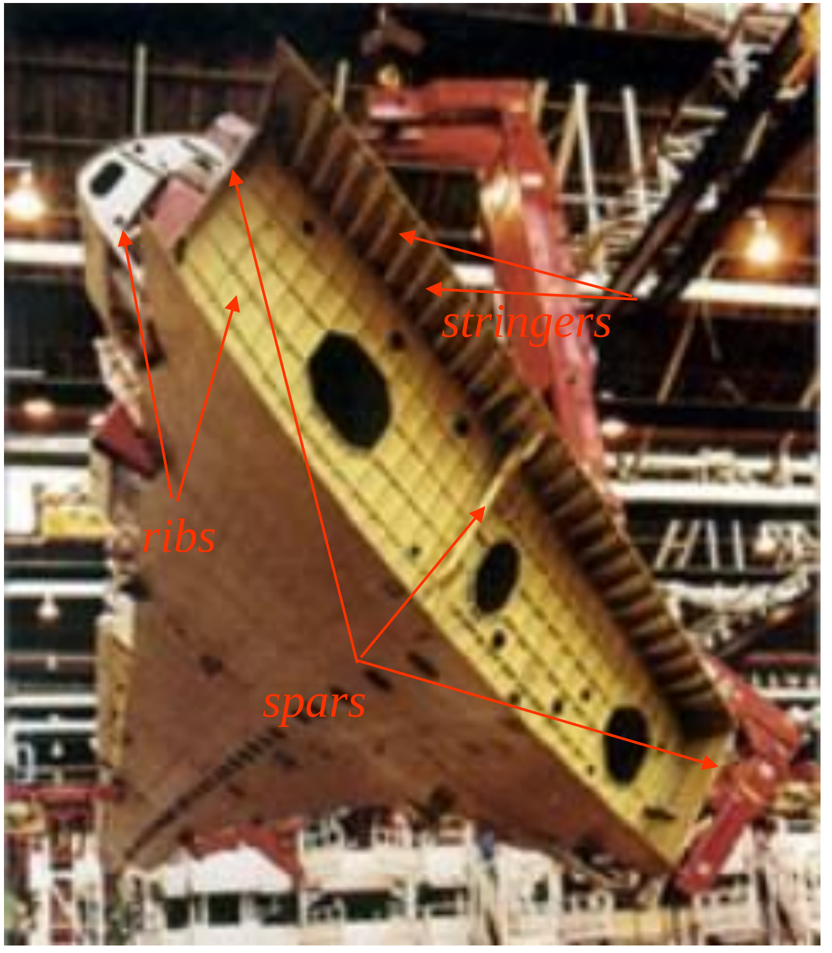
with appropriate shear loads ($\boldsymbol{T}_z^l$ & $\boldsymbol{T}_z^r$)

    - And as $\boldsymbol{T}_z^l - W = \boldsymbol{T}_z^r$
  - The shear flux at the periphery of the frame $q_f = q^l - q^r$ is also obtained

Using $q_o(s) = -\frac{I_{zz}T_z - I_{yz}T_y}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\boldsymbol{\sigma}} z ds + \sum_{i:\ s_i \leq s} z_i A_i\right] - \frac{I_{yy}T_y - I_{yz}T_z}{I_{yy}I_{zz} - I_{yz}^2}\left[\int_0^s t_{\text{direct }\boldsymbol{\sigma}} y ds + \sum_{i:\ s_i \leq s} y_i A_i\right]$

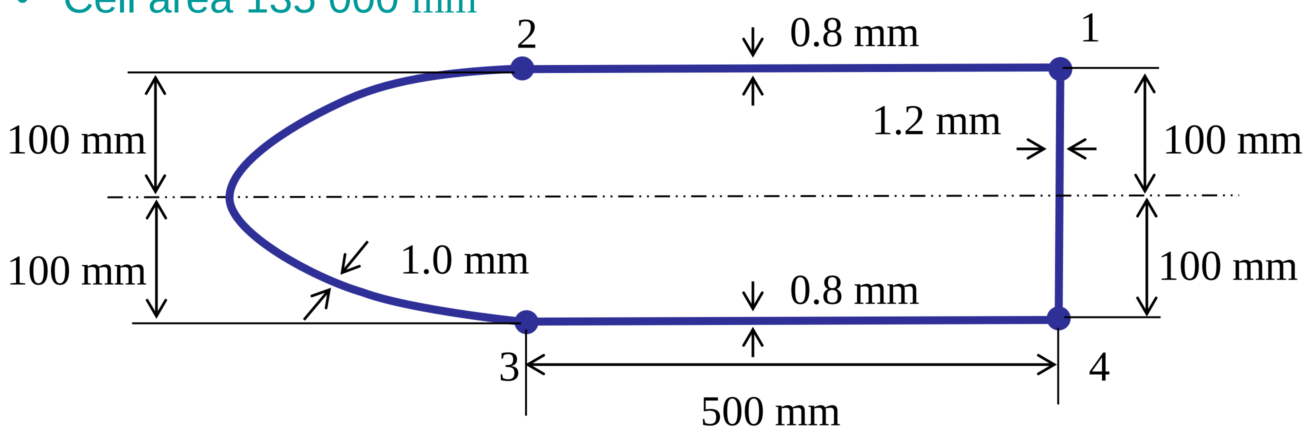
With $\boldsymbol{T}_z$ and $\boldsymbol{T}_y$ the loads applied on the frame itself (W here)

- Wing ribs
  - Purposes
    - Maintain the shape of airfoil
    - Transfer loads to wing skin
    - Provide column support for stringers
  - Structure
    - Unsymmetrical
    - Continuous webs (except holes for control runs)
  - Shear loads in the ribs
    - Periphery shear flow is obtained from the loading discontinuity
    - As for fuselage frame



Exercise 1: Single cell

- Cross-section of a single cell
  - Thin walled cell
    - Cell area 135 000 mm^2

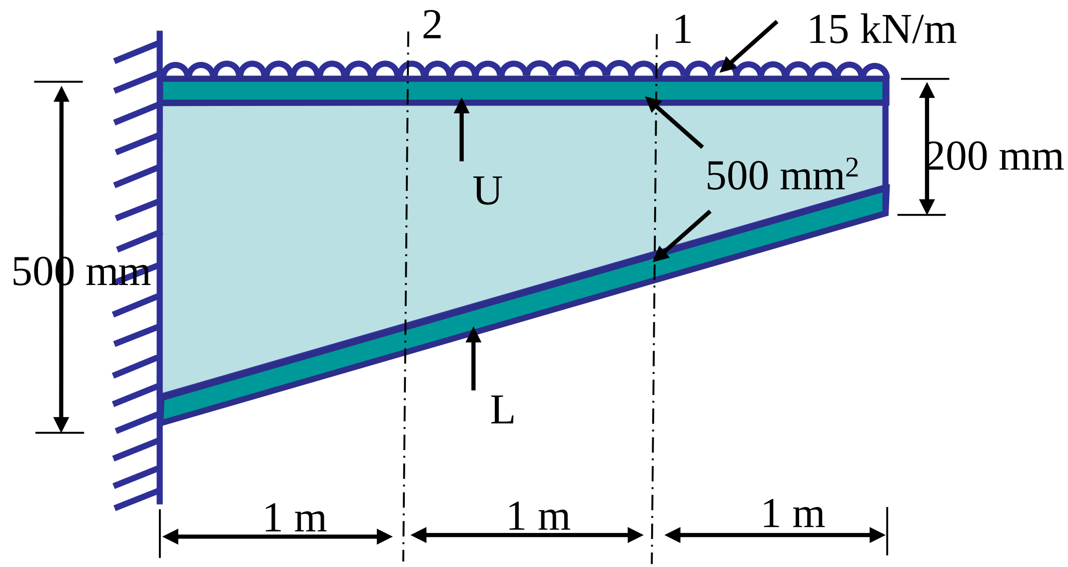


  - Horizontal axis of symmetry
    - Direct stress carrying booms 1 to 4
      - $A_1 = A_4 = 450$ mm^2
      - $A_2 = A_3 = 550$ mm^2
    - Panels which are assumed to carry only shear stresses
    - Constant shear modulus
  - Shear centre?

Wall	Length (m)	Thickness (mm)
12, 34	0.5	0.8
23	0.58	1.0
41	0.2	1.2

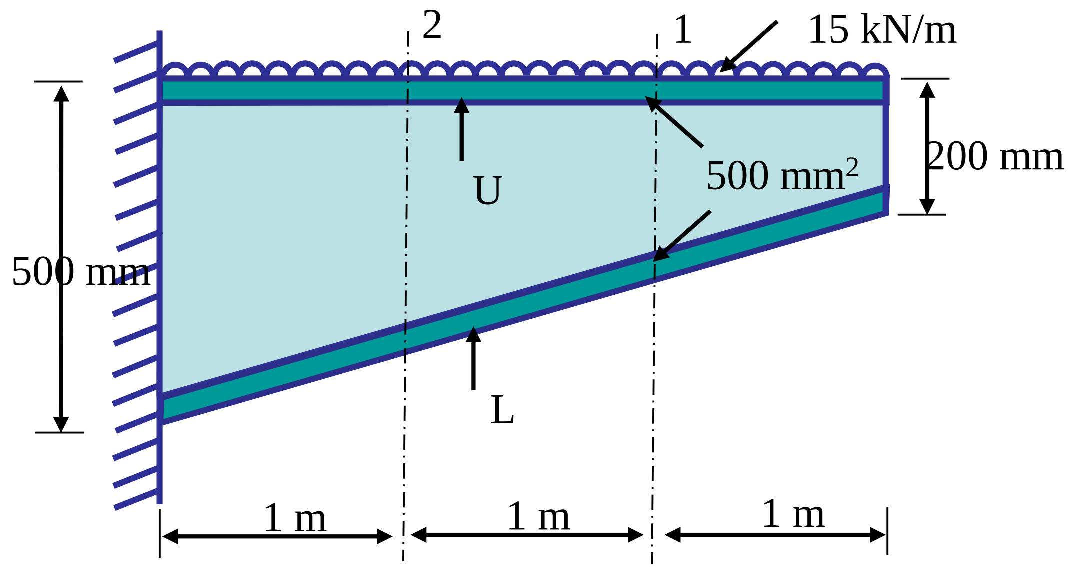
Exercise 2: Wing spar

- Spar
  - Distributed load
    - 15 N/m
  - Upper & lower flanges
    - Area 500 mm^2
    - Resist all the direct loads
  - Spar web
    - Effective only in shear
  - At section 1 and 2
    - Flange loads?
    - Shear flows in web?

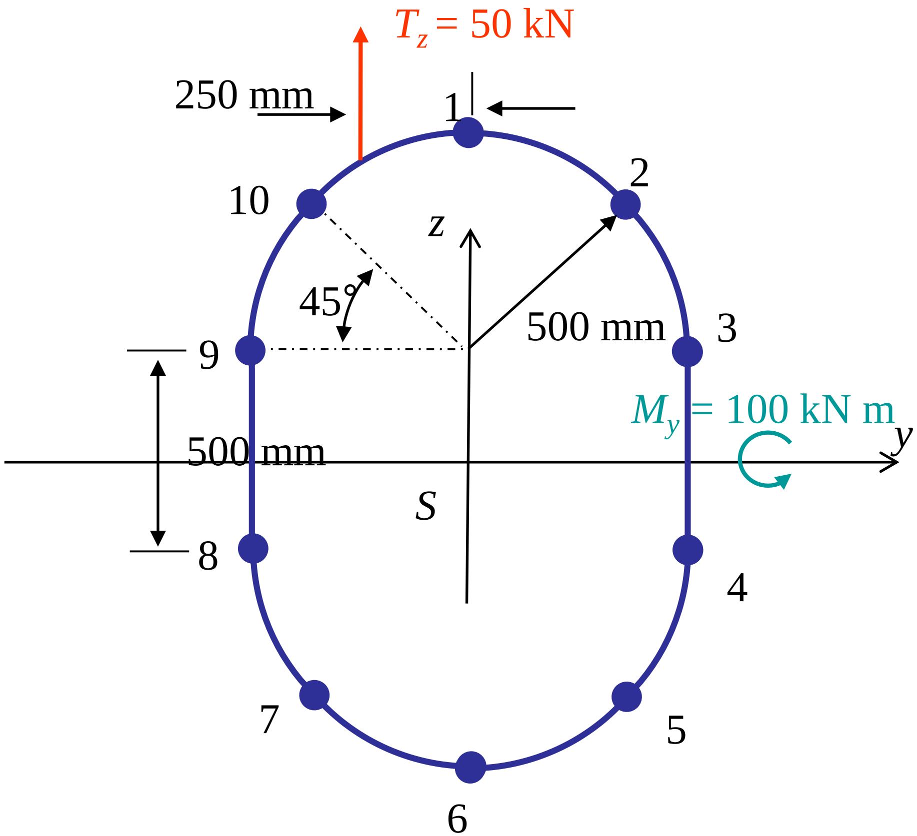


Exercise 3: Wing spar

- Spar 2
  - Distributed load
    - 15 N/m
  - Upper & lower flanges
    - Area 500 mm^2
    - Resist direct loads only
  - Spar web
    - Effective in shear
    - **Resist direct load**
    - Thickness of 2 mm
  - At section
    - Flange loads?
    - Shear flows in web?



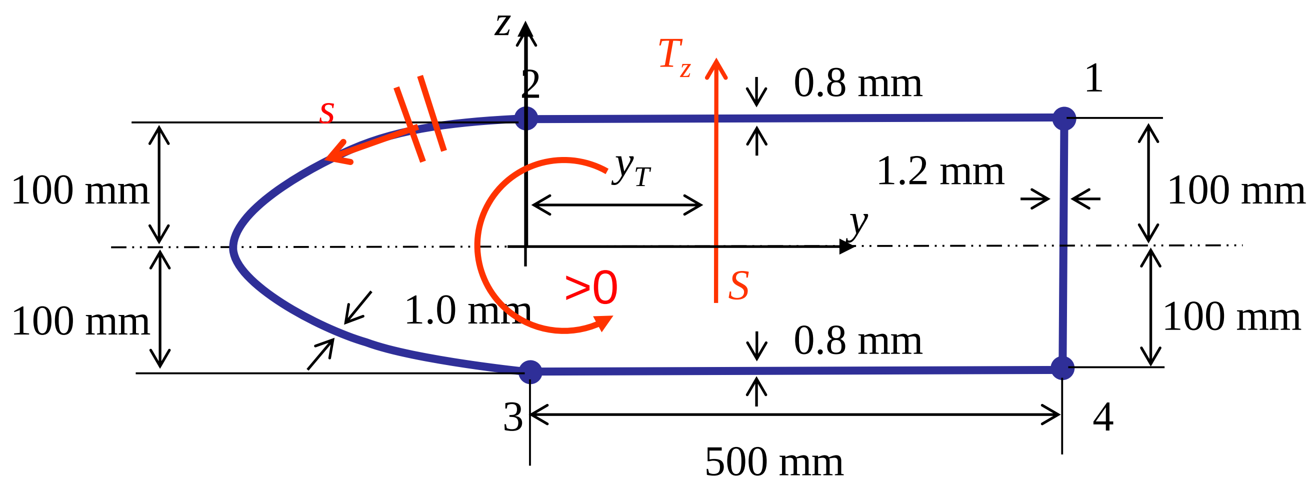
- Idealized fuselage section
  - Direct stress carrying booms
    - Area of each boom 150 mm^2
    - Direct load?
  - Shear stress carrying skin panels
    - Shear flow?



References

- Lecture notes
  - Aircraft Structures for engineering students, T. H. G. Megson, Butterworth-Heinemann, An imprint of Elsevier Science, 2003, ISBN 0 340 70588 4
- Other references
  - Books
    - Mécanique des matériaux, C. Massonet & S. Cescotto, De boek Université, 1994, ISBN 2-8041-2021-X

- As shear center lies on Oy, by symmetry we consider only T_Z



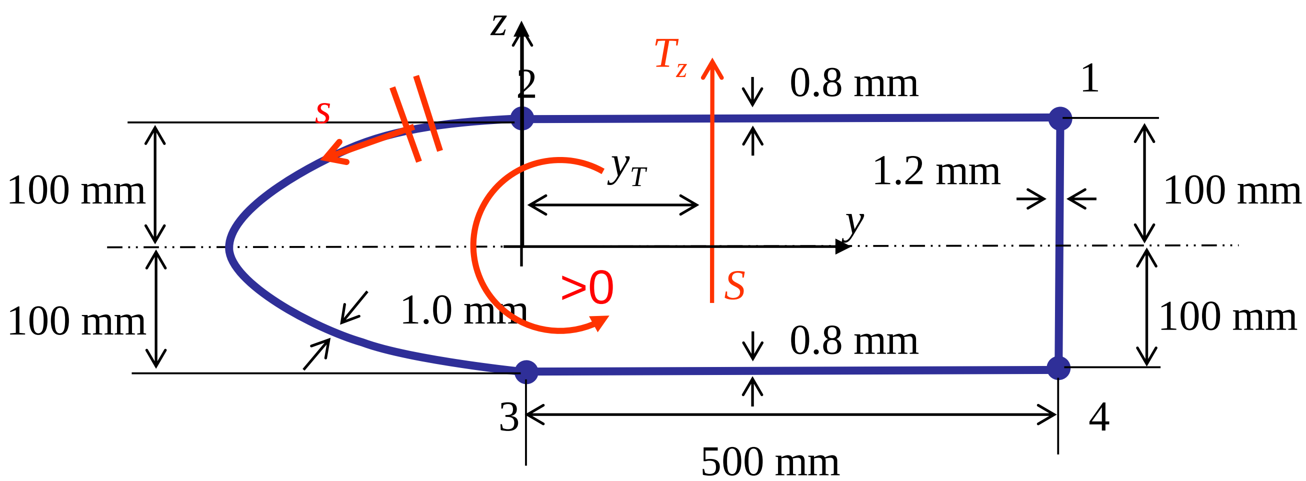
  – Compute shear flux: cut a wall

$$q(s) = -\frac{T_z}{I_{yy}} \sum_{i:\, s_i \leq s} z_i A_i + q(0)$$

  – Section already idealized as only booms resist direct stress

    - By symmetry, centroid on Oy
    - $I_{yy} = \sum_{i=1}^{4} A_i z_i^2 = 2 \times 450 \times 100^2 + 2 \times 550 \times 100^2 = 20 \times 10^6\ mm^4$

- Open shear flow



– $q_o(s) = -\frac{T_z}{I_{yy}} \sum_{i:\, s_i \le s} z_i A_i$

$$\Rightarrow \begin{cases} q_0^{23} = 0 \\ q_0^{34} = -\frac{T_z}{I_{yy}} \times 550 \times (-100) = 2.75 \times 10^{-3}\, T_z \\ q_0^{41} = q_0^{34} - \frac{T_z}{I_{yy}} \times 450 \times (-100) = 5 \times 10^{-3}\, T_z \\ q_0^{12} = 2.75 \times 10^{-3}\, T_z \quad \text{(by symmetry)} \end{cases}$$

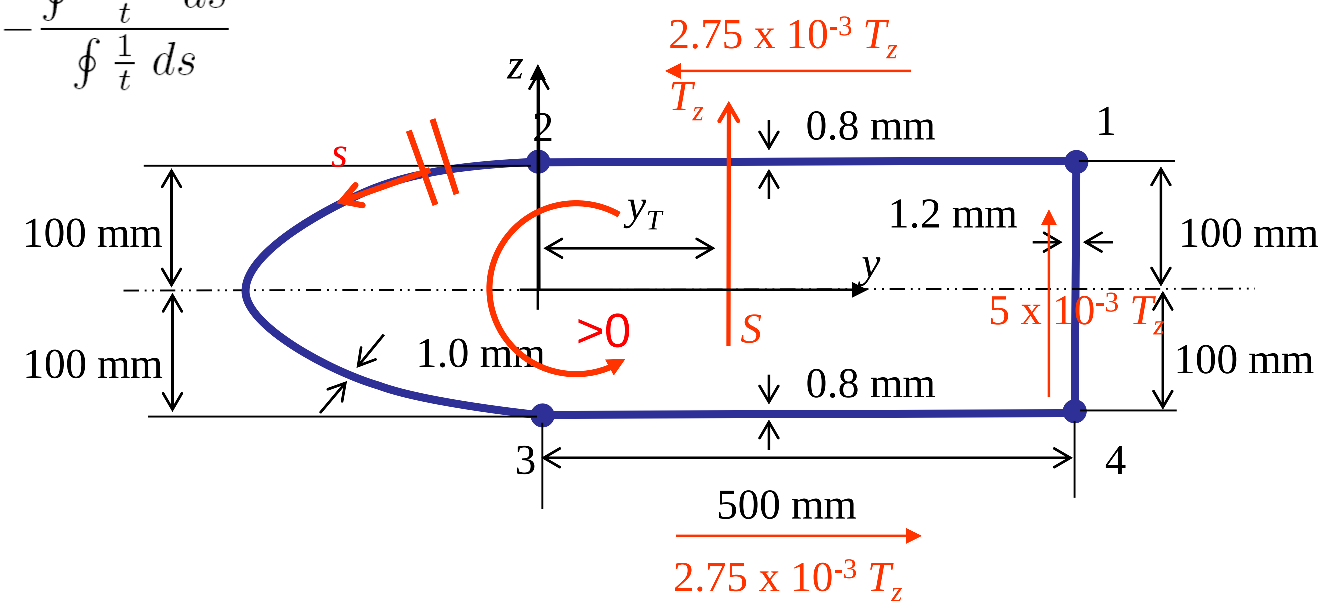
- Constant shear flow
  - Load through the shear center

$\Longrightarrow$ no torsion

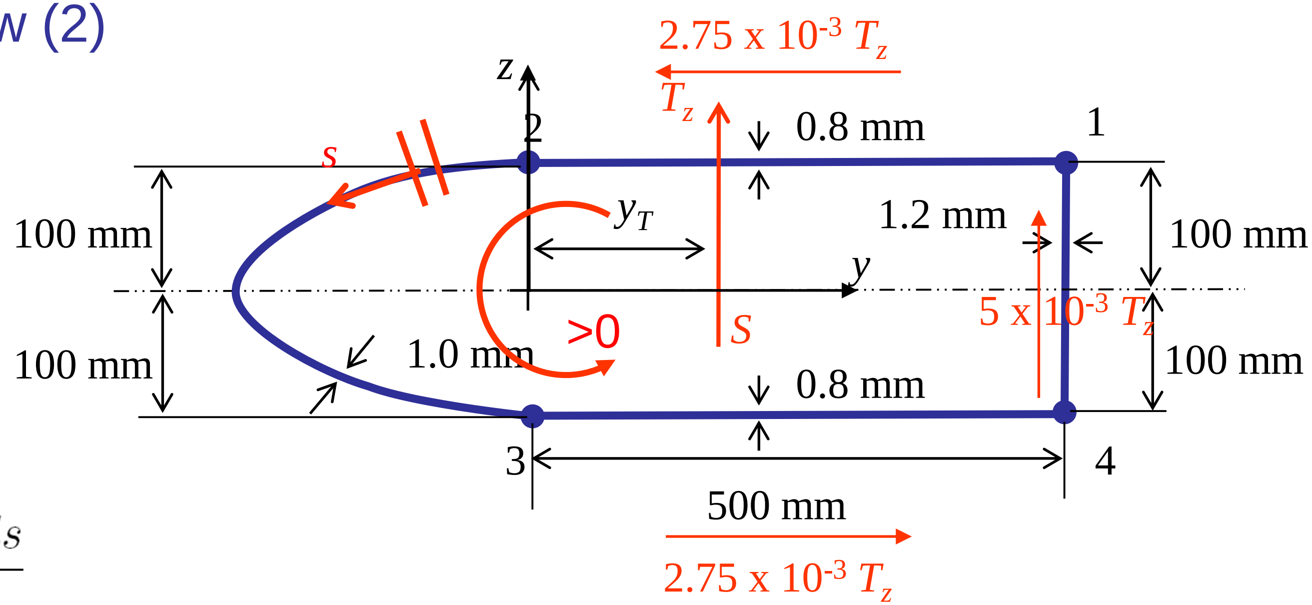
$$\Longrightarrow \oint \frac{q}{\mu t} ds = 2A_h \frac{\partial \theta}{\partial x}$$

$$\Longrightarrow \oint \frac{q}{\mu t} ds = \oint \frac{q_o(s) + q(0)}{\mu t} ds = 0$$

$$\Longrightarrow q(0) = -\frac{\oint \frac{q_0(s)}{t} ds}{\oint \frac{1}{t} ds}$$



- Constant shear flow (2)



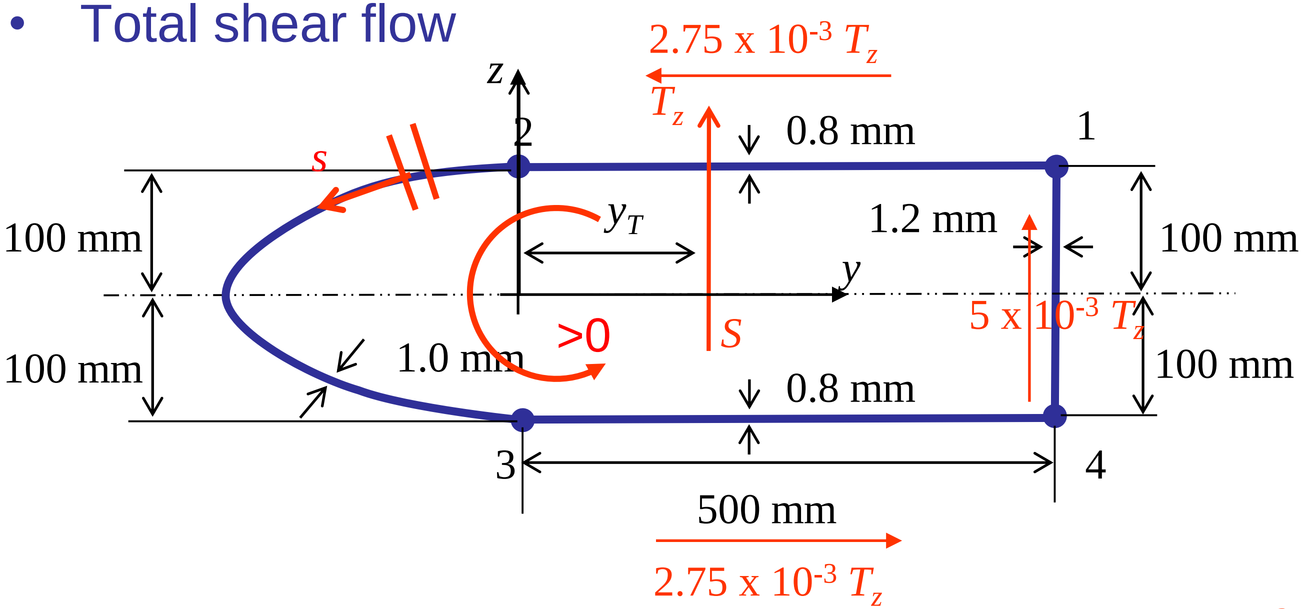
– $$q(0) = -\frac{\oint \frac{q_0(s)}{t}\, ds}{\oint \frac{1}{t}\, ds}$$

$$\begin{cases} \oint \frac{1}{t}\, ds = \frac{580}{1} + 2 \times \frac{500}{0.8} + \frac{200}{1.2} = 1996.7 \\ \oint \frac{q_0(s)}{t}\, ds = 2 \times q_0^{34} \times \frac{l^{34}}{t_{34}} + q_0^{41} \times \frac{l^{41}}{t_{41}} \end{cases}$$

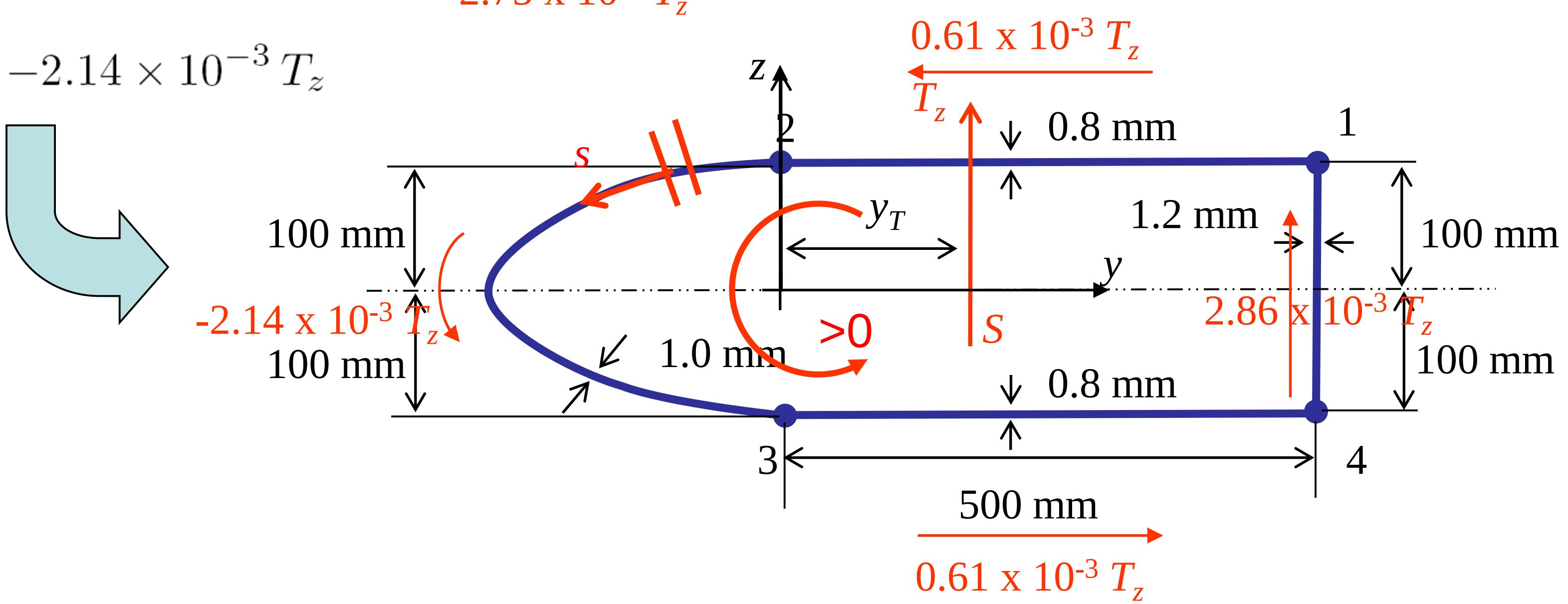
$$= 2 \times 2.75 \times 10^{-3}\, T_z \times \frac{500}{0.8} + 5 \times 10^{-3}\, T_z \times \frac{200}{1.2} = 4.2708\, T_z$$

$$\Longrightarrow \quad q(0) = -2.14 \times 10^{-3}\, T_z$$

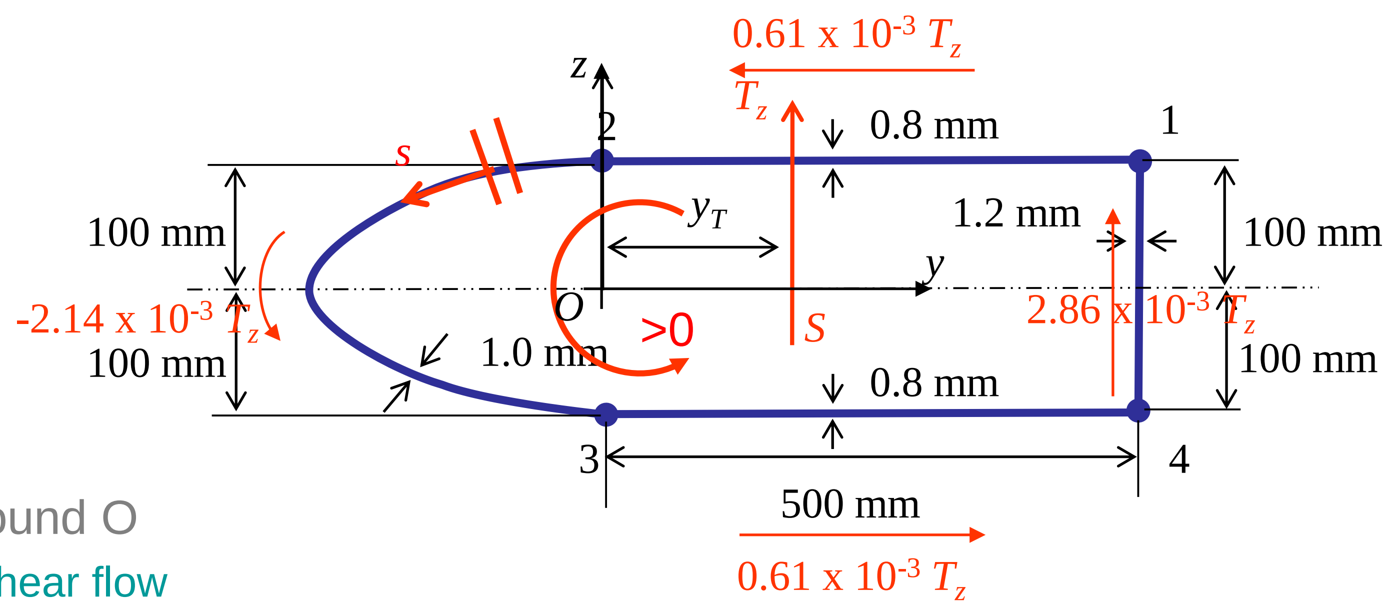
- Total shear flow



$$q\left(0\right)=-2.14\times10^{-3}\,T_z$$



- Shear center



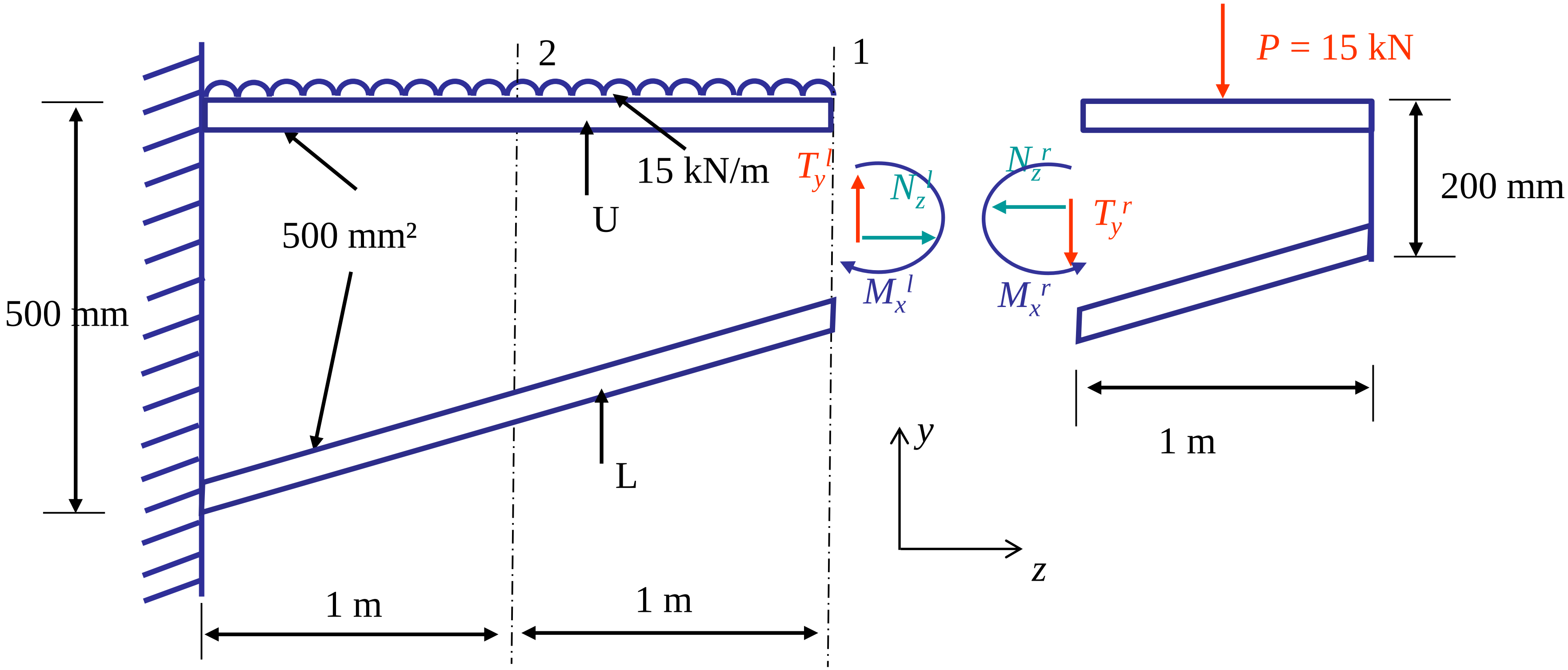
  – Moment around O
    - Due to shear flow
    - Should be balanced by the external load

$$T_z y_T = 2 \times 0.61 \times 10^{-3}\, T_z \times 500 \times 100 + 2.86 \times 10^{-3}\, T_z \times 200 \times 500$$
$$-2.14 \times 10^{-3}\, T_z \times 2 \underbrace{(135000 - 500 \times 200)}_{A_{23}}$$

$\Longrightarrow$ $y_T = 197.2\ mm$

Exercise 2: Wing spar

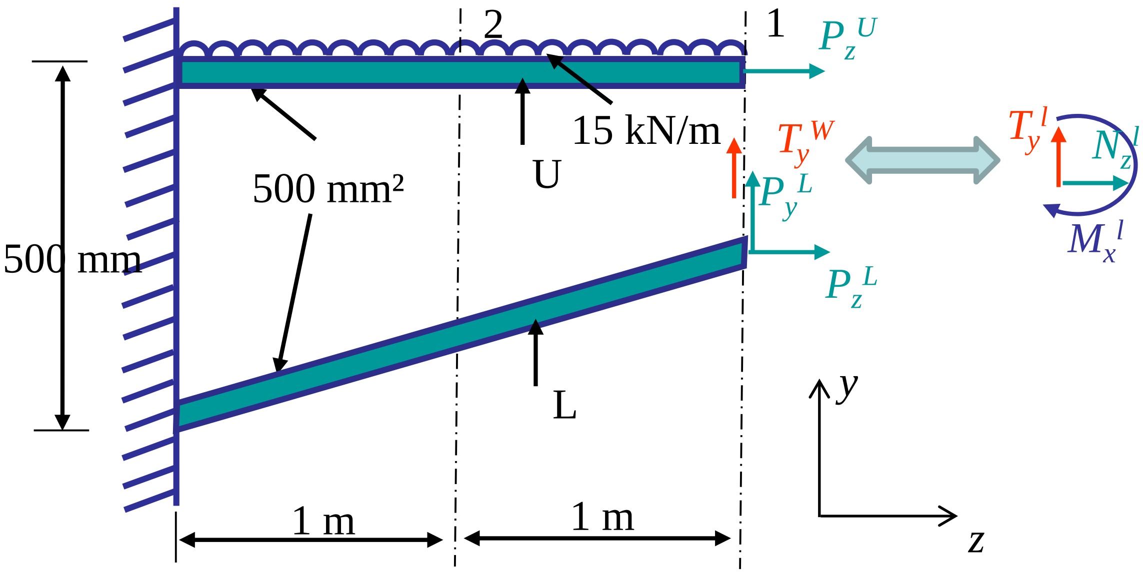
- Cut at section 1



  – Right part equilibrium

$$\begin{cases} N_z^r = 0 \\ T_y^r + P = 0 \\ M_x^r - P * 0.5 = 0 \end{cases} \Leftrightarrow \begin{cases} N_z^r = 0 \\ T_y^r = -15 \text{ kN} \\ M_x^r = 7.5 \text{ kN m} \end{cases} \Rightarrow \begin{cases} N_z^l = 0 \\ T_y^l = -15 \text{ kN} \\ M_x^l = 7.5 \text{ kN m} \end{cases}$$

- Section 1



$$\begin{cases} N_z^l = & 0 \\ T_y^l = & -15 \text{ kN} \\ M_x^l = & 7.5 \text{ kN m} \end{cases}$$

– Flanges in section 1

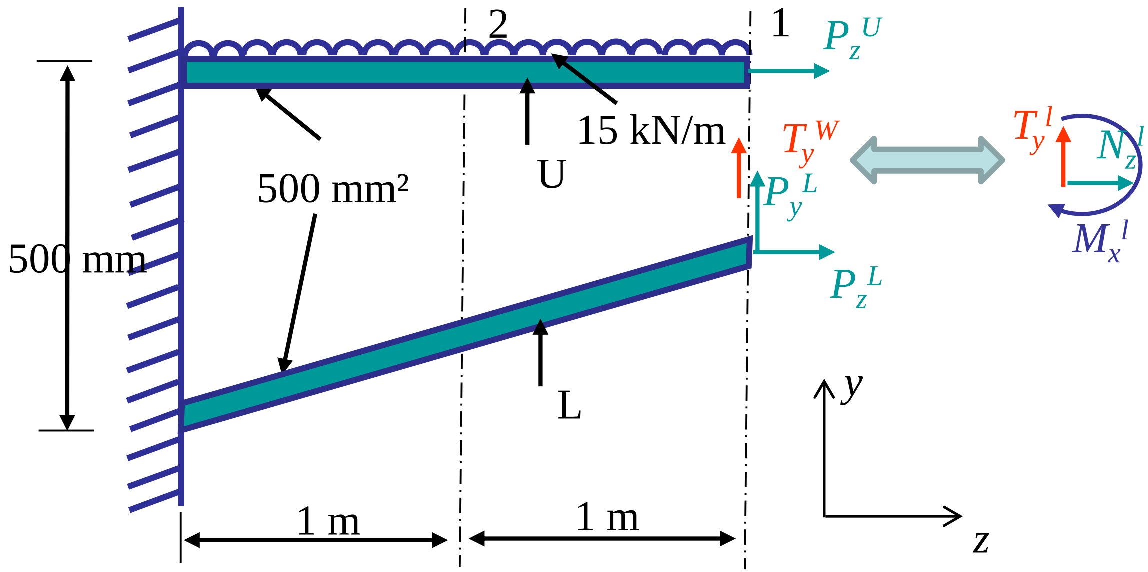
- Flanges carry all the direct stress

$$\begin{cases} P_z^L + P_z^U = 0 \\ 300 * P_z^U = M_x^l \end{cases} \Leftrightarrow \begin{cases} P_z^L = -25 \text{ kN} \\ P_z^U = 25 \text{ kN} \end{cases}$$

- No shearing in flanges

$$\Rightarrow \begin{cases} P_y^L = P_z^L \frac{100}{1*10^3} \\ P_y^U = 0 \end{cases} \Rightarrow \begin{cases} P_y^L = -2.5 \text{ kN} \\ P_y^U = 0 \text{ kN} \end{cases}$$

- Section 1 (2)



$$\begin{cases} N_z^l = & 0 \\ T_y^l = & -15 \text{ kN} \\ M_x^l = & 7.5 \text{ kN m} \end{cases}$$

- Flanges in section 1 (2)
  - Resultant loading

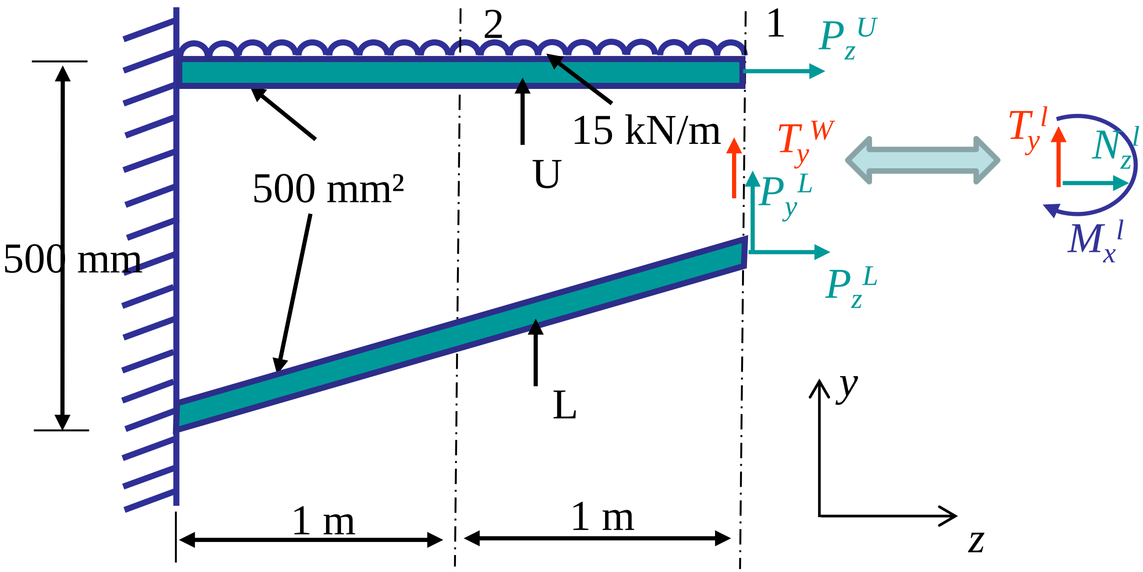
$$\begin{cases} P_z^L = -25 \text{ kN} \\ P_z^U = 25 \text{ kN} \end{cases}$$

$$\begin{cases} P_y^L = -2.5 \text{ kN} \\ P_y^U = 0 \text{ kN} \end{cases}$$

$$\Rightarrow \begin{cases} P_U = \sqrt{\left(P_z^U\right)^2 + \left(P_y^U\right)^2} \\ P_L = -\sqrt{\left(P_z^L\right)^2 + \left(P_y^L\right)^2} \end{cases}$$

$$\Leftrightarrow \begin{cases} P_U = & 25 \text{ kN} & \text{tension} \\ P_L = & -25.1 \text{ kN} & \text{compression} \end{cases}$$

- Section 1 (3)



$$\begin{cases} N_z^l = & 0 \\ T_y^l = & -15 \text{ kN} \\ M_x^l = & 7.5 \text{ kN m} \end{cases}$$

- Web
  - Carry shear load only

$$T_y^{web} + P_y^L = T_y^l \Rightarrow T_y^{web} = -15 + 2.5 = -12.5 \text{ kN}$$

$$\Leftrightarrow |q| = \frac{T_y^{web}}{L_{web}} = \frac{12.5 * 10^3}{300} = 41.7 \text{ N/mm}$$

  - This is an average value

- Section 2
  - Similar developments
  - Section loading

$$\begin{cases} N_z^l = & 0 \\ T_y^l = & -30 \text{ kN} \\ M_x^l = & 30 \text{ kN m} \end{cases}$$

  - Flanges

$$\begin{cases} P_z^L = -75 \text{ kN} \\ P_z^U = 75 \text{ kN} \end{cases}$$

$$\begin{cases} P_y^L = -7.5 \text{ kN} \\ P_y^U = 0 \text{ kN} \end{cases}$$

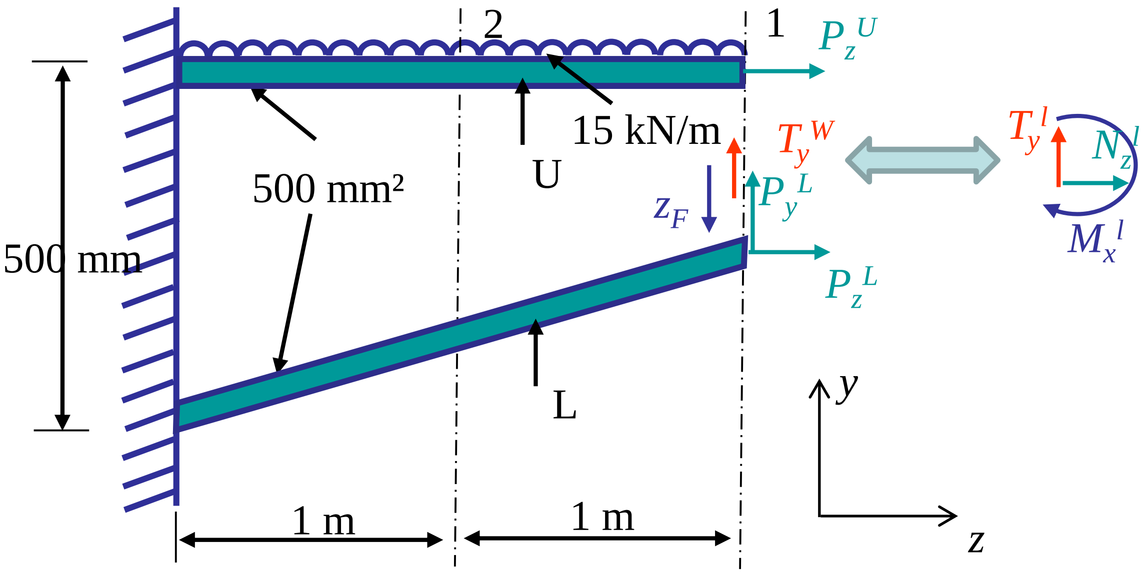
$$\begin{cases} P_U = & 75 \text{ kN} & \text{tension} \\ P_L = & -75.4 \text{ kN} & \text{compression} \end{cases}$$

  - Web

$$T_y^{web} = -30 + 7.5 = -22.5 \text{ kN}$$

$$|q| = \frac{T_y^{web}}{L_{web}} = \frac{22.5 * 10^3}{400} = 56.3 \text{ N/mm}$$

- Section 1 with web resisting direct load



$$\begin{cases} N_z^l = 0 \\ T_y^l = -15\text{ kN} \\ M_x^l = 7.5\text{ kN m} \end{cases}$$

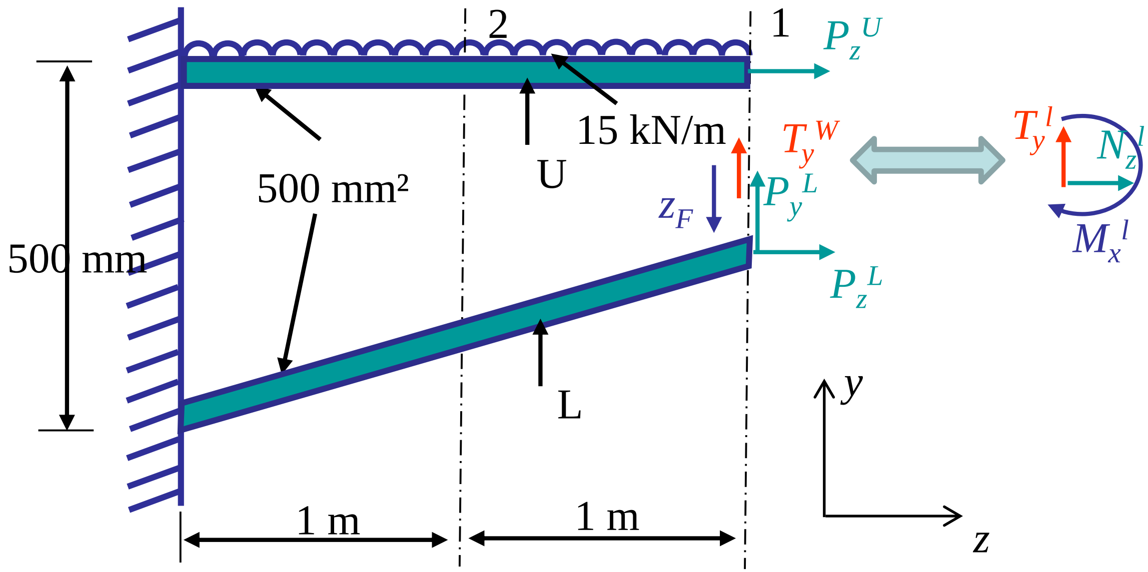
– Flanges in section 1

- Direct stress obtained from $\sigma_z = \frac{M_x^l y}{I_{xx}}$
- As flanges and web carry direct stress

$$I_{xx} = 2 \times A_{\text{flange}} \times z_F^2 + I_{xx}^{\text{web}}$$

$$\Rightarrow I_{xx} = 2 \times 500 \times 150^2 + \frac{2 \times 300^3}{12} = 27 \times 10^6 \text{ mm}^4$$

- Section 1 with web resisting direct load (2)



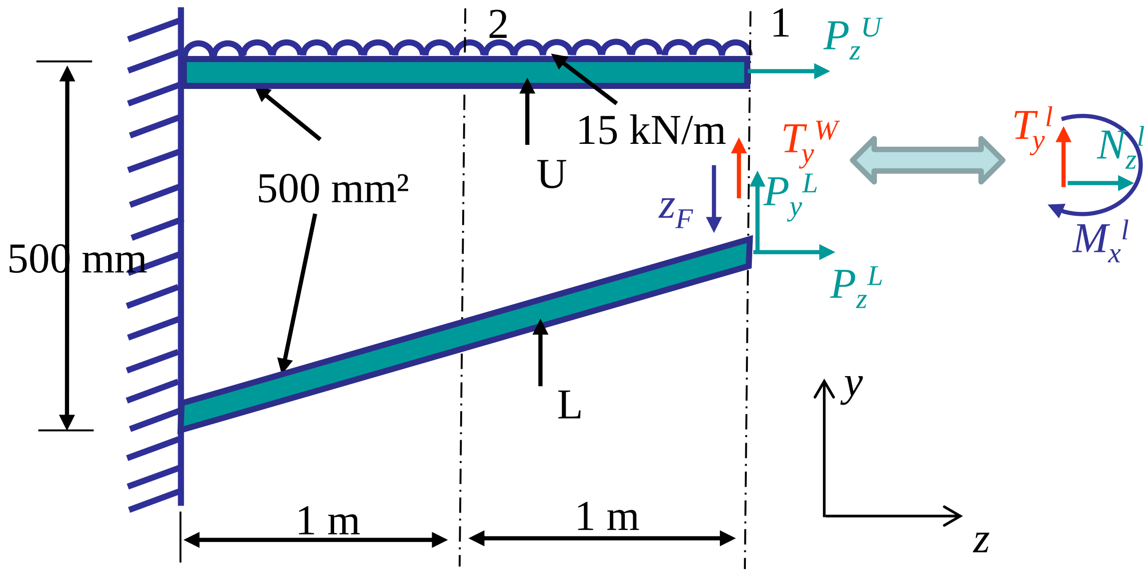
$$\begin{cases} N_z^l = \quad 0 \\ T_y^l = -15\ \text{kN} \\ M_x^l = 7.5\ \text{kN m} \end{cases}$$

- Flanges in section 1 (2)
  - Direct stress obtained from $\sigma_z = \dfrac{M_x^l y}{I_{xx}}$ & $I_x = 27 \times 10^6\ \text{mm}^4$

$$\begin{cases} \sigma_{z,U} = \dfrac{7.5 \times 10^6 \times 150}{2.7 \times 10^7} \\ \sigma_{z,L} = \dfrac{7.5 \times 10^6 \times -150}{2.7 \times 10^7} \end{cases}$$

$$\Leftrightarrow \begin{cases} \sigma_{z,U} = 41.7\ \text{N/mm}^2 \\ \sigma_{z,L} = -41.7\ \text{N/mm}^2 \end{cases}$$

- Section 1 with web resisting direct load (3)



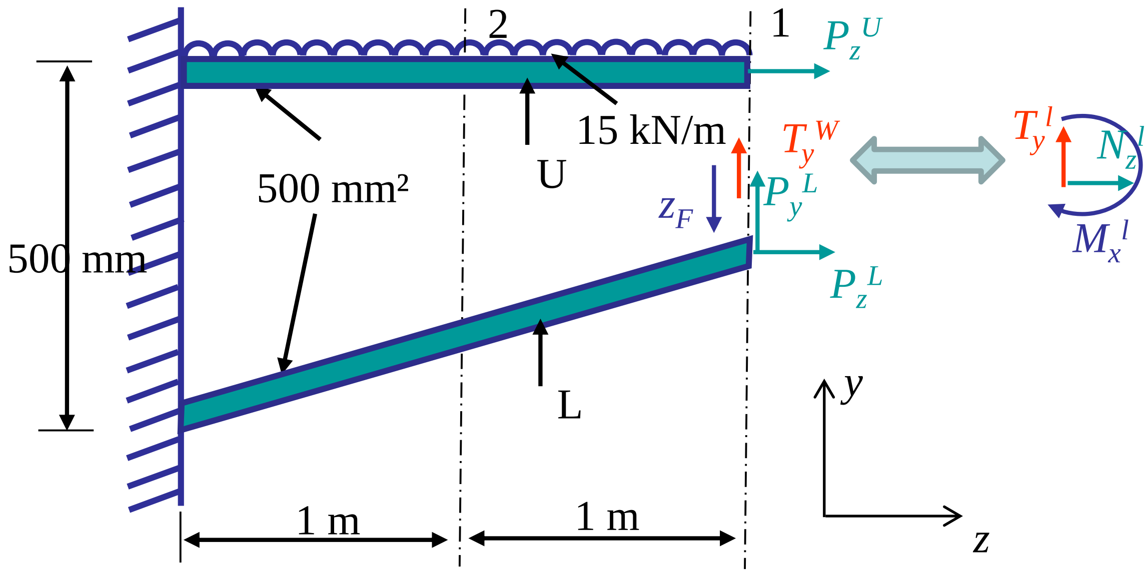
$$\begin{cases} N_z^l = & 0 \\ T_y^l = & -15 \text{ kN} \\ M_x^l = & 7.5 \text{ kN m} \end{cases}$$

– Loading in flanges

- $\begin{cases} \sigma_{z,U} = 41.7 \text{ N/mm}^2 \\ \sigma_{z,L} = -41.7 \text{ N/mm}^2 \end{cases} \Leftrightarrow \begin{cases} P_z^L = \sigma_{z,L} \times 500 \\ P_z^U = \sigma_{z,U} \times 500 \end{cases} \Leftrightarrow \begin{cases} P_z^L = -20.850 \\ P_z^U = 20.850 \end{cases}$
- $\begin{cases} P_y^L = P_z^L \dfrac{100}{1 \times 10^3} \\ P_y^U = 0 \end{cases} \Leftrightarrow \begin{cases} P_y^L = -2.085 \text{ kN} \\ P_y^U = 0 \text{ kN} \end{cases}$

$$\begin{cases} P_U = \sqrt{\left(P_z^U\right)^2 + \left(P_y^U\right)^2} \\ P_L = -\sqrt{\left(P_z^L\right)^2 + \left(P_y^L\right)^2} \end{cases} \Leftrightarrow \begin{cases} P_U = 20.85 \text{ kN} & \text{tension} \\ P_L = -20.954 \text{ kN} & \text{compression} \end{cases}$$

- Section 1 with web resisting direct load (4)



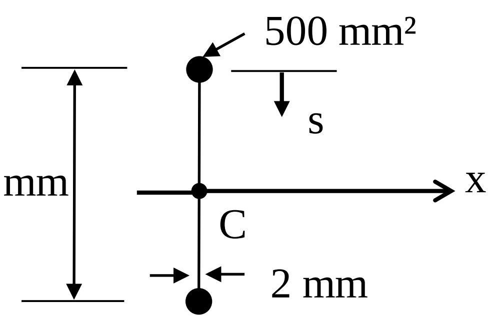
$$\begin{cases} N_z^l = & 0 \\ T_y^l = & -15 \text{ kN} \\ M_x^l = & 7.5 \text{ kN m} \end{cases}$$

– Shear force in web

- $\begin{cases} P_y^L = -2.085 \text{ kN} \\ P_y^U = \quad 0 \text{ kN} \end{cases}$
- $T_y^{web} + P_y^L = T_y^l$

$\Leftrightarrow$ $T_y^{web} = -15 + 2.085 = -12.915 \text{ kN}$

- Section 1 with web resisting direct load (5)
  - Shear flow in web
    - As $T_y^{web} = -15 + 2.085 = -12.915\text{ kN}$ & as web carries direct stress



$$\Rightarrow q(s) = -\frac{T_y}{I_{zz}}\left[\int_0^s t_{\text{direct}\,\boldsymbol{\sigma}}y\,ds + \sum_{i:\,s_i\leq s} y_i A_i\right]$$

$$\Leftrightarrow q = \frac{12915}{2.7\times 10^7}\left[\int_0^s 2\,(150-s)\,ds + 500\times 150\right]$$

$$\Leftrightarrow q = 4.8\times 10^{-4}\,\left(300s - s^2 + 75000\right)$$

- Maximum value of q occurs when s = 150 mm, i.e. $q_{\max} = 46.8\text{ N/mm}$

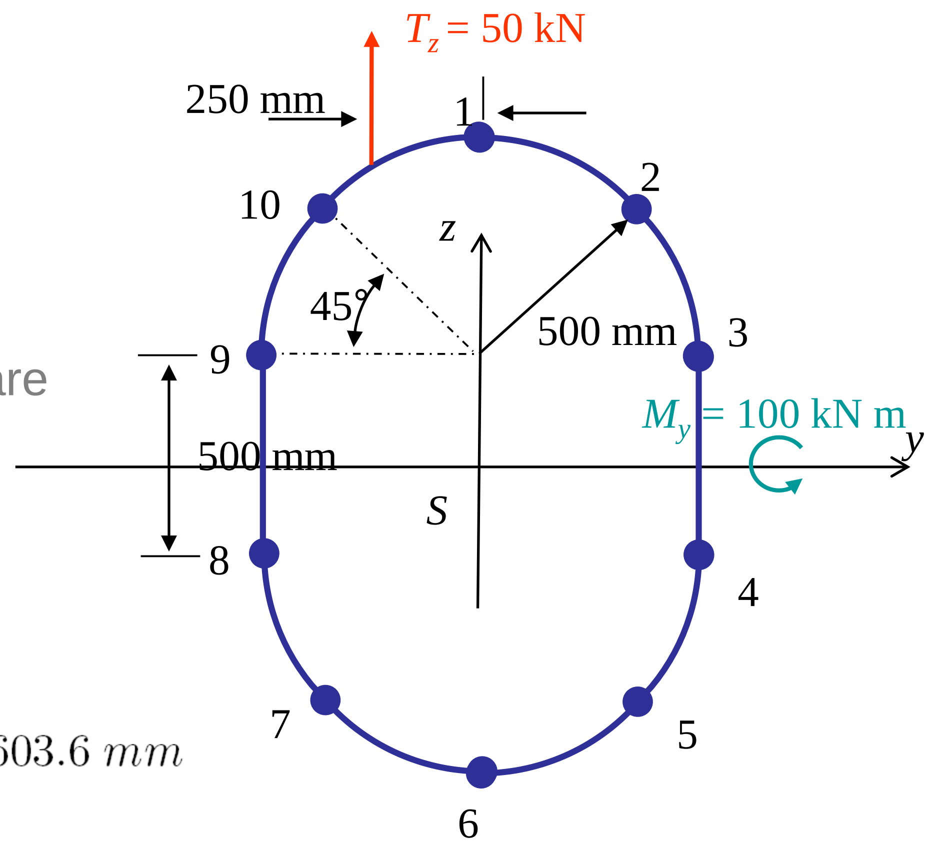
- Comparison without/with web resisting direct load
  - Section 1 without web resisting direct load

$$\begin{cases} P_U = \quad 25 \text{ kN} & \text{tension} \\ P_L = -25.1 \text{ kN} & \text{compression} \end{cases} \qquad |q| = \frac{T_y^{web}}{L_{web}} = \frac{12.5 * 10^3}{300} = 41.7 \text{ N/mm}$$

  - Section 1 with web resisting direct load

$$\begin{cases} P_U = \quad 20.85 \text{ kN} & \text{tension} \\ P_L = -20.954 \text{ kN} & \text{compression} \end{cases} \qquad \begin{aligned} q &= 4.8 \times 10^{-4} \left(300s - s^2 + 75000\right) \\ q_{\text{max}} &= 46.8 \text{ N/mm} \end{aligned}$$

- Bending
  - Second moment of area
    - Doubly symmetrical section
    - I_{yz}=0
    - Only idealized booms are carrying direct stress
  - The z-coordinates of the booms are



$$z_1 = -z_6 = 750\ mm$$

$$z_3 = z_9 = -z_4 = -z_8 = 250\ mm$$

$$z_2 = z_{10} = -z_5 = -z_7$$
$$= 250 + 500 \sin(45°) = 603.6\ mm$$

  - The second moment of area is

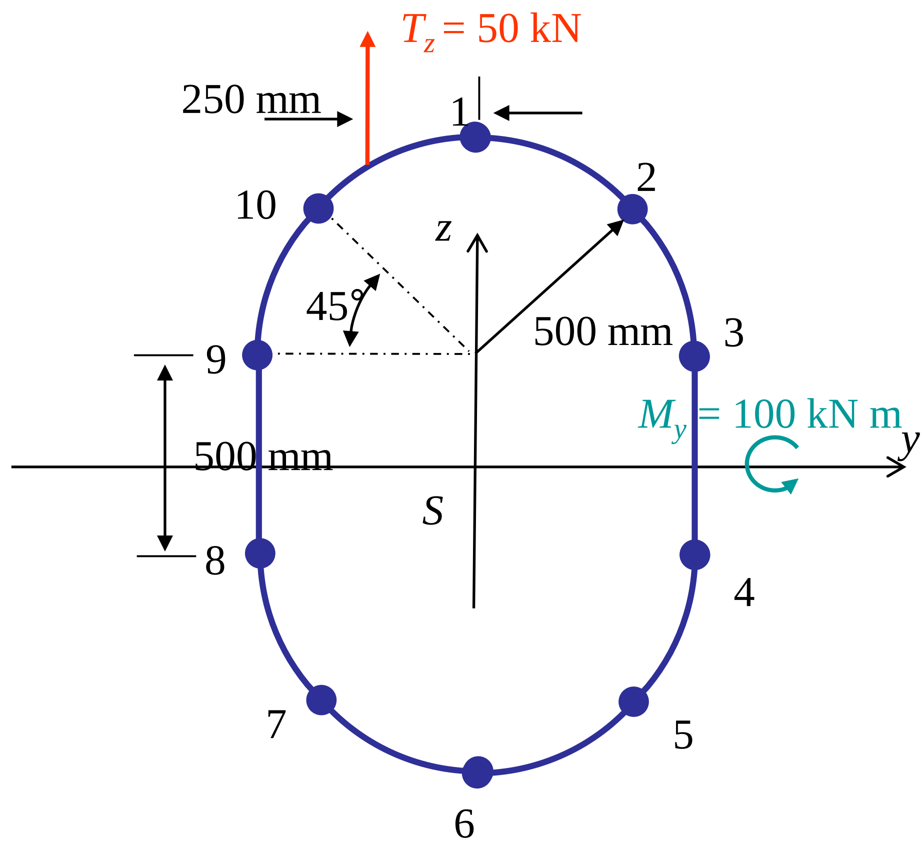
$$I_{yy} = \sum_{i=1}^{10} A_i z_i^2 = 2 \times \left(2 \times 150 \times 250^2 + 2 \times 150 \times 603.6^2 + 150 \times 750^2\right)$$
$$= 425 \times 10^6\ mm^4$$

- Bending (2)
  - Direct stress in the principal axes

$$\boldsymbol{\sigma}^i_{xx} = \frac{M_y}{I_{xx}} z^i$$

$$\Rightarrow \sigma^i_{xx} = \frac{100 \times 10^6}{425 \times 10^6} z^i$$

$$\Rightarrow \sigma^i_{xx} = 0.24 z^i$$



Boom	1	2 & 10	3 & 9	4 & 8	5 & 7	6
σ_{xx} (N/mm²)	180	144.9	60	-60	-144.9	-180

Exercise 4: Fuselage

- Shear flow
  - Shear centre S lies on both axes of symmetry
  - Shear flow reads

$$q(s) = -\frac{T_z}{I_{yy}} \sum_{i:\, s_i \leq s} z_i A_i + q(0)$$

  - Open shear flux

$$q_o(s) = -\frac{T_z}{I_{yy}} \sum_{i:\, s_i \leq s} z_i A_i$$

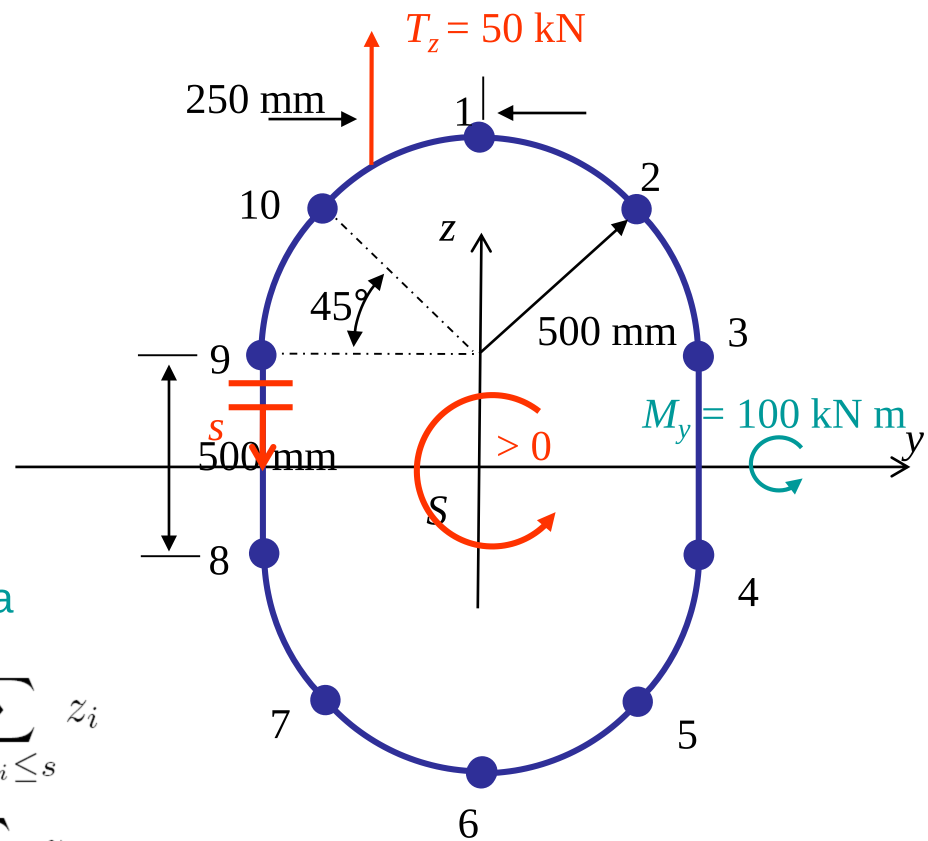
    - As all booms have the same area

$$q_o(s) = -\frac{50 \times 10^3 \times 150}{425 \times 10^6} \sum_{i:\, s_i \leq s} z_i$$

$$\Rightarrow \quad q_o(s) = -0.018 \sum_{i:\, s_i \leq s} z_i$$

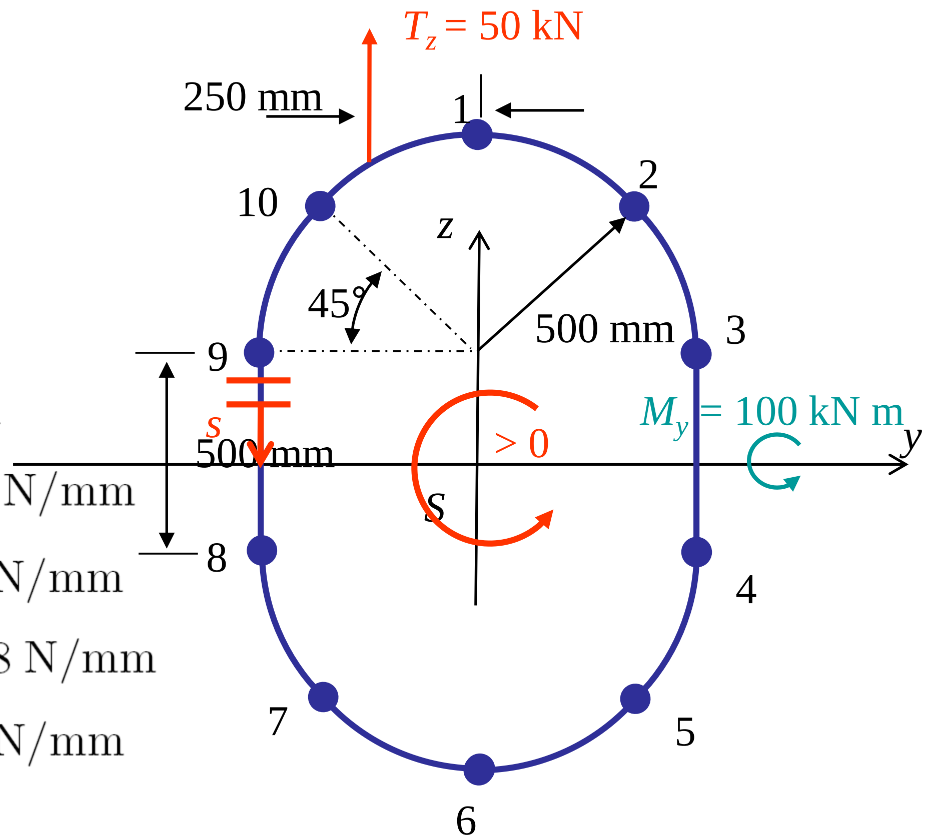
    - Cut between booms 8 and 9

$$q_o^{98} = 0$$



- Open shear flow

$$q_o(s) = -0.018 \sum_{i:\, s_i \leq s} z_i$$



  – Cut between booms 8 and 9

$$q_o^{98} = 0$$

$$q_o^{87} = -0.018 \times (-250) = 4.5 \text{ N/mm}$$

$$q_o^{76} = 4.5 - 0.018 \times (-603.6) = 15.4 \text{ N/mm}$$

$$q_o^{65} = 15.4 - 0.018 \times (-750) = 28.9 \text{ N/mm}$$

$$q_o^{54} = 28.9 - 0.018 \times (-603.6) = 39.8 \text{ N/mm}$$

$$q_o^{43} = 39.8 - 0.018 \times (-250) = 44.3 \text{ N/mm}$$

  – By symmetry

$$q_o^{32} = q_o^{54}$$

$$q_o^{21} = q_o^{65}$$

$$q_o^{1\,10} = q_o^{76}$$

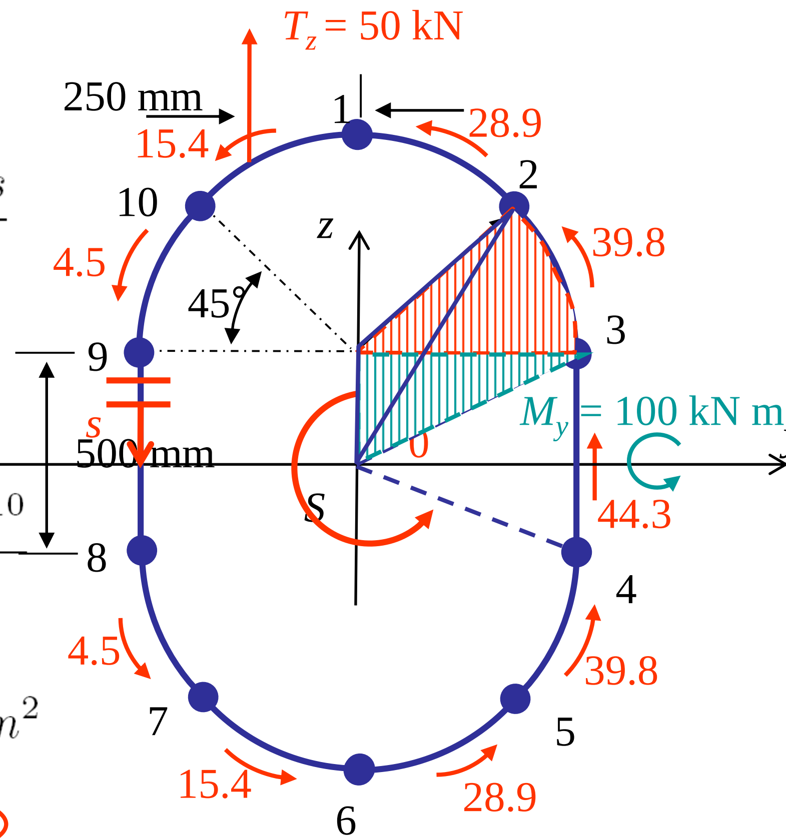
$$q_o^{10\,9} = q_o^{87}$$

- Constant part of shear flow
  - Anticlockwise s, q

$$q\left(s=0\right)=\frac{y_T T_z - z_T T_y - \oint p\left(s\right) q_o\left(s\right) ds}{2A_h}$$

  - As q is constant between 2 booms
    - By defining $A_{i\,i+1}$ the area swept from boom i to $i+1$

$$q\left(0\right)=\frac{T_z y_T - 2\times\sum_{i=1}^{9} A_{i+1\,i}\, q_o^{i+1\,i} - 2\times A_{1\,10}\, q_o^{1\,10}}{2A_h}$$



- With

$$A_{43}=2\times\frac{1}{2}\times 500\times 250 = 125000\; mm^2$$

$$A_{32} = \frac{1}{2}\times 500\times 250 + \frac{\pi}{4}\times\frac{500^2}{2} - \frac{1}{2}\times 250\times 500\cos\left(\frac{\pi}{4}\right) = 116474.8\; mm^2$$

$$A_{21}=\frac{1}{2}\times 250\times 500\cos\left(\frac{\pi}{4}\right)+\frac{\pi}{4}\times\frac{500^2}{2} = 142374.8\; mm^2$$

- Remaining areas by symmetry

- Constant part of shear flow (2)
  - As

$$A_{43} = 2 \times \frac{1}{2} \times 500 \times 250 = 125000\ mm^2$$

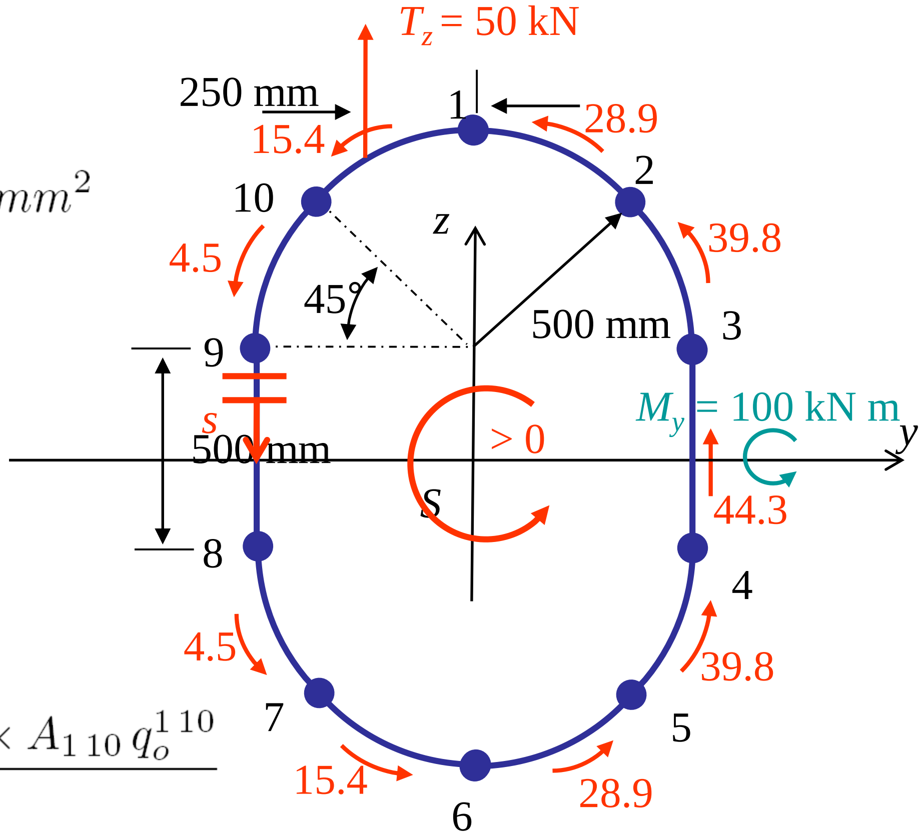
$$A_{32} = 116474.8\ mm^2$$

$$A_{21} = 142374.8\ mm^2$$

$$A_h = 500 \times 1000 + 2 \times \pi \times \frac{500^2}{2}$$

$$= 1285398.2\ mm^2$$

  - Constant flux becomes



$$q(0) = \frac{T_z y_T - 2 \times \sum_{i=1}^{9} A_{i+1\,i}\, q_o^{i+1\,i} - 2 \times A_{1\,10}\, q_o^{1\,10}}{2A_h}$$

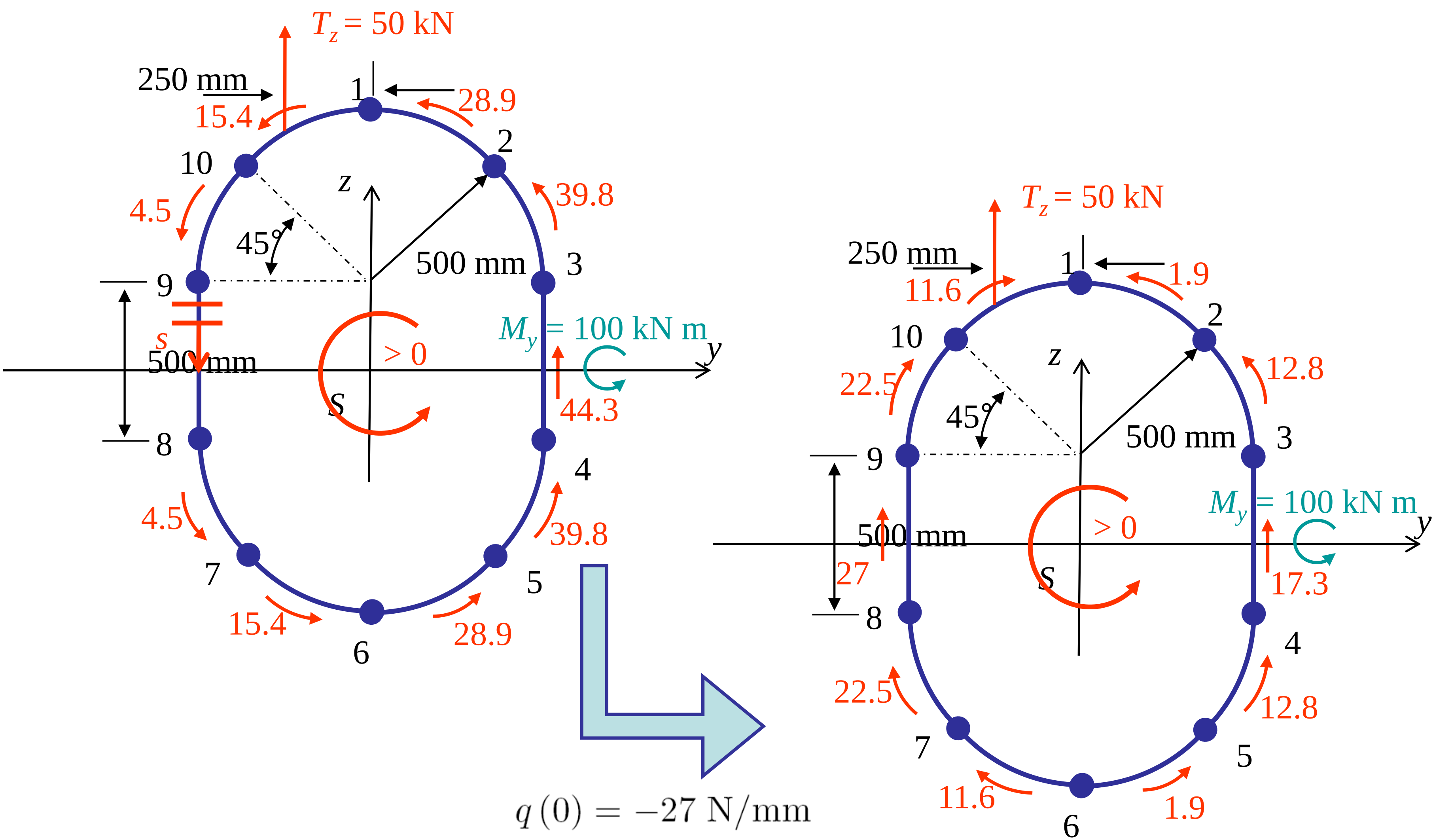
$$\Rightarrow q(0) = \frac{50 \times 10^3 \times (-250) - 2 \times (2 \times 28.9 A_{21} + 2 \times 39.8 A_{32} + 44.3 A_{43} + \ldots}{2A_h}$$

$$\frac{\ldots 2 \times 4.5 A_{10\,9} + 2 \times 15.4 A_{1\,10})}{2A_h}$$

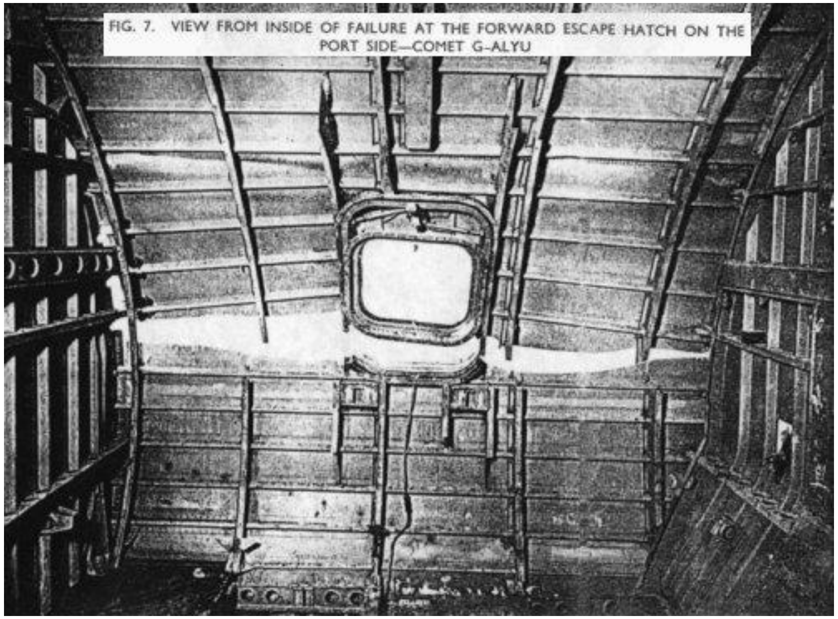
$$\Rightarrow q(0) = -27\ \mathrm{N/mm}$$

Exercise 4: Fuselage

- Total shear flow (in N/mm)



- Cuts outs
  - There are opening in
    - Doors, windows, …
  - This leads to discontinuities on the otherwise closed stiffened shells
    - This affects loading
      - In stringers
      - In skin, …
    - These regions have to be heavily reinforced



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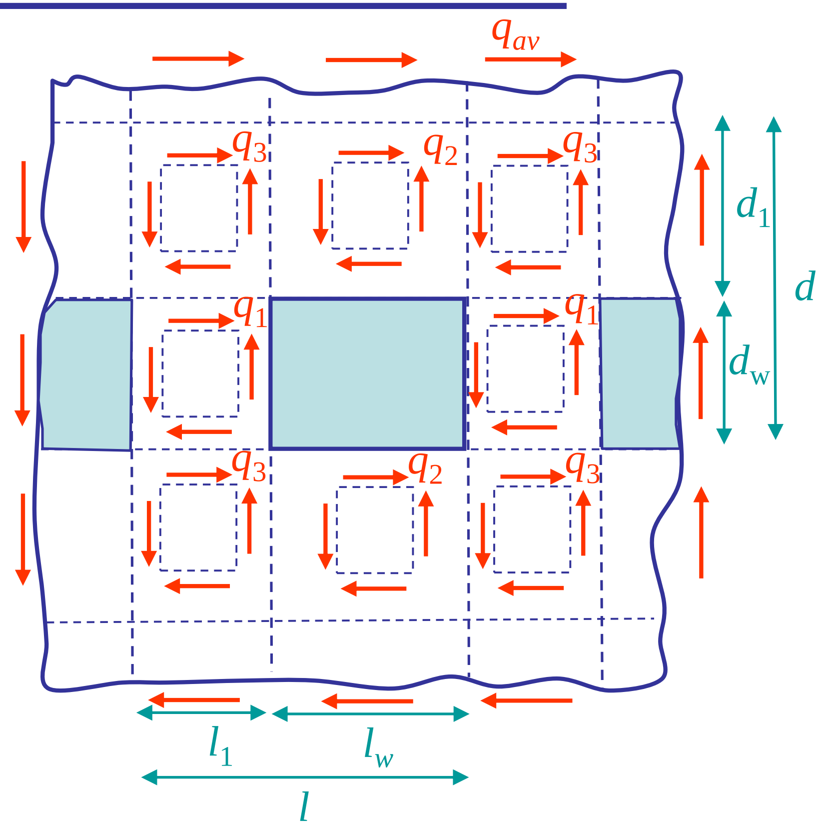
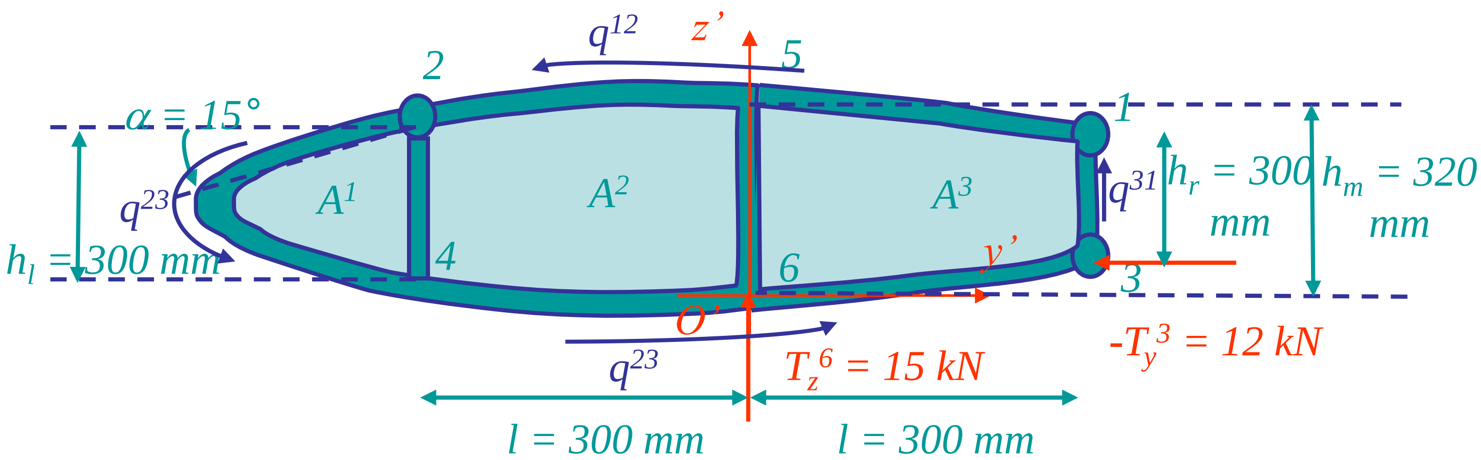
- Windows cut outs approximation
  - Windows spaced a distance l
  - Panel subjected to an average shear flux q_{av}
    - Shear value in the panel without cut outs
  - Consider shear flux balance on
    - An horizontal line through the cut outs
      - Average shear q_1: $q_{\text{av}} l = q_1 l_1$
    - A vertical line through the cut outs
      - Average shear q_2: $q_{\text{av}} d = q_2 d_1$
    - An horizontal line on top of the cut outs
      - Average shear fluxes q_2 & q_3: $q_{\text{av}} l = q_2 l_w + q_3 l_1$
    - A vertical line beside of the cut outs
      - Average shear fluxes q_1 & q_3: $q_{\text{av}} d = q_3 d_1 + q_1 d_w$



$$\Rightarrow \quad q_1 = \frac{l}{l_1} q_{\text{av}} \ , \ q_2 = \frac{d}{d_1} q_{\text{av}} \quad \& \quad q_3 = q_{\text{av}} \left(1 - \frac{d_w l_w}{d_1 l_1} \right)$$

- Example
  - Three-cell symmetrical section
    - $A^1 = 0.05\ \mathrm{m}^2$
    - $A^2 = A^3 = 0.095\ \mathrm{m}^2$
  - Shear flow at periphery
    - Is discontinuous where there are stringers (booms 1, 2 & 3) as direct stress is provided by the booms only
    - Results from discontinuous shear flow on each side of the rib due to the external loading applied in the rib plane
  - Shear flow in the web panels due to this shear flow discontinuity?
    - Effective only in shearing
  - Axial loads in the flanges?

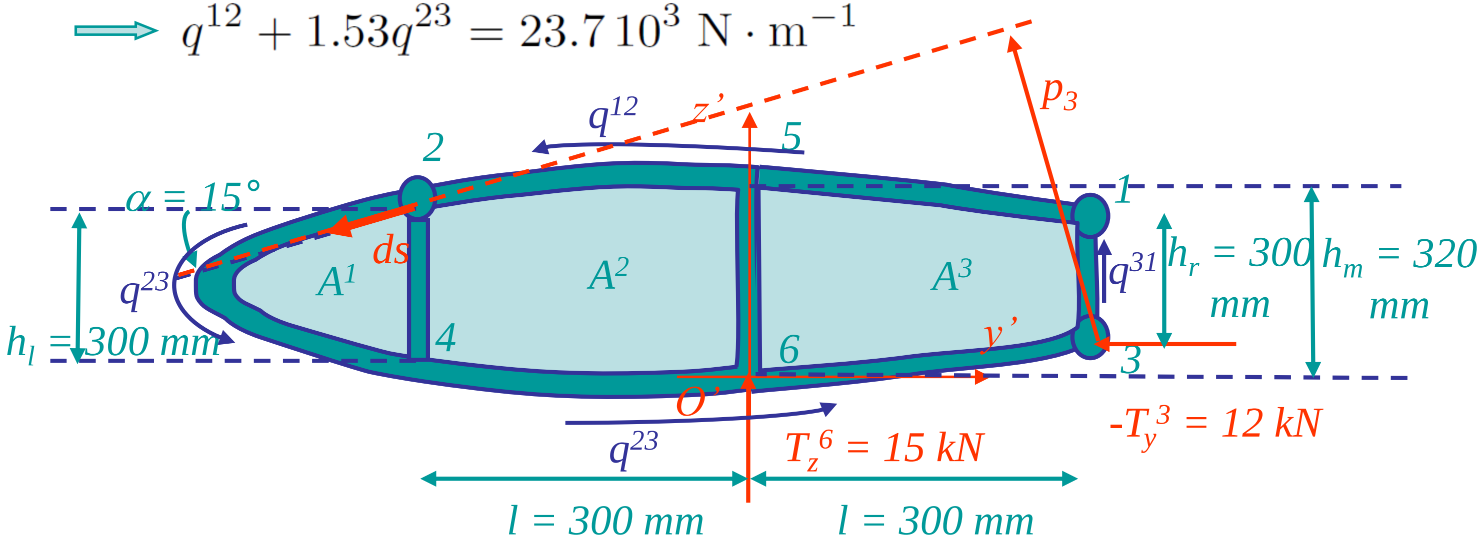


- Shear flow at periphery
  - Equilibrium of applied forces on the flanges
    - $2l\left(q^{23}-q^{12}\right)+\boldsymbol{T}_y^3=0 \Longrightarrow q^{23}-q^{12}=\frac{12\,10^3}{0.6}=20\,10^3\ \mathrm{N}\cdot\mathrm{m}^{-1}$
    - $h_r\, q^{31}-h_l q^{23}+\boldsymbol{T}_z^6=0 \Longrightarrow q^{23}-q^{31}=\frac{15\,10^3}{0.3}=50\,10^3\ \mathrm{N}\cdot\mathrm{m}^{-1}$
    - Moment around boom 3

$$\int_1^2 q^{12}p_3ds+\int_2^3 q^{23}p_3ds-\boldsymbol{T}_z^6 l=0 \Longrightarrow q^{12}\int_1^2 p_3ds+q^{23}\int_2^3 p_3ds=\boldsymbol{T}_z^6 l$$

$$\Longrightarrow q^{12}2\frac{A^2+A^3}{2}+q^{23}2\left(A^1+\frac{A^2+A^3}{2}\right)=\boldsymbol{T}_z^6 l$$

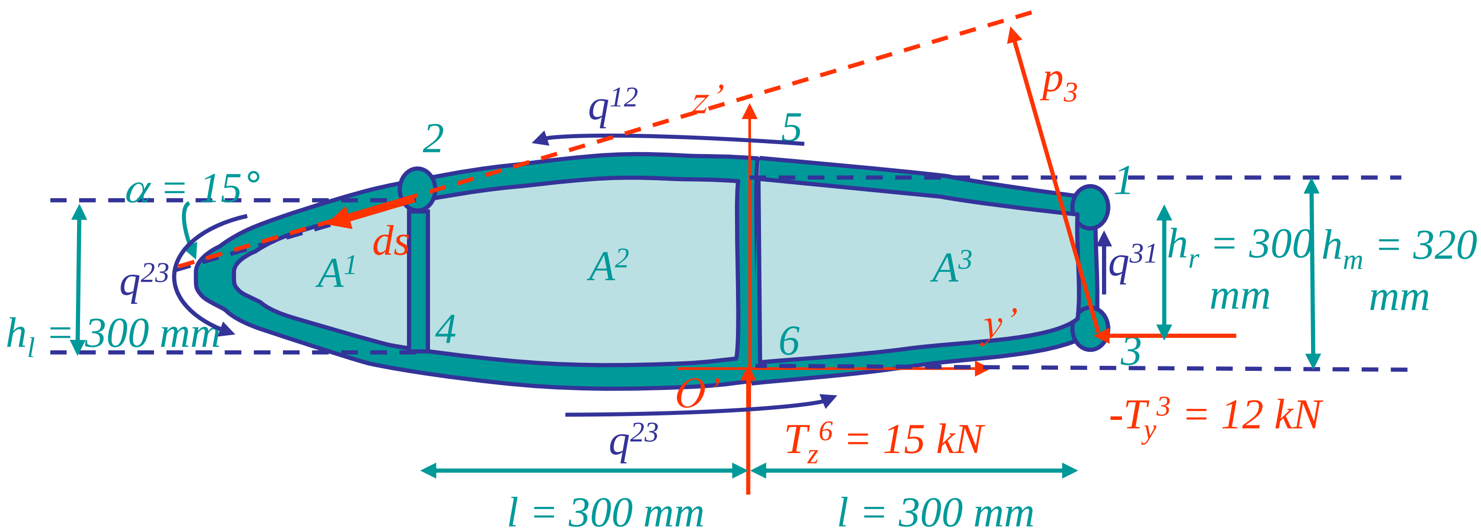
$$\Longrightarrow q^{12}+1.53q^{23}=23.7\,10^3\ \mathrm{N}\cdot\mathrm{m}^{-1}$$



- Shear flow at periphery (2)
  - Equilibrium (2)
    - $q^{23} - q^{31} = \frac{15\,10^3}{0.3} = 50\,10^3\ \mathrm{N \cdot m^{-1}}$
    - $q^{23} - q^{12} = \frac{12\,10^3}{0.6} = 20\,10^3\ \mathrm{N \cdot m^{-1}}$
    - $q^{12} + 1.53 q^{23} = 23.7\,10^3\ \mathrm{N \cdot m^{-1}}$

$$\Longrightarrow 2.53 q^{23} = 43.7\,10^3\ \mathrm{N \cdot m^{-1}}$$

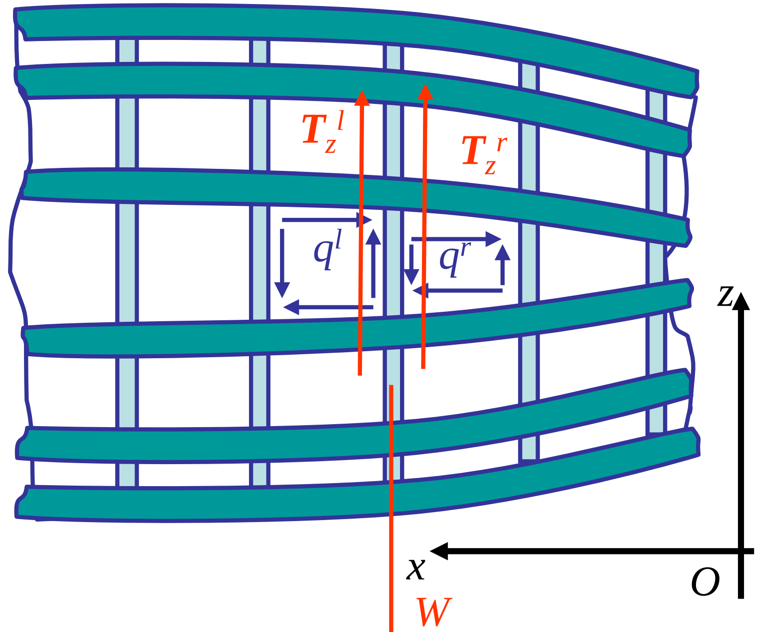
$$\Longrightarrow \begin{cases} q^{23} = 17.2\,10^3\ \mathrm{N \cdot m^{-1}} \\ q^{12} = q^{23} - 20\,10^3 = -2.8\,10^3\ \mathrm{N \cdot m^{-1}} \\ q^{31} = q^{23} - 50\,10^3 = -32.8\,10^3\ \mathrm{N \cdot m^{-1}} \end{cases}$$

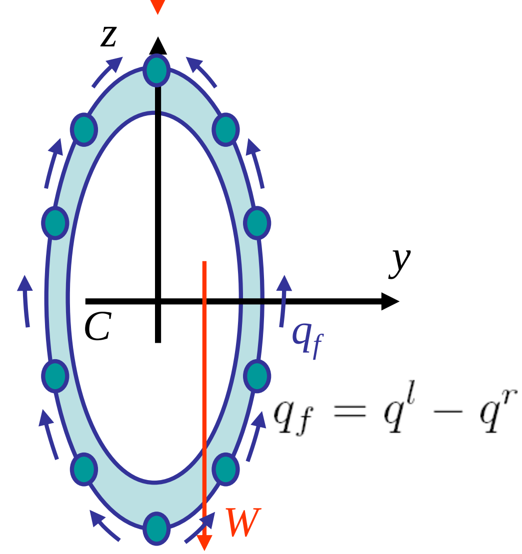


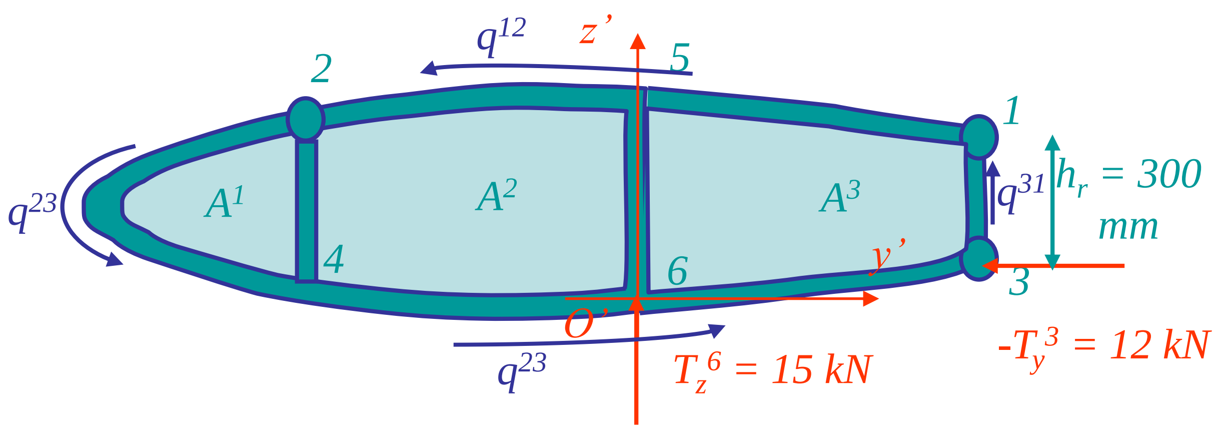
- Shear flow at periphery (3)
  - Solution

$$\begin{cases} q^{23} = 17.2\,10^3\ \mathrm{N}\cdot\mathrm{m}^{-1} \\ q^{12} = q^{23} - 20\,10^3 = -2.8\,10^3\ \mathrm{N}\cdot\mathrm{m}^{-1} \\ q^{31} = q^{23} - 50\,10^3 = -32.8\,10^3\ \mathrm{N}\cdot\mathrm{m}^{-1} \end{cases}$$

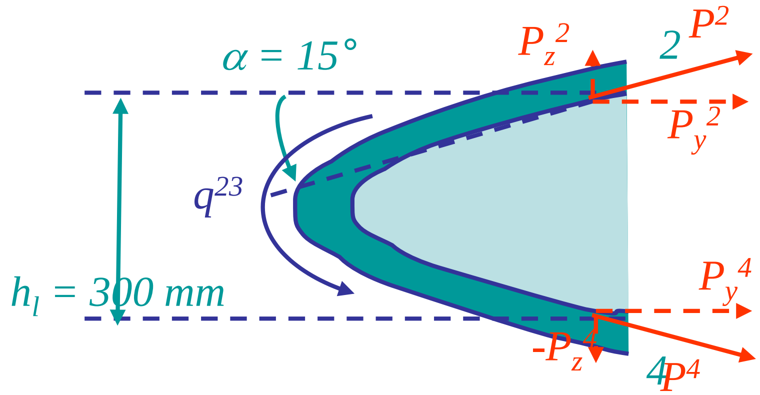
  - This corresponds to the peripheral forces applied to the rib-flanges due to the neighboring shells and the discontinuities resulting from rib loading

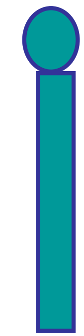






- Web panel 1
  - Axial loading in rib-flange
    - Equilibrium of structure just before stiffener 24
    - Moment around point 4
      - Shearing in panel has no component along *Oy*





$$\int_2^4 q^{23} p_4 ds = P_y^2 h_l$$

$$\Longrightarrow \quad q^{23} \int_2^4 p_4 ds = P_y^2 h_l \Longrightarrow \quad 2A^1 q^{23} = P_y^2 h_l$$

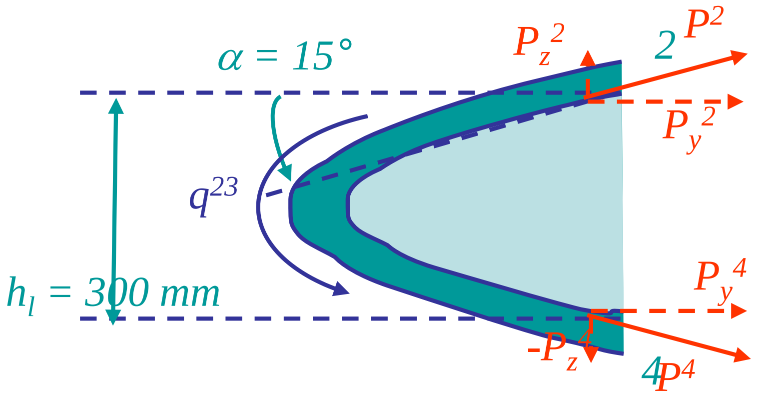
$$\Longrightarrow P_y^2 = \frac{2A^1 q^{23}}{h_l} = \frac{2\ 0.05\ 17.2\ 10^3}{0.3} = 5.7\ 10^3\ \text{N}$$

$$\Longrightarrow P_z^2 = P_y^2 \tan 15\,\text{deg} = 5.7\ 10^3\ \tan 15\,\text{deg} = 1.5\ 10^3\ \text{N}$$

$$\Longrightarrow P^2 = \frac{P_y^2}{\cos 15\,\text{deg}} = \frac{5.7\ 10^3}{\cos 15\,\text{deg}} = 5.9\ 10^3\ \text{N}$$

      - Flange in traction at point 2

- Web panel 1 (2)
  - Axial load in rib-flange (2)
    - Equilibrium of structure just before stiffener 24
    - Moment around point 2



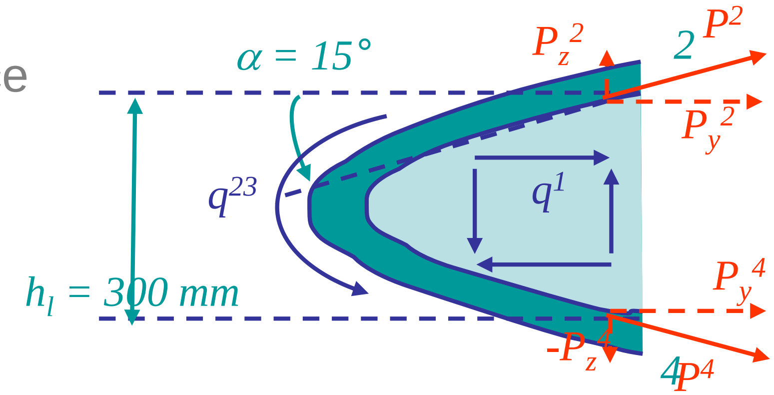
$$P_y^4 = -\frac{2A^1 q^{23}}{h_l} = -\frac{2\ 0.05\ 17.2\ 10^3}{0.3} = -5.7\ 10^3\ \mathrm{N}$$

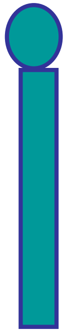
$$P_z^4 = P_y^4 \tan -15\,\mathrm{deg} = 5.7\ 10^3\ \tan 15\,\mathrm{deg} = 1.5\ 10^3\ \mathrm{N}$$

$$P^4 = \frac{P_y^4}{\cos 15\,\mathrm{deg}} = -\frac{5.7\ 10^3}{\cos 15\,\mathrm{deg}} = -5.9\ 10^3\ \mathrm{N}$$

- Flange in compression at point 4

- Web panel 1 (3)
  - Web shearing should balance the loading in the flanges
    - Equilibrium just before stiffener 24



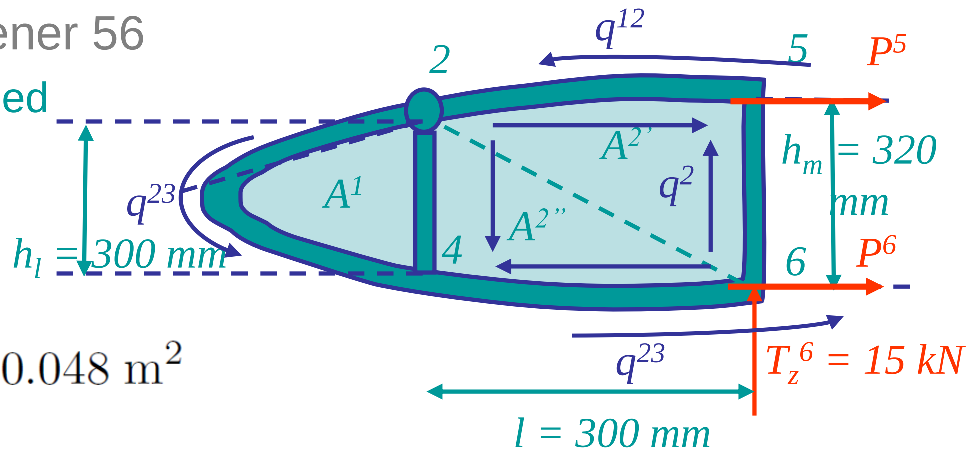


$$q^1 h_l + P_z^2 + P_z^4 - q^{23} h_l = 0$$

$$\Rightarrow q^1 = -\frac{P_z^2 + P_z^4}{h_l} + q^{23}$$

$$\Rightarrow q^1 = -\frac{1500 + 1500}{0.3} + 17200 = 7.2\,10^3 \text{ N} \cdot \text{m}^{-1}$$

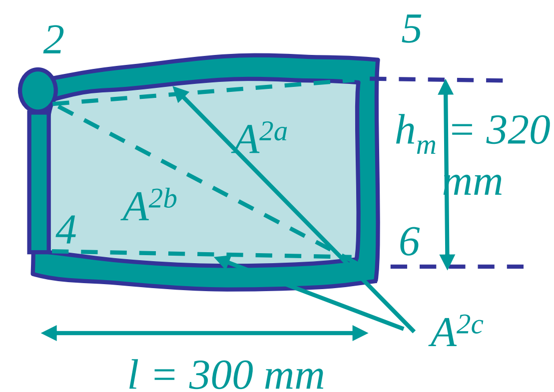
- Web panel 2
  - Loading in flange just before stiffener 56
    - More geometric information needed
    - Flange has zero-slope at points 5 & 6
    - Areas



– $A^{2a} = \dfrac{h_m l}{2} = \dfrac{0.32\ 0.3}{2} = 0.048\ \mathrm{m}^2$

– $A^{2b} = \dfrac{h_l l}{2} = \dfrac{0.3\ 0.3}{2} = 0.045\ \mathrm{m}^2$

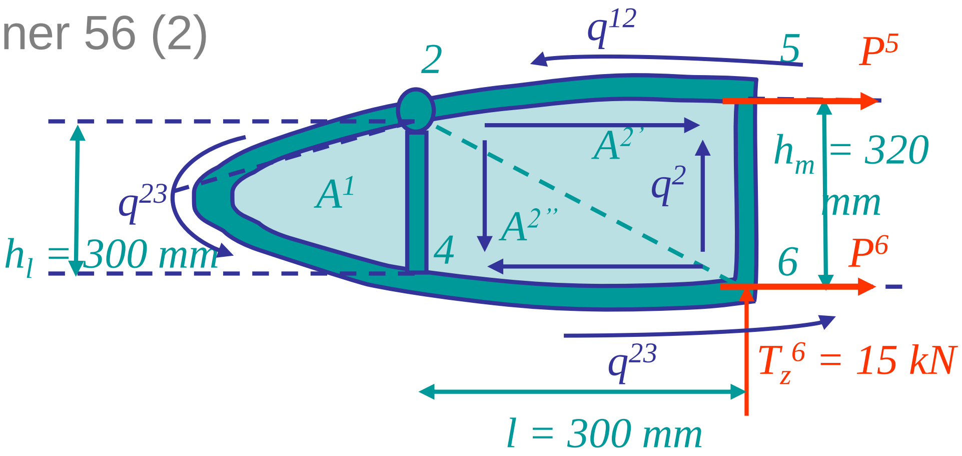
– $$A^{2c} = \frac{A^2 - A^{2a} - A^{2b}}{2} = \frac{0.095 - 0.045 - 0.048}{2} = 0.001\ \mathrm{m}^2$$



$$\Longrightarrow A^{2'} = A^{2a} + A^{2c} = 0.048 + 0.001 = 0.049\ \mathrm{m}^2$$

$$\Longrightarrow A^{2''} = A^{2b} + A^{2c} = 0.045 + 0.001 = 0.046\ \mathrm{m}^2$$

- Web panel 2 (2)
  - Loading in flange just before stiffener 56 (2)
    - Moment around Point 6



$$\int_5^2 q^{12} p_6 ds + \int_2^6 q^{23} p_6 ds = P^5 h_m$$

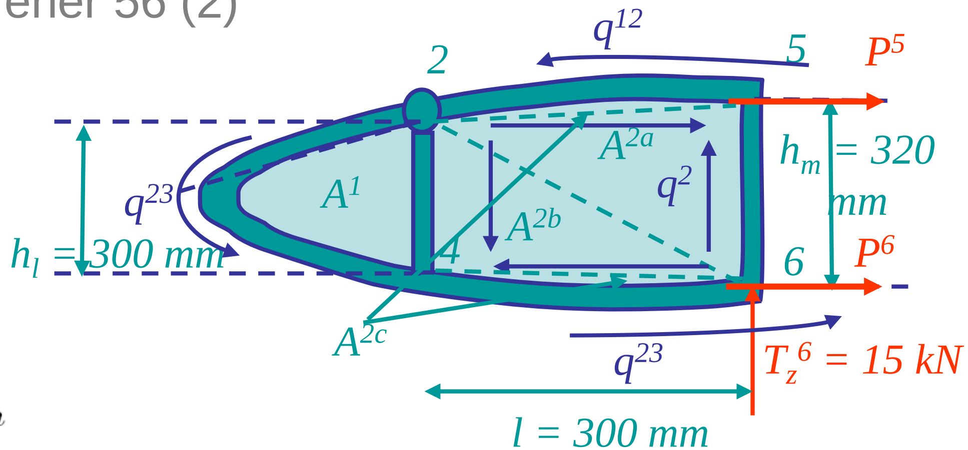
$$\Rightarrow q^{12} \int_5^2 p_6 ds + q^{23} \int_2^6 p_6 ds = P^5 h_m$$

$$\Rightarrow P^5 = \frac{2A^{2'} q^{12} + q^{23}\left(2A^{2''} + 2A^1\right)}{h_m}$$

$$\Rightarrow P^5 = \frac{-2\ 0.049\ 2.8\ 10^3 + 2\ \ (0.046 + 0.05)\ 17.2\ 10^3}{0.32} = 9.5\ 10^3\ \mathrm{N}$$

  - Traction in flange at point 5

- Web panel 2 (3)
  - Loading in flange just before stiffener 56 (2)
    - Moment around Point 5



$$\int_5^2 q^{12} p_5 ds + \int_2^6 q^{23} p_5 ds = -P^6 h_m$$

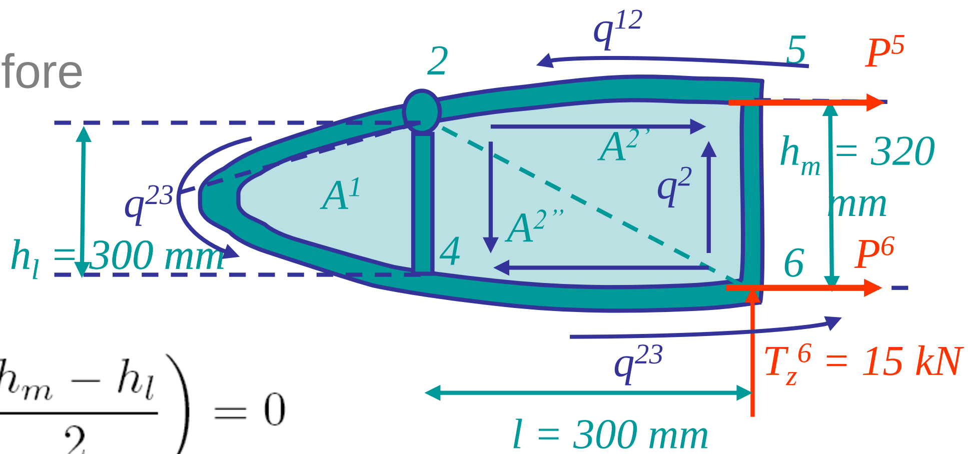
$$\Longrightarrow q^{12} \int_5^2 p_5 ds + q^{23} \int_2^6 p_5 ds = -P^6 h_m$$

$$\Longrightarrow P^6 = -\frac{2A^{2c} q^{12} + q^{23} \left(2A^{2a} + 2A^{2b} + 2A^{2c} + 2A^1\right)}{h_m}$$

$$\Longrightarrow P^6 = -\frac{-2\ 0.001\ 2.8\ 10^3 + 2\ \ (0.048 + 0.045 + 0.001 + 0.05)\ 17.2\ 10^3}{0.32} = -15.5 10^3 \text{ N}$$

- Compression in flange at point 6

- Web panel 2 (4)
  - Web shearing should balance the loading in the flanges just before stiffener 56
  - As P^5 and P^6 have no vertical component



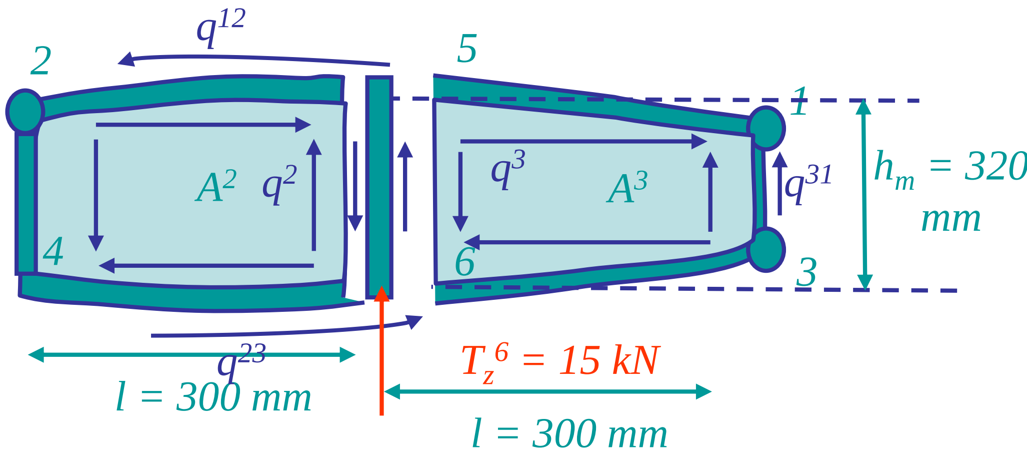
$$q^2 h_m - q^{12}\frac{h_m - h_l}{2} - q^{23}\left(h_l + \frac{h_m - h_l}{2}\right) = 0$$

$$\Longrightarrow q^2 = q^{12}\frac{h_m - h_l}{2h_m} + q^{23}\frac{h_m + h_l}{2h_m}$$

$$\Longrightarrow q^2 = -2.8\,10^3\frac{0.2}{0.64} + 17.2\,10^3\frac{0.62}{0.64} = 15.8\,10^3\ \mathrm{N}\cdot\mathrm{m}^{-1}$$

- This is the shear flow along stiffener 56 (not constant in web)
- Shear flow along stiffener 24 was equal to q^1 (as no vertical load on this stiffener)

- Web panel 3
  - Equilibrium of stiffener 56



$$T_z^6 - \left(q^2 - q^3\right) h_m = 0$$

$$\Longrightarrow \quad q^3 = q^2 - \frac{T_z^6}{h_m}$$

$$\Longrightarrow \quad q^3 = 15.8\,10^3 - \frac{15\,10^3}{0.32} = -31.1\,10^3\ \mathrm{N \cdot m^{-1}}$$

- This is the shear flow along stiffener 56 (not constant in web)

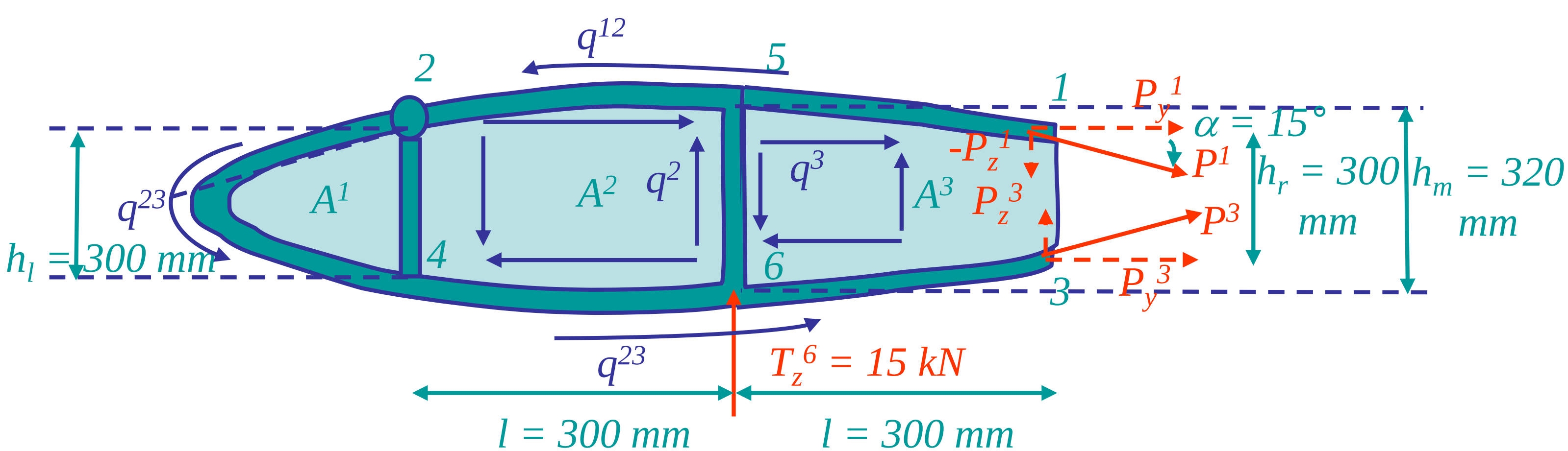
- Web panel 3 (2)
  - Loading in flange just before stiffener 13
    - Moment around Point 3

$$\int_1^2 q^{12} p_3 ds + \int_2^3 q^{23} p_3 ds - l\, T_z^6 = P_y^1 h_r$$

$$\Longrightarrow P_y^1 = \frac{q^{12}\left(A^3 + A^2\right) + q^{23}\left(2A^1 + A^2 + A^3\right) - l\, T_z^6}{h_r}$$

$$\Longrightarrow P_y^1 = \frac{-2.8\, 10^3\, (0.095 + 0.095) + 17.2\, 10^3\, (0.1 + 0.095 + 0.095) - 0.3\, 15\, 10^3}{0.3}$$

$$\Longrightarrow P_y^1 = -146 \text{ N}$$



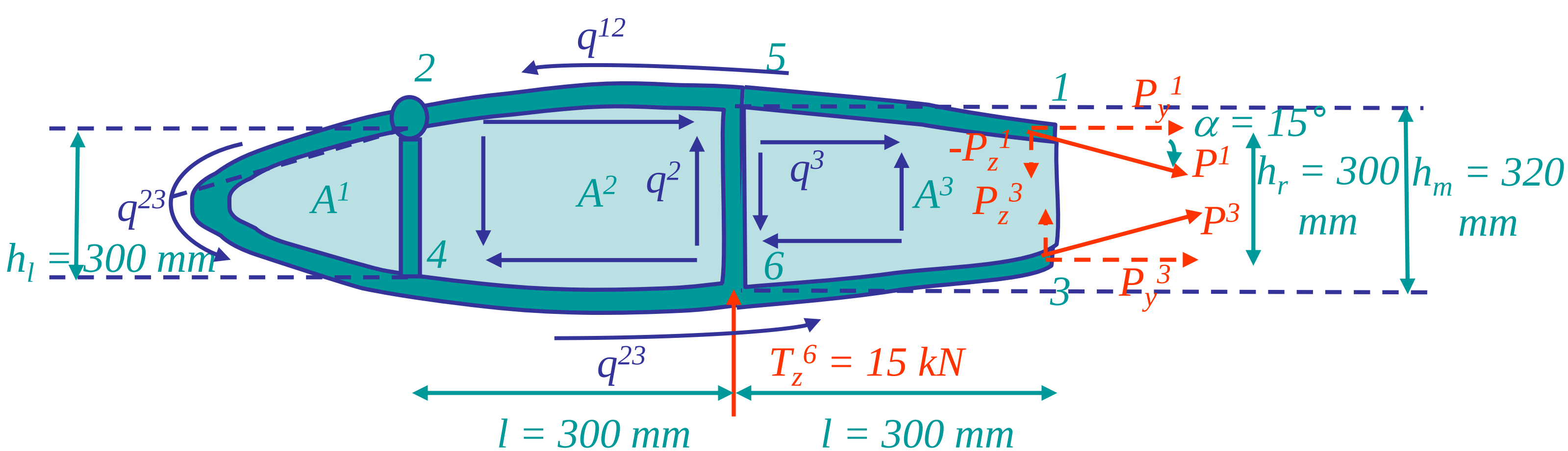
- Web panel 3 (3)
  - Loading in flange just before stiffener 13 (2)
    - Moment around Point 3 (2)

$$P_y^1 = -146 \text{ N}$$

$$\Longrightarrow P_z^1 = P_y^1 \tan(-15 \deg) = 146 \ \tan 15 \deg = 39 \text{ N}$$

$$\Longrightarrow P^1 = \frac{P_y^1}{\cos(-15 \deg)} = -\frac{146}{\cos 15 \deg} = -151 \text{ N}$$

  - Compression in flange at point 1



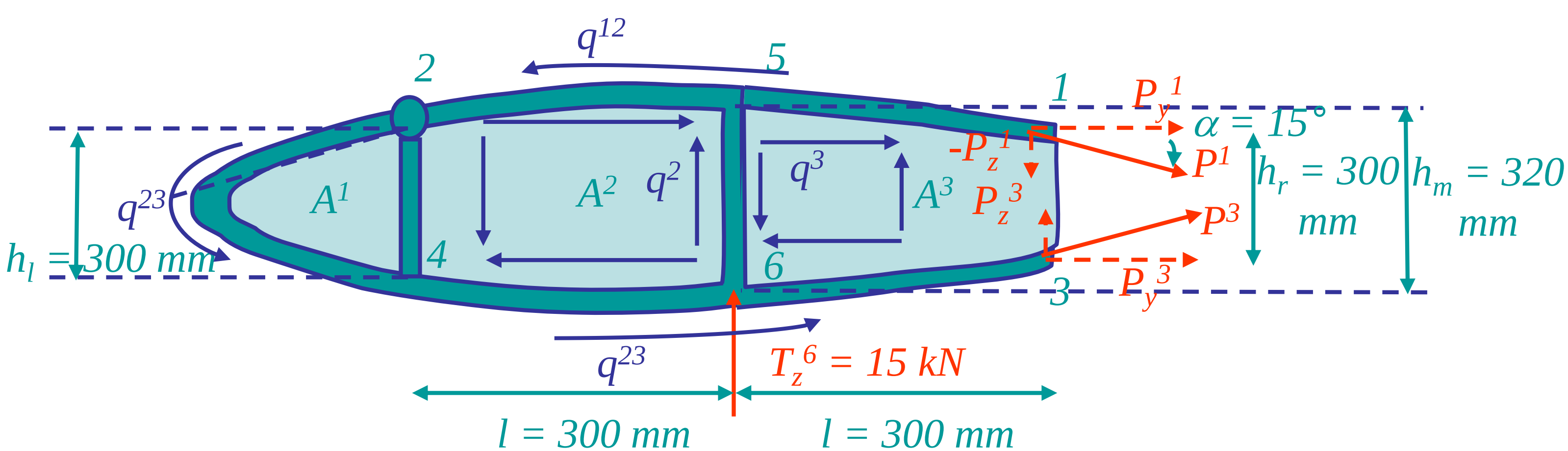
- Web panel 3 (4)
  - Loading in flange just before stiffener 13 (3)
    - Moment around Point 1

$$\int_1^2 q^{12} p_1 ds + \int_2^3 q^{23} p_1 ds - l\, T_z^6 = -P_y^3 h_r$$

$$\Rightarrow P_y^3 = \frac{-q^{12}\left(2A^{2c} + 2A^{2c}\right) - q^{23}\left(2A^1 + 2A^2 - 2A^{2c} + 2A^3 - 2A^{2c}\right) + l\, T_z^6}{h_r}$$

$$\Rightarrow P_y^3 = \frac{2.8\, 10^3\ 4\, 0.001 - 17.2\, 10^3\, (0.1 + 4\, 0.095 - 4\, 0.001) + 0.3\, 15\, 10^3}{0.3}$$

$$\Rightarrow P_y^3 = -12.2\, 10^3 \text{ N}$$



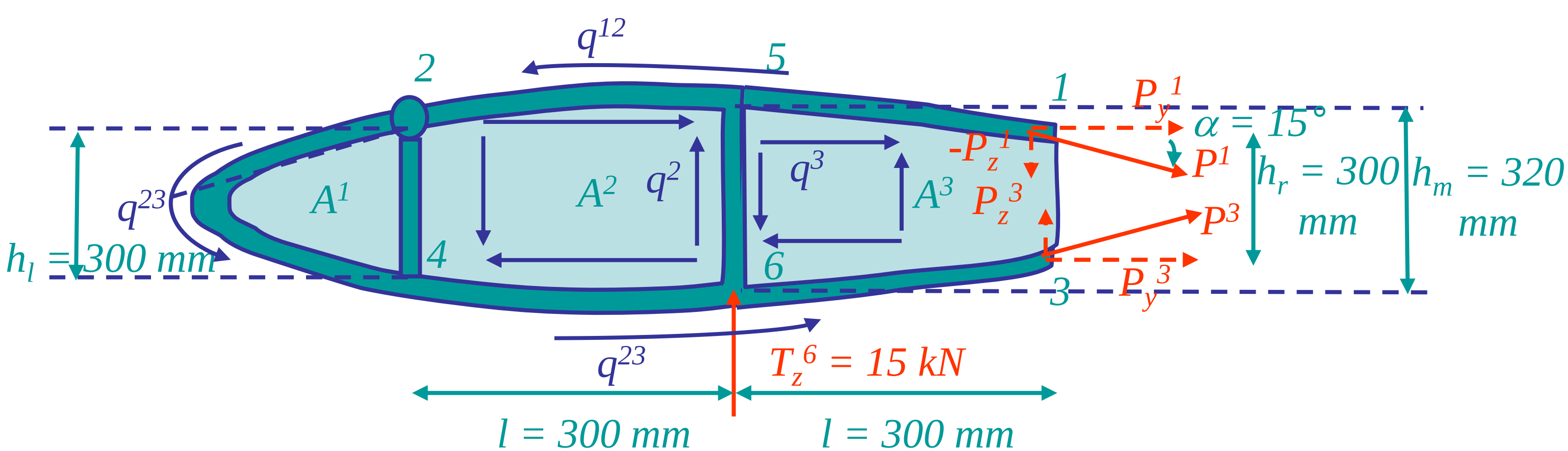
- Web panel 3 (5)
  - Loading in flange just before stiffener 13 (4)
    - Moment around Point 1 (2)

$$P_y^3 = -12.2\,10^3 \text{ N}$$

$$\Longrightarrow P_z^3 = P_y^3 \tan 15\,\text{deg} = -12.2\,10^3 \;\tan 15\,\text{deg} = -3.28\,10^3 \text{ N}$$

$$\Longrightarrow P^3 = \frac{P_y^3}{\cos 15\,\text{deg}} = -\frac{12.2\,10^3}{\cos 15\,\text{deg}} = -12.7\,10^3 \text{ N}$$

  - Compression in flange at point 3

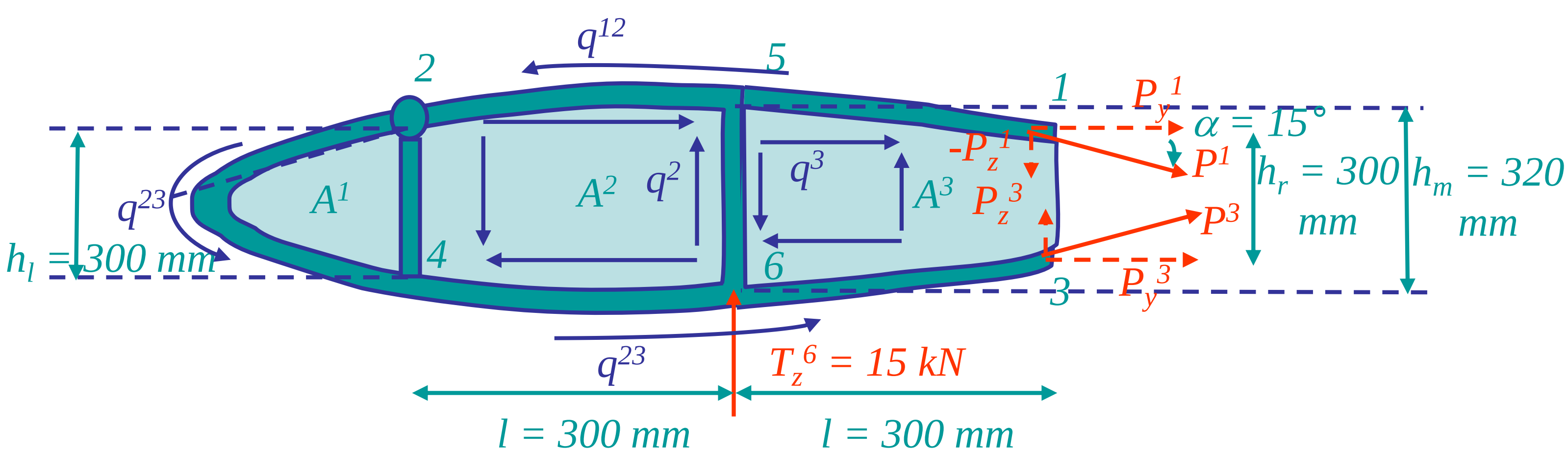


- Web panel 3 (6)
  - Shearing in web panel 3 just before stiffener 13
    - Equilibrium of the left part

$$q^3 h_r + P_z^1 + P_z^3 - q^{23} h_l + T_z^6 = 0 \Longrightarrow q^3 = \frac{-T_z^6 + q^{23} h_l - P_z^1 - P_z^3}{h_r}$$

$$\Longrightarrow q^3 = \frac{-15\,10^3 + 17.2\,10^3\,0.3 - 39 + 3.28\,10^3}{0.3}$$

$$\Longrightarrow q^3 = -22\,10^3\ \mathrm{N \cdot m^{-1}}$$



Annex 2: Stiffener/web construction

- Web panel 3 (7)
  - Shearing in web panel 3 just before stiffener 13 (2)
    - Verification: Equilibrium of the right part
      - Vertical equilibrium

$$q^3 h_r + P_z^1 + P_z^3 = q^{31} h_r$$

$$\Longrightarrow q^3 = q^{31} - \frac{P_z^1 + P_z^3}{0.3}$$

$$\Longrightarrow q^3 = -32.8\,10^3 - \frac{39 - 3.28\,10^3}{0.3} = -22\,10^3\ \mathrm{N \cdot m^{-1}}$$

      - Horizontal equilibrium

$$P_y^1 = -146\ \mathrm{N}$$?

$$P_y^3 = -12.2\,10^3\ \mathrm{N}$$